

Permutation-Equivariant Quantum K-theory of the Quintic Singularity

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Abstract

We compute the genus-0 permutation-equivariant quantum K-theory of the singularity of the quintic polynomial, parallel to Givental–Lee’s quantum K-theory of the quintic threefold. We obtain a generating function which recovers the entire 25-dimensional space of solutions to the q -difference equation satisfied by the permutation-equivariant K-theoretic I -function of the quintic.

Introduction

The quintic threefold occupies a central place in Gromov–Witten (GW) theory since its very definition [CDLOGP91]. Still, even if major mirror symmetry conjectures about it have been proven [Giv96, LLY99], and the genus $g = 1$ and 2 have been successfully treated [Zin08, GJR17], the full Gromov–Witten potential remains unknown. Several methods to compute GW invariants have been found [MP06, CLLL22, ABPZ23], but it is still impossible to reach the conjectural formulae by Huang–Klemm–Quackenbush for $g \leq 51$ [HKQ09]. Remarkably, we even lack conjectures beyond $g = 51$.

In order to overcome this difficulty, following ideas of Witten [Wit93], an alternative approach has been extensively developed in the last decades. In simple terms, instead of focusing on the hypersurface X within the complex projective space $\mathbb{P}^4$, we regard the associated affine cone yielding a singularity at the origin of $\mathbb{A}^5$. The quantum theory of this singularity, referred to in physics literature as the Landau–Ginzburg model, is the so-called FJRW cohomological field theory (Co-hFT), constructed by Fan, Jarvis and Ruan in the analytic category [FJR13], and by Polishchuk and Vaintrob in algebro-geometric terms [PV16]. As Witten explains, this new point of view arises from a change of stability condition in geometric invariant theory for the action of a reductive group G on a vector space enriched with a G -equivariant complex-valued function W . This is a general setup referred to as the gauged linear sigma model (GLSM), which in principle recovers the geometry of the quintic hypersurface in $\mathbb{P}^4$ and that of the singularity in $\mathbb{A}^5$ via a change of stability condition. Therefore, it is natural to expect the so-called Landau–Ginzburg/Calabi–Yau (LG/CY) correspondence between the FJRW invariants of the singularity (LG side) and the GW invariants of the quintic (CY side). Interestingly, the two theories are radically different in their moduli spaces and in the cohomology classes involved; therefore, they are expected to shed new light to each other. This idea found confirmation in a series of results, among which we can mention a few that will play a role in the development illustrated here. Chiodo, Ruan and Iritani [CR10, CIR14]

casted it within the framework of the LG/CY correspondence, compatibly with Orlov equivalence. Fan–Jarvis–Ruan [FJR18] constructed a mathematical theory of the GLSM, and Chang–Li–Li–Liu [CLLL22] provided an algorithm computing both GW and FJRW invariants.

The goal of this paper is to extend these methods to quantum K-theory, an analogue of GW theory introduced by A. Givental and Y.-P. Lee [Lee04]. It turns out that on the singularity side the relevant K-theoretic invariant are already defined: the definition of the virtual class in FJRW theory factors through K-theory since its early definition. The first example was the theory of the A_{r-1} singularity which was identified to the theory of r th roots $\mathcal{L}$ of the (log) canonical bundle; there, Polishchuk–Vaintrob [PV01] and Chiodo [Chi06] provided a definition of the relevant intersection numbers directly in K-theory. The FJRW algebraic construction of [PV16] is a generalisation of these and, as argued by Guéré [Gué23, §3, “The K-theoretic FJRW theory”], it is essentially a K-class. In genus 0, in the case of the quintic hypersurface, such a K-class boils down to the Euler characteristics of the K-theoretic Euler class λ_{-1} of the vector bundle $R^1\pi_*\mathcal{L}^{\oplus 5}$, where $\pi: \mathcal{C} \rightarrow \mathcal{M}$ is the universal (twisted) curve on the moduli space $\mathcal{M}$ of 5th roots $\mathcal{L}$ of the (log) canonical bundle, see (2). In this paper, we deal with a refinement of quantum K-theory introduced by Givental under the name of *permutation-equivariant quantum K-theory* [Giv04]: its invariants encode not only the Euler characteristics, but rather the full S_n -module structure of the virtual fundamental sheaf (see Definition 1.12).

For these permutation-equivariant K-theoretic invariants we provide a full computation encoded in a generating function I_{FJRW}^K , which recovers the entire space of solutions of the q -difference equation satisfied by the generating function I_{GW}^K of the quantum K-theoretic invariants of the quintic (see [Giv04])

$$\left[(1 - q^{Q\partial_Q})^5 - Q \prod_{k=1}^5 (1 - q^{5Q\partial_Q+k}) \right] P^{l_q(Q)} I_{\text{GW}}^K(q, Q) = 0. \quad (1)$$

This is an interesting improvement over the case of the quintic threefold, where the generating function I_{GW}^K only provides a 5-dimensional subspace of solutions of the degree-25 equation (1). Here, 25-dimensions are spanned, and this is made possible by the fact that the FJRW invariants are naturally defined over the 25-dimensional state space $\mathbb{C}[\hat{\mu}_5]^5$; *i.e.*, the K-theory of the inertia stack of the target $B\mu_5$. Furthermore, these functions match the basis of solutions already identified by Yaoxiong Wen via an analytic continuation of I_{GW}^K [Wen22].

A large scale picture beyond the case of the quintic incorporating permutation-equivariant quantum K-theory, mirror symmetry and the LG/CY correspondence is still lacking, but it is also getting more and more precise thanks to a number of recent papers over the past ten years. We refer to the work of Konstantin Aleshkin and Melissa Liu [AL22, AL23] and references therein. There, the authors used the above mentioned framework of GLSM in order to deduce I -functions satisfying q -difference equations. Under some Calabi–Yau conditions they obtain wall crossing statements [AL22, Thm. 4.3, Prop. 4.6] similar to the LG/CY correspondence. One can hope that this approach and the present paper can contribute to recast the permutation-equivariant K-theory in a global mirror symmetry framework as it happens for Gromov–Witten theory in [CIR14].

Overview of the main results. We set up the theory for a CY hypersurface in $\mathbb{P}^{r-1}$. The K-theoretic FJRW invariants of the polynomial $X_0^r + \dots + X_r^r$ are defined via the moduli space $\overline{\mathcal{M}}_{0,n}^r$ of r -spin curves, which parametrizes twisted curves together with an r th root $\mathcal{L}$ of the log-canonical bundle $\omega_{\log}$. The universal curve $\pi: \overline{\mathcal{C}}_{0,n}^r \rightarrow \overline{\mathcal{M}}_{0,n}^r$ carries a universal r th root $\mathcal{L}$, and we define the

virtual class as the K-theoretic Euler class of the higher direct image of $\mathcal{L}$

$$\lambda_{-1} (R^1 \pi_* \mathcal{L}^{\oplus r}) \in K^0(\overline{\mathcal{M}}_{0,n}^r). \quad (2)$$

The K-theoretic FJRW invariants are defined as the Euler characteristics of this K-theoretic class

$$\langle \phi_{a_1}, \dots, \phi_{a_n} \rangle_{0,n} = \chi \left(\overline{\mathcal{M}}_{0,\underline{a}}^r; (\lambda_{-1} R^1 \pi_* \mathcal{L}_{\underline{a}})^{\otimes r} \right),$$

or, in their permutation-equivariant version, as the S_n -module $H^* \left(\overline{\mathcal{M}}_{0,n}^r; (\lambda_{-1} R^1 \pi_* \mathcal{L})^{\otimes r} \right)$.

The computation of these invariants is parallel to Givental–Tonita’s computation of the genus-0 quantum K-theory [GT11, Givb] via Lefschetz’s trace formula, which is an instance of the Grothendieck–Riemann–Roch theorem for stacks [Kaw79, Toe99]. If g is a finite-order automorphism of a smooth proper stack X , and F is an equivariant coherent sheaf, then the trace of g on the cohomology groups of F is given by

$$\mathrm{tr}_g(H^*(X, F)) = \int_{X^g} \mathrm{ch} \left(\frac{\mathrm{Tr}(F)}{\mathrm{Tr}(\lambda_{-1} \mathcal{N}^\vee)} \right) \mathrm{td}(T),$$

where X^g is the fixed-points stack, $\mathcal{N}$ is the normal bundle to the map $X^g \rightarrow X$, $\mathrm{Tr}(F)$ is the trace bundle, and $\mathrm{td}(T)$ is the Todd class of tangent sheaf. This formula is used to compute all the K-FJRW invariants by recursion on the number of markings. Indeed, the integral above takes place on the fixed-points stack of $\overline{\mathcal{M}}_{0,n}^r$, which is the disjoint union of r components of dimension $\dim \overline{\mathcal{M}}_{0,n} = n - 3$, and other lower-dimensional boundary strata with the usual Deligne–Mumford recursive structure. Such a recursive structure is the core of Givental’s formalism where the invariants take the form of a generating function in a polarized symplectic space. Thus, the full computation derives from inserting the top-dimensional classes $\mathrm{ch}(\lambda_{-\xi} R^1(\pi_* \mathcal{L}^{\oplus r}))$ in Givental’s formalism. These classes are usually referred to as the fake quantum K-theory, and come here in r variants indexed by $\xi \in \mu_r$. Givental’s symplectic space is $\mathcal{K} = K^0(\mathcal{IB}\mu_r)(q^{1/r})$, and the generating function $J : \mathcal{K}_+ \rightarrow \mathcal{K}$ is defined by

$$J(t(q^{1/r})) = 1 - q + t(q^{1/r}) + \sum_{a,\xi} \sum_{n \geq 2} \frac{r}{\mathfrak{e}(a)} \phi^a \otimes e_\xi \left\langle \frac{\phi_a \otimes e_\xi}{1 - q^{1/\mathfrak{e}(a)} \mathcal{L}_0}, t(\mathcal{L}_1), \dots, t(\mathcal{L}_n) \right\rangle_{0,n+1}^{S_n}.$$

The state space also carries a natural μ_r -action, and we obtain an analogue of Givental–Tonita’s adelic characterization theorem [GT11].

Theorem A. *Let f be a μ_r -invariant element of $\mathcal{K}$. Then f lies in the image of the J function if and only if*

- f has poles only at $q = 0, \infty$, and at the roots of unity;
- the expansion of f at $q = 1$ is a value of the fake J -function;
- for all $\xi_0 \in \mu_\infty$ such that ξ_0^r has order m , we have

$$\Phi_{\xi_0} \left(f(q^{\frac{1}{rm}}) \right) \in \square_{\xi_0} \Delta^{-1} \mathcal{T} L^{\mathrm{fake}}$$

where $\mathcal{T}L^{\text{fake}}$ is the tangent space at the point $\Phi_0 f \in L^{\text{fake}}$, Δ is the operator of the fake theory (Proposition 2.4), and $\square_{\xi_0}$ is defined by

$$\begin{aligned}\square_{\xi_0}(\phi_{a+rl} \otimes e_{\xi}) &= \exp \left(r \sum_{k \geq 1} \frac{s^k \xi^k \xi_0^{rk}}{k} \frac{q^{k \frac{a}{rm}}}{\xi_0^{-rk} q^{k/m} - 1} \right) \phi_{a+rl} \otimes e_{\xi} \quad \text{if } 1 \leq a \leq r-1, \text{ and} \\ \square_{\xi_0}(\phi_{rl} \otimes e_{\xi}) &= \exp \left(r \sum_{k \geq 1} \frac{s^k \xi^k \xi_0^{rk}}{k} \frac{q^{\frac{k}{m}}}{\xi_0^{-rk} q^{k/m} - 1} \right) \phi_{rl} \otimes e_{\xi}.\end{aligned}$$

We deduce a simpler characterization of the J -function in terms of untwisted invariants.

Corollary. *Let L_{FJRW}^K denote the image of the J -function, and let L_{un}^K denote the range of the untwisted J -function. Then the μ_r -invariant points of L_{FJRW}^K and L_{un}^K coincide*

$$(L_{\text{FJRW}}^K)^{\mu_r} = \Delta(L_{\text{un}}^K)^{\mu_r}.$$

This formulation is quite useful because the untwisted invariants coincide with the permutation-equivariant quantum K-theory of a point, explicitly computed by Givental [Giva]. In particular, we deduce from it the following theorem.

Theorem B. *The following function lies on L_{FJRW}^K*

$$I_{\text{FJRW}}^K(x, q) = (1 - q) \sum_{\xi \in \mu_r} \sum_{a=0}^{r-1} \sum_{n \geq 0} \frac{\prod_{0 \leq k < n} \left(1 - \xi q^{\frac{a+1}{r} + k} \right)^r}{\prod_{k=1}^{rn+a} (1 - q^k)} x^{rn+a} \phi_{a+1} \otimes e_{\xi}.$$

If $r = 5$, we decompose the I -function into 25 functions

$$I_{\text{FJRW}}^K(x, q) = \sum_{a, \xi} x^a I_{a, \xi}(x, q) \phi_{a+1} \otimes e_{\xi}.$$

We introduce the modification

$$\tilde{I}_{a, \xi}(x, q^{-1}) = e_{q, q^{\frac{a+1}{r}} \xi^{-1}}(x) I_{a, \xi}(x^{1/5}, q^{-1}).$$

Then the functions $\tilde{I}_{a, \xi}(x, q^{-1})$ form a basis of solutions of the q -difference equation

$$\left[\prod_{k=1}^5 (1 - q^{-k+x\partial_x}) - x q^{10+20x\partial_x} (1 - q^{x\partial_x})^5 \right] I = 0, \quad (3)$$

which is (1) after the change of variable $x = Q^{-1}$.

Outline of the paper. In the first section, we recall the definition of the moduli space of r -spin curves, and we define the permutation-equivariant K-theoretic FJRW invariants. In the second section, we define and compute the fake invariants by using Chiodo-Zvonkine's theorem [CZ09]. In the third section, we define and compute the spine CohFT, which is another building block of the FJRW invariants, related to r -spin curves with symmetries. In the fourth section we use Lefschetz's formula to prove the adelic characterization theorem, which recursively determines the FJRW invariants. In the last section we give an alternative description of the J -function in terms of untwisted invariants, and we use it to find a point I_{FJRW} in the image of the J -function.

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Notations and conventions

All schemes and stacks are of finite type over $\mathbb{C}$. The Chow rings are taken with rational or complex coefficients

- $\overline{\mathcal{M}}_{0,n}^r, \widetilde{\mathcal{M}}_{0,n}^r$: moduli stack of r -spin curves, and moduli space of r -spin curves with trivialized marked points,
- $\overline{\mathcal{M}}_{0,n}^{r,s}$: moduli stack of r th roots of $\omega_{\log}^{\otimes s}$,
- $\mathcal{L}$: universal r th root of $\omega_{\log}$,
- $\mathcal{L}_i$: tautological line bundles over $\widetilde{\mathcal{M}}_{0,n}^r$,
- $\mathcal{I}X$: inertia stack of X ,
- $\underline{a}, \underline{b}, \underline{\gamma}$: multi-indices,
- BG : classifying stack of the group G ,
- ϵ ramification index and cardinality of isotropy group,
- $f_{(\xi)}$: if f is a Laurent polynomial in $q^{1/r}$, $f_{(\xi)}$ is the expansion of f at $q^{1/r} = \xi$.

1 Defining the invariants

In this section we give the definition of the FJRW invariants in the permutation-equivariant case. We first define a K -theoretic class Λ over the moduli space of r -spin curves. The symmetric group S_n acts by permuting the marked points, and the class Λ is S_n -equivariant in a natural way. Thus, the cohomology groups $H^*(\Lambda)$ form S_n -modules. To encode this representation in a way independent of n , we follow Getzler–Kapranov [GK98] and Givental [Giva] and use Schur–Weyl duality. The representation $H^*(\Lambda)$ of S_n is encoded in a symmetric function in infinitely many variables, which we take as a definition of the FJRW invariant. More generally, we give a definition of the FJRW invariants with value in a chosen λ -ring R .

This procedure allows us to define generating functions for the FJRW invariants.

1.1 Spin curves and the fundamental class

Definition 1.1. Let X be a Deligne–Mumford stack, and let $\mathcal{L}$ be a line bundle over X . We say that $\mathcal{L}$ is *faithful* if, for any geometric point x of X , the induced representation $\text{Aut}(x) \rightarrow \mathbb{G}_m$ is faithful.

If X is of finite type, $\mathcal{L}$ is faithful if and only if the induced morphism $X \rightarrow B\mathbb{G}_m$ is representable (see [AV01, 4.4.3.]).

Definition 1.2. Let $n \geq 3$ and $r \geq 1$ be integers. An r -spin curve with n marked points is the data of $(C, (\Sigma_1, \dots, \Sigma_n), \mathcal{L}, \alpha)$, where

- $(C, (\Sigma_1, \dots, \Sigma_n))$ is a balanced twisted curve in the sense of [ACV03],
- $\mathcal{L}$ is a faithful line bundle over C ,
- $\alpha : \mathcal{L}^{\otimes r} \rightarrow \omega_{\log}$ is an isomorphism.

The moduli stack of r -spin curves $\overline{\mathcal{M}}_{g,n}^r$ is the stack classifying families of r -spin curves. There is a universal curve $\pi : \overline{\mathcal{C}}_{g,n}^r \rightarrow \overline{\mathcal{M}}_{g,n}^r$, and a universal line bundle $\mathcal{L}_{g,n} \rightarrow \overline{\mathcal{C}}_{g,n}^r$. The marked points $\Sigma_i \subset \overline{\mathcal{C}}_{g,n}^r$ are closed substacks of the universal curve.

We also define the stack $\widetilde{\mathcal{M}}_{0,n}^r$ classifying r -spin curves together with a section σ_i at each marked point, namely,

$$\widetilde{\mathcal{M}}_{0,n}^r = \Sigma_1 \times_{\overline{\mathcal{M}}_{g,n}^r} \cdots \times_{\overline{\mathcal{M}}_{g,n}^r} \Sigma_n.$$

The forgetful map $p : \widetilde{\mathcal{M}}_{g,n}^r \rightarrow \overline{\mathcal{M}}_{g,n}^r$ is a μ_r^n -gerbe. The universal curve of $\widetilde{\mathcal{M}}_{0,n}^r$ is $\widetilde{\mathcal{C}}_{g,n}^r = \overline{\mathcal{C}}_{g,n}^r \times_{\overline{\mathcal{M}}_{g,n}^r} \widetilde{\mathcal{M}}_{g,n}^r$, and we still denote by $\pi : \widetilde{\mathcal{C}}_{g,n}^r \rightarrow \widetilde{\mathcal{M}}_{g,n}^r$ the projection. The stack $\widetilde{\mathcal{M}}_{0,n}^r$ is equipped with the tautological line bundles $\mathcal{L}_i = \sigma_i^* \omega_\pi$, where ω_π is the dualizing line bundle.

1.1.1 Multiplicities

At each marked point x_i of an r -spin curve C , the stabilizer is canonically isomorphic to μ_{e_i} , for some $e_i | r$. The line bundle $\mathcal{L}|_{x_i}$ is a representation of μ_{e_i} , given by some element $d_i \in \mathbb{Z}_{e_i}$, with $d_i \wedge e_i = 1$. We refer to $a_i := d_i \frac{r}{e_i} \in \mathbb{Z}/r\mathbb{Z}$ as the *multiplicity* of $\mathcal{L}$ at x_i , and we denote by $\text{mult}_{x_i}(\mathcal{L})$. We associate to each r spin curve the multi-index of its multiplicities $\underline{a} = (a_1, \dots, a_n) \in (\mathbb{Z}/r\mathbb{Z})^n$. The marked points with multiplicity 0 are called *broad marked points*.

Definition 1.3. Let $\underline{a} \in (\mathbb{Z}/r\mathbb{Z})^n$ be a multi-index, and let $\overline{\mathcal{M}}_{g,\underline{a}}^r$ (resp. $\widetilde{\mathcal{M}}_{g,\underline{a}}^r$) be the stack of r -spin curves $(C, \mathcal{L})$ such that the multiplicities of $\mathcal{L}$ are given by $\underline{a}$. Then $\overline{\mathcal{M}}_{g,\underline{a}}^r$ (resp. $\widetilde{\mathcal{M}}_{g,\underline{a}}^r$) is a connected component of $\overline{\mathcal{M}}_{g,n}^r$ (resp. $\widetilde{\mathcal{M}}_{g,n}^r$).

Over $\widetilde{\mathcal{M}}_{\underline{a}}^r$, the pullbacks $\sigma_i^*(\omega_{\log})$ are canonically trivial by the residue map. Thus, $\sigma_i^* \mathcal{L}$ is an r -th root of the trivial bundle, and defines a morphism $\widetilde{\mathcal{M}}_{\underline{a}}^r \rightarrow B\mu_r$. By taking the multiplicities into account, we define the evaluation morphism

$$\text{ev}_i : \widetilde{\mathcal{M}}^r \rightarrow \mathcal{I}B\mu_r = \bigsqcup_{a=0}^{r-1} B\mu_r,$$

by sending $\widetilde{\mathcal{M}}_{0,\underline{a}}^r$ to the a_i -th copy of $B\mu_r$.

1.1.2 Fundamental class

In order to define the fundamental class, we need $R^1\pi_* \mathcal{L}$ to be a vector bundle. This is the case, for example, if $H^0(C, \mathcal{L}) = 0$ for each curve over $\text{Spec}(\mathbb{C})$. However, when a spin curve has at least 2 broad points, it may happen that $H^0(C, \mathcal{L})$ is non-zero. To fix this, we twist $\mathcal{L}$ by the divisor of broad marked points before pushing forward to $\overline{\mathcal{M}}_{0,n}^r$.

Lemma 1.4. Let $E \subset \overline{\mathcal{C}}_{\underline{a}}^r$ be the divisor of broad marked points, that is, $E = \bigsqcup_{a_i=0} \Sigma_i$. Then we have $\pi_* \mathcal{L}_{\underline{a}}(-E) = 0$, and $R^1\pi_* \mathcal{L}(-E)$ is a vector bundle over $\overline{\mathcal{M}}_{0,\underline{a}}^r$.

Proof. See [CR10, lem 4.1.1.]. □

Remark 1.5. This twist has a mild impact on the Chern classes. Indeed, we have the exact sequence

$$0 \rightarrow \mathcal{L}(-E) \rightarrow \mathcal{L} \rightarrow \mathcal{L}|_E \rightarrow 0.$$

For a broad marked point Σ_i , we have $p_*(\mathcal{L}|_{\Sigma_i}) = \sigma_i^* \mathcal{L}$, which shows that $c(p_*(\mathcal{L}|_{\Sigma_i})) = 1$ (in $A_{\mathbb{Q}}^\bullet$). Thus we have $c(R\pi_* \mathcal{L}) = c(R\pi_* \mathcal{L}(-E))$, and $\text{ch}(R\pi_* \mathcal{L}) = \text{ch}(R\pi_* \mathcal{L}(-E)) + m$, where m is the number of broad marked points.

Proposition 1.6. *Let S_n act on $\widetilde{\mathcal{M}}_{0,n}^r$ and $\overline{\mathcal{M}}_{0,n}^r$ by permuting of the marked points. Then the sheaf $R^1\pi_* \mathcal{L} \in \text{Coh}(\overline{\mathcal{M}}_{0,n}^r)$ is naturally S_n -equivariant, and so is its pullback to $\widetilde{\mathcal{M}}_{0,n}^r$.*

Remark 1.7. Note that the different connected components of $\widetilde{\mathcal{M}}_{0,n}^r$ may be permuted by the S_n -action.

Definition 1.8 (The fundamental class). The fundamental class is the S_n -equivariant K-theoretic class

$$\Lambda_n = (\lambda_{-1}(R^1\pi_* \mathcal{L}(-E)))^{\otimes r} \in K_{S_n}^0(\overline{\mathcal{M}}_{0,n}^r).$$

More generally, we define

$$\Lambda_n(s) = (\lambda_{-s}(R^1\pi_* \mathcal{L}(-E)))^{\otimes r} \in K_{S_n}^0(\overline{\mathcal{M}}_{0,n}^r)[[s]].$$

1.2 A symplectic space

Following Givental ([Giv04], [GT11]) we define an infinite-dimensional symplectic space $\mathcal{K}$, referred to as the loop space. This space has a natural polarization $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$, and is crucial to define the relevant generating functions.

Definition 1.9. The state space of the FJRW theory is $K^0(\mathcal{IB}\mu_r)_{\mathbb{C}}$. Since $\mathcal{IB}\mu_r \simeq \bigsqcup_{a=0}^{r-1} B\mu_r$, there is an isomorphism of *vector spaces*

$$K^0(\mathcal{IB}\mu_r)_{\mathbb{C}} \simeq \mathbb{C}[\mathbb{Z}_r] \otimes_{\mathbb{C}} \mathbb{C}[\widehat{\mu}_r],$$

and the product is given by

$$\phi_a \otimes [d] \cdot \phi_{a'} \otimes [d'] = \delta_{a,a'} \phi_a \otimes [d + d'].$$

The state space is equipped with the orbifold pairing twisted by the fundamental class. Explicitly, let e_{ξ} be the virtual representation

$$e_{\xi} = \frac{1}{r} \sum_{k=0}^{r-1} \xi^{-1}[k] \in K^0(B\mu_r)_{\mathbb{C}}.$$

Then, the elements $\{\phi_a \otimes e_{\xi} | ah \in \mathbb{Z}_r, \xi \in \mu_r\}$ form a basis of the state space, and the pairing is given by

$$\langle \phi_a \otimes e_{\xi}, \phi_{a'} \otimes e_{\xi'} \rangle = \begin{cases} \frac{1}{r} \delta_{a,-a'} \delta_{\xi,\xi'} & \text{if } a \neq 0, \\ \frac{(1-\xi s)^r}{r} \delta_{0,a'} \delta_{\xi,\xi'} & \text{otherwise.} \end{cases} \quad (4)$$

Finally, the Adams operation of K-theory are the ring morphisms

$$\Psi^m(\phi_a \otimes e_{\xi}) = \sum_{\zeta^m = \xi} \phi_a \otimes e_{\zeta}.$$

For each element $a \in \mathbb{Z}_r$, let $\mathfrak{e}(a)$ be the order of the subgroup generated by a in $\mathbb{Z}_r$. If $\mathcal{L}$ has multiplicity a at a marked point x_i , then its automorphism group $\text{Aut}(x_i)$ is isomorphic to $\mathbb{Z}/\mathfrak{e}(a)\mathbb{Z}$. We decompose $\mathbb{C}[\mathbb{Z}_r]$ according to this order $\mathfrak{e}$:

$$\mathbb{C}[\mathbb{Z}_r] = \bigoplus_{\mathfrak{e}|r} V_{\mathfrak{e}}, \quad (5)$$

with $V_{\mathfrak{e}} = \bigoplus_{\mathfrak{e}(a)=\mathfrak{e}} \mathbb{C} \cdot \phi_a$.

Definition 1.10. The loop space of the FJRW theory is the space of rational functions

$$\mathcal{K} = \bigoplus_{\mathfrak{e}|r} V_{\mathfrak{e}}(q^{\frac{1}{\mathfrak{e}}}) \otimes \mathbb{C}[\widehat{\boldsymbol{\mu}}_r]. \quad (6)$$

The loop space $\mathcal{K}$ is equipped with the symplectic form

$$\Omega(f, g) = [\text{Res}_{q=0} + \text{Res}_{q=\infty}] \langle f(q), g(q^{-1}) \rangle \frac{dq}{q}. \quad (7)$$

We define a polarization of this symplectic vector space by setting

$$\mathcal{K}_+ = \bigoplus_{\mathfrak{e}} V_{\mathfrak{e}} \left[q^{\frac{1}{\mathfrak{e}}}, q^{\frac{-1}{\mathfrak{e}}} \right] \quad \mathcal{K}_- = \{f \in \mathcal{K} | f(0) \neq \infty \text{ and } f(\infty) = 0\}.$$

1.3 The invariants

Definition 1.11. Let t be an element of $\mathcal{K}_+$, and $i \in \{1, \dots, n\}$. We introduce the class $t(\mathcal{L}_i) \in K^0(\widetilde{\mathcal{M}}_{0,n}^r)$, which is defined on elementary tensors by

$$E \otimes q^{j/r} \mapsto \text{ev}_i^* E \otimes \mathcal{L}_i^j.$$

For any $t \in \mathcal{K}_+$, the class $\Lambda_n(s) \otimes \bigotimes_{i=1}^n t(\mathcal{L}_i)$ is naturally an S_n -equivariant class, and its cohomology groups form an S_n -module denoted by

$$[t(\mathcal{L}_1), \dots, t(\mathcal{L}_n)]_n = H^* \left(\widetilde{\mathcal{M}}_{0,n}^r; p^* \Lambda_n(s) \otimes \bigotimes_{i=1}^n t(\mathcal{L}_i) \right).$$

More generally, let $n = k_1 + \dots + k_s$ be a partition of n , and let $H \subset S_n$ be the subgroup $S_{k_1} \times \dots \times S_{k_s}$. Let us denote $x_{i,k}$ ($i \in \{1, \dots, s\}$ and $k \in \{1, \dots, k_s\}$) the n marked points. Then, for a sequence of inputs $t^{(1)}, \dots, t^{(s)} \in \mathcal{K}_+$, the cohomology groups

$$H^* \left(\widetilde{\mathcal{M}}_{0,n}^r; p^* \Lambda_n(s) \otimes \bigotimes_{k=1}^s \bigotimes_{l=1}^{k_s} t^{(k)}(\mathcal{L}_{k,i}) \right)$$

are H -modules, denoted by

$$\left[t^{(1)}(\mathcal{L}_{1,1}), \dots, t^{(1)}(\mathcal{L}_{1,k_1}); \dots, t^{(s)}(\mathcal{L}_{s,k_s}) \right]_n.$$

In order to define a generating function for these S_n -modules, we use the ring of symmetric function ([GK98], [Giva]). Each representation of (V, ρ) of S_n yields a degree n symmetric function f_ρ (in infinitely many variables) in the following way. For each $m \geq n$, the group $\mathrm{GL}_m(\mathbb{C})$ acts on $V \otimes_{S_n} (\mathbb{C}^m)^{\otimes n}$. The symmetric function f_ρ is the function such that for all m , $f(x_1, \dots, x_m, 0, \dots)$ is the character of the GL_m -representation $V \otimes_{S_n} (\mathbb{C}^m)^{\otimes n}$. The ring of symmetric function allows us to encode representations of S_n for various n in a single ring, and thus to define generating functions.

More generally, let R be a λ -ring over $\mathbb{C}$. We assume that R is equipped with the $\mathcal{I}$ -adic topology for an ideal $\mathcal{I}$ of R , such that

- R is Hausdorff,
- for all $m \geq 0$, $\Psi^m(\mathcal{I}) \subset \mathcal{I}^m$.

In that case, the completion $\widehat{R}$ of R remains a λ -ring. The main examples are the ring of symmetric functions, and $\mathbb{C}[X]$, and $\mathcal{I}$ is the ideal of functions with positive degree.

We extend the scalar to R in Definition 1.10, and complete the resulting ring with respect to the $\mathcal{I}$ -adic topology (see [CCIT09, Appendix B] for a detailed construction of the loop space). In particular, $\mathcal{K}_+$ is made of functions t which, modulo any power of $\mathcal{I}$, are Laurent polynomials.

We now come to the definition of the FJRW invariants.

Definition 1.12. We keep the notations of Definition 1.11. For any elements $\nu_1, \dots, \nu_s \in R$, we define

$$\left\langle t^{(1)}(\mathcal{L}_{1,1}) \otimes \nu_1, \dots, t^{(s)}(\mathcal{L}_{s,k_s}) \otimes \nu_s \right\rangle_{0,n}^H = \frac{1}{\prod_i k_i!} \sum_{h \in H} \mathrm{tr}_h \left[t^{(1)}(\mathcal{L}_{1,1}), \dots, t^{(s)}(\mathcal{L}_{s,k_s}) \right]_n \prod_{i=1}^s \prod_{j=1}^{\infty} \Psi^r(\nu_i)^{l_j(h)}, \quad (8)$$

where $l_j(h)$ is the number of cycles of length j in h .

Remark 1.13. In the case where $H = S_n$, and R is the ring of symmetric functions, (8) yields the usual symmetric function associated to the S_n -module $[t(\mathcal{L}_1), \dots, t(\mathcal{L}_n)]$.

Remark 1.14 (Vanishing). With the same notations as in the previous definition, let us choose $t^{(i)} = q^{j/\mathfrak{e}(a)} \phi_a \otimes [d]$ (with $d \in \widehat{\mu}_r \simeq \mathbb{Z}_r$). Then, the invariant vanishes unless $dd_i = j \pmod{\mathfrak{e}_i}$.

Indeed, consider the forgetful map $p : \widetilde{\mathcal{M}}_{0,n}^r \rightarrow \overline{\mathcal{M}}_{0,n}^r$. The line bundle $\mathcal{L}_i^j \otimes \mathrm{ev}_i^*[d]$ carries the $\mu_{\mathfrak{e}_i}$ representation $\zeta \mapsto \zeta^{-j} \zeta^{dd_i}$, where $\mu_{\mathfrak{e}_i}$ is the group of 2-automorphisms of the section σ_i . Then, $p_*(\mathcal{L}_i^j \mathrm{ev}_i^*[d]) = 0$ if this representation is non-trivial.

Proposition 1.15 (poly-linearity). *For $t, t' \in \mathcal{K}_+$, and $\nu, \nu' \in R$, we have*

$$\langle t \otimes \nu + t' \otimes \nu', \dots, t \otimes \nu + t' \otimes \nu' \rangle_{0,n}^{S_n} = \sum_{k+l=n} \langle t \otimes \nu, \dots, t \otimes \nu, t' \otimes \nu', \dots, t' \otimes \nu' \rangle_{0,n}^{S_k \times S_l}. \quad (9)$$

We use this formula to extend Definition 1.12 to inputs $t \in \mathcal{K}_+$.

Proof. See [Giva] example 5. □

Assumption 1.16. From now on we will assume that the input t belongs to $\mathcal{JK}_+$. This ensures that the following formal series are well-defined.

Definition 1.17. The genus-0 permutation-equivariant potential is the formal function $\mathcal{F}_0$, defined over $\mathcal{JK}_+$ by

$$\mathcal{F}_0(t) = \sum_{n \geq 0} \langle t(\mathcal{L}_1), \dots, t(\mathcal{L}_n) \rangle_{0,n}^{S_n} \quad (10)$$

We also consider the mixed potential

$$\mathcal{F}_0(x, t) = \sum_{n+k \geq 3} \frac{1}{k!} \langle x(\mathcal{L}_1), \dots, x(\mathcal{L}_k), t(\mathcal{L}_{k+1}), \dots, t(\mathcal{L}_{n+k}) \rangle_{0,k+n}^{S_n}. \quad (11)$$

We now introduce the J -function, which will be defined as the derivative of the mixed potential with respect to the first variable.

Definition 1.18. The J -function is the formal function $\mathcal{JK}_+ \rightarrow \mathcal{K}$ defined by

$$J(t) = (1-q)\phi_1 \otimes [0] + t + \sum_{\substack{n \geq 2 \\ a \in \mathbb{Z}_r \\ \xi \in \mu_r}} \frac{r}{\mathfrak{e}(a)} \phi^a \otimes e_\xi \left\langle \frac{\phi_a \otimes e_\xi}{1 - q^{\frac{1}{\mathfrak{e}(a)}} \mathcal{L}_0}, t(\mathcal{L}_1), \dots, t(\mathcal{L}_n) \right\rangle_{0,n+1}^{S_n}, \quad (12)$$

with $\phi^a = \phi_{-a}$.

2 The fake theories

This section is devoted to the definition and computation of the so-called fake theories, which can be seen as building block for the K-theoretic invariants. Using the theory of twisted invariants developed by Coates, Givental, and Tonita in [Giv04],[CG07],[Ton14], and a theorem by Chiodo and Zvonkine [CZ09], we are able to fully compute these fake theories.

2.1 The fake invariants

Definition 2.1. Let $\mathcal{A}_\xi$, $\mathcal{B}$, and $\mathcal{C}$ be invertible multiplicative classes, and let Z denote the singular locus in the universal curve $\overline{\mathcal{C}}_{0,n}^r \rightarrow \overline{\mathcal{M}}_{0,n}^r$. Define the following classes

$$\begin{aligned} \mathcal{A}_{\xi,n}(\underline{a}) &= r \mathcal{A}_\xi(R\pi_* \mathcal{L}_{\underline{a}}(-E)) \in H^* \left(\overline{\mathcal{M}}_{\underline{a}}^r \right) \\ \mathcal{B}_{0,n} &= \mathcal{B} \left(\pi_* \left(\omega_{\log}^{-1} - 1 \right) \right) \\ \mathcal{C}_{0,n} &= \mathcal{C} \left(\pi_* \mathcal{O}_Z \right). \end{aligned}$$

The fake invariants are defined by

$$\left\langle \phi_{a_1} \otimes e_{\xi_1} \mathcal{L}_1^{k_1}, \dots, \phi_{a_n} \otimes e_{\xi_n} \mathcal{L}_n^{k_n} \right\rangle_{0,n}^{\text{fake}} = \begin{cases} \int_{\overline{\mathcal{M}}_{0,\underline{a}}^r} \mathcal{A}_{\xi,n}(\underline{a}) \mathcal{B}_{0,n} \mathcal{C}_{0,n} \prod_{i=1}^n \text{ch}(\mathcal{L}_i^{k_i}) & \text{if } \xi_i = \xi \forall i, \\ 0 & \text{otherwise.} \end{cases} \quad (13)$$

Remark 2.2. Notice a slight abuse of notation in the definition above. Indeed, the tautological line bundles $\mathcal{L}_i$ do not live on $\overline{\mathcal{M}}_{0,n}^r$, but rather on $\mathcal{M}_{0,n}^r$. Thus, in the definition above, $\text{ch}(\mathcal{L}_i)$ should be interpreted as $e^{\frac{\psi_i}{\epsilon_i}}$, where ψ_i is the usual ψ -class pulled back from $\overline{\mathcal{M}}_{0,n}$.

For the rest of this section, we choose

$$\begin{aligned}\mathcal{B}(L) &= \text{td}^{-1}(L) \\ \mathcal{C}(L) &= \text{td}^{-1}(L)\end{aligned}$$

With this definition, the fake invariants become

$$\left\langle \phi_{a_1} \otimes e_\xi \mathcal{L}_1^{k_1}, \dots, \phi_{a_n} \otimes e_\xi \mathcal{L}_n^{k_n} \right\rangle_{0,n}^{\text{fake}} = \int_{\overline{\mathcal{M}}_{0,\underline{a}}^r} \mathcal{A}_{\xi,n}(\underline{a}) \prod_{i=1}^n \text{ch}(\mathcal{L}_i^{k_i}) \text{td}(\mathcal{T}),$$

where $\mathcal{T}$ is the tangent space.

2.2 Fake J -functions

Following the work of Givental [Giv04] and Tonita [Ton14] we organize these invariants in a suitable generating function called the J -function.

Definition 2.3. Let $\mathcal{K}^{\text{fake}}$ be the vector space

$$\mathcal{K}^{\text{fake}} = \bigoplus_{\epsilon} V_{\epsilon} \otimes \mathbb{C}[\widehat{\mu}_r] \otimes \mathbb{C}[[q^{1/\epsilon} - 1, (q^{1/\epsilon} - 1)]], \quad (14)$$

equipped with the symplectic form

$$\Omega^{\text{fake}}(f, g) = r \text{Res}_{q^{1/r}=1} \left\langle f(q^{-1/r}), g(q^{1/r}) \right\rangle^{\mathcal{A}} \frac{dq^{1/r}}{q^{1/r}}, \quad (15)$$

where the inner product $\langle \cdot, \cdot \rangle^{\mathcal{A}}$ is defined over $\mathbb{C}[\mathbb{Z}_r] \otimes \mathbb{C}[\widehat{\mu}_r]$ by

$$\langle \phi_a \otimes e_\xi, \phi_b \otimes e_\zeta \rangle^{\mathcal{A}} = \begin{cases} \delta_{a,-b} \delta_{\xi,\zeta} & \text{if } a \neq 0, \\ \delta_{0,b} \delta_{\xi,\zeta} \mathcal{A}_{\xi}^{-1}(\mathcal{O}) & \text{otherwise.} \end{cases}$$

The dual of the element $\phi_a \otimes e_\xi$ is $\phi^a \otimes e_\xi = \phi_{-a} \otimes e_\xi$. We equip this symplectic space with the polarization

$$\begin{aligned}\mathcal{K}_+^{\text{fake}} &= \bigoplus_{\epsilon} V_{\epsilon} \otimes \mathbb{C}[\widehat{\mu}_r][[q^{1/\epsilon} - 1]], \\ \mathcal{K}_-^{\text{fake}} &= \bigoplus_{\epsilon} V_{\epsilon} \otimes \mathbb{C}[\widehat{\mu}_r][(q^{1/\epsilon} - 1)^{-1}].\end{aligned}$$

The potential of the fake theory is the formal function defined on $\mathcal{K}_+^{\text{fake}}$ by

$$\mathcal{F}^{\text{fake}}(t) = \sum_{n \geq 3} \frac{1}{n!} \langle t(\mathcal{L}_1), \dots, t(\mathcal{L}_n) \rangle_{0,n}^{\text{fake}}. \quad (16)$$

The fake J -function is the shifted graph of the differential of $\mathcal{F}$ inside $\mathcal{K}^{\text{fake}}$

$$J(t) = 1 - q + t + \sum_{\substack{n \geq 2 \\ a \in \mathbb{Z}_r}} \frac{\phi^a \otimes e_\xi}{\epsilon(a)n!} \left\langle \frac{\phi_a \otimes e_\xi}{1 - q^{1/\epsilon(a)} \mathcal{L}_0}, t(\mathcal{L}_1), \dots, t(\mathcal{L}_n) \right\rangle_{0,n+1}^{\text{fake}}. \quad (17)$$

2.3 Lagrangian cones

The image of the J -function is a Lagrangian cone in $\mathcal{K}^{\text{fake}}$, which can be explicitly computed, as we now explain. The collection of classes $\mathcal{A}_{\xi,n}(\underline{a})$ form the genus-0 part of a CohFT over the state space $\mathbb{C}[\mathbb{Z}_r] \otimes \mathbb{C}[\widehat{\mu}_r]$. Its associated Lagrangian cone $L^{\mathcal{A}}$ (see [Giv04]) lies in the symplectic space $(\mathcal{H}^{\mathcal{A}}, \Omega^{\mathcal{A}})$ given by

$$\mathcal{H}^{\mathcal{A}} = V[[z, z^{-1}]] \quad \Omega^{\mathcal{A}}(f(z), g(z)) = \text{Res}_{z=0} \langle f(-z), g(z) \rangle^{\mathcal{A}} dz$$

For $\mathcal{A}_{\xi} = 1$, the resulting cone is called the *untwisted cone* L^{un} , and can be explicitly computed from the cohomological J -function of a point.

Proposition 2.4 (Chiodo–Zvonkine [CZ09]). *Let $w_{\xi}(z) = \sum_{g \geq 0} w_{d,\xi} z^d \in \mathbb{C}[[z]]$ be power series, and let $\mathcal{A}_{\xi}$ be the multiplicative classes*

$$\mathcal{A}_{\xi}(E) = \exp \left(\sum_{d \geq 0} w_{d,\xi} \text{ch}_d(E) \right). \quad (18)$$

Let Δ be the operator acting on $\mathcal{H}^{\mathcal{A}}$ such that for all $0 \leq a \leq r-1$ we have

$$\Delta(\phi_{a+1} \otimes e_{\xi}) = \exp \left(\sum_d w_{d,\xi} \frac{B_{d+1}(\frac{a+1}{r})}{(d+1)!} z^d \right) \phi_{a+1}. \quad (19)$$

Then we have

$$\Delta L^{\text{un}} = L^{\mathcal{A}}. \quad (20)$$

Remark 2.5. The shift in the definition of Δ happens because of the twist by the divisor of broad marked points $\mathcal{L}(-E)$.

Finally, it is a consequence of [Ton14] that the cones L^{fake} and $L^{\mathcal{A}}$ coincide.

Proposition 2.6 ([Ton14]). *Let ch be the morphism*

$$\begin{aligned} \text{ch} : \mathcal{K}^{\text{fake}} &\rightarrow \mathcal{H}^{\mathcal{A}} \\ q^{j/r} \phi_a &\mapsto e^{jz/r} \phi_a. \end{aligned}$$

Then we have

$$L^{\text{fake}} = \text{ch}^{-1}(L^{\mathcal{A}}). \quad (21)$$

Since ch is an isomorphism, we identify L^{fake} and $L^{\mathcal{A}}$, and write $L^{\text{fake}} = L^{\mathcal{A}}$. We now apply these results to the classes

$$\mathcal{A}_{\xi}(E) = \text{ch}(\lambda_{-s\xi-1} E)^{-r}.$$

Proposition 2.7. *For $\mathcal{A}_{\xi}$ as above, we have*

$$w_{d,\xi} = \sum_{k \geq 1} \frac{s^k \xi^k k^d}{k},$$

and

$$\Delta(\phi_a \otimes e_{\xi}) = \begin{cases} \exp \left(\sum_{k \geq 1} \frac{s^k \xi^k}{k} \frac{q^{\frac{ka}{r}}}{q^{\frac{k}{r}-1}} \right) \phi_a \otimes e_{\xi} & \text{if } a \neq 0, \\ \exp \left(\sum_{k \geq 1} \frac{s^k \xi^k}{k} \frac{q^k}{q^{k-1}} \right) \phi_0 \otimes e_{\xi} & \text{otherwise.} \end{cases}$$

2.4 An extension of the fake theory

In order to deal with permutations of the marked points in the next section, we need to generalize slightly the previous definition to include rm th roots of $\omega_{\log}^{\otimes m}$, and rm th roots of unity. Indeed, an r -spin curve with an automorphism of order m naturally yields an rm -root of $\omega_{\log}$ on the quotient curve (see Section 4.5). The moduli space of rm th roots of $\omega_{\log}^{\otimes m}$ is $\overline{\mathcal{M}}_{0,n}^{rm,m}$, and has a forgetful map $\epsilon : \overline{\mathcal{M}}_{0,n}^{rm,m} \rightarrow \overline{\mathcal{M}}_{0,n}$. The universal curve $\overline{\mathcal{C}}_{0,n}^{rm,m}$ carries the universal rm th root $\mathcal{L}$.

Definition 2.8. For multiplicative classes $\mathcal{A}_\xi = \exp(\sum w_{d,\xi} \text{ch}_d)$, $\xi \in \mu_{rm}$, and $\underline{a} \in (\mathbb{Z}/rm\mathbb{Z})^n$, we define

$$\mathcal{A}_{\xi,n}(\underline{a}) = rm\epsilon_* \mathcal{A}_\xi(R\pi_* \mathcal{L}_{\underline{a}}(-E)), \quad (22)$$

where $\mathcal{L}_{\underline{a}}$ is the universal rm th root of $\omega_{\log}^m$, with multiplicity $\underline{a}$. These classes form a genus-0 CohFT over the state space $W = \mathbb{C}[\mathbb{Z}_{rm}] \otimes \mathbb{C}[\widehat{\mu}_{rm}]$. We write W as the direct sum

$$\mathbb{C}[\mathbb{Z}/rm\mathbb{Z}] = \bigoplus_{\mathfrak{e} | rm} W_{\mathfrak{e}},$$

where $W_{\mathfrak{e}}$ is spanned by the basis elements ϕ_a such that the order of a in $\mathbb{Z}/r\mathbb{Z}$ is $\mathfrak{e}$.

For the remaining part of this article, we fix

$$\mathcal{A}_\xi(E) = \text{ch}(\lambda_{-s\xi} E)^{-r},$$

for $\xi \in \mu_{rm}$. By [CZ07], the associated Lagrangian cone of this CohFT is equal to ΔL^{un} , with

$$\begin{aligned} \Delta(\phi_{a+1} \otimes e_\xi) &= \exp \left(r \sum_{k \geq 1} \frac{s^k \xi^k k^d}{k(d+1)!} z^d B_{d+1} \left(\frac{a+1}{rm} \right) \right) \phi_{a+1} \otimes e_\xi \\ &= \exp \left(r \sum_{k \geq 1} \frac{(s\xi)^k}{k} \frac{q^{\frac{k(a+1)}{rm}}}{q^k - 1} \right) \phi_{a+1} \otimes e_\xi \quad \text{for } 0 \leq a \leq rm - 1 \end{aligned}$$

We extend the fake invariants to $\mathbb{C}[\mathbb{Z}_{rm}] \otimes \mathbb{C}[\widehat{\mu}_{rm}]$ by setting

$$\left\langle \phi_{a_1} \otimes e_{\xi_1} \mathcal{L}_1^{k_1}, \dots, \phi_{a_n} \otimes e_{\xi_n} \mathcal{L}_n^{k_n} \right\rangle_{0,n}^{\text{fake}} = \begin{cases} \int_{\overline{\mathcal{M}}_{0,\underline{a}}}^r \mathcal{A}_{\xi,n}(\underline{a}) \mathcal{B}_n \mathcal{C}_n \prod_{i=1}^n \text{ch}(\mathcal{L}_i^{k_i}) & \text{if } \xi_i = \xi \forall i, \\ 0 & \text{otherwise.} \end{cases} \quad (23)$$

The associated Lagrangian cone L^{fake} lies in the symplectic space

$$\mathcal{K}_{rm} = \mathbb{C}[\mathbb{Z}_{rm}] \otimes \mathbb{C}[\widehat{\mu}_{rm}] \llbracket q-1, (q-1)^{-1} \rrbracket,$$

By [Ton14], the polarization of $\mathcal{K}_{rm}$ is given by

$$\begin{aligned} (\mathcal{K}_{rm})_+ &= \bigoplus_{\mathfrak{e}} W_{\mathfrak{e}} \otimes \mathbb{C}[\widehat{\mu}_r] \llbracket 1 - q^{1/\mathfrak{e}} \rrbracket \\ (\mathcal{K}_{rm})_- &= \bigoplus_e W_{\mathfrak{e}} \otimes \mathbb{C}[\widehat{\mu}_r] \left[\left(1 - q^{1/\mathfrak{e}} \right)^{-1} \right], \end{aligned}$$

where $\mathfrak{e}(a)$ denotes the order of a in $\mathbb{Z}_r$. With this choice of polarization, we have

$$L^{\text{fake}} = \Delta L^{\text{un}}.$$

We still denote L^{fake} the Lagrangian cone associated to the fake invariants. This notation is justified by the following proposition.

Proposition 2.9. *Let Φ_0 be the inclusion morphism*

$$\begin{aligned} \Phi_0 : \mathcal{K}^{\text{fake}} &\rightarrow \mathcal{K}_{rm} \\ \phi_a \otimes e_\xi &\mapsto \phi_{ma} \otimes e_\xi. \end{aligned}$$

Then Φ_0 is an isomorphism of polarized symplectic spaces onto its image, and we have $\Phi_0(L^{\text{fake}}) \subset L^{\text{fake}}$.

3 The spine CohFT

This section is devoted to the definition and computation of the spine CohFT. This CohFT is designed to reproduce the moduli space of heads (see 4.8), and recover the spine contribution of Section 4. Given an r -spin curve $(C, \mathcal{L})$ with an automorphism $g \in \text{Aut}(C)$ of order m and an isomorphism $\phi : g^*\mathcal{L} \rightarrow \mathcal{L}$ (compatible with the spin structure) there is a line bundle $\tilde{\mathcal{L}}$ on the quotient curve $D = C/g$ constructed by descent. The line bundle is canonically an rm th root of $\omega_{\log}^{\otimes m}$ on D . Thus, D is equipped with this rm th root $\tilde{\mathcal{L}}$, and the m th root T of the trivial bundle corresponding to the $\mathbb{Z}_m$ -cover $C \rightarrow D$. We take this situation as a definition, and we consider the moduli space

$$\overline{\mathcal{M}}_0^{rm}(B\mathbb{Z}_m; \underline{a}, \underline{b}) = \overline{\mathcal{M}}_{0, \underline{a}}^{rm, m} \times_{\overline{\mathcal{M}}_{0, n}} \overline{\mathcal{M}}_{0, \underline{b}}^{m, 0}.$$

Objects of this stack are twisted curves with an rm th root $\tilde{\mathcal{L}}$ of $\omega_{\log}^m$, and an m th root T of the trivial line bundle. The universal curve carries two line bundles: $\tilde{\mathcal{L}}$ and T . For a collection of multiplicative classes $\mathcal{A}_0, \dots, \mathcal{A}_{m-1}$, we construct the spine CohFT with the classes

$$\Lambda^{\text{spine}}(\underline{a}, \underline{b}) := rm^2 \prod_{j=0}^{m-1} \mathcal{A}_j(R\pi_* \tilde{\mathcal{L}} \otimes T^j) \in H^*(\overline{\mathcal{M}}_0^{rm}(B\mathbb{Z}_m; \underline{a}, \underline{b})).$$

3.1 Stable maps to $B\mathbb{Z}_m$ and roots of the trivial bundle

Recall that for a multi-index $\underline{\gamma} \in (\mathbb{Z}_m)^n$, the space $\overline{\mathcal{M}}_{g, n}(B\mathbb{Z}_m, \underline{\gamma})$ parametrizes stable maps to $B\mathbb{Z}_m$ with holonomy $\underline{\gamma}$. This stack admits an other description in terms of m -th roots of the trivial bundle. For a curve $\bar{D}$, such a map is given by a $\mathbb{Z}_m$ cover $p : C \rightarrow \bar{D}$, with holonomy $\underline{\gamma}$. Let σ be the canonical generator of $\mathbb{Z}_m$, and $\zeta = e^{\frac{2i\pi}{m}}$. At a marked point $x \in p^{-1}(x_i)$ the stabilizer is $G_x = \mathbb{Z}/\mathfrak{e}_i\mathbb{Z}$, where $\mathfrak{e}_i$ is the ramification index. We identify G_x with $\mathbb{Z}/\mathfrak{e}_i\mathbb{Z}$ via the generator $\sigma_i = \sigma^{m/\mathfrak{e}_i}$. We write $\gamma_i = \sigma_i^{k_i}$, where k_i and $\mathfrak{e}_i$ are co-prime. The action of $\mathbb{Z}_m$ induces a character χ_x of G_x via its action on the tangent space at x . This character is given by the holonomy data

$$\chi_x(\sigma_i) = \zeta^{\nu_i \frac{m}{\mathfrak{e}_i}},$$

where ν_i is the inverse of k_i in Z_{ϵ_i} . The algebra $p_*\mathcal{O}_C$ is a locally free sheaf of rank m with a $\mathbb{Z}_m$ -action, and admits a decomposition into isotypical factor

$$p_*\mathcal{O}_C = \bigoplus_{j=0}^{m-1} T_j,$$

where T_j is the subsheaf of sections s such that $\sigma^*s = \zeta^j s$.

Lemma 3.1. *There is a canonical morphism $T_1^{\otimes m} \rightarrow \mathcal{O}_D$, which is an isomorphism. The multiplicity of T_1 at x_i (i.e. the representation $\mu_{\epsilon_i} \rightarrow \mathbb{G}_m$ given by $L|_{x_i}$) is $-\nu_i \in \mathbb{Z}_{\epsilon_i}$.*

Thus, we obtain an isomorphism

$$\overline{\mathcal{M}}_{g,n}(B\mathbb{Z}_m, \underline{\gamma}) \simeq \overline{\mathcal{M}}_{g,\underline{b}}^{m,0},$$

where $\underline{b}$ is given by $b_i = -\nu_i \frac{m}{\epsilon_i}$. The description of $\overline{\mathcal{M}}_{g,n}(B\mathbb{Z}_m)$ in terms of roots of the trivial bundle is more convenient in the next section.

3.2 The spine CohFT

Let $\underline{a} \in (\mathbb{Z}/rm\mathbb{Z})^n$ and $\underline{b} \in (\mathbb{Z}/m\mathbb{Z})^n$ be multi-indices. Define the space $\overline{\mathcal{M}}_0^{rm}(B\mathbb{Z}_m, \underline{a}, \underline{b})$ as

$$\overline{\mathcal{M}}_0^{rm}(B\mathbb{Z}_m; \underline{a}, \underline{b}) = \overline{\mathcal{M}}_{0,\underline{a}}^{rm,m} \times_{\overline{\mathcal{M}}_{0,n}} \overline{\mathcal{M}}_0^{m,0}(\underline{b}).$$

Objects of this stack are given by a curve D , together with an rm th root $\overline{\mathcal{L}}$ of $\omega_{\log}^{\otimes m}$, and an m th root T of the trivial line bundle. For the invariants to match with the Lefschetz's formula, we impose that the stabilizer of a node has order $\text{lcm}(m \text{ord}(ma), \text{ord}(b))$, where $\bar{a} \in \mathbb{Z}_{rm}$ is the multiplicity of $\overline{\mathcal{L}}$, and $b \in \mathbb{Z}_m$ is the multiplicity of T . The universal curve carries two universal line bundles $\overline{\mathcal{L}}_{\underline{a}}$, and $T_{\underline{b}}$. Let $\mathcal{A}_0, \dots, \mathcal{A}_{m-1}$ be invertible multiplicative classes, with $\mathcal{A}_j = \exp(\sum_{d \geq 0} w_d^j \text{ch}_d)$, and let E be the divisor of broad points in the universal curve $E = \bigsqcup_{a_i=0} x_i$. The projections to $\overline{\mathcal{M}}_{0,n}$ of the classes

$$\Lambda^{\text{spine}}(\underline{a}, \underline{b})_{0,n} = rm^2 \prod_{j \in \mathbb{Z}_m} \mathcal{A}_j \left(R\pi_* \overline{\mathcal{L}}_{\underline{a}}(-E) \otimes T_{\underline{b}}^j \right), \quad (24)$$

form the genus-0 part of a cohomological field theory with state space $\mathbb{C}[\mathbb{Z}_{rm}] \otimes H^*(\mathcal{IB}\mathbb{Z}_m, \mathbb{C})$, and basis $\{\phi_a \otimes [b] | a \in \mathbb{Z}_{rm}, b \in \mathbb{Z}_m\}$.

Proposition 3.2. *Let L^{spine} be the Lagrangian cone associated to the CohFT above. Then we have that*

$$L^{\text{spine}} = \square L^{\text{un}},$$

where L^{un} is the Lagrangian cone for $\mathcal{A}_j = 1$, and $\square$ is the operator given by

$$\square \cdot \phi_a \otimes [b] = \prod_{j \in \mathbb{Z}_m} \exp \left(\sum_{d \geq 0} w_d^j \frac{z^d}{(d+1)!} \tilde{B}_{d+1} \left(\frac{a}{rm} + \frac{b}{m} \right) \right)$$

Proof. This is a straightforward generalization of [CZ07], theorem 1.2.2.. □

We now apply the previous result to the classes encountered in Lefschetz's formula. Let s be a formal variable, and $\mathcal{A}_{\xi,j}$ be the classes

$$\mathcal{A}_{\xi,j}(E) = \text{ch} \left(\lambda_{-s\xi e^{-\frac{2i\pi j}{m}} E} \right)^{-r}. \quad (25)$$

Then we have

$$w_d^j = r \sum_{k \geq 1} \frac{\left(s e^{-\frac{2i\pi j}{m}} \right)^k \xi^k k^d}{k}.$$

Definition 3.3. Set

$$\begin{aligned} \Lambda_{\xi}^{\text{spine}}(\underline{a}, \underline{b}) &= rm^2 \prod_j \mathcal{A}_{\xi,j} \left(R\pi_* \overline{\mathcal{L}_{\underline{a}}}(-E) \otimes T_{\underline{b}}^j \right), \\ \mathcal{B}_{0,n} &= \text{td}^{-1}(\pi_* \omega_{\log}^{-1} - 1) \prod_{k=1}^{m-1} \text{td}_{\zeta^k} \left(\pi_* \left(1 - \omega_{\log}^{-1} \otimes T^k \right) \right), \\ \mathcal{C}_{0,n} &= \text{td}(-\pi_* \mathcal{O}_Z), \end{aligned}$$

with $\zeta = \exp(\frac{2i\pi}{m})$ and

$$\text{td}_{\zeta}(L) = \frac{1}{1 - \zeta \exp(-c_1(L))}.$$

The state space of the spine invariants is $\mathbb{C}[\mathbb{Z}_{rm}] \otimes \mathbb{C}[\widehat{\mu}_{rm}] \otimes \mathbb{C}[\mathbb{Z}_m]$, and the spine invariants are defined by

$$\left\langle \phi_{a_1} \otimes e_{\xi} \otimes [b_1] \mathcal{L}_1^{k_1}, \dots, \phi_{a_n} \otimes e_{\xi} \otimes [b_n] \mathcal{L}_n^{k_n} \right\rangle_{0,n}^{\text{spine}} = \int_{\overline{\mathcal{M}}^{rm}(B\mathbb{Z}_m, \underline{a}, \underline{b})} \Lambda_{\xi}^{\text{spine}}(\underline{a}, \underline{b})_{0,n} \mathcal{B}_n \mathcal{C}_n. \quad (26)$$

Proposition 3.4. *The $\mathcal{B}$ twist change the dilaton shift to $1 - q^m$. In the 0 sector, the $\mathcal{C}$ twist changes to polarization to*

$$\mathcal{K}_{-}^{\text{spine},0} = \text{Span} \left\{ \frac{q^{mk/\epsilon(a)}}{(1 - q^{m/\epsilon(a)})^k} \phi_a \otimes e_{\xi} \otimes [0] | k \in \mathbb{N} \right\}. \quad (27)$$

For $\nu_0 \in \mathbb{Z}_m^{\times}$, the polarization in the ν_0 sector is given by

$$\mathcal{K}_{-}^{\text{spine},-\nu_0} = \text{Span} \left\{ \frac{q^{k/\epsilon(a)m}}{(1 - q^{1/\epsilon(a)m})^k} \phi_a \otimes e_{\xi} \otimes [\nu_0] | k \in \mathbb{N} \right\}, \quad (28)$$

where ϵ denotes the order of $a \bmod r$.

Proof. See [GT11, Section 7]. □

Thus, the J -function of the spine invariants is defined by

$$J^{\text{spine}}(t) = 1 - q^m + t + \sum \frac{1}{m\epsilon(a)n!} \phi_a \otimes e_{\xi} \otimes [k] \left\langle \frac{\phi_a \otimes e_{\xi} \otimes [-k]}{1 - q^{\frac{1}{m\epsilon}} \mathcal{L}_0}, t(\mathcal{L}_1), \dots, t(\mathcal{L}_n) \right\rangle_{0,n+1}^{\text{spine}}. \quad (29)$$

Proposition 3.5. *The image of the J -function is given by*

$$L^{\text{spine}} = \square L^{\text{un}}.$$

In the context of expanding the J -function at $\xi_0 = \exp \frac{2i\pi k_0}{rm}$, the state space has two sector of particular interest : $b = 0$ and $b = \nu_0 = \frac{1}{k_0} \bmod m$, the last case corresponding to an automorphism g such that $\text{tr}(\mathcal{L}_0) = \xi_0$. The restrictions of $\square$ to these two subspaces are denoted $\square_0$ and $\square_{\xi_0}$ respectively, and are given by

$$\begin{aligned} \square_0 \cdot \phi_a \otimes e_\xi &= \exp \left(r \sum_{k,d,j} \frac{\left(se^{-\frac{2i\pi j}{m}} \right)^k \xi^k k^d}{k(d+1)!} z^d B_{d+1} \left(\frac{a}{rm} \right) \right) \phi_a \otimes e_\xi \\ &= \exp \left(\sum_{k,d} rm \frac{(s\xi)^{mk} (km)^d}{mk(d+1)!} z^d B_{d+1} \left(\frac{a}{rm} \right) \right) \phi_a \otimes e_\xi \\ &= \exp \left(r \sum_k \frac{s^{mk} \xi^{mk}}{k} \frac{e^{\frac{akmz}{rm}}}{e^{kmz} - 1} \right) \phi_a \otimes e_\xi \\ &= \exp \left(r \sum_k \frac{s^{mk} \xi^{mk}}{k} \frac{q^{\frac{ak}{r}}}{q^{km} - 1} \right) \phi_a \otimes e_\xi \quad \text{if } a \neq 0, \\ \square_0 \cdot \phi_0 \otimes e_\xi &= \exp \left(r \sum_k \frac{s^{mk} \xi^{mk}}{k} \frac{q^{ak}}{q^{km} - 1} \right) \phi_0 \otimes e_\xi. \end{aligned}$$

Let $\tilde{B}_d(x)$ denote the Bernoulli polynomial restricted to $]0;1]$, and expanded $\mathbb{Z}$ -periodically. For $a \notin m\mathbb{Z}_{rm}$, let us write $a = \mathbf{a} + rl$, with $1 \leq \mathbf{a} \leq r-1$. Then we have

$$\begin{aligned} \square_{\xi_0} \cdot \phi_a \otimes e_\xi &= \exp \left(\sum_{k,d,j} \frac{s^k \xi^k e^{-\frac{2i\pi kj}{m}} (kz)^d}{k(d+1)!} \tilde{B}_{d+1} \left(\frac{a}{rm} + \frac{\nu_0 j}{m} \right) \right) \phi_a \otimes e_\xi \\ &= \exp \left(r \sum_{k,d,j} \frac{s^k \xi^k \xi_0^{-rjk} (kz)^d}{k(d+1)!} \tilde{B}_{d+1} \left(\frac{a}{rm} + \frac{j}{m} \right) \right) \phi_a \otimes e_\xi \\ &= \exp \left(r \sum_{k,d,j} \frac{s^k \xi^k \xi_0^{-rjk} \xi_0^{rlk} (kz)^d}{k(d+1)!} B_{d+1} \left(\frac{\mathbf{a}}{rm} + \frac{j}{m} \right) \right) \phi_a \otimes e_\xi \\ &= \exp \left(r \sum_{k,j} \frac{s^k \xi^k \xi_0^{-rjk} \xi_0^{rlk}}{k} \frac{q^{k \frac{\mathbf{a} + rj}{rm}}}{q^k - 1} \right) \phi_a \otimes e_\xi \\ &= \exp \left(r \sum_k \frac{s^k \xi^k \xi_0^{rlk}}{k} \frac{q^{k \frac{\mathbf{a}}{rm}}}{q^{k/m} \xi_0^{-rk} - 1} \right) \phi_a \otimes e_\xi \end{aligned}$$

Similarly, for $a = rl$ we have

$$\begin{aligned}
\Box_{\xi_0} \cdot \phi_{rl} \otimes e_\xi &= \exp \left(\sum_{j=0}^{m-1} \sum_{k,d} \frac{s^k \xi^k \xi_0^{-rkj}}{k(d+1)!} z^d \tilde{B}_{d+1} \left(\frac{l+j}{m} \right) \right) \phi_{rl} \otimes e_\xi \\
&= \exp \left(r \sum_{j=1}^m \sum_{k,d} \frac{s^k \xi^k \xi_0^{rk(l-j)} (kz)^d}{k(d+1)!} B_{d+1} \left(\frac{j}{m} \right) \right) \phi_{rl} \otimes e_\xi \\
&= \exp \left(r \sum_{j=1}^m \sum_{k \geq 1} \frac{s^k \xi^k \xi_0^{rk(l-j)}}{k} \frac{q^{kj/m}}{q^{-k} - 1} \right) \phi_{rl} \otimes e_\xi \\
&= \exp \left(r \sum_{k \geq 1} \frac{s^k \xi^k \xi_0^{rkl}}{k} \frac{q^{k/m}}{\xi_0^{-rk} q^{k/m} - 1} \right) \phi_{rl} \otimes e_\xi
\end{aligned}$$

The spine invariants enjoy a natural symmetry, arising from the cyclic permutation of the line bundles $\bar{\mathcal{L}} \otimes T_j$ and from our choice of $\mathcal{A}_{\xi,j}$.

Lemma 3.6. *We have*

$$\Lambda_\xi^{\text{spine}}(\underline{a}, \underline{b}) = \Lambda_{\xi e^{-\frac{2i\pi}{m}}}^{\text{spine}}(\underline{a} + r\underline{b}, \underline{b}). \quad (30)$$

In particular, if T is an element of $\mathcal{K}_+^{\text{fake}}$, $\zeta \in \mu_{rm}$, and $a' \in \mathbb{Z}_{rm}$, then the following correlator does not depend on $k \in \mathbb{Z}_m$

$$\left\langle \phi_{a+rk} \otimes e_{\xi \xi_0^{a+rk}} \otimes [-\nu_0], \sum_{k'=0}^{m-1} \phi_{a'+rk'} \otimes e_{\zeta \xi_0^{-a'-rk'}} \otimes [\nu_0], \Psi^m(\Phi_0(T)), \dots, \Psi^m(\Phi_0(T)) \right\rangle_{0,n+2}^{\text{spine}}. \quad (31)$$

3.3 Comparison with the fake theory

We show that the 0-sector of the spine cone L^{spine} contains $\Psi^m(L^{\text{fake}})$.

Lemma 3.7. *The untwisted cone $L^{\text{un}} \subset \mathcal{K}^{\text{spine}}$ is stable by the transformation $q \mapsto q^m$.*

Proof. The cohomological cone of a point is

$$L_{\text{pt}} = z \bigcup_{\tau \in \mathbb{C}} \exp \left(\frac{\tau}{z} \right) \mathcal{H}_+,$$

which is obviously invariant by the transformation $z \mapsto mz$. The result follows for the untwisted cone. $\square$

Lemma 3.8. *Let $\Phi_0 : \mathcal{K}^{\text{fake}} \rightarrow \mathcal{K}^{\text{spine}}$ be the morphism $\phi_a \otimes e_\xi \mapsto \phi_{ma} \otimes e_\xi \otimes [0]$. Then, $\Psi^m \circ \Phi_0$ is a morphism of polarized symplectic spaces.*

Proof. This follows from Definition 3.3. $\square$

Proposition 3.9. *In $\mathcal{K}^{\text{spine}}$ we have that*

$$\Psi^m(\Delta\Phi_0 L^{\text{un}}) = \Psi^m(\Phi_0 \Delta L^{\text{un}}) \subset \square_0 L^{\text{un}}. \quad (32)$$

In particular for any element t of $\mathcal{K}_+^{\text{fake}}$, we have

$$J^{\text{spine}}(\Psi^m \Phi_0 t) = \Psi^m \Phi_0 J^{\text{fake}}(t).$$

Proof. The first assertion comes from a direct computation, together with Lemma 3.7. The second assertion is deduced from the first one, together with the change in dilaton shift from $1 - q$ to $1 - q^m$, and Lemma 3.8. $\square$

Corollary 3.10. *Let $J_{(1)}(t)$ be the image in $\mathcal{K}^{\text{fake}}$ of $J(t)$, and let $T = [J_{(1)}]_+ - 1 + q$, where $[\cdot]_+$ denotes the projection on $\mathcal{K}_+^{\text{fake}}$. Then,*

$$J^{\text{spine}}(\Psi^m \Phi_0 T) = \Psi^m \Phi_0 (J^{\text{fake}}(T)) = \Psi^m \Phi_0 J_{(1)}(t).$$

4 Adelic characterization

In this section, we use Lefschetz's fixed point formula to compute the J -function in terms of the fake and spine theories (see Section 3.2 for the definition of the spine CohFT). We first recall the Lefschetz formula for equivariant sheaves, and deduce a natural μ_r -action on the loop space making the J -function equivariant. This action plays a critical role on taking care of the ghost automorphisms of twisted curves. Then, we use Lefschetz's formula to compute the expansion of the J -function at each root of unity. The results are expressed in the adelic characterization theorem, which characterizes values of the J -function in terms of their expansions at each root of unity.

4.1 Lefschetz formula and the μ_r -action

Let X be a proper smooth Deligne–Mumford stack over $\mathbb{C}$, let h be an automorphism of X of finite order, and let F be an equivariant coherent sheaf. Then Lefschetz's formula reads

$$\text{tr}_h(H^*(X, f)) = \int_{X^h} \text{ch} \left(\frac{\text{Tr}_h(F)}{\text{Tr}_g \lambda_{-1} \mathcal{N}^\vee} \right) \text{Td}(T_{X^h}), \quad (33)$$

where X^h is the fixed-points stack, and $\mathcal{N}$ is the normal bundle to the morphism $X^h \rightarrow X$. Finally, $\text{Tr}_h(F)$ is obtained by decomposing $F|_{X^h}$ into isotypical factors $F = \bigoplus_\lambda F^\lambda$, and multiplying each factor by λ

$$\text{Tr}_h(F) = \sum_\lambda \lambda F^\lambda \in K^0(X) \otimes \mathbb{C}. \quad (34)$$

The first consequence of this formula is that the potential $\mathcal{F}$, and the J -function are equivariant with respect to some action of μ_r on $\mathcal{K}$. Indeed, for all $h \in S_n$, objects of the fixed-points stack $(\widetilde{\mathcal{M}}_{0,n+1}^r)^h$ are given by the data $(C, \mathcal{L}, (\sigma_i)_{i=0}^n, g, \phi, (\eta_i)_{i=0}^n)$, where

- $(C, \mathcal{L}, (\sigma_i)_{i=0}^n)$ is an object of $\widetilde{\mathcal{M}}_{0,n+1}^r$ over S ,
- g is an automorphism of C such that $g(x_i) = x_{h(i)}$,

- $\phi : g^* \mathcal{L} \rightarrow \mathcal{L}$ is an isomorphism compatible with the spin structure,
- $\eta_i : g \circ \sigma_i \rightarrow \sigma_{h(i)}$ is a 2-isomorphism.

We let μ_r act on $(\widetilde{\mathcal{M}}_{0,n+1}^r)^h$ by changing the 2-isomorphisms $\eta_0, \dots, \eta_n$. Explicitly, if ζ is an element of μ_r , an object $((C, \mathcal{L}, (\sigma_i)_{i=0}^n, g, \phi, (\eta_i)_{i=0}^n)$ is sent to

$$\zeta \cdot (C, \mathcal{L}, (\sigma_i)_{i=0}^n, g, \phi, (\eta_i)_{i=0}^n) = (C, \mathcal{L}, (\sigma_i)_{i=0}^n, g, \phi, \zeta^{-\frac{r}{\epsilon_i}} \circ \eta_i), \quad (35)$$

where ϵ_i denotes the cardinality of the stabilizer at the i th marked point.

This action has a natural analogue on the state space.

Definition 4.1. The μ_r -action on $\mathcal{K}$ is defined by

$$\zeta \cdot (q^{j/r} \phi_a \otimes e_\xi) = \zeta^j q^{j/r} \phi_a \otimes e_{\xi \zeta^a}. \quad (36)$$

Proposition 4.2. The genus-0 potential is μ_r -invariant, and the J -function is μ_r -equivariant :

$$\zeta \cdot J(t) = J(\zeta \cdot t). \quad (37)$$

Proof. Let us begin with the statement about the potential. Recall (remark 1.14) that the invariants associated to $t = q^{j/r} \phi_a[l]$ vanish unless $ld = j \pmod{\epsilon}$, where $d = \frac{a\epsilon}{r}$.

On the other hand, we compute that

$$\begin{aligned} \zeta \cdot (q^{j/\epsilon(a)} \phi_a \otimes [l]) &= \zeta \cdot \left(q^{j/\epsilon(a)} \phi_a \otimes \sum_{\xi \in \mu_r} \xi^l e_\xi \right) \\ &= \zeta^{\frac{j r}{\epsilon(a)}} q^{j/\epsilon(a)} \phi_a \otimes \sum_{\xi \in \mu_r} \zeta^{-al} \zeta^{al} \xi^l e_{\xi \zeta^a} \\ &= \zeta^{\frac{j r}{\epsilon(a)}} \zeta^{-al} q^{j/\epsilon(a)} \phi_a \otimes [l]. \end{aligned}$$

Let us denote $\zeta' = \zeta^{r/\epsilon} \in \mu_\epsilon$. Then we have

$$\zeta \cdot (q^{j/\epsilon(a)} \phi_a \otimes [l]) = \zeta' j \zeta'^{-dl} q^{j/\epsilon(a)} \phi_a \otimes [l].$$

Thus, the subspace of invariant elements in $\mathcal{K}_+$ is exactly the vector space spanned by elements of the form $q^{j/\epsilon(a)} \phi_a \otimes [l]$, with $j = d(a)l \pmod{\epsilon(a)}$. Thus, the result follows from the poly-linearity of the invariants.

Now, let us discuss the case of the J -function. It is enough to show that $J(t)_-$ is μ_r -invariant. To do so, we show that the classes appearing in Lefschetz's formula are μ_r -equivariant with respect to the μ_r action on $(\widetilde{\mathcal{M}}^r)^h$.

First, the class $\text{ch}(\text{Tr}(\Lambda_n))$ is obviously invariant, since it is defined on the space $\overline{\mathcal{M}}_{0,n}^r$. Next, the class $\text{ch}(\text{Tr}(\text{ev}_i^*(e_\xi)))$ has value 1 if $\text{tr}(\sigma_i^* \mathcal{L}_i) = \xi$, and 0 elsewhere. Composing η_i with $\zeta^{-r/\epsilon_i} \in \mu_{\epsilon_i}$ multiplies the trace by ζ^{-a_i} , so we have $\zeta^* (\text{ch}(\text{Tr}(e_{\xi \zeta^{-a_i}}))) = \text{ch}(\text{Tr}(\text{ev}_i^* e_\xi))$. Similarly, we have $\zeta^* \text{ch} \text{Tr} \left(\frac{1}{1 - q^{1/\epsilon} \mathcal{L}_0} \right) = \frac{1}{1 - \zeta_0 \zeta^{r/\epsilon} q^{1/\epsilon} \text{ch}(\text{Tr} \mathcal{L}_0)}$. Finally, $\zeta^* (\text{ch}(\text{Tr}(\mathcal{L}_i \otimes \text{ev}_i^* e_{\xi \zeta^{-a_i}}))) = \text{ch}(\text{Tr}(\zeta^{r/\epsilon_i} \mathcal{L}_i \otimes e_\xi))$.

Summing up, we have shown that

$$\left\langle \frac{\phi_a \otimes e_\xi}{1 - q^{1/\epsilon} \zeta^{r/\epsilon} \mathcal{L}_0}, t, \dots, t \right\rangle_{0,n+1}^{S_n} = \left\langle \frac{\phi_a \otimes e_{\xi \zeta^{-a}}}{1 - q^{1/\epsilon} \mathcal{L}_0}, \zeta \cdot t, \dots, \zeta \cdot t \right\rangle_{0,n+1}^{S_n}. \quad (38)$$

Thus we have

$$\begin{aligned} \zeta \cdot J(t) &= 1 - q + \zeta \cdot t + \sum \frac{r \phi^a \otimes e_{\xi \zeta^{-a}}}{\mathfrak{e}(a)} \left\langle \frac{\phi_a \otimes e_\xi}{1 - \zeta^{r/\epsilon(a)} q^{1/\epsilon(a)} \mathcal{L}_0}, t(\mathcal{L}_1), \dots, t(\mathcal{L}_n) \right\rangle_{0,n+1}^{S_n} \\ &= 1 - q + \zeta \cdot t + \sum \frac{r \phi^a \otimes e_{\xi \zeta^{-a}}}{\mathfrak{e}(a)} \left\langle \frac{\phi_a \otimes e_{\xi \zeta^{-a}}}{1 - q^{1/\epsilon(a)} \mathcal{L}_0}, \zeta t(\mathcal{L}_1), \dots, \zeta t(\mathcal{L}_n) \right\rangle_{0,n+1}^{S_n} \\ &= J(\zeta t)(q^{1/r}) \end{aligned}$$

□

Corollary 4.3. *For all $t \in \mathcal{K}_+$, the projection of $[J(t)]_-$ of $J(t)$ to $\mathcal{K}_-$ parallel to $\mathcal{K}_+$ is an invariant vector.*

4.2 Adelic characterization

Following [GT11] and [Givb], we apply Lefschetz's formula to the J -function to find recursion relations. More precisely, we compute the expansion of the J -function at each root of unity ξ_0 , and show that it corresponds to the fake and spine theories.

Definition 4.4. Let ξ_0 be a root of unity, let m be the order of ξ^r , and let $\mathcal{K}_{rm}$ be the symplectic space defined in Section 2.4.

We define the following linear maps

$$\begin{aligned} \Phi_0 : \mathcal{K} &\rightarrow \mathcal{K}_{rm} \\ \phi_a \otimes e_\xi &\mapsto \phi_{ma} \otimes e_\xi \\ \Phi_{\xi_0} : \mathcal{K} &\rightarrow \mathcal{K}_{rm} \\ q^{j/r} \phi_a \otimes e_\xi &\mapsto \xi_0^j q^{j/r} \sum_{l=0}^{m-1} \phi_{a+rl} \otimes e_{\xi \xi_0^{-a-rl}} \end{aligned}$$

We think of Φ_0 (resp. Φ_{ξ_0}) as an embedding of $\mathcal{K}$ in the 0 sector (resp. the k_0 sector) of the spine Cohft.

Theorem 4.5 (Adelic characterization). *Let $f(q^{1/r})$ be a μ_r -invariant element of $\mathcal{K}$. Then f lies in the image of the J -function if and only if*

- The poles of f belong to $\mu_\infty \cup \{0, \infty\}$,
- the expansion $f_{(1)}$ at $q^{1/r} = 1$ belongs to the fake cone L^{fake} ,
- For all $\xi_0 \in \mu_\infty$ such that ξ_0^r has order m , we have

$$\Phi_{\xi_0} \left(f(q^{\frac{1}{rm}}) \right) \in \square_{\xi_0} \Delta^{-1} \mathcal{T} L^{\text{fake}}$$

where $\mathcal{TL}^{\text{fake}}$ is the tangent space at the point $\Phi_0 f \in L^{\text{fake}}$, Δ is the operator of the fake theory (Proposition 2.4), and $\square_{\xi_0}$ is defined by

$$\begin{aligned}\square_{\xi_0}(\phi_{a+rl} \otimes e_{\xi}) &= \exp \left(r \sum_{k \geq 1} \frac{s^k \xi^k \xi_0^{rlk}}{k} \frac{q^{k \frac{a}{rm}}}{\xi_0^{-rk} q^{k/m} - 1} \right) \phi_{a+rl} \otimes e_{\xi} \text{ if } a \in \{1, \dots, r-1\} \\ \square_{\xi_0}(\phi_{rl} \otimes e_{\xi}) &= \exp \left(r \sum_{k \geq 1} \frac{s^k \xi^k \xi_0^{rlk}}{k} \frac{q^{\frac{k}{m}}}{\xi_0^{-rk} q^{k/m} - 1} \right) \phi_{rl} \otimes e_{\xi}.\end{aligned}$$

Remark 4.6. We showed in Section 4.1 that the J function is μ_r -equivariant, and that for all $t \in \mathcal{K}_+$, we have

$$[J(t)]_- = [J(\bar{t})]_-,$$

where $\bar{t} = \frac{1}{r} \sum_{\xi \in \mu_r} \xi \cdot t$ is the projection of t on the subspace of invariant elements. Thus, the theorem above characterizes all possible values of the J -function.

The rest of this section is devoted to the proof of Theorem 4.5. We first show that these 3 conditions are necessary. The first item is obviously necessary, and the other two follow from Lefschetz's formula, which allows us to compute the expansion of $J(t)$ at $q^{1/r} = 1$, and $q^{1/r} = \xi_0$.

Definition 4.7. Let $\xi_0 \in \mu_{\infty}$ be a root of unity, and let $n \geq 2$ be an integer. The symmetric group S_n acts on $\widetilde{\mathcal{M}}_{0,n+1}^r$ by permuting the last n marked points. We define $\mathcal{M}_{n+1}(\xi_0)$ as the substack of $\bigsqcup_{h \in S_n} (\widetilde{\mathcal{M}}_{0,n+1}^r)^h$ made of the curves such that $\text{tr}(\mathcal{L}_0) = \xi_0^{r/\epsilon_0}$ (where ϵ_0 is the order of $\text{mult}_{x_0}(\mathcal{L})$ in $\mathbb{Z}/r\mathbb{Z}$).

The polar part at $q^{1/r} = \xi_0^{-1}$ is precisely the contribution of $\mathcal{M}(\xi_0)$ to Lefschetz's formula.

4.3 Expansion at $q^{1/r} = 1$

The expansion at $q^{1/r} = 1$ of the J -function has the form

$$J_{(1)}(t) = 1 - q + t + \sum_{\xi_0 \neq 1} \text{Cont}(\xi_0)(q) + \text{Cont}(1)(q), \quad (39)$$

where $\text{Cont}(\xi_0)$ denotes the contribution of $\mathcal{M}(\xi_0)$. The function $J_{(1)}$ is an element of the space $\mathcal{K}^{\text{fake}}$ (see Definition 2.3). By definition, the only pole of $\text{Cont}(\xi_0)$ is at $q^{1/r} = \xi_0^{-1}$. Thus, in formula 39, the term $1 - q + t + \sum_{\xi_1 \neq 1} \text{Cont}(\xi_1)(q)$ lies in $\mathcal{K}_+^{\text{fake}}$, while the term $\text{Cont}(1)(q)$ lies in $\mathcal{K}_-^{\text{fake}}$.

Definition 4.8. Let C be an object of $\mathcal{M}_n(1)$. The *head* of C is the largest connected subcurve C^{head} such that

- $x_0 \in C^{\text{head}}$,
- C^{head} is g -stable, and $g|_{C^{\text{head}}} = \text{id}$.

The moduli stack of heads $\mathcal{M}_{N,n}^{\text{head}}$ is the stack parametrizing r -spin curves $(C, \mathcal{L}, \sigma_i)$ with $n + N + 1$ marked points (and section), together with automorphisms η_i of σ_i for $i = N + 1, \dots, N + n + 1$, and an isomorphism $\phi : \mathcal{L} \rightarrow \mathcal{L}$ compatible with the spin structure.

Remark 4.9. The head of a curve $C \in \mathcal{M}_n(1)(S)$ is always non-empty. Indeed, since $\text{tr}(\mathcal{L}_0) = 1$, the restriction of g to the irreducible component containing x_0 is the identity.

Notice that on each connected component of $\mathcal{M}^{\text{head}}$, ϕ is the multiplication by some r -th root of unity ξ . Thus, the stack of heads has a natural decomposition

$$\mathcal{M}_{n,N}^{\text{head}} = \bigsqcup_{\xi \in \mu_r} \mathcal{M}_{n,N}^{\text{head}}(\xi). \quad (40)$$

Definition 4.10. An *arm*, is an object of the stack

$$\mathcal{M}^{\text{arm}} = \bigsqcup_{\xi_0 \neq 1} \mathcal{M}(\xi_0). \quad (41)$$

An arm C such that $\text{mult}_{x_0}(\mathcal{L}) = 0$ is called a *broad arm*. We further decompose the moduli stack of arms by taking into consideration the g -action on $\sigma_0^* \mathcal{L}$:

$$\mathcal{M}^{\text{arm}}(\zeta_0)(S) := \{(C, \mathcal{L}, g) \in \mathcal{M}^{\text{arm}}(S) \mid \text{tr}(\sigma_0^* \mathcal{L}) = \zeta_0\}. \quad (42)$$

Let C be an object of $\mathcal{M}_n(1)$. Then, C is the union of the head and N other curves $C_1, \dots, C_N$, attached to the head at the nodes $p_1, \dots, p_n$. Let D_i be the divisor of $\widetilde{\mathcal{M}}_{0,n}^r$ defined by the node p_i , and let $\mathcal{N}_i$ be its normal bundle. We still denote $\mathcal{N}_i$ its pullback to S . By definition of the head, the action of g on $\mathcal{N}_i$ is non trivial. Thus, the curves C_i , (together with the restriction of g) are arms, that is, objects of $\bigsqcup_{\xi_0 \neq 1} \mathcal{M}(\xi_0)$.

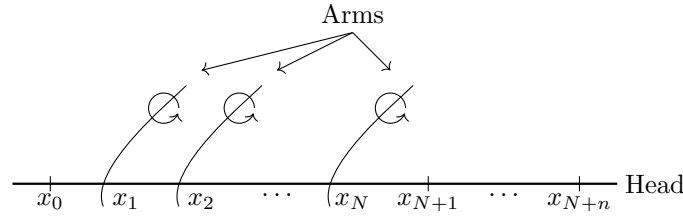


Figure 1: Decomposition into head and arms

Proposition 4.11. The decomposition into head and arms yields a morphism of stacks

$$\bigsqcup_{\substack{n \geq 0 \\ N \geq 0 \\ n+N \geq 2}} \mathcal{M}_{N,n}^{\text{head}}(\xi) \times_{\mathcal{IB}\mu_r} \underbrace{\mathcal{M}^{\text{arm}}(\xi) \times_{\mathcal{IB}\mu_r} \cdots \times_{\mathcal{IB}\mu_r} \mathcal{M}^{\text{arm}}(\xi)}_{N \text{ times}} \rightarrow \mathcal{M}(1), \quad (43)$$

where the morphisms are

$$\begin{aligned} \text{ev}_i : \mathcal{M}_{N,n}^{\text{head}} &\rightarrow \mathcal{IB}\mu_r & \text{ev}_0 : \mathcal{M}^{\text{arm}} &\rightarrow \mathcal{IB}\mu_r \\ (C, \mathcal{L}) &\mapsto (\mathcal{L}|_{x_i}, \text{mult}_{x_i}(\mathcal{L})) & (C, \mathcal{L}) &\mapsto (\mathcal{L}_{x_1}, -\text{mult}_{x_1}(\mathcal{L})) \end{aligned}$$

Taking the union over all $\xi \in \mu_r$ yields an isomorphism.

Proof. Let $(C^{\text{head}}, C_1, \dots, C_N, \mathcal{L}^{\text{head}}, \mathcal{L}^{(1)}, \dots, \mathcal{L}^{(N)})$ be an object of $\mathcal{M}^{\text{head}}(\xi)_{n,N} \times_{\mathcal{I}B\mu_r} \mathcal{M}^{\text{arm}}(\xi) \times_{\mathcal{I}B\mu_r} \dots \times_{\mathcal{I}B\mu_r} \mathcal{M}^{\text{arm}}(\xi)$ over a connected base scheme S . Let $x_0, \dots, x_{N+n}$ be the marked points of C^{head} , and let y_i be the first marked point of C_i . Each x_i, y_i is the trivial μ_{ϵ_i} -gerbe over S (with ϵ_i the order of $\text{mult}_{x_i}(\mathcal{L})^{\text{head}}$), so we have a canonical isomorphism $x_i \simeq y_i$. We define C to be the gluing of the curves along this isomorphism (it exists by [AGV08], prop.A.1.1). Moreover, we also have canonical isomorphisms $\mathcal{L}_{|x_i}^{\text{head}} \simeq \mathcal{L}_{|y_i}^{(i)}$, so the line bundles glue and yield a line bundle $\mathcal{L}$ over C . We also have isomorphisms $(\mathcal{L}^{\text{head}})^{\otimes r} \simeq (\omega_{\log})_{|C^{\text{head}}}$ and $\mathcal{L}^{(i)} \rightarrow (\omega_{\log})_{|C_i}$ which glue. This is a consequence of the fact that the restriction of $\omega_{\log}$ to a node or a marked point is canonically trivial, and that $\mathcal{L}_{|x_i}^{\text{head}}$ and $\mathcal{L}_{|y_i}^{(i)}$ are isomorphic as maps to $B\mu_r$. Finally, the linearizing maps $\phi^{\text{head}}, \phi_i$ glue into a global isomorphism $g^*\mathcal{L} \rightarrow \mathcal{L}$ because they coincide at each node. $\square$

Proposition 4.12. *Let $\mathcal{Y}$ be a connected component of $\mathcal{M}(1)$ such that the head of the universal curve carries $n+1$ marked points and N arms. Let $\xi^{-1} \in \mu_r$ be the r th root of unity corresponding to the morphism $\phi : \mathcal{L}^{\text{head}} \rightarrow \mathcal{L}^{\text{head}}$, i.e., $\text{tr}(\sigma_0^*\mathcal{L}) = \xi$. Let m be the number broad arms, Λ^{head} be the virtual class on $\mathcal{M}^{\text{head}}$, and Λ_i be the virtual class on each copy of the moduli space of arms. Then over $\mathcal{Y}$, the virtual class factorizes as follows*

$$\text{ch}(\text{Tr}(\Lambda)) = \text{ch} \circ \text{Tr}(\Lambda^{\text{head}}) \times \prod_{i=1}^N \text{ch}(\text{Tr}(\Lambda_i)) \frac{1}{(1 - s\xi)^{rm}}.$$

Moreover, we have

$$\text{ch}(\text{Tr}(\Lambda^{\text{head}})) = \text{ch}(\lambda_{-s\xi} R^1\pi_*\mathcal{L}^{\text{head}}(-E))^r. \quad (45)$$

In other words, the head contribution corresponds to the fake theory.

Proof. Let $x_1, \dots, x_N$ be the nodes connecting the head to the arms, and let $C^\nu = C^{\text{head}} \sqcup \bigsqcup C_i$ be the partial normalization of the curve at these points. Let $p : C^\nu \rightarrow C$ be the projection, $\mathcal{L}^\nu = p_*p^*\mathcal{L}$, and let E^ν be the divisor of broad marked points on C^ν . We write $E^\nu = E_{\text{head}} \bigsqcup_{i=1}^N E_i$, where E_{head}, E_i are the divisors of broad marked points in C^{head} and C_i respectively. The divisors E_ν and $p^*(E)$ may differ because new broad points may arise from the normalization. There is an exact sequence

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{L}^\nu \rightarrow \bigoplus_{i=1}^N \mathcal{L}_{|x_i} \rightarrow 0. \quad (46)$$

The pushforward $\pi_*\mathcal{L}_{|x_i}$ is non-zero only if $\text{mult}_{x_i}(\mathcal{L}) = 0$. Remember that $\pi_*\mathcal{L}(-E) = 0$ so we have the long exact sequence

$$0 \rightarrow \pi_*\mathcal{L}^\nu(-E) \rightarrow \bigoplus_{\epsilon_i=0} \pi_*\mathcal{L}_{|x_i} \rightarrow R^1\pi_*\mathcal{L}(-E) \rightarrow R^1\pi_*\mathcal{L}^\nu(-E) \rightarrow 0 \quad (47)$$

Similarly, we have a short exact sequence $0 \rightarrow p_*p^*\mathcal{L}(-E_\nu) \rightarrow \mathcal{L}^\nu(-E) \rightarrow \bigoplus_{\epsilon_i=0} \mathcal{L}_{|x_i}^{\oplus 2} \rightarrow 0$. We get

$$\begin{aligned} \lambda_{-s}(R\pi_*\mathcal{L}(-E)) &= \lambda_{-s}(R\pi_*\mathcal{L}^\nu(-E)) \bigotimes_{\epsilon_i=0} \lambda_{-s}(R\pi_*\mathcal{L}_{|x_i})^{-1} \\ &= \lambda_{-s}(R\pi_*\mathcal{L}^\nu(-E_\nu)) \bigotimes_{\epsilon_i=0} \lambda_{-s}(R\pi_*\mathcal{L}_{|x_i}) \end{aligned}$$

$$= (\lambda_{-s} R^1 \pi_* \mathcal{L}^{\text{head}}(-E_{\text{head}}))^{-1} \bigotimes_i (\lambda_{-s} R^1 \pi_* \mathcal{L}^{(i)}(-E_i))^{-1} \bigotimes_{\mathbf{e}_i=0} (1 - sp_* \mathcal{L}|_{x_i}).$$

Since all the sequences above are exact sequences of equivariant sheaves, we may take the trace bundle and the Chern character to get the first statement

$$\text{ch} \circ \text{Tr}(\Lambda) = (\text{ch} \circ \text{Tr}(\lambda_{-s} R \pi_* \mathcal{L}))^{-r} = \text{ch} \circ \text{Tr}(\lambda_{-s} \Lambda^{\text{head}}) \bigotimes_i \text{ch} \circ \text{Tr}(\Lambda_i) \bigotimes_{\mathbf{e}_i=0} \frac{1}{(1 - \xi s)^r}.$$

The second statement follows immediately from the assumption that ϕ is given by ξ^{-1} . $\square$

Proposition 4.13. *For all $t \in \mathcal{K}_+$, we have*

$$J_{(1)}(t) \in \mathcal{L}^{\text{fake}}. \quad (48)$$

Proof. We use the decomposition into head and arms to express the terms in Lefschetz's formula as an integral on the moduli space of heads. Recall that the expansion of the J -function at $q^{1/r} = 1$ is

$$J_1(t) = 1 - q + t + \sum_{\zeta \neq 1} \text{Cont}(\zeta) + \text{Cont}(1). \quad (49)$$

Let $\tilde{t}$ denote

$$\tilde{t} = t + \sum_{\zeta \neq 1} \text{Cont}(\zeta) = [J_1(t)]_+ - 1 + q. \quad (50)$$

Then we have $\tilde{t} \in \mathcal{K}_+^{\text{fake}}$, and Proposition 4.12 implies that $J_1(t) = J^{\text{fake}}(\tilde{t})$. $\square$

4.4 Expansion at other roots of unity

Let $\xi_0 \in \mu_\infty$, and let $m(\xi_0)$ (or simply m when ξ_0 is fixed) be the order of ξ_0^r . Because of the μ_r -invariance, we assume that $m > 1$. The polar part of $\mathcal{J}$ at $q^{1/r} = \xi_0^{-1}$ comes from the contribution of $\mathcal{M}(\xi_0)$ to the Lefschetz formula. Since $m \neq 1$ the automorphism g acts non-trivially on the connected component of x_0 , namely, g acts on this component by a rotation of order m . This action allows us to decompose the curve into a spine, some legs, and a tail. A decomposition of $\mathcal{M}_n(\xi_0)$ as a union of products follows, and Lefschetz's formula factorizes as a product of classes called the spine/leg/tail contributions. This allows us to recognize the expansion J_{ξ_0} as a tangent vector to the Lagrangian cone of the spine CohFT.

Definition 4.14. Let C be an object of $\mathcal{M}(\xi_0)$ over a connected scheme. The spine of C is the largest connected subcurve $C^{\text{spine}} \subset C$, such that

- $x_0 \in C^{\text{spine}}$,
- C^{spine} is g -stable, and its nodes are balanced with respect to g ,
- $g|_{C^{\text{spine}}}^m = \text{id}$.

Lemma 4.15. *The marked points of C^{spine} are either fixed by g , or have an orbit of cardinality m . More precisely, the spine of a curve is isomorphic to a balanced chain of (orbifold) $\mathbb{P}^1$ s with standard $\mathbb{Z}_m$ action, with m -tuples of curves or marked points attached. Over the spine curve, the automorphism g has exactly 2 smooth fixed points denoted by x_0 and x_∞ .*

Proof. Over the irreducible component containing the first marked point, the Riemann-Hurwitz formula implies that g has exactly 2 ramification points, with maximal ramification index. If the second ramification point is a node, then we apply the same argument to the irreducible component attached to it. $\square$

The marked points of the spine consist of one of the following

- m tuples of permuted marked points,
- m tuples of nodes,
- the fixed points x_0 and x_∞ .

Definition 4.16. Let C be a curve in $\mathcal{M}_n(\xi_0)$. If x_∞ is a node of C , the connected component attached to C^{spine} at x_∞ is the *tail* of C . The complement of the spine and the tail is a union of *legs*. A leg is a m -tuple of spin curves, cyclically permuted by g .

Thus the curve C decomposes as the union of

- a spine with $mN + 2$ marked points,
- a (potentially empty) tail attached at x_∞ , and
- N legs, i.e. N sets of m cyclically permuted spin curves.

We know show that the decomposition of curves into spine, legs and tail leads to a decomposition of the fixed-points stack $(\mathcal{M}_{0,n}^r)^h$ as a (union of) products of the corresponding moduli stacks.

Definition 4.17 (spines). The moduli space of *spine curves* $\mathcal{M}_{n+2}^{\text{spine}}(\xi_0, \zeta_0, \zeta_\infty)$ is the stack parameterizing

- an r -spin curve $(C, \mathcal{L})$ with $mn + 2$ marked points,
- a balanced automorphism g of order m fixing x_0 , and permuting the other marked points by some permutation h whose conjugacy class is given by the partition $(m, \dots, m, 1, 1)$,
- an isomorphism $\phi : g^* \mathcal{L} \rightarrow \mathcal{L}$ compatible with the spin structure
- 2-isomorphisms $\eta_0 : g \circ \sigma_0 \rightarrow \sigma_0$, and $\eta_\infty : g \circ \sigma_\infty \rightarrow \sigma_\infty$, such that $\text{tr}(\mathcal{L}_0) = \xi_0$, $\text{tr}(\mathcal{L}_\infty) = \xi_0^{-1}$, $\text{tr}(\sigma_0^* \mathcal{L}) = \zeta_0$, $\text{tr}(\sigma_\infty^* \mathcal{L}) = \zeta_\infty$.

We will always assume that the marked points $x_1, \dots, x_n$ lie in different $\mathbb{Z}_m$ -orbits, and we use the same notation for their image in the quotient curve $D = C/g$. Notice that, on each connected component of $\mathcal{M}^{\text{spine}}$, the morphism $\phi^m : (g^m)^* \mathcal{L} \rightarrow \mathcal{L}$ is the multiplication by an r -th root of unity ξ .

The moduli space of spines is equipped with its virtual class

$$\Lambda^{\text{spine}} = (\lambda_{-1}(R^1 \pi_* \mathcal{L}(-E)))^{\otimes r} \in K_{\mathbb{Z}_m}^0(\mathcal{M}^{\text{spine}}(\xi_0)), \quad (51)$$

where E is the divisor of broad marked points in the universal curve.

Definition 4.18 (legs). The moduli space $\mathcal{M}_{n+1}^{\text{leg}}$ is the stack parameterizing

- an m -tuple of r -spin curves $(C_i, \mathcal{L}^{(i)})$ indexed by $\mathbb{Z}_m$,
- isomorphisms of spin curves (with section) $g_i : (C_i, \mathcal{L}^{(i)}) \rightarrow (C_{i+1}, \mathcal{L}^{(i+1)})$ such that the trace of $g_{m-1} \circ \dots \circ g_0$ on the first cotangent line of C_0 is non-trivial.

Let $\xi \in \mu_r$ be an r th root of unity. We denote by $\mathcal{M}^{\text{leg}}(\xi)$ the substack where $\text{tr}_{g^m}(\sigma_0^* \mathcal{L}) = \xi$. The moduli space of legs is equipped with its virtual class

$$\Lambda^{\text{leg}} = (\lambda_{-1} (R^1 \pi_* \mathcal{L}(-E)))^{\otimes r}. \quad (52)$$

The moduli stack of legs admits a natural description in terms of arms.

Lemma 4.19. *Let γ be the automorphism of $(\mathcal{M}_{n+1}^{\text{arm}})^m$ permuting the different copies of $\mathcal{M}_{n+1}^{\text{arm}}$. Then, the moduli stack of legs is the fixed-point stack*

$$\mathcal{M}_{n+1}^{\text{leg}} \simeq ((\mathcal{M}_{n+1}^{\text{arm}})^m)^{\gamma}. \quad (53)$$

Definition 4.20 (tails). The moduli stack of tails is

$$\mathcal{M}^{\text{tail}}(\xi_0, \zeta) = \bigsqcup_{\xi_0 \neq \xi_1} \mathcal{M}(\xi_1, \zeta). \quad (54)$$

The moduli space of tails is equipped with its virtual class

$$\Lambda^{\text{tail}} = (\lambda_{-1} (R^1 \pi_* \mathcal{L}(-E)))^{\otimes r}. \quad (55)$$

On the stack of spines $\mathcal{M}^{\text{spine}}(\xi_0, \zeta)$, let ξ be the locally constant function corresponding to the morphism

$$\phi^m : (g^m)^* \mathcal{L} \rightarrow \mathcal{L}.$$

Lemma 4.21. *We have the equality*

$$\zeta^m = \xi^{-1} \xi_0^{-am},$$

where $a = \text{mult}_{x_0} \mathcal{L}$.

Proof. Both sides of the equality correspond to the trace of g^m on the line bundle $\sigma_0^* \mathcal{L}$. $\square$

We introduce a tool to glue legs to spines.

Definition 4.22. The gluing stack G is defined by

$$G = ((\mathcal{I}B\mu_r)^m)^{\gamma},$$

where γ is the automorphism permuting the different copies of $\mathcal{I}B\mu_r$.

Objects of G over a connected scheme S are given by the data $(a, (L_i)_{i=0}^{m-1}, (\alpha_i)_{i=0}^{m-1})$, where

- $a \in \mathbb{Z}_r$ denotes a connected component of $\mathcal{I}B\mu_r$,
- L_i is a line bundle on S ,

- $\alpha_i : L_i^{\otimes r} \rightarrow \mathcal{O}_S$ is an isomorphism,
- $f_i : L_i \rightarrow L_{i+1}$ is an isomorphism compatible with α_i (with $L_m = L_0$).

There are morphisms $\mathcal{M}^{\text{spine}} \rightarrow G$ and $\mathcal{M}^{\text{leg}} \rightarrow G$ given by the line bundles $\sigma_i^* \mathcal{L}$ at a $\mathbb{Z}_m$ -orbit of marked points.

Proposition 4.23. *There is a morphism of stacks*

$$\mathcal{M}^{\text{spine}}(\xi_0, \zeta_0, \zeta_\infty)_{N+2} \times_G \underbrace{\mathcal{M}^{\text{leg}}(\xi) \times_G \cdots \mathcal{M}^{\text{leg}}(\xi)}_{N \text{ times}} \times_{\mathcal{IB}\mu_r} \mathcal{M}^{\text{tail}}(\xi_0, \zeta_\infty) \rightarrow \mathcal{M}(\xi_0), \quad (56)$$

where $\xi = \zeta_0^m \xi_0^{a_0 m}$, and the morphisms are

- $\text{ev}_i : \mathcal{M}^{\text{spine}} \rightarrow G$, $i = 1, \dots, N$,
- $\overline{\text{ev}}_0 : \mathcal{M}_{n_i+1}^{\text{leg}} \rightarrow G$,
- $\overline{\text{ev}}_0 : \mathcal{M}_{m+1}^{\text{tail}} \rightarrow \mathcal{IB}\mu_r$.

Proposition 4.24. *Let $\mathcal{Y}$ be a connected component of $\mathcal{M}^{\text{spine}}$, and let l (resp. t) be the number of broad legs (resp. tails) over $\mathcal{Y}$. Then, the virtual class factorizes as*

$$\text{ch}(\text{Tr } \Lambda) = \text{ch} \left(\text{Tr}(\Lambda^{\text{spine}}) \bigotimes_i \text{Tr } \Lambda_i^{\text{leg}} \otimes \text{Tr } \Lambda^{\text{tail}} \right) \frac{1}{(1 - s^m \xi)^{rl} (1 - s \zeta_\infty)^{rt}} \quad (57)$$

Proof. Similar to the proof of Proposition 4.12. $\square$

We now detail detail the contribution to Lefschetz's formula coming from the different terms in the product.

4.5 Spine contribution

We now explicitly compute the term $\text{Tr}(\Lambda^{\text{spine}})$ appearing in Proposition 4.24. A spine curve C is an r -spin curve together with a $\mathbb{Z}_m$ -symmetry. Quotienting by $\mathbb{Z}_m$, we get a curve D , equipped with an rm th root of $\omega_{\log}^{\otimes m}$. This allows us to relate the spine contribution to the spine CohFT of Section 3.2.

Let C be an object of $\mathcal{M}^{\text{spine}}$, and let $p : C \rightarrow D = C/\mathbb{Z}_m$, be the quotient map, where the generator of $\mathbb{Z}_m$ acts by the automorphism g . The algebra $p_* \mathcal{O}_C$ has a decomposition $p_* \mathcal{O}_C = \bigoplus_{j=0}^{m-1} T^j$, where T is the line bundle of functions f such that $g^* f = e^{2i\pi/m} f$.

Lemma 4.25. *Let us write $\xi_0 = e^{\frac{2i\pi k_0}{rm}}$, and let ν_0 be the inverse of k_0 in $\mathbb{Z}_m$. T has non trivial multiplicities only at the two ramification points x_0 and x_∞ , and*

- $\text{mult}_{x_0}(T) = \nu_0 \in \mathbb{Z}_m$,
- $\text{mult}_{x_\infty}(T) = -\nu_0 \in \mathbb{Z}_m$.

The couple $(\mathcal{L}, \phi)$ is not always an equivariant sheaf. However, if we define $\tilde{\phi}$ by $\tilde{\phi} = \zeta \xi_0^a \phi$, then $(\mathcal{L}, \tilde{\phi})$ is a $\mathbb{Z}_m$ -equivariant line bundle (see Lemma 4.21). Let $\tilde{\mathcal{L}}$ be the line bundle over D obtained by descent. The morphism $\alpha : \mathcal{L}^{\otimes r} \rightarrow \omega_{\log}$ induces an isomorphism

$$\bar{\alpha} : \tilde{\mathcal{L}}^{\otimes rm} \simeq \omega_{\log}^m. \quad (58)$$

Proposition 4.26. *The data $(D, \bar{\mathcal{L}}, T)$ above defines a morphism*

$$f : \mathcal{M}_{2+n}^{\text{spine}}(a_0, a_1, \dots, a_\infty) \rightarrow \mathcal{M}^{rm, m}(a_0, ma_1, \dots, ma_n, \bar{a}_\infty) \times_{\mathcal{M}_{0, n}} \mathcal{M}^{m, 0}(\nu, 0, \dots, 0, -\nu),$$

where $\mathcal{M}^{rm, m}$ is the moduli space of rm -th roots of $\omega_{\log}^m$, and $\bar{a}_\infty$ is determined by the degree condition

$$mn - a_0 - \bar{a}_\infty - \sum ma_i = 0 \pmod{rm}.$$

This morphism is bijective on closed points, and has degree $\frac{1}{r}$. Moreover, under this morphism we have

$$R\pi_* \mathcal{L} = f^* \bigoplus_{j=0}^{m-1} R\pi_*(\bar{\mathcal{L}} \otimes T^j), \quad (59)$$

and the trace of g is given by

$$\text{Tr}_g(R\pi_* \mathcal{L}) = \bigoplus_{j=0}^{m-1} \zeta \xi_0^a e^{\frac{-2i\pi j}{m}} R\pi_*(\bar{\mathcal{L}} \otimes T^j). \quad (60)$$

In particular, the spine contribution coincides with a value of the spine CohFT (see Section 3.2)

$$\text{ch} \circ \text{Tr}(\Lambda^{\text{spine}}) = \Lambda_{\zeta_0 \xi_0}^{\text{spine}}(\underline{a}, \underline{b}),$$

with $\underline{a} = (a_0, ma_1, \dots, \bar{a}_\infty)$ and $\underline{b} = (\nu_0, 0, \dots, 0, -\nu_0)$.

4.6 Leg contribution

We compute the trace bundle $\text{Tr}(\Lambda^{\text{leg}})$, following Givental's computation in [Giv17]. More precisely, we want to compute the following generating function, which we call the *leg contribution* :

$$\sum_{n \geq 2} \frac{1}{n!} \int_{\mathcal{M}_{n+1}^{\text{leg}}} \frac{\text{ch}(\text{Tr}(\Lambda^{\text{leg}})) \text{ch}(\text{Tr}(\bigotimes_{i=1}^n t(\mathcal{L}_i)))}{\text{ch}\left(\text{Tr}\left(\bigotimes_{k=0}^{m-1} (1 - q^{1/\epsilon} \mathcal{L}_{0,k})\right)\right) \text{ch Tr}(\lambda_{-1} \mathcal{N}^\vee)} \text{td}(\mathcal{T}), \quad (61)$$

where $\mathcal{N}$ is the normal bundle to the morphism $\mathcal{M}_{n+1}^{\text{leg}} \rightarrow (\mathcal{M}_{0, n+1}^r)^m$, $\mathcal{L}_{0,k}$ are the cotangent lines at the first marked point of each component of the universal curve, and e is the order of the stabilizer of the first marked point.

Let $(C_0, \dots, C_{m-1})$ be an object of $\mathcal{M}^{\text{leg}}(\xi)$. The pushforward $R^1\pi_* \mathcal{L}$ is the direct sum

$$R^1\pi_* \mathcal{L} = \bigoplus_{i \in \mathbb{Z}_m} E_i, \quad (62)$$

where $E_i = R^1\pi_*(\mathcal{L}(-E))|_{C_i}$. We begin with a general lemma.

Lemma 4.27. *Let X be a smooth stack over $\mathbb{C}$, let $E = \bigoplus_{i \in \mathbb{Z}_m} E_i$ be a $\mathbb{Z}_m$ -graded vector bundle over X , and let $g_i : E_i \rightarrow E_{i+1}$ be isomorphisms such that $g = g_{m-1} \circ g_{m-2} \circ \dots \circ g_0 \in \text{Aut}(E_0)$ has finite order. Then we have*

$$\text{Tr}_{g_\bullet} \Psi^k(E) = \begin{cases} m \Psi^m(\text{Tr}_g \Psi^{k/m} E_0) & \text{if } m|k \\ 0 & \text{otherwise.} \end{cases} \quad (63)$$

Proof. If $g = \text{id}$, then this is the situation in [Giv17]. Let us recall the argument. There is an isomorphism of $\mathbb{Z}_m$ -bundles $E \simeq E_0 \otimes \mathcal{O}_X[\mathbb{Z}_r]$, and we compute that $\Psi^k(\mathcal{O}_X[\mathbb{Z}_r]) = \mathcal{O}_X^m$ is $m|k$, and $\Psi^k(\mathcal{O}_X[\mathbb{Z}_r]) = 0$ otherwise.

If $g \neq \text{id}$, we can decompose each E_i into the sum of eigenspaces for g . These eigenspaces are preserved by the g_i , so we may assume that $g = \lambda \text{id}$ for some $\lambda \in \mathbb{C}^*$. Then we have

$$\text{Tr}_{g_\bullet}(\Psi^k(E)) = \begin{cases} m\lambda^{k/m}\Psi^k(E_0) & \text{if } m|k, \\ 0 & \text{otherwise.} \end{cases} \quad (64)$$

□

Corollary 4.28. *We extend the Adams operations to $K^0(X)[[s]]$ by setting*

$$\Psi^m(s) = s^m.$$

Then, with the same notations as in the previous lemma, we have

$$\text{Tr}(\lambda_{-s}E) = \Psi^m(\text{Tr}(\lambda_{-s}E_0)). \quad (65)$$

Proof. Recall that $\lambda_{-s}(E) = \exp\left(-\sum_{k \geq 1} \frac{s^k \Psi^m(E)}{k}\right)$. Thus, using Lemma 4.27 we get

$$\begin{aligned} \text{Tr}(\lambda_{-s}E) &= \exp\left(-\sum_k \frac{s^k \text{Tr} \Psi^k(\bigoplus E_i)}{k}\right) \\ &= \exp\left(-\sum_k \frac{s^{km} \Psi^m \text{Tr} \Psi^k(E_0)}{k}\right) \\ &= \Psi^m(\text{Tr}_g \lambda_{-s}E_0). \end{aligned}$$

□

Proposition 4.29 ([Giv17] lemma p.5). *Let X be a proper smooth Deligne-Mumford stack over $\mathbb{C}$, and let $\pi : X \rightarrow \text{Spec}(\mathbb{C})$ be the projection. Let E, T be vector bundles over X , and let E_i, T_i be the pullbacks $E_i = p_i^*E$, and $T_i = p_i^*T$, where $p_i : X^m \rightarrow X$ is the i -th projection. Let $\phi : X^m \rightarrow X^m$ be the cyclic permutation of factors. We choose finite order isomorphisms $g_i : E \rightarrow E$ and $f_i : T \rightarrow T$, which induce isomorphisms $g_i : \phi^*E_i \rightarrow E_{i+1}$ and $f_i : \phi^*T_i \rightarrow T_{i+1}$. We equip $E_\bullet = \bigoplus_{i=0}^{m-1} E_i$ and $T_\bullet = \bigoplus_{i=0}^{m-1} T_i$ with the induced $\mathbb{Z}_m$ equivariant structure. Finally, let $g : E \rightarrow E$ (resp. $f : T \rightarrow T$) be the composition $g = g_{m-1} \circ \dots \circ g_0$ (resp. $f = f_{m-1} \circ \dots \circ f_0$).*

Then we have

$$\text{tr } H^*\left(X^m; \left(\Psi^k\left(\frac{E_\bullet}{k}\right)\right) \otimes T_\bullet\right) = \begin{cases} \Psi^m\left(H^*\left(X; \text{Tr}_g \frac{\Psi^{k/m} E}{k/m} \otimes \text{Tr}_f T\right)\right) & \text{if } m|k, \\ 0 & \text{otherwise.} \end{cases} \quad (66)$$

Proof. Use Lefschetz's formula to compute the left hand-side, and Adams–Riemann–Roch for the right hand-side, combined with Lemma 4.27. □

We apply the previous results to the space $\mathcal{M}^{\text{leg}} \simeq ((\mathcal{M}^{\text{arm}})^m)^\phi$ (see Lemma 4.19). Let pr denote the projection to the first factor.

Corollary 4.30. *We have*

$$\mathrm{tr} \left(H^*(\Lambda^{\mathrm{leg}}) \right) = \Psi^m \left(H^*(\mathrm{Tr}(\Lambda^{\mathrm{arm}})) \right) \in \mathbb{C}[[s]]. \quad (67)$$

More generally, the leg contribution is given by

$$\Psi^m \left([J_{(1)}(t)]_+ \right),$$

where $[\cdot]_+$ denotes the projection to $\mathcal{K}_+^{\mathrm{fake}}$, parallel to $\mathcal{K}_-^{\mathrm{fake}}$.

4.7 Conclusion

We now explain how the previous results imply that J_{ξ_0} is a tangent vector to the cone L^{spine} of the spine CohFT. The spine CohFT (defined in the Section 3.2) encodes the spine contribution. Its state space is $K^0(IB\mu_{rm})_{\mathbb{C}} \otimes_{\mathbb{C}} H^*(IB\mathbb{Z}_m, \mathbb{C})$, with pairing given by the product of the orbifold pairing on each factor. The factorization of the virtual class means that we can view the leg and tail contributions as “inputs of the spine CohFT”.

Let us decompose $H^*(IB\mathbb{Z}_m, \mathbb{C}) = \bigoplus_{d \in \mathbb{Z}_m} \mathbb{C} \cdot [d]$. We refer to the subspaces $K^0(IB\mu_{rm})_{\mathbb{C}} \otimes [d]$ as sectors of the state space. We define two embeddings Φ_0, Φ_{ξ_0} of $\mathcal{K}$ into $\mathcal{K}_{rm}$, corresponding to the sectors 0 and $-\nu_0 = -\frac{1}{k_0} \bmod m$ respectively (recall that $\xi_0 = \exp(\frac{2i\pi k_0}{rm})$).

Proposition 4.31. *The image of $J(t)(q^{1/rm})$ by Φ_{ξ_0} is a tangent vector to the spine cone :*

$$\Phi_{\xi_0} \left(J(t)(q^{1/rm}) \right) \in \mathcal{T}L^{\mathrm{spine}}. \quad (68)$$

The tangency point is $J^{\mathrm{spine}}(T)$, with

$$T = \Psi^m \Phi_0(J_{(1)}(t)).$$

Lemma 4.32. *Let C be an object of $\mathcal{M}_{mn+2}^{\mathrm{spine}}(\xi_0, \zeta_0, \zeta_{\infty})$, and let $\underline{a} \in \mathbb{Z}_r^n$ be the multi-index of multiplicities of $\mathcal{L}$ at each $\mathbb{Z}_m$ -orbit of marked points. Then we have*

$$\zeta_{\infty} = \zeta_0 \xi_0^m \sum a_i - mn \quad (69)$$

Proof. First, let us assume that $\xi_0 = e^{\frac{2i\pi k_0}{rm}}$ is a primitive rm th root of unity, and let ν be the inverse of k_0 modulo rm . We may also assume that $(\mathcal{L}, \phi)$ is a $\mathbb{Z}_m$ -equivariant bundle, and that C is smooth, that is, C is isomorphic to a stacky $\mathbb{P}^1$. Let $\bar{L}$ be the bundle over $D = C/\mathbb{Z}_m$ obtained by descent, and let $\bar{a}_0, \bar{a}_{\infty}, \bar{a}_i$ be its multiplicities at the marked points.

Let $p : U \rightarrow C$ be the $\mathbb{Z}_r$ -cover ramified over x_0 and x_{∞} . Then, there exists a lift $\bar{g}$ of g such that

- $\bar{g}$ has order rm , and its trace on the tangent space at x_0 and x_{∞} is ξ_0 and ξ_0^{-1} respectively,
- $p^*\mathcal{L}$ is $\mathbb{Z}_{rm}$ -equivariant, and descends to $\bar{\mathcal{L}}$ on the quotient $U/\mathbb{Z}_{rm} \simeq D$.

Then ζ_0 and ζ_{∞} are the trace of $\bar{g}$ on $p^*\mathcal{L}$ at x_0 and x_{∞} respectively. Thus, we have that

$$\begin{aligned} \zeta_0^{-\nu} &= e^{\frac{2i\pi \bar{a}_0}{rm}} \\ \zeta_{\infty}^{\nu} &= e^{\frac{2i\pi \bar{a}_{\infty}}{rm}}. \end{aligned}$$

Finally, we use the fact that $\bar{a}_0 + \bar{a}_\infty = \sum_i m a_i - mn$ to obtain

$$\begin{aligned}\zeta_\infty &= \xi_0^{\bar{a}_\infty} \\ &= \zeta_0 \xi_0^m \sum a_i - mn\end{aligned}$$

The proof is similar for a general ξ_0 . □

Proof of Proposition 4.31. Let δt be the tail contribution, namely

$$\delta t = \sum_{\substack{a \in \mathbb{Z}_r \\ \xi \in \mu_r}} \sum_{n \geq 2} \frac{r \phi^a \otimes e_\xi}{\mathfrak{e}(a)n!} \int_{\mathcal{M}^{\text{tail}}(\xi_0)} \frac{\text{ch}(\text{Tr}(\Lambda^{\text{tail}} \otimes \text{ev}_0^*(\phi_a \otimes e_\xi) \otimes t(\mathcal{L}_i)))}{(1 - q^{1/m \mathfrak{e}(a)} \text{Tr}(\mathcal{L}_0)) \text{ch}(\text{Tr}(\lambda_{-1} \mathcal{N}^\vee))} \text{td}(\mathcal{T}), \quad (70)$$

where $\mathcal{N}$ is the normal bundle to the morphism $\mathcal{M}_{n+1}^{\text{tail}} \rightarrow \overline{\mathcal{M}}_{0,n+1}^r$, and $\mathcal{T}$ is the tangent bundle. Then we have

$$\delta t(q^m) = \left[J(t)(q^{1/r} \xi_0^{-1}) \right]_+.$$

We claim that

$$\Phi_{\xi_0} \left(J(t)(q^{1/rm}) \right) = \Phi_{\xi_0}(\delta t) + \sum_{\substack{a \in \mathbb{Z}_{rm} \\ \xi \in \mu_{rm}}} \sum_{n \geq 1} \frac{\phi^a \otimes e_\xi \otimes [-\nu_0]}{m \mathfrak{e}(a)n!} \left\langle \frac{\phi_a \otimes e_\xi \otimes [\nu_0]}{1 - q^{1/\mathfrak{e}(a)m}}, T, \dots, T, \Phi_{\xi_0}(\delta t) \right\rangle_{0,n+2}^{\text{spine}}, \quad (71)$$

with $T = \Psi^m \left(\Phi_0 \left([J_{(1)}(t)]_+ \right) \right)$, and $\mathfrak{e}(a)$ is the order of a in $\mathbb{Z}_r$. This claim follows from the factorization of the virtual class and Lemma 3.6. Thus, $\Phi_{\nu_0}(J(t))$ is a tangent vector to L^{spine} at $\Psi^m(\Phi_0 J(t))$. □

Let $\mathcal{T}^{\nu_0}$ denote the tangent space to L^{spine} at $\Psi^m \Phi_0(J_1(t))$ intersected with the $-\nu_0$ sector. Then we have

$$\mathcal{T}^{\nu_0} = \square_1 \mathcal{T} L^{\text{un}}, \quad (72)$$

where the second tangent space is computed at $\square_0^{-1} \Psi^m \Phi_0(J_1(t)) = \Delta^{-1} \Phi_0(J_1(t))$ (see Corollary 3.10), and L^{un} is the untwisted cone in $\mathcal{K}_{rm}$.

Proposition 4.33. *We have*

$$\Phi_{\xi_0} J(t)(q^{1/rm}) \in \square_1 \Delta^{-1} \mathcal{T}_{\Phi_0 J_{(1)}(t)} L^{\text{fake}}$$

Proof. This is a consequence of Proposition 3.9. □

4.8 Reconstruction

So far, we proved that the values of the J -function satisfy the 3 conditions in Theorem 4.5. We now explain how these properties allow to reconstruct the J -function. We follow the proof of [GT11, prop. 4]. Recall that the ground λ -ring R is supposed to carry a Hausdorff $\mathcal{J}$ -adic topology such that $\Psi^m(\mathcal{J}) \subset \mathcal{J}^m$.

Let f be a μ_r -invariant element of $\mathcal{K}_+$ such that $f = 1 - q \pmod{\mathcal{J}}$. We write $f = 1 - q + t + f_-$, where $1 - q + t \in \mathcal{K}_+$, and $f_- \in \mathcal{K}_-$. Notice that the last 2 conditions of Theorem 4.5 are stable by

base change. In particular, if f is an element satisfying those conditions, then so does the image of f modulo $\mathcal{J}^n$. We will show by recursion that for all $n \in \mathbb{N}$, f is a value of the J -function modulo $\mathcal{J}^n$.

For $n = 1$, we assumed that $f = 1 - q \pmod{\mathcal{J}}$, which is a value of the J -function.

Now, suppose that $f = J(t) \pmod{\mathcal{J}^n}$ for some $n \geq 1$, and let us show that $f = J(t) \pmod{\mathcal{J}^{n+1}}$. We just need to check that $\tilde{t}$ is the arm contribution modulo $\mathcal{J}^{n+1}$, i.e., that we have $\tilde{t} = [J_{(1)}(t)]_+ - 1 + q - t$. By definition the arm contribution is the sum of the polar parts of $J(t)$ at all the non-trivial roots of unity, so we need to show that for all $\xi_0 \in \mu_\infty$, the polar part of f at $q^{1/r} = \xi_0^{-1}$ matches the polar part of $J(t)$. Let us begin with the r th roots of unity. First, notice that $[f_{(1)}]_- \pmod{\mathcal{J}^{n+1}}$ is determined by $[f_{(1)}]_+ \pmod{\mathcal{J}^n}$ by the formula $f_{(1)} = J^{\text{fake}}([f_{(1)}]_+ - 1 + q)$. The μ_r -invariance of f implies that the polar part at $q^{1/r} = \xi^{-1} \in \mu_r$ is exactly $\xi \cdot [f_{(1)}]_-$. Since $J(t)$ is also μ_r -invariant, its polar part at the r th roots of unity coincide with that of f . Now, suppose that ξ_0 is some root of unity such that the order of ξ_0^r is $m \geq 2$. Then, the polar part of f at $q^{1/r} = \xi_0^{-1}$ is determined by $[\Phi_{\xi_0} f]_- \pmod{\mathcal{J}^{n+1}}$, which only depends on the tail contribution modulo $\mathcal{J}^n$, and on $[f_{(1)}]_+ \pmod{\mathcal{J}^{\lfloor \frac{n+1}{m} \rfloor + 1}}$ because of the third condition of the adelic characterization. Thus, the induction hypothesis allows us to conclude that the polar parts of f and $J(t)$ coincide modulo $\mathcal{J}^{n+1}$ at any root of unity.

Finally, we conclude that $f = J(t)$, which concludes the proof of Theorem 4.5.

5 I -function and difference equation

In this section, we use the adelic characterization to give a simpler description of the image of the J -function using “untwisted” invariants. Then, we use this description to find a specific value of the J -function, following the method given by Coates, Corti, Iritani and Tseng in [CCIT09]. In this section we choose R to be the λ -ring $\mathbb{C}[X]$, with Adam’s operations $\Psi^k(X) = X^k$.

5.1 FJRW invariants from untwisted invariants

Definition 5.1. Let $n = k_1 + \dots + k_s$ be a partition of n , and let $H \subset S_n$ be the subgroup $H = S_{k_1} \times \dots \times S_{k_s}$. For a sequence $t^{(1)}, \dots, t^{(s)}$ of elements of $\mathcal{K}_+$, the cohomology groups

$$\left[t^{(1)}(\mathcal{L}_{1,1}), \dots, t^{(1)}(\mathcal{L}_{1,k_1}); \dots, t^{(s)}(\mathcal{L}_{s,k_s}) \right]_n^{\text{un}} = H^* \left(\widetilde{\mathcal{M}}_{0,n}^r; \bigotimes_{k=1}^s \bigotimes_{l=1}^{k_s} t^{(k)}(\mathcal{L}_{k,i}) \right) \quad (73)$$

is an H -module.

For any elements $\nu_1, \dots, \nu_s \in R$, we define

$$\left\langle t^{(1)}(\mathcal{L}_{1,1}) \otimes \nu_1, \dots, t^{(s)}(\mathcal{L}_{s,k_s}) \otimes \nu_s \right\rangle_{0,n}^{\text{un},H} = \frac{1}{\prod_i k_i!} \sum_{h \in H} \text{tr}_h \left[t^{(1)}(\mathcal{L}_{1,1}), \dots, t^{(s)}(\mathcal{L}_{s,k_s}) \right]_n^{\text{un}} \prod_{i=1}^s \prod_{r=1}^{\infty} \Psi^r(\nu_i)^{l_r(h)}. \quad (74)$$

The associated J -function is

$$J^{K,\text{un}}(t) = 1 - q + t + \sum_{a,\xi} \frac{r}{\mathfrak{e}(a)} \phi_a \otimes e_\xi \left\langle \frac{\phi^a \otimes e_\xi}{1 - q^{1/\mathfrak{e}(a)} \mathcal{L}_0}, t(\mathcal{L}_1), \dots, t(\mathcal{L}_n) \right\rangle_{n+1}^{S,\text{un}},$$

and the image of the J -function is the subvariety

$$L_{\text{un}}^K \subset \mathcal{K}.$$

Theorem 5.2 (Adelic characterization). *Let $f = 1 - q + t + f_-$ be a μ_r -invariant element of $\mathcal{K}$. Then, f belongs to L_{FJRW}^K if and only if*

- the poles of f are at $q = 0$, $q = \infty$, and the roots of unity,
- $f_{(1)} \in L_{\text{un}}^H$,
- $\Phi_{\xi_0} f(q^{\frac{1}{rm}}) \in T$, where T is the tangent space to the untwisted cohomological cone at the point $J^{H,\text{un}}(t)_{(1)}$.

Proof. Follow the proof of the previous adelic characterization, replacing Λ with 1. $\square$

Theorem 5.3. *The invariant points of L coincide with the invariant points of ΔL_{un}^K :*

$$(L_{\text{FJRW}}^K)^{\mu_r} = (\Delta L_{\text{un}}^K)^{\mu_r}, \quad (75)$$

where Δ is the operator of the fake theory.

Lemma 5.4. *Let f be a point of $\mathcal{K}$, and ξ_0 a root of unity such that ξ_0^m has order m . Then we have*

$$\xi^{-1} \cdot \Phi_{\xi_0} \left((\Delta f)(q^{1/m}) \right) = \square_{\xi_0} \xi^{-1} \cdot \Phi_{\xi_0} f(q^{1/m}). \quad (76)$$

Proof. This is a straightforward computation. $\square$

Proof of Theorem 5.3. We check the criteria of the adelic characterization. First, a direct computation shows that Δ sends μ_r -invariant points to μ_r -invariant points. The first 2 items of the adelic characterization are obviously satisfied, and the last one follows from Lemma 5.4. $\square$

5.2 The I -function

Recall [Giva] that the J -function of a point is

$$J_{\text{pt}}(x) = 1 - q + (1 - q) \sum_{n \geq 1} \frac{x^n}{\prod_{k=1}^n (1 - q^k)}. \quad (77)$$

Thus, we have that

$$I^{\text{un}} := J^{S,\text{un}}(x\phi_2 \otimes [0]) = (1 - q)\phi_1 \otimes [0] + (1 - q) \sum_{n \geq 1} \frac{x^n}{\prod_{k=1}^n (1 - q^k)} \phi_{n+1} \otimes [0]. \quad (78)$$

We now follow the computation in [CCIT09],[CR10]. We define $w_{\xi}(z) = \sum_{d \geq 0} w_{\xi,d} \frac{z^d}{d!}$ with

$$w_{\xi,d} = r \sum_{k \geq 1} \frac{\xi^k s^k k^d}{k}.$$

Notice that $\exp(-w_\xi(z)) = (1 - \xi sq)^r$. We also define the functions $G_{y,\xi}$ by

$$G_{y,\xi}(x, z) = \sum_{k,l} w_{\xi,k+l-1} \frac{B_l(y) x^k z^{l-1}}{k!l!}.$$

These functions satisfy two equations :

$$\begin{aligned} G_{y,\xi}(x, z) &= G_{0,\xi}(x + yz, z), \\ G_{0,\xi}(x + z, z) &= G_{0,\xi}(x, z) + w_\xi(x). \end{aligned}$$

Let ∇ be the vector field $\nabla = \frac{x}{r} \partial_x$.

Proposition 5.5. *The function $\exp(-G_{1/r}(z\nabla, z)) I^{\text{un}}$ is a value of the untwisted permutation-equivariant J -function $J^{S,\text{un}}$.*

Proof. We check the conditions of the adelic characterization. Noting that $[0] = \sum_{\xi \in \mu_r} e_\xi$, we compute that

$$\exp(-G_{1/r}(z\nabla, z)) x^n \phi_{n+1} \otimes e_\xi = \exp\left(-\sum_{k \geq 1} \frac{\xi^k s^k}{k} \frac{q^{k \frac{n+1}{r}}}{q^k - 1}\right) x^n \phi_{n+1} \otimes e_\xi. \quad (79)$$

This shows that $\exp(-G_{1/r}(z\nabla, z)) I^{\text{un}}$ is μ_r -invariant, and has poles at $0, \infty$, and the roots of unity.

By a theorem Coates–Corti–Iritani–Tseng [CCIT09, thm. 4.6], the operator $\exp(-G_{1/r}(z\nabla, z))$ preserves the cone $L^{H,\text{un}}$. Thus, the second condition is also satisfied. The same argument also applies to show that the third conditions holds. $\square$

Theorem 5.6. *The function*

$$I_{\text{FJRW}}^K = (1 - q) \sum_{\xi \in \mu_r} \sum_{n \geq 0} \frac{\prod_{0 \leq k < \lfloor d/r \rfloor} \left(1 - \xi q^{\left\{\frac{n}{r}\right\} + \frac{1}{r} + k}\right)^r x^n}{\prod_{k=1}^n (1 - q^k)} \phi_{n+1} \otimes e_\xi$$

is a point of the the image L_{FJRW}^K of the J -function.

Proof. We compute $I_{\text{FJRW}}^K := \Delta \exp(-G_{1/r}(z\nabla, z)) I^{\text{un}}$. For $n+1 \notin r\mathbb{Z}$ we have $\Delta \phi_{n+1} \otimes e_\xi = \exp(G_{0,\xi}(\frac{n+1}{r}z, z))$, so

$$\begin{aligned} \Delta \exp(-G_{1/r,\xi}(z\nabla, z)) x^n \phi_{n+1} \otimes e_\xi &= \exp\left(G_{0,\xi}\left(\left\{\frac{n+1}{r}\right\}z, z\right) - G_{0,\xi}\left(\frac{n+1}{r}z, z\right)\right) x^n \phi_{n+1} \otimes e_\xi \\ &= \prod_{k=0}^{\lfloor n/r \rfloor - 1} \exp\left(-w_\xi\left(\left\{\frac{n+1}{r}\right\}z + kz\right)\right) x^n \phi_{n+1} \otimes e_\xi \\ &= \prod_{k=0}^{\lfloor n/r \rfloor - 1} \left(1 - s\xi q^{\left\{\frac{n}{r}\right\} + \frac{1}{r} + k}\right)^r x^n \phi_{n+1} \otimes e_\xi. \end{aligned}$$

For $n + 1 \in r\mathbb{Z}$, we have

$$\Delta\phi_0 \otimes e_\xi = \exp(G_{0,\xi}(z, z)).$$

Thus we have

$$\begin{aligned} \Delta \exp(-G_{1/r,\xi}(z\nabla, z)) x^n \phi_0 \otimes e_\xi &= \exp\left(G_{0,\xi}(z, z) - G_{0,\xi}\left(\frac{n+1}{r}z, z\right)\right) x^n \phi_{n+1} \otimes e_\xi \\ &= \prod_{k=0}^{\lfloor n/r \rfloor - 1} \exp(-w_\xi(z + kz)) x^n \phi_{n+1} \otimes e_\xi \\ &= \prod_{k=0}^{\lfloor n/r \rfloor - 1} (1 - s\xi q^{1+k})^r. \end{aligned}$$

Finally, we take the limit $s \mapsto 1$ to get the result. $\square$

Remark 5.7 (Dual class). We could also define FJRW invariants using the dual of the fundamental class

$$\Lambda_n^\vee = \lambda_{-1}(R^1\pi_*\mathcal{L}(-E)^{\oplus r})^\vee.$$

All the proofs adapt easily to this case, and we also obtain an I -function for these invariants

$$I_{\text{FJRW}}^{K,\vee} = (1-q) \sum_{\xi \in \mu_r} \sum_{n \geq 0} \frac{\prod_{0 \leq k < \lfloor d/r \rfloor} \left(1 - \xi^{-1} q^{-\{\frac{n}{r}\} - \frac{1}{r} - k}\right)^r x^n}{\prod_{k=1}^n (1 - q^k)} \phi_{n+1} \otimes e_\xi. \quad (80)$$

5.3 Difference equation

In this section we assume that $r = 5$, we describe the difference equation satisfied by the I -function. We expand the I -function with respect to the basis $\phi_a \otimes e_\xi$, and we write $I_{\text{FJRW}}(x, q) = \sum_{a=0}^{r-1} \sum_{\xi \in \mu_r} x^a I_{a,\xi}(x, q) \phi_{a+1} \otimes e_\xi$.

Proposition 5.8. *The function $I_{a,\xi}(x^{1/5}, q^{-1})$ is a solution of the equation*

$$\left[\prod_{k=0}^4 (1 - q^{-k+5x\partial_x+a}) - xq^{14+4a+20x\partial_x} (1 - \xi^{-1} q^{\frac{a+1}{5}+x\partial_x})^5 \right] G(x) = 0.$$

Proof. For $0 \leq a \leq 4$, we have

$$I_{a,\xi}(x^{1/5}, q) = \sum_{d \geq 0} \frac{\prod_{0 \leq k < \lfloor \frac{5d+a}{5} \rfloor} \left(1 - \xi q^{\{\frac{a}{5}\} + \frac{1}{5} + k}\right)^5}{\prod_{k=1}^{5d+a} (1 - q^k)} x^d \quad (81)$$

$$= \sum_{d \geq 0} \frac{\prod_{0 \leq k < d} \left(1 - \xi q^{\frac{a+1}{5} + k}\right)^5}{\prod_{k=1}^{5d+a} (1 - q^k)} x^d \quad (82)$$

Thus, it satisfies the equation

$$\left[\prod_{k=0}^4 (1 - q^{-k+5x\partial_x+a}) - x(1 - \xi q^{\frac{a+1}{5}+x\partial_x})^5 \right] I_{a,\xi}(x^{1/5}, q) = 0.$$

Thus, $I_{a,\xi}(x^{1/5}, q^{-1})$ satisfies

$$\left[\prod_{k=0}^4 (1 - q^{-k+5x\partial_x+a}) - xq^{15+4a-1+20x\partial_x} (1 - \xi^{-1} q^{\frac{a+1}{5}+x\partial_x})^5 \right] I_{a,\xi}(x^{1/5}, q^{-1}) = 0 \quad (83)$$

□

For $\lambda \in \mathbb{C}$, the q -character $e_{q,\lambda}(x)$ is the function

$$e_{q,\lambda}(x) = \prod_{k=0}^{\infty} \frac{(1 + xq^{-k-1})(1 + x^{-1}q^{-k})}{(1 + q^{-k-1}\lambda^{-1}x)(1 + x^{-1}\lambda q^{-k})}.$$

The q -character satisfies the difference equation $q^{x\partial_x} e_{q,\lambda}(x) = \lambda e_{q,\lambda}(z)$. Let us define $\tilde{I}_{a,\xi}(x, q^{-1}) := e_{q,q^{\frac{a+1}{5}}\xi^{-1}}(x) I_{a,\xi}(x^{1/5}, q^{-1})$.

Proposition 5.9. *For $a \in \{0, \dots, r-1\}$, the function, $\tilde{I}_{a,\xi}(x, q^{-1})$ is a solution to the difference equation*

$$\left[\prod_{k=1}^5 (1 - q^{-k+5x\partial_x}) - xq^{10+20x\partial_x} (1 - q^{x\partial_x})^5 \right] F(x) = 0 \quad (84)$$

Moreover, the functions $\tilde{I}_{a,\xi}(x, q^{-1})$ coincide with the 25 solutions $W_{l,m}$ obtained by Y. Wen in [Wen22]

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