Nonlinear and degenerate discounted approximation in discrete weak KAM theory

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À la mémoire de Nicolas Bergeron
un collègue en or
un mathématicien sensationnel
une incarnation de générosité.

Abstract

In this paper, we introduce a discrete version of the nonlinear implicit Lax-Oleinik operator as studied for instance in [21]. We consider the associated vanishing discount problem with a non-degenerate condition and prove convergence of solutions as the discount factor goes to 0. We also discuss the uniqueness of the discounted solution. This paper can be thought as the discrete version of [5], and a generalization of [6] and [26, Chapter 3]. The convergence result is a selection principle for fixed points of a family of nonlinear operators.

Keywords. Mather measures, Weak KAM theory, Discretization.

Introduction

0.1 Brief history of the problem

The discounted approximation appeared for Hamilton-Jacobi equations in Lions Papanicolaou and Varadhan's celebrated preprint [17]. The goal is to solve[3] an equation on the torus of the form

$$H(x, d_x u) = c_0, \quad x \in \mathbb{T}^N \tag{0.1}$$

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[3] In this introduction, solutions are intended in the viscosity sense but no knowledge of this notion is required to understand the paper.

where $u : \mathbb{T}^N \to \mathbb{R}$ and $c_0 \in \mathbb{R}$ are the unknown and the Hamiltonian $H : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfies some coercivity condition. The problem of the previous equation is that if it admits solutions, they are not unique (the set of solutions is invariant by addition of constants). To deal with this problem, the authors approximate the equation as follows: given $\lambda > 0$, they solve the discounted equation

$$\lambda u_\lambda(x) + H(x, d_x u_\lambda) = 0,$$

and establish that the solution u_λ is unique, that the family $(u_\lambda)_{\lambda\in(0,1)}$ is equicontinuous, that λu_λ converges to a constant $(-c_0)$ as $\lambda \to 0^+$ and that $(u_\lambda + c_0/\lambda)_{\lambda\in(0,1)}$ is bounded. Therefore, taking a converging subsequence $u_{\lambda_n} + c_0/\lambda_n \to u_0$ provides a solution u_0 to (0.1).

The first convergence result for the whole family $(u_\lambda + c_0/\lambda)_{\lambda\in(0,1)}$ was established in [16] for a particular case followed by [7] for a result in full generality under Tonelli type hypotheses on H. A discrete version of this result was published in [6] around the same time. The necessity of convexity of Hamiltonian in the convergence result is given by a counterexample in [28].

This convergence phenomenon was followed by many generalizations (see [10, 15] for non-compact cases, see [13, 14, 18, 27] for second order cases, see [9, 11, 12] for weakly coupled systems, and see [2] for mean field games). Amongst the ones that are of interest to us here, let us cite also the papers the papers [3, 4, 22, 20] that prove similar results for equations of the form $G\big(x, \lambda u_\lambda(x), d_x u_\lambda(x)\big) = 0$ where $G(x, u, p)$ verifies Tonelli type hypotheses in the variables (x, p) and is increasing in u. This is the nonlinear version of the problem, the results in [7] corresponding to the particular case $G(x, u, p) = u + H(x, p)$. The degenerate aspect was studied in [25] for Hamiltonians of the form $G(x, u, p) = \alpha(x)u + H(x, p)$ where α is a continuous nonnegative function that verifies some non degeneracy condition but is allowed to vanish on large portions of $\mathbb{T}^N$ (a discrete version of this results is presented in [26]). Both those settings were merged in a nonlinear degenerate setting in [5] where general Hamiltonians $G(x, u, p)$ are considered, verifying Tonelli type hypotheses in the variables (x, p) and being non-decreasing in u. The nondegeneracy hypothesis consists in prescribing that G is increasing in some regions. When the equation is not non-decreasing in u, the asymptotic behavior is not clear yet. One can refer to [8, 23] for the approximation process when $\lambda \to 0^-$, and see [19] for the asymptotic behavior of a particular non-monotone case.

0.2 The discretization

The philosophy of the discrete problem stems from Lax-Oleinik type formulas. In the previously mentioned results, if u_λ solves $G\big(x, \lambda u_\lambda(x), d_x u_\lambda(x)\big) = 0$ then the solution u_λ verifies

$$\forall x \in \mathbb{T}^N, \ \ \forall t > 0, \quad u_\lambda(x) = \inf_\gamma u_\lambda\big(\gamma(-t)\big) + \int_{-t}^0 L_G\big(\gamma(s), u_\lambda\big(\gamma(s)\big), \dot\gamma(s)\big)ds,$$

where $L_G : \mathbb{T}^N \times \mathbb{R} \times \mathbb{R}^N$ is a function related to G that is non-increasing in u, and the infimum is taken amongst absolutely continuous curves $\gamma : \mathbb{T}^N \to \mathbb{R}$ such that $\gamma(0) = x$. The idea is then to fix $t > 0$ (small in spirit) and to consider an approximation of the integral by a function that may depend on $\gamma(0) = x$ and $\gamma(-t)$ but also possibly on the values $u_\lambda(x)$ and $u_\lambda\big(\gamma(-t)\big)$. When the approximation function depends linearly on $u_\lambda\big(\gamma(-t)\big)$, and does not depend on $u_\lambda(x)$, the whole discrete system then reduces to the case considered in [6] and [26, Chapter 3]. One can refer to the example given in Section 3 below.

0.3 Setting and statement of results

One advantage of the discretization is that it is non longer necessary to have a differentiable structure. We then work on (X,d) a compact metric space and consider a continuous function that is C^1 with respect to the last two variables[4], $\ell : X \times X \times \mathbb{R} \times \mathbb{R}$ such that

(1) there is a constant $\kappa_u > 0$ such that for all (x,y,u,v), $0 \geqslant \partial_u \ell(x,y,u,v) \geqslant -\kappa_u$,

(2) there is a constant $\kappa_v > 0$ such that for all (x,y,u,v), $0 \geqslant \partial_u \ell(x,y,u,v) \geqslant -\kappa_v$,

(3) $\int_{X\times X} \big(\partial_u \ell(z,x,0,0)+\partial_v \ell(z,x,0,0)\big)d\mu(z,x) < 0$ for all Mather measures μ of $\ell(\cdot,\cdot,0,0)$.

The notion of Mather measure will be detailed later in the paper. Let us already stress that it may happen (quite often actually) that there is only one Mather measure. Therefore if this is the case, this last nondegeneracy condition only requires that $\partial_u \ell(z,x,0,0)$ or $\partial_v \ell(z,x,0,0)$ is negative somewhere on the support of this Mather measure.

Let us denote $c_0 \in \mathbb{R}$ the critical constant of the function $\ell(\cdot,\cdot,0,0)$ (its precise definition is given later). Given this cost function (or discrete Lagrangian) we introduce an implicit Lax-Oleinik operator:

Proposition 0.1. *There is a $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, if $\varphi \in C^0(X,\mathbb{R})$ there is a unique $T_\lambda\varphi \in C^0(X,\mathbb{R})$ such that for all $x \in X$,*

$$T_\lambda\varphi(x) = \min_{z\in X} \big\{\varphi(z) + \ell\big(z,x,\lambda\varphi(z),\lambda T_\lambda\varphi(x)\big)\big\} + c_0.$$

The implicit Lax-Oleinik semigroup was studied in [21] in the continuous setting. It corresponds to the viscosity solutions of Hamilton-Jacobi equations depending on the unknown function. It is also meaningful in the optimal control theory of systems with a non-holonomic constraint, see [1]. Thus, our discrete semigroup here can be thought of as an approximation of the cost function of a class of optimal control systems.

We then solve the discounted equation:

Theorem 1. *For $\lambda \in (0,\lambda_0)$ the operator T_λ has a fixed point u_λ. Moreover if we set S_λ the set of fixed points of T_λ, then the family $(S_\lambda)_{\lambda\in(0,\lambda_0)}$ is made of equicontinuous and equibounded functions.*

Finally we prove the convergence of solutions of the discounted equations:

[4]Our results actually require less regularity.

Theorem 2. *The family* $(S_\lambda)_{\lambda\in(0,\lambda_0)}$ *converges to a singleton as* $\lambda \to 0$ *in the sense that there exists* $u_0 : X \to \mathbb{R}$ *such that for any choice* $u_\lambda \in S_\lambda$ *for* $\lambda \in (0, \lambda_0)$ *the (uniform) convergence* $u_\lambda \to u_0$ *holds as* $\lambda \to 0^+$.

In establishing those results we also prove two characterizations for the limit u_0. We also address the issue of uniqueness of fixed points $u_\lambda \in S_\lambda$ under quite natural assumptions. Actually, in the simplified setting presented above, we prove that S_λ is a singleton for λ small enough.

0.4 Organization of the paper

- In the first Section 1 we recall some needed facts on discrete weak KAM solutions (corresponding to $\lambda = 0$).
- In the following Section 2 we introduce a general theory of implicit Lax-Oleinik operators.
- Then in Section 3 we define and study solutions to the discounted equations.
- Finally in Section 4 we prove the convergence as $\lambda \to 0$ of solutions to the discounted equations.
- The last section adresses the uniqueness issue.

1 Classical discrete weak KAM theory

We briefly recall classical results that will be used in the rest of the paper. References are, amongst many others, [26, 24, 6]. Let (X, d) be a compact metric space and $\ell_0 : X \times X \to \mathbb{R}$ a continuous function sometimes called cost function. The discrete Lax-Oleinik semigroup is

Definition 1.1. The discrete Lax-Oleinik semigroup is the operator $T_0 : C^0(X, \mathbb{R}) \to C^0(X, \mathbb{R})$ which to $f : X \to \mathbb{R}$ associates

$$T_0 f : x \mapsto T_0 f(x) = \min_{y\in X} f(y) + \ell_0(y, x).$$

It can be checked that T_0 is non decreasing, 1-Lipschitz for the sup-norm and commutes with addition of constants. This allows to prove the discrete weak KAM theorem:

Theorem 3. *There exists a unique constant* c_0 *such that there is a function* $u : X \to \mathbb{R}$ *verifying* $u = T_0 u + c_0$.

The constant c_0 is called the critical constant of ℓ_0. A function u verifying $u = T_0 u + c_0$ is called a weak KAM solution. Weak KAM solutions are not unique, for instance, if $K \in \mathbb{R}$ then $u + K$ is also a weak KAM solution. Note also that the critical constant for the cost function $\tilde{\ell}_0 = \ell_0 + c_0$ is $\tilde{c}_0 = 0$.

By definition of T_0, a weak KAM solution u verifies $u(y) - u(x) \leqslant \ell_0(x, y) + c_0$ for all $x, y \in X$. This motivates the definition

Definition 1.2. A function $v : X \to \mathbb{R}$ is called a subsolution if it verifies

$$\forall (x,y) \in X \times X, \quad v(y) - v(x) \leqslant \ell_0(x,y) + c_0$$

This is equivalent to $T_0 v + c_0 \geqslant v$.

An easy but important fact is that

Proposition 1.1. *The set of subsolutions is closed (under uniform convergence but also under pointwise convergence) and it is convex.*

An important tool we will use is that of **Mather measure**. In all the paper, all measures are Borel measures even if not explicitly stated.

Definition 1.3. A Borel measure μ on $X \times X$ is closed if its marginals coincide: $\pi_{1*}\mu = \pi_{2*}\mu$, where $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$.

If $v : X \to \mathbb{R}$ is a continuous subsolution, integrating the family of inequalities $v(y) - v(x) \leqslant \ell_0(x,y) + c_0$ against a closed probability measure μ, we discover that $\int_{X\times X} \ell_0(x,y)d\mu(x,y) \geqslant -c_0$. This leads to the notion of Mather measures (or equivalently of minimizing measures):

Definition 1.4. A Mather measure (or minimizing measure) μ is a probability measure on $X \times X$ that is closed and verifies

$$\int_{X\times X} \ell_0(x,y)d\mu(x,y) = -c_0.$$

We will denote by $\mathfrak{M}_0$ the set of Mather measures.

Finally we will need an important function associated to ℓ_0 called Peierls' barrier. If $n > 0$, let

$$\forall (x,y) \in X \times X, \quad h_n(x,y) = \min_{\substack{(x_0,\cdots,x_n)\in X^{n+1} \\ x_0=x, x_n=y}} \sum_{k=0}^{n-1} \ell_0(x_k, x_{k+1}).$$

Definition 1.5. *Peierls' barrier is the function $h : X \times X \to \mathbb{R}$ defined by*

$$\forall (x,y) \in X \times X, \quad h(x,y) = \liminf_{n\to+\infty} h_n(x,y) + nc_0.$$

Here are some key properties of Peierls' barrier

Proposition 1.2. *1. The function h is finite valued and continuous on $X \times X$.*

2. *For all $x \in X$, the function $h(x,\cdot)$ is a weak KAM solution and the function $-h(\cdot,x)$ is a subsolution.*

3. *If $v : X \to \mathbb{R}$ is any subsolution, then*

$$\forall (x,y) \in X \times X, \quad v(y) - v(x) \leqslant h(x,y).$$

A crucial set in weak KAM theory is the projected Aubry set:

Definition 1.6. The projected Aubry set is $\mathfrak{A} = \{x \in X, \ \ h(x,x) = 0\}$.

This set is proven to be non-empty.

We end this section with a comparison principle:

Proposition 1.3. *Let $u : X \to \mathbb{R}$ be a weak KAM solution and $v : X \to \mathbb{R}$ be a subsolution. Assume that $u_{|\mathfrak{A}} \geq v_{|\mathfrak{A}}$ then $u \geqslant v$ on X.*

2 Discrete version of implicit semigroup

Assume (X, d) is a compact metric space where $d : X \times X \to \mathbb{R}$ is the distance function. In the paper, $c : X \times X \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. Hypotheses that will be needed are the following

(Lu) for each $(z, x, v) \in X \times X \times \mathbb{R}$, $u \mapsto c(z, x, u, v)$ is κ_u-Lipschitz continuous and $\kappa_u \leq 1$.

(Lv) for each $(z, x, u) \in X \times X \times \mathbb{R}$, $v \mapsto c(z, x, u, v)$ is κ_v-Lipschitz continuous and $\kappa_v < 1$.

Proposition 2.1. *Assume c verifies hypothesis* (Lv). *For each continuous function $\varphi : X \to \mathbb{R}$, there is a unique continuous function $T\varphi : X \to \mathbb{R}$ satisfying*

$$T\varphi(x) = \min_{z \in X} c\big(z, x, \varphi(z), T\varphi(x)\big).$$

The operator $T : (C^0(X, \mathbb{R}), \|\cdot\|_\infty) \to (C^0(X, \mathbb{R}), \|\cdot\|_\infty)$ is continuous and compact.

Moreover, if the family of functions $x \mapsto c(z, x, u, v)$ is locally equi-Lipschitz continuous, $T\varphi(x)$ is Lipschitz continuous.

Proof. We first prove that $T\varphi(x)$ exists. For a continuous function $f : X \to \mathbb{R}$, define

$$\mathcal{A}f(x) = \min_{z \in X} c\big(z, x, \varphi(z), f(x)\big).$$

By the continuity of c and f and compactness of X, we see that $\mathcal{A}$ is an operator from $C^0(X, \mathbb{R})$ to itself. Indeed, $\mathcal{A}f$ is an infimum of equicontinuous functions. We are going to find a fixed point of $\mathcal{A}$. We take two continuous functions f and g on X. By compactness of X, let z be a minimal point realizing the minimum in the definition of $\mathcal{A}g(x)$, then we have

$$\mathcal{A}f(x) - \mathcal{A}g(x) \leq c\big(z, x, \varphi(z), f(x)\big) - c\big(z, x, \varphi(z), g(x)\big) \leq \kappa_v \|f - g\|_\infty.$$

Exchanging the role of f and g, we get that $\mathcal{A}$ is a contraction in $(C^0(X, \mathbb{R}), \|\cdot\|_\infty)$, since $\kappa_v < 1$. By the Banach fixed point theorem, there is a unique fixed point of $\mathcal{A}$, which is $T\varphi(x)$.

Then we prove the operator T is compact. Let $r > 0$ and $\varphi \in C^0(X, \mathbb{R})$ such that $\|\varphi\|_\infty < r$. Consider the sequence $(f_n)_{n \in \mathbb{N}}$ with $f_0 = 0$ and

$$f_{n+1}(x) = \min_{z \in X} c\big(z, x, \varphi(z), f_n(x)\big).$$

Since $\mathcal{A}$ is a contraction, the sequence f_n converges to $T\varphi$ uniformly. We also have

$$\begin{aligned}\|f_{n+1}\|_\infty &\leq \sum_{i=0}^{n} \|f_{i+1} - f_i\|_\infty \leq \sum_{i=0}^{n} (\kappa_v)^i \|f_1\|_\infty \\ &\leq \frac{1}{1-\kappa_v}\|f_1\|_\infty \leq \frac{1}{1-\kappa_v} \max_{\substack{z\in X, x\in X\\ |u|\leq r}} |c(z,x,u,0)| =: R_0.\end{aligned}$$

Define $R := \max\{R_0, r\}$. Let ω be a modulus of continuity of c restricted to the compact set $X \times X \times [-R,R]^2$. For each $x, y \in X$, let z_y be a minimal point in the definition of $f_1(y)$, we have

$$f_1(x) - f_1(y) \leq c(z_y, x, \varphi(z_y), 0) - c(z_y, y, \varphi(z_y), 0) \leq \omega\big(d(x,y)\big).$$

Exchanging x and y, we have

$$|f_1(x) - f_1(y)| \leq \omega\big(d(x,y)\big).$$

More generally, if $f \in C^0(X, \mathbb{R})$ is such that $\|f\|_\infty < R$, and if ω_f is a modulus of continuity of f, then for $x, y \in X$,

$$\begin{aligned}\mathcal{A}f(x) - \mathcal{A}f(y) &\leqslant c\big(z_y, x, \varphi(z_y'), f(x)\big) - c\big(z_y', y, \varphi(z_y), f(y)\big) \\ &= c\big(z_y, x, \varphi(z_y'), f(x)\big) - c\big(z_y, x, \varphi(z_y'), f(y)\big) \\ &\quad + c\big(z_y, x, \varphi(z_y'), f(y)\big) - c\big(z_y', y, \varphi(z_y), f(y)\big) \\ &\qquad\qquad \leqslant \kappa_v \omega_f(x,y) + \omega(x,y).\end{aligned}$$

It follows, by exchanging the roles of x and y, that $\kappa_v\omega_f + \omega$ is a modulus of continuity of $\mathcal{A}f$. Applying to the sequence $(f_n)_n$ we obtain by induction that f_{n+1} has $(1 + \kappa_v + \cdots + \kappa_v^n)\omega$ as modulus of continuity. Hence the whole sequence is equicontinuous with modulus $\frac{\omega}{1-\kappa_v}$ and so is $T\varphi$. As $\|T\varphi\|_\infty \leqslant R_0$ and R_0 only depends on r, this proves that T is compact by the Arzela-Ascoli theorem. Next we prove that T is continuous. Let $(\varphi_n)_n$ be a sequence converging to φ. By the previous point, the sequence $(T\varphi_n)_n$ is precompact. Let $(k_n)_n$ be an extraction such that $(T\varphi_{k_n})_n$ uniformly converges to a function ψ. Then if $x \in X$ we can pass to the limit in the relations

$$T\varphi_{k_n}(x) = \min_{z\in X} c\big(z, x, \varphi_{k_n}(z), T\varphi_{k_n}(x)\big)$$

to obtain

$$\psi(x) = \min_{z\in X} c\big(z, x, \varphi(z), \psi(x)\big)$$

and by uniqueness, $\psi = T\varphi$. This proves that T is continuous.

Now we prove the Lipschitz continuity of $T\varphi$ under the additional Lipschitz assumption of c with respect to x. Let κ_x^R be the Lipschitz constant of $x \mapsto c(z,x,u,v)$ for $|u|$ and $|v|$ bounded by $R > 0$. Applying the previous method, we obtain that if f_n is κ_n-Lipschitz, then f_{n+1} is κ_{n+1}-Lipschitz with $\kappa_{n+1} = \kappa_x^R + \kappa_v\kappa_n$. Therefore, $(f_n)_n$ is equi-Lipschitz continuous with constant $\frac{\kappa_x^R}{1-\kappa_v}$, and uniformly converges to $T\varphi$. Then $T\varphi$ is Lipschitz continuous. □

Now we assume

(M) $u \mapsto c(z,x,u,v)$ is non-decreasing and $v \mapsto c(z,x,u,v)$ is non-increasing.

Proposition 2.2. *(Order preserving). If $f \le g$, then $Tf \le Tg$.*

Proof. We argue by contradiction. Assume there is $x \in X$ such that $Tf(x) > Tg(x)$. Let z_g be a point realizing the definition of $Tg(x)$, then we have

$$Tf(x) \le c\big(z_g, x, f(z_g), Tf(x)\big) \le c\big(z_g, x, g(z_g), Tg(x)\big) = Tg(x),$$

which leads to a contradiction. □

To end this section, assume now that the three hypotheses (Lu), (Lv) and (M) are satisfied:

Proposition 2.3. *(Non-expensiveness). For each f, g, we have $\|Tf - Tg\|_\infty \le \|f - g\|_\infty$.*

Proof. We are going to prove $Tf(x) - \|f-g\|_\infty - Tg(x) \le 0$ for each $x \in X$. We argue by contradiction. Assume there is $x \in X$ such that $Tf(x) - \|f-g\|_\infty - Tg(x) > 0$. Let z_g be a point realizing the definition of $Tg(x)$. Then we have

$$\begin{aligned}
&Tf(x) - \|f-g\|_\infty - Tg(x) \\
&\le c\big(z_g, x, f(z_g), Tf(x)\big) - \|f-g\|_\infty - c\big(z_g, x, g(z_g), Tg(x)\big) \\
&\le c\big(z_g, x, g(z_g), Tf(x)\big) - c\big(z_g, x, g(z_g), Tg(x)\big) \le 0,
\end{aligned}$$

which leads to a contradiction. For the last inequality, we use (Lu). Exchanging f and g, and then the proof is complete. □

3 Discounted solutions

Now we consider the discounted problem. Let $\ell : X \times X \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy

(l1) $\ell(z,x,u,v)$ is κ_u-Lipschitz in u and κ_v-Lipschitz in v.

(l2) $\ell(z,x,u,v)$ is non-increasing in u and v.

(l3) $\partial_u \ell(z,x,0,0)$ and $\partial_v \ell(z,x,0,0)$ exist, and

$$|\ell(z,x,v,u) - \ell(z,x,0,0) - \partial_u \ell(z,x,0,0)u - \partial_v \ell(z,x,0,0)v| \le \frac{|u|+|v|}{2}\eta(\frac{|u|+|v|}{2}),$$

where η is a modulus of continuity.

(l4) $\int_{X\times X} \big(\partial_u \ell(z,x,0,0) + \partial_v \ell(z,x,0,0)\big) d\mu(z,x) < 0$ for all Mather measures μ of $\ell(\cdot,\cdot,0,0)$.

We assume the critical value of $(x,z) \mapsto \ell(z,x,0,0)$ **equals zero**[5]**.**

Note that under hypothesis (l4), both functions $(z,x) \mapsto \partial_u \ell(z,x,0,0)$ and $(z,x) \mapsto \partial_v \ell(z,x,0,0)$ are continuous as uniform limits of continuous functions.

Let $\lambda_0 > 0$ such that $\lambda_0 \max(\kappa_u, \kappa_v) < 1$. For $\lambda_0 > \lambda > 0$, $u + \ell(z,x,\lambda u, \lambda v)$ satisfies the basic assumptions (Lu), (Lv) and (M) for $c(z,x,u,v)$ in Section 2. Define for $\varphi \in C^0(X,\mathbb{R})$

$$T_\lambda \varphi(x) = \min_{z \in X} \left\{ \varphi(z) + \ell\big(z,x,\lambda\varphi(z), \lambda T_\lambda \varphi(x)\big) \right\}.$$

This is well defined by Proposition 2.1.

<u>**Example**</u>: If $\ell(z,x,u,v)$ is of the form $\ell_0(z,x) - \alpha(z)u$, where $\alpha : X \to \mathbb{R}$ is a non-negative function, we have

$$T_\lambda \varphi(x) = \min_{z \in X} \left\{ \big(1 - \lambda\alpha(z)\big)\varphi(z) + \ell_0(z,x) \right\} = T_0\big((1-\lambda\alpha)\varphi\big)(x),$$

which is the degenerate vanishing discount problem as treated in [26].

Proposition 3.1. *For $\lambda < \lambda_0$, the operator T_λ admits at least one fixed point u_λ.*

Moreover, the family of all such fixed points $(u_\lambda)_{\lambda \in (0,\lambda_0)}$ is uniformly bounded and equicontinuous.

Proof. **Step 1.** We first prove the existence of u_λ for $\lambda < \lambda_0$. By the discrete weak KAM theorem, T_0 has a fixed point u (recall the critical constant is 0). Since

$$T_0(u+k) = T_0 u + k = u + k, \quad \forall k \in \mathbb{R},$$

we can choose $\bar{u} \geq 0$ with $T_0 \bar{u} = \bar{u}$. We prove that $T_\lambda \bar{u} \leq \bar{u}$. Assume there is a point $x \in X$ such that $T_\lambda \bar{u}(x) > \bar{u}(x)$. Let z be a point realizing the minimum in the definition of $T_0 \bar{u}(x)$, we have

$$T_\lambda \bar{u}(x) \leq \bar{u}(z) + \ell\big(z,x,\lambda\bar{u}(z), \lambda T_\lambda \bar{u}(x)\big) \leq \bar{u}(z) + \ell(z,x,0,0) = T_0 \bar{u}(x) = \bar{u}(x),$$

which leads to a contradiction. By Proposition 2.2, we have

$$\bar{u} \geq T_\lambda \bar{u} \geq T_\lambda \circ T_\lambda \bar{u} \geq \dots.$$

Similarly, let $\underline{u}$ be a negative weak KAM solution, we have

$$\underline{u} \leq T_\lambda \underline{u} \leq T_\lambda \circ T_\lambda \underline{u} \leq \dots.$$

Since $\underline{u} \leq \bar{u}$, we have for all $n \geqslant 0$,

$$\underline{u} \leq T_\lambda^n \underline{u} \leq T_\lambda^n \bar{u} \leq \bar{u}.$$

[5] If this is not the case, all our results apply to the function $\tilde{\ell} = \ell - c_0$ where c_0 is the critical constant of $(x,z) \mapsto \ell(z,x,0,0)$.

Now we show that $T_\lambda^n \underline{u}$ is equi-continuous for all $n \geq 1$ and $\lambda < \lambda_0$. Let ω be a modulus of continuity of ℓ restricted to $X \times X \times [-M, M]^2$ where $M > \lambda_0 \max(\|\bar{u}\|_\infty, \|\underline{u}\|_\infty)$. By symmetry of the roles of x and y, assume without loss of generality $T_\lambda^n \underline{u}(x) \geq T_\lambda^n \underline{u}(y)$, let z be a minimal point in the definition of $T_\lambda(T_\lambda^{n-1}\underline{u})(y)$, then by (11) we have

$$\begin{aligned}
|T_\lambda^n \underline{u}(x) - T_\lambda^n \underline{u}(y)| &= T_\lambda^n \underline{u}(x) - T_\lambda^n \underline{u}(y) \\
&\leq \ell\big(z, x, \lambda T_\lambda^{n-1}\underline{u}(z), \lambda T_\lambda^n \underline{u}(x)\big) - \ell\big(z, y, \lambda T_\lambda^{n-1}\underline{u}(z), \lambda T_\lambda^n \underline{u}(y)\big) \\
&\leq \ell\big(z, x, \lambda T_\lambda^{n-1}\underline{u}(z), \lambda T_\lambda^n \underline{u}(y)\big) - \ell\big(z, y, \lambda T_\lambda^{n-1}\underline{u}(z), \lambda T_\lambda^n \underline{u}(y)\big) \\
&\leq \omega\big(d(x,y)\big).
\end{aligned}$$

We finally get the equi-continuity of $T_\lambda^n \underline{u}$. Then $T_\lambda^n \underline{u}$ uniformly converges to a function $\tilde{u}_\lambda$. We have

$$\begin{aligned}
\|T_\lambda \tilde{u}_\lambda - \tilde{u}_\lambda\|_\infty &\leq \|T_\lambda \tilde{u}_\lambda - T_\lambda^n \underline{u}\|_\infty + \|T_\lambda^n \underline{u} - \tilde{u}_\lambda\|_\infty \\
&\leq \|\tilde{u}_\lambda - T_\lambda^{n-1}\underline{u}\|_\infty + \|T_\lambda^n \underline{u} - \tilde{u}_\lambda\|_\infty \to 0.
\end{aligned}$$

Then $\tilde{u}_\lambda$ is a fixed point of T_λ, and $\underline{u} \leq \tilde{u}_\lambda \leq \bar{u}$.

Step 2. We prove for $\lambda < \lambda_0$, all such fixed points u_λ are uniformly bounded, more precisely, $\underline{u} \leq u_\lambda \leq \bar{u}$. . We prove that $u_\lambda \leq \bar{u}$, the lower bound of u_λ is similar. Assume there is $x_0 \in X$ such that

$$u_\lambda(x_0) - \bar{u}(x_0) = \max_{x \in X} \big(u_\lambda(x) - \bar{u}(x)\big) > 0.$$

Let $(x_{-k})_{k \in \mathbb{N}}$ be a sequence obtained inductively such that for all $k \geqslant 0$, x_{-k-1} is a point realizing the minimum in the definition of $T_0 \bar{u}(x_{-k})$. It follows that for all $k \geqslant 0$, equatlity $\bar{u}(x_{-k}) - \bar{u}(x_{-k-1}) = \ell(x_{-k-1}, x_{-k}, 0, 0)$ holds. We first show that if $\lambda < \lambda_0$, we have $u_\lambda(x_{-k}) > \bar{u}(x_{-k})$ for all $k \geq 0$. Assume $u_\lambda(x_{-1}) \leq \bar{u}(x_{-1})$. By (11) and (12), we have

$$\begin{aligned}
u_\lambda(x_0) - u_\lambda(x_{-1}) &\leq \ell\big(x_{-1}, x_0, \lambda u_\lambda(x_{-1}), \lambda u_\lambda(x_0)\big) \\
&\leq \ell\big(x_{-1}, x_0, \lambda u_\lambda(x_{-1}), \lambda \bar{u}(x_0)\big) \\
&\leq \ell\big(x_{-1}, x_0, \lambda \bar{u}(x_{-1}), \lambda \bar{u}(x_0)\big) + \lambda \kappa_u (\bar{u} - u_\lambda)(x_{-1}) \\
&\leq \ell(x_{-1}, x_0, 0, 0) + \lambda \kappa_u (\bar{u} - u_\lambda)(x_{-1}) \\
&= \bar{u}(x_0) - \bar{u}(x_{-1}) + \lambda \kappa_u (\bar{u} - u_\lambda)(x_{-1}),
\end{aligned}$$

which implies that

$$(1 - \lambda \kappa_u)(u_\lambda - \bar{u})(x_{-1}) \geq u_\lambda(x_0) - \bar{u}(x_0) > 0,$$

which leads to a contradiction as $0 < 1 - \lambda \kappa_u < 1$. Then $u_\lambda(x_{-1}) > \bar{u}(x_{-1})$.

Note that by (12), we have

$$\begin{aligned}
\bar{u}(x_0) - \bar{u}(x_{-1}) &= \ell(x_{-1}, x_0, 0, 0) \\
&\geq \ell\big(x_{-1}, x_0, \lambda \bar{u}(x_{-1}), \lambda \bar{u}(x_0)\big) \\
&\geq \ell\big(x_{-1}, x_0, \lambda u_\lambda(x_{-1}), \lambda u_\lambda(x_0)\big) \\
&\geq u_\lambda(x_0) - u_\lambda(x_{-1}),
\end{aligned}$$

which implies $u_\lambda(x_{-1}) - \bar{u}(x_{-1}) \geq u_\lambda(x_0) - \bar{u}(x_0)$. By the definition of x_0, this must be an equality, that is, $u_\lambda(x_{-1}) - \bar{u}(x_{-1}) = u_\lambda(x_0) - \bar{u}(x_0)$. Moreover, as all previous inequalities are equalities, we obtain that $\ell(x_{-1}, x_0, 0, 0) = \ell\big(x_{-1}, x_0, \lambda\bar{u}(x_{-1}), \lambda\bar{u}(x_0)\big)$.

By induction, the same proof then shows that

$$u_\lambda(x_{-k}) - \bar{u}(x_{-k}) = \max_{x\in X}\big(u_\lambda(x) - \bar{u}(x)\big) > 0, \quad \forall k \geq 0,$$

and that $\ell(x_{-k-1}, x_{-k}, 0, 0) = \ell\big(x_{-k-1}, x_{-k}, \lambda\bar{u}(x_{-k-1}), \lambda\bar{u}(x_{-k})\big)$ for all $k \geqslant 0$.

Define the probability measure on $X \times X$, for $N > 0$,

$$\mu_N := N^{-1} \sum_{k=-N}^{-1} \delta_{(x_k, x_{k+1})}.$$

By weak compactness of measures on $X \times X$ let $N_n \to +\infty$ be an extraction and μ a probability measure on $X \times X$ such that $\mu_{N_n} \to \mu$, as $n \to +\infty$.

Let $f \in C^0(X, \mathbb{R})$. Since

$$\int_{X\times X} \big(f(x) - f(y)\big) d\mu_N = \frac{f(x_0) - f(x_{-N})}{N} \leq \frac{2\|f\|_\infty}{N} \to 0,$$

the measure μ is closed. We also have

$$\int_{X\times X} \ell(z, x, 0, 0) d\mu_N = \frac{\bar{u}(x_0) - \bar{u}(x_{-N})}{N} \to 0.$$

Thus, μ is a Mather measure. Since $\mu_{N_n} \to \mu$, for each $(z, x) \in \text{supp}(\mu)$, there is a sequence $(z_n, x_n) \in \text{supp}(\mu_{N_n})$ with $(z_n, x_n) \to (z, x)$. We have known that $u_\lambda(z_n) - \bar{u}(z_n)$ equals a constant $M > 0$. Therefore, $u_\lambda(z) - \bar{u}(z) = M > 0$. Similarly, we have $u_\lambda(x) - \bar{u}(x) = M > 0$.

By the same argument, from $\ell(z_n, x_n, 0, 0) = \ell\big(z_n, x_n, \lambda\bar{u}(z_n), \lambda\bar{u}(x_n)\big)$ we obtain $\ell(z, x, 0, 0) = \ell\big(z, x, \lambda\bar{u}(z), \lambda\bar{u}(x)\big)$.

Since $u_\lambda > \bar{u} \geq 0$ on supp(μ), by (12), we have

$$\ell(z, x, r, s) = \ell\big(z, x, \lambda u_\lambda(z), \lambda u_\lambda(x)\big),$$

for all $(z, x) \in \text{supp}(\mu)$ and $r \in [0, \lambda u_\lambda(z)]$, $s \in [0, \lambda u_\lambda(x)]$. Then

$$\partial_{u,v}\ell(z, x, 0, 0) = 0, \quad \forall (z, x) \in \text{supp}(\mu),$$

which contradicts (14).

Step 3. We finally prove the equi-continuity of u_λ. Let $x, y \in X$. Assume without loss of generality that $u_\lambda(x) \geq u_\lambda(y)$, let z the a minimal point of $u_\lambda(y)$, then by (11) we have

$$\begin{aligned}
u_\lambda(x) - u_\lambda(y) &\leq \ell\big(z, x, \lambda u_\lambda(z), \lambda u_\lambda(x)\big) - \ell\big(z, y, \lambda u_\lambda(z), \lambda u_\lambda(y)\big) \\
&\leq \ell\big(z, x, \lambda u_\lambda(z), \lambda u_\lambda(y)\big) - \ell\big(z, y, \lambda u_\lambda(z), \lambda u_\lambda(y)\big) \\
&\leq \omega\big(d(x, y)\big).
\end{aligned}$$

$\square$

4 Vanishing discount convergence

Let $\lambda \to 0$, by Proposition 3.1, there is a sequence $\lambda_n \to 0$ such that u_{λ_n} uniformly converges. Let u_* be a limit function of the family $(u_\lambda)_{\lambda\in(0,\lambda_0)}$. The vanishing discount problem concerns the uniqueness of u_*.

Notation: Let us define $\ell_0 : X \times X \to \mathbb{R}$ the function defined by $(z,x) \mapsto \ell(z,x,0,0)$.

Let $\mathcal{S}_0$ be the set of subsolutions w of ℓ_0 that satisfy

$$\int_{X\times X} \big(\partial_u \ell(z,x,0,0)w(z) + \partial_v \ell(z,x,0,0)w(x)\big)d\mu(z,x) \geq 0, \tag{4.1}$$

for all Mather measures μ of ℓ_0. The set $\mathcal{S}_0$ is non-empty since negative weak KAM solutions fullfill (4.1).

Theorem 4. *Let $\lambda \to 0$, u_λ uniformly converges to*

$$u_0 := \sup_{w\in\mathcal{S}_0} w,$$

where the supremum is taken pointwise. The function u_0 is therefore a fixed point of T_0.

We also establish an alternative formula for the limit u_0:

Theorem 5. *The following holds for all $x \in X$:*

$$u_0(x) = \min_{\mu\in\mathfrak{M}_0} \frac{\int_{X\times X}\Big(\partial_u \ell(z,y,0,0)h(z,x) + \partial_v \ell(z,y,0,0)h(y,x)\Big)d\mu(z,y)}{\int_{X\times X}\Lambda(z,y)d\mu(z,y)},$$

where $\mathfrak{M}_0$ denotes the set of Mather measures of ℓ_0,

$$\Lambda(z,y) := \partial_u \ell(z,y,0,0) + \partial_v \ell(z,y,0,0),$$

and $h(z,x)$ is Peierls' barrier of ℓ_0.

Remark 4.1.
1. When ℓ satisfies that $\partial_v \ell(\cdot,\cdot,0,0)$ is constant, the previous equality reduces to
$$u_0(x) = \min_{\mu\in\mathfrak{M}_0}\left(\int_{X\times X}\Lambda(z,y)d\mu(z,y)\right)^{-1}\int_{X\times X}\Lambda(z,y)h(z,x)d\mu(z,y).$$
2. Symmetrically, when ℓ satisfies that $\partial_u \ell(\cdot,\cdot,0,0)$ is constant, then
$$u_0(x) = \min_{\mu\in\mathfrak{M}_0}\left(\int_{X\times X}\Lambda(y,z)d\mu(y,z)\right)^{-1}\int_{X\times X}\Lambda(z,y)h(y,x)d\mu(z,y).$$

Proposition 4.1. *For each Mather measure μ of ℓ_0, we have*

$$\int_X \big(\partial_u \ell(z,x,0,0)u_*(z) + \partial_v \ell(z,x,0,0)u_*(x)\big)d\mu(z,x) \geq 0.$$

Proof. Let μ be a Mather measure, Since $T_\lambda u_\lambda = u_\lambda$, recalling that $\int \ell_0 d\mu = 0$, we have

$$\begin{aligned}
&\int_{X\times X} \big(u_\lambda(x) - u_\lambda(z)\big) d\mu(z,x) \\
&\leq \int_{X\times X} \ell\big(z, x, \lambda u_\lambda(z), \lambda u_\lambda(x)\big) d\mu(z,x) \\
&\leq \int_{X\times X} \big(\ell(z,x,0,0) + \lambda\partial_u \ell(z,x,0,0)u_\lambda(z) + \lambda\partial_v\ell(z,x,0,0)u_\lambda(x)\big) d\mu(z,x) + \lambda\varepsilon(\lambda) \\
&= \int_{X\times X} \big(\lambda\partial_u \ell(z,x,0,0)u_\lambda(z) + \lambda\partial_v\ell(z,x,0,0)u_\lambda(x)\big) d\mu(z,x) + \lambda\varepsilon(\lambda),
\end{aligned}$$

where $\varepsilon(\lambda) = \|u_\lambda\|_\infty \eta(\lambda \|u_\lambda\|_\infty)$. Since μ is closed, we have

$$\int_{X\times X} \big(u_\lambda(x) - u_\lambda(z)\big) d\mu(z,x) = 0.$$

Therefore, we have

$$\int_{X\times X} \big(\partial_u \ell(z,x,0,0)u_\lambda(z) + \partial_v\ell(z,x,0,0)u_\lambda(x)\big) d\mu(z,x) \geq -\varepsilon(\lambda).$$

Letting $\lambda \to 0$ along the sequence $(\lambda_n)_n$, we then get the result. $\square$

Remark 4.2. As observed in the proof of Proposition 3.1, since u_λ is a fixed point of T_λ, for each $x \in X$, there is a sequence $(x_n)_{-n\in\mathbb{N}}$ with $x_0 = x$, such that

$$\forall n \leqslant 0, \quad u_\lambda(x_n) = u_\lambda(x_{n-1}) + \ell\big(x_{n-1}, x_n, \lambda u_\lambda(x_{n-1}), \lambda u_\lambda(x_n)\big).$$

Here we note that the sequence $(x_n)_{-n\in\mathbb{N}}$ depends on x and λ.

Lemma 4.1. *For $-n \in \mathbb{N}_+$, we define*

$$\beta_n = \frac{\prod_{i=n}^{-1}\big(1 + \lambda\partial_u\ell(x_i, x_{i+1}, 0, 0)\big)}{\prod_{i=n}^{0}\big(1 - \lambda\partial_v\ell(x_{i-1}, x_i, 0, 0)\big)}, \quad \beta_0 = \frac{1}{1 - \lambda\partial_v\ell(x_{-1}, x, 0, 0)}.$$

Since $(x_n)_{-n\in\mathbb{N}}$ depends on x and λ, the sequence $(\beta_n)_{-n\in\mathbb{N}}$ also depends on x and λ. For each integer $N > 0$, we have

$$\begin{aligned}
u_\lambda(x) = &\sum_{n=-N+1}^{0} \beta_n \Big(\ell(x_{n-1}, x_n, 0, 0) + \theta_k(\lambda)\Big) \\
&+ \big(1 + \lambda\partial_u\ell(x_{-N}, x_{-N+1}, 0, 0)\big)\beta_{-N+1} u_\lambda(x_{-N}),
\end{aligned} \tag{4.2}$$

where $|\theta_k(\lambda)| \leq \lambda\varepsilon(\lambda)$.

Proof. Since u_λ is a fixed point of T_λ, we have

$$\begin{aligned}
u_\lambda(x) &= u_\lambda(x_{-1}) + \ell\big(x_{-1}, x, \lambda u_\lambda(x_{-1}), \lambda u_\lambda(x)\big) \\
&= u_\lambda(x_{-1}) + \ell(x_{-1}, x, 0, 0) + \partial_u\ell(x_{-1}, x, 0, 0)\lambda u_\lambda(x_{-1}) + \partial_v\ell(x_{-1}, x, 0, 0)\lambda u_\lambda(x) + \theta_0(\lambda),
\end{aligned}$$

where $|\theta_0(\lambda)| \le \lambda\varepsilon(\lambda)$, which implies

$$\begin{aligned} u_\lambda(x) &= \frac{1+\lambda\partial_u\ell(x_{-1},x,0,0)}{1-\lambda\partial_v\ell(x_{-1},x,0,0)}u_\lambda(x_{-1}) + \frac{1}{1-\lambda\partial_v\ell(x_{-1},x,0,0)}\big(\ell(x_{-1},x,0)+\theta_0(\lambda)\big) \\ &= \big(1+\lambda\partial_u\ell(x_{-1},x,0,0)\big)\beta_0 u_\lambda(x_{-1}) + \beta_0\big(\ell(x_{-1},x,0)+\theta_0(\lambda)\big). \end{aligned}$$

We also have

$$u_\lambda(x_{-1}) = u_\lambda(x_{-2}) + \ell\big(x_{-2},x_{-1},\lambda u_\lambda(x_{-2}),\lambda u_\lambda(x_{-1})\big),$$

which implies

$$\begin{aligned} &\big(1+\lambda\partial_u\ell(x_{-1},x,0,0)\big)\beta_0 u_\lambda(x_{-1}) \\ &= \big(1+\lambda\partial_u\ell(x_{-2},x_{-1},0,0)\big)\beta_{-1}u_\lambda(x_{-2}) + \beta_{-1}\big(\ell(x_{-2},x_{-1},0,0)+\theta_{-1}(\lambda)\big), \end{aligned}$$

where $|\theta_{-1}(\lambda)| \le \lambda\varepsilon(\lambda)$. Letting this procedure go on, and adding all equalities up, we get (4.2). □

Proposition 4.2. *There is $r>0$ such that for each $\lambda\in(0,r)$, there is $K>0$, independent of λ and x, such that $\lambda\sum_{k\le 0}\beta_k \le K$.*

Proof. We argue by contradiction. Assume there is a sequence $(\lambda_n)_{n\in\mathbb{N}}\to 0$ and $(x_n)_{n\in\mathbb{N}}\in X^{\mathbb{N}}$, and for all n, a minimal sequence $(x_k^n)_{-k\in\mathbb{N}}$ associated to $u_{\lambda_n}(x_n)$ and an integer $N_n>0$ such that

$$\lambda_n\sum_{k=-N_n}^{-1}\beta_{k+1}^n \to +\infty.$$

Here $(\beta_k^n)_{-k\in\mathbb{N}}$ is the sequence associated to $(x_k^n)_{-k\in\mathbb{N}}$ as defined in Lemma 4.1, which depends on x_n and λ_n. As for k fixed, $\beta_k^n\to 1$ as $n\to+\infty$, we have $N_n\to+\infty$.

Define the probability measure

$$\mu_n := C_n^{-1}\sum_{k=-N_n}^{-1}\beta_{k+1}^n\delta_{(x_k^n,x_{k+1}^n)},$$

where $C_n=\sum_{k=-N_n}^{-1}\beta_{k+1}^n$. Up to an extraction, we assume $\mu_n\to\mu$.

μ **is closed:** Let $f\in C^0(X,\mathbb{R})$, we have

$$\begin{aligned} &\Big|\int_{X\times X}\big(f(x)-f(z)\big)d\mu_n(z,x)\Big| \\ &= C_n^{-1}\Big|\sum_{k=-N_n}^{-1}\beta_{k+1}^n\big(f(x_{k+1}^n)-f(x_k^n)\big)\Big| \\ &= C_n^{-1}\Big|\sum_{k=-N_n}^{-1}(\beta_k^n-\beta_{k+1}^n)f(x_k^n) - \beta_{-N_n}^n f(x_{-N_n}^n) + \beta_0^n f(x_n)\Big| \\ &\le C_n^{-1}\Big(\sum_{k=-N_n}^{-1}(\beta_{k+1}^n-\beta_k^n)\|f\|_\infty + 2\|f\|_\infty\Big) \le 4C_n^{-1}\|f\|_\infty, \end{aligned}$$

where we use the fact $\beta_k^n \leq \beta_{k+1}^n$. Since $C_n \to +\infty$, μ is closed.

μ **is minimizing:** By definition we have

$$\begin{aligned}
&\left| \int_{X\times X} \ell(z,x,0,0) d\mu_n(z,x) \right| \\
&= C_n^{-1} \left| \sum_{k=-N_n}^{-1} \beta_{k+1}^n \ell(x_k^n, x_{k+1}^n, 0, 0) \right| \\
&= C_n^{-1} \left| u_{\lambda_n}(x_n) - \sum_{k=-N_n}^{-1} \beta_{k+1}^n \theta_{k+1}^n(\lambda_n) - \big(1 + \lambda \partial_u \ell(x_{-N}^n, x_{-N+1}^n, 0, 0)\big) \beta_{-N+1}^n u_\lambda(x_{-N}^n) \right| \\
&\leq 2C_n^{-1} \|u_{\lambda_n}\|_\infty + \lambda_n \omega(\lambda_n) \to 0.
\end{aligned}$$

Now using

$$\frac{1+x}{1-y} = 1 + \frac{x+y}{1-y} \leq \exp\left\{ \frac{x+y}{1-y} \right\} \leq \exp\{x+y\}, \quad \text{for } \ y \leq 0,$$

we get

$$\begin{aligned}
-\int_{X\times X} \Lambda(z,x) d\mu_n(z,x) &= -C_n^{-1} \sum_{k=-N_n}^{-1} \beta_{k+1}^n \Lambda(x_k^n, x_{k+1}^n) \\
&\leq -C_n^{-1} \sum_{k=-N_n}^{-1} \exp\{\lambda_n \sum_{i=k+1}^{-1} \Lambda(x_i^n, x_{i+1}^n)\} \Lambda(x_k^n, x_{k+1}^n) \\
&\leq -C_n^{-1} \exp\{\|\Lambda\|_\infty\} \sum_{k=-N_n}^{-1} \exp\{\lambda_n \sum_{i=k}^{-1} \Lambda(x_i^n, x_{i+1}^n)\} \Lambda(x_k^n, x_{k+1}^n) \\
&\leq C_n^{-1} \exp\{\|\Lambda\|_\infty\} \int_0^{+\infty} e^{-\lambda_n x} dx = \frac{\exp\{\|\Lambda\|_\infty\}}{\lambda_n C_n} \to 0,
\end{aligned}$$

which contradicts (l4). $\square$

In the following, let $(x_n)_{-n\in\mathbb{N}}$ and $(\beta_n)_{-n\in\mathbb{N}}$ be the sequences defined in Remark 4.2 and Lemma 4.1 respectively, associated to the pair $x_0 \in X$ and $\lambda \in (0,r)$.

Proposition 4.3. *For each $\lambda \in (0,r)$ and $x_0 \in X$, we have*

$$u_\lambda(x_0) = \sum_{n\leq 0} \beta_n \ell(x_{n-1}, x_n, 0, 0) + \Omega(\lambda), \tag{4.3}$$

where

$$\lim_{\lambda\to 0} \Omega(\lambda) \to 0.$$

Proof. Since $\lambda \sum_{n\leq 0} \beta_n \leq K$, we have $\beta_{-N+1} \to 0$ and

$$\left| \sum_{n=-N+1}^{0} \beta_n \theta_n(\lambda) \right| \leq \sum_{n\leq 0} \beta_n \lambda \varepsilon(\lambda) \leq K\varepsilon(\lambda) \to 0.$$

By (4.2) we get (4.3), where $\Omega(\lambda) := \sum_{n\leq 0} \beta_n \theta_n(\lambda)$. $\square$

Definition 4.1. We define the following probability measures on $X \times X$

$$\mu_\lambda^1 = C_\lambda^{-1} \sum_{k\le -1} \frac{\beta_{k+1}}{1-\lambda\partial_v \ell(x_{k-1},x_k,0,0)} \delta_{(x_k,x_{k+1})},$$

and

$$\mu_\lambda^2 = C_\lambda^{-1} \sum_{k\le -1} \frac{\beta_{k+1}}{1-\lambda\partial_v \ell(x_{k-1},x_k,0,0)} \delta_{(x_{k-1},x_k)},$$

where

$$C_\lambda := \sum_{k\le -1} \frac{\beta_{k+1}}{1-\lambda\partial_v \ell(x_{k-1},x_k,0,0)}.$$

Note that as x_0 will be fixed in what follows, we only specify explicitly the dependance of those measures in λ but they also depend on x_0.

Since the functions $(z,x)\mapsto \partial_v\ell(z,x,0,0)$ and $(z,x)\mapsto \partial_u\ell(z,x,0,0)$ are bounded respectively by κ_v and κ_u, it is easily observed that $C_\lambda \to +\infty$ (each term of the sum converges to 1 as $\lambda\to 0$). Moreover $\lambda C_\lambda \le \lambda \sum_{k\le 0}\beta_k \le K$.

Proposition 4.4. *For each subsolution w of ℓ_0 and $\lambda\in(0,r)$, we have*

$$\begin{aligned} u_\lambda(x) \ge{}& \beta_0 w(x) \\ &+ \lambda C_\lambda \int_{X\times X} \partial_u\ell(z,x,0,0) w(z)\, d\mu_\lambda^1(z,x) \\ &+ \lambda C_\lambda \int_{X\times X} \partial_v\ell(z,x,0,0) w(x)\, d\mu_\lambda^2(z,x) + \Omega(\lambda). \end{aligned}$$

Proof. By (4.3) we have

$$\begin{aligned} u_\lambda(x) &\ge \sum_{n\le 0} \beta_n\big(w(x_n)-w(x_{n-1})\big) + \Omega(\lambda) \\ &= \beta_0 w(x) + \lambda \sum_{k\le -1} \frac{\beta_{k+1}}{1-\lambda\partial_v\ell(x_{k-1},x_k,0,0)} \partial_u\ell(x_k,x_{k+1},0,0) w(x_k) \\ &\quad + \lambda \sum_{k\le -1} \frac{\beta_{k+1}}{1-\lambda\partial_v\ell(x_{k-1},x_k,0,0)} \partial_v\ell(x_{k-1},x_k,0,0) w(x_k) + \Omega(\lambda) \\ &\ge \beta_0 w(x) + \lambda C_\lambda \int_{X\times X} \partial_u\ell(z,x,0,0) w(z)\, d\mu_\lambda^1 \\ &\quad + \lambda C_\lambda \int_{X\times X} \partial_v\ell(z,x,0,0) w(x)\, d\mu_\lambda^2(z,x) + \Omega(\lambda). \end{aligned}$$

□

Lemma 4.2. *The limits of μ_λ^1 and μ_λ^2 coincide in the weak* topology as $\lambda\to 0$. That is, if there is a sequence $\lambda_n\to 0$ such that $\mu_{\lambda_n}^1\to\mu$, then $\mu_{\lambda_n}^2\to\mu$.*

Proof. For all $f \in C^0(X \times X, \mathbb{R})$, we have

$$\begin{aligned}
&\Big| \int_{X\times X} f(z,x) d(\mu_\lambda^1 - \mu_\lambda^2) \Big| \\
&= C_\lambda^{-1} \Big| \sum_{k\le -1} \frac{\beta_{k+1}}{1-\lambda\partial_v \ell(x_{k-1},x_k,0,0)} \big(f(x_k,x_{k+1}) - f(x_{k-1},x_k) \big) \Big| \\
&= C_\lambda^{-1} \Big| \frac{\beta_0}{1-\lambda\partial_v \ell(x_{-2},x_{-1},0,0)} f(x_{-1},x_0) + \sum_{k\le -2} \frac{\beta_{k+1}}{1-\lambda\partial_v \ell(x_{k-1},x_k,0,0)} f(x_k,x_{k+1}) \\
&\quad - \sum_{k\le -2} \frac{\beta_{k+1}}{1+\lambda\partial_u \ell(x_{k+1},x_{k+2},0,0)} f(x_k,x_{k+1}) \Big| \\
&\le C_\lambda^{-1} \|f\|_\infty \\
&\quad + \lambda C_\lambda^{-1} \Big| \sum_{k\le -2} \frac{\partial_u \ell(x_{k+1},x_{k+2},0,0) + \partial_v \ell(x_{k-1},x_k,0,0)}{\big(1-\lambda\partial_v \ell(x_{k-1},x_k,0,0)\big)\big(1+\lambda\partial_u \ell(x_{k+1},x_{k+2},0,0)\big)} \beta_{k+1} f(x_k,x_{k+1}) \Big| \\
&\le C_\lambda^{-1} \|f\|_\infty + \lambda C_\lambda^{-1} \frac{\kappa_u+\kappa_v}{1-\lambda\kappa_u} \sum_{k\le -1} \beta_{k+1} \|f\|_\infty \\
&\le \left(1 + \frac{\kappa_u+\kappa_v}{1-\lambda\kappa_u} K \right) C_\lambda^{-1} \|f\|_\infty \to 0.
\end{aligned}$$

Indeed, recall that $C_\lambda \to +\infty$ as $\lambda \to 0$. □

Proposition 4.5. *Any limit μ of $d\mu_{\lambda_n}^{1,2}$ as $\lambda_n \to 0$ is a Mather measure of $\ell(z,x,0,0)$.*

Proof. We first prove that μ is **closed**. Let $f \in C^0(X,\mathbb{R})$, then

$$\begin{aligned}
&\int_{X\times X} \big(f(x) - f(z)\big) d\mu_\lambda^1(z,x) \\
&= C_\lambda^{-1} \sum_{k\le -1} \frac{\beta_{k+1}}{1-\lambda\partial_v \ell(x_{k-1},x_k,0,0)} \big(f(x_{k+1}) - f(x_k) \big) \\
&= C_\lambda^{-1} \frac{\beta_0}{1-\lambda\partial_v \ell(x_{-2},x_{-1},0,0)} f(x) \\
&\quad + \lambda C_\lambda^{-1} \sum_{k\le -1} \beta_{k+1} \frac{\partial_u \ell(x_{k-2},x_{k-1},0,0) + \partial_v \ell(x_k,x_{k+1},0,0)}{\big(1-\lambda\partial_v \ell(x_{k-2},x_{k-1},0,0)\big)\big(1-\lambda\partial_v \ell(x_{k-1},x_k,0,0)\big)} f(x_k).
\end{aligned}$$

Since $-\kappa_v < \partial_v \ell(z,x,0,0) \le 0$ and $-\kappa_v \le \partial_u \ell(z,x,0,0) \le 0$, we have

$$\begin{aligned}
&\Big| \int_{X\times X} \big(f(x) - f(z)\big) d\mu_\lambda^1(z,x) \Big| \\
&\qquad\qquad \le C_\lambda^{-1} \|f\|_\infty (1 + (\kappa_u+\kappa_v)\lambda \sum_{k\le 0} \beta_k) \\
&\qquad\qquad\qquad \le (1 + (\kappa_u+\kappa_v)K) C_\lambda^{-1} \|f\|_\infty \to 0.
\end{aligned}$$

Then we prove μ is **minimizing**. By (4.3) we have

$$\begin{aligned}
&\Big|\int_{X\times X}\ell(z,x,0,0)\big(1-\lambda\partial_v\ell_0(z,x)\big)d\mu_\lambda^2(z,x)-C_\lambda^{-1}u_\lambda(x_0)\Big|\\
&\quad=C_\lambda^{-1}\Big|\sum_{k\le -1}(\beta_{k+1}-\beta_k)\ell(x_{k-1},x_k,0,0)-\beta_0\ell_0(x_{-1},x)-\Omega(\lambda)\Big|\\
&\quad\le C_\lambda^{-1}\Bigg(\Big|\lambda\sum_{k\le -1}\frac{\partial_u\ell_0(x_k,x_{k+1},0,0)+\partial_v\ell(x_{k-1},x_k,0,0)}{1-\lambda\partial_v\ell(x_{k-1},x_k,0,0)}\beta_{k+1}\ell_0(x_{k-1},x_k)\Big|\\
&\qquad\qquad+\|\ell_0\|_\infty+K\varepsilon(\lambda)\Bigg)\\
&\quad\le C_\lambda^{-1}\big(\lambda\sum_{k\le 0}(\kappa_u+\kappa_v)\beta_k\|\ell_0\|_\infty+\|\ell_0\|_\infty+K\varepsilon(\lambda)\big)\\
&\quad\le C_\lambda^{-1}\big(((\kappa_u+\kappa_v)K+1)\|\ell_0\|_\infty+K\varepsilon(\lambda)\big)\to 0.
\end{aligned}$$

Let $\lambda\to 0$, we get $\int_{X\times X}\ell_0(z,x)d\mu_\lambda(z,x)=0$. □

We now turn to the proof of the main theorem. Recall that $\mathcal{S}_0$ is the set of subsolutions w of ℓ_0 that satisfy

$$\int_{X\times X}\big(\partial_u\ell(z,x,0,0)w(z)+\partial_v\ell(z,x,0,0)w(x)\big)d\mu(z,x)\ge 0,$$

for all Mather measures μ of ℓ_0.

Proof of Theorem 4. We first show that u_0 is well-define, that is, functions in $\mathcal{S}_0$ are uniformly bounded from above. Assume there is $w\in\mathcal{S}_0$ such that $w\ge\delta>0$, then by (4.1) we have

$$\begin{aligned}
0&\le\int_{X\times X}\big(\partial_u\ell(z,x,0,0)w(z)+\partial_v\ell(z,x,0,0)w(x)\big)d\mu(z,x)\\
&\le\delta\int_{X\times X}\big(\partial_u\ell(z,x,0,0)+\partial_v\ell(z,x,0,0)\big)d\mu(z,x),
\end{aligned}$$

which contradicts (l4). Therefore, for all $w\in\mathcal{S}_0$, there is $x_0\in X$ such that $w(x_0)\le 0$. By the equi-continuity of subsolutions of ℓ_0, the result follows.

Recall that we consider a decreasing sequence $\lambda_n\to 0$ such that $u_{\lambda_n}\to u_*$ uniformly.

Since for all $(x,z)\in X\times X$,

$$u_\lambda(x)-u_\lambda(z)\le\ell\big(z,x,\lambda u_\lambda(x),\lambda u_\lambda(z)\big).$$

Let $\lambda\to 0$ we get

$$u_*(z)-u_*(x)\le\ell(z,x,0,0),$$

which means u_* is a subsolution of ℓ_0. By Proposition 4.1, $u_*\in\mathcal{S}_0$, which implies $u_*\le u_0$.

Let now $x_0 \in X$ and up to a further extraction, assume that the associated measures converge. By Lemma 4.2 the limits are the same and $\lim\limits_{n\to+\infty} \mu^1_{\lambda_n} = \lim\limits_{n\to+\infty} \mu^2_{\lambda_n} = \mu$ is a Mather measure by Proposition 4.5. For each $w \in \mathcal{S}_0$, by Proposition 4.4, we have

$$\begin{aligned} u_*(x_0) \geq & \lim_{n\to+\infty} \beta_0 w(x_0) \\ & + \limsup_{n\to+\infty} \lambda_n C_{\lambda_n} \Big(\int_{X\times X} \partial_u \ell(z,x,0,0) w(z) d\mu^1_{\lambda_n} + \int_{X\times X} \partial_v \ell(z,x,0,0) w(x) d\mu^2_{\lambda_n} \Big) \\ = & w(x_0) + \limsup_{n\to+\infty} \lambda_n C_{\lambda_n} \int_{X\times X} \big(\partial_u \ell(z,x,0,0) + \partial_v \ell(z,x,0,0)\big) w(x) d\mu \\ \geq & w(x_0), \end{aligned}$$

where we have used that $w \in \mathcal{S}_0$. Therefore, $u_*(x_0) \geq \sup_{w\in\mathcal{S}_0} w(x_0) = u_0(x_0)$. We finally get $u_* = u_0$.

Now we prove that u_0 is a fixed point of T_0. We have seen that $u_0 = u_*$ is a subsolution. Let $x_0 \in X$. Since X is compact, let z_n be a point realizing the minimum in $u_{\lambda_n}(x_0)$ and up to extracting, assume $z_{\lambda_n} \to z_*$. By

$$u_{\lambda_n}(x_0) - u_{\lambda_n}(z_{\lambda_n}) = \ell\big(z_{\lambda_n}, x_0, \lambda_n u_{\lambda_n}(z_{\lambda_n}), \lambda_n u_{\lambda_n}(x_0)\big),$$

we get

$$u_0(x_0) - u_0(z_*) = \ell(z_*, x_0, 0, 0).$$

Thus, u_0 is a fixed point of T_0. □

Remark 4.3. As a byproduct of the previous proof, we have also proven that $u_0 \in \mathcal{S}_0$.

We finish this section by the alternative representation formula of u_0.

Proof of Theorem 5. Define

$$\hat{u}_0(x) = \min_{\mu\in\mathfrak{M}_0} \frac{\int_{X\times X} \Big(\partial_u \ell(z,y,0,0) h(z,x) + \partial_v \ell(z,y,0,0) h(y,x)\Big) d\mu(z,y)}{\int_{X\times X} \Lambda(z,y) d\mu(z,y)},$$

where $\mathfrak{M}_0$ denotes the set of Mather measures of ℓ_0. Note first that for each $\mu \in \mathfrak{M}_0$, the function

$$x \mapsto \frac{\int_{X\times X} \Big(\partial_u \ell(z,y,0,0) h(z,x) + \partial_v \ell(z,y,0,0) h(y,x)\Big) d\mu(z,y)}{\int_{X\times X} \Lambda(z,y) d\mu(z,y)},$$

is a subsolution of ℓ_0. Indeed each $-h(z,\cdot)$ is a subsolution (Proposition 1.2) hence the integral is a barycenter of subsolutions (Proposition 1.1). Hence $\hat{u}_0$ is also a subsolution for ℓ_0 as an infimum of subsolutions.

proof that $u_0 \leq \hat{u}_0$: Let $x \in X$ and $\mu \in \mathfrak{M}_0$. Integrating the inequalities $u_0(x) \leqslant u_0(z) + h(z,x)$ recalled in Proposition 1.2, we find that

$$u_0(x) \int_{X\times X} \partial_u \ell(z,y,0,0) d\mu(z,y) \geq \int_{X\times X} \partial_u \ell(z,y,0,0) \big(u_0(z) + h(z,x)\big) d\mu(z,y), \quad (4.4)$$

$$u_0(x)\int_{X\times X}\partial_v\ell(z,y,0,0)d\mu(z,y)\geq\int_{X\times X}\partial_v\ell(z,y,0,0)\big(u_0(y)+h(y,x)\big)d\mu(z,y).\quad(4.5)$$

Recall that $u_0\in\mathcal{S}_0$ so that

$$\int_{X\times X}\big(\partial_u\ell(z,y,0,0)u_0(z)+\partial_v\ell(z,y,0,0)u_0(y)\big)d\mu(z,y)\geq 0.$$

Therefore, summing (4.4) and (4.5) and using the previous inequality, we obtain

$$u_0(x)\int_{X\times X}\Lambda(z,y)d\mu(z,y)\geq\int_{X\times X}\Big(\partial_u\ell(z,y,0,0)h(z,x)+\partial_v\ell(z,y,0,0)h(y,x)\Big)d\mu(z,y).$$

Dividing by $\int_{X\times X}\Lambda(z,y)d\mu(z,y)<0$ and taking a minimum over $\mu\in\mathfrak{M}_0$ yields the desired $u_0(x)\leqslant\hat{u}_0(x)$.

proof that $u_0\geq\hat{u}_0$: We first show $v_y(\cdot):=-h(\cdot,y)+\hat{u}_0(y)\in\mathcal{S}_0$ for all $y\in X$. Let $\mu\in\mathfrak{M}_0$. We get

$$\begin{aligned}
&\int_{X\times X}\big(\partial_u\ell(z,x,0,0)v_y(z)+\partial_v\ell(z,x,0,0)v_y(x)\big)d\mu(z,x)\\
&\quad=-\int_{X\times X}\Big(h(z,y)\partial_u\ell(z,x,0,0)+h(x,y)\partial_v\ell(z,x,0,0)\Big)d\mu(z,x)\\
&\qquad+\int_{X\times X}\Lambda(z,x)\hat{u}_0(y)d\mu(z,x)\\
&\quad=-\int_{X\times X}\Big(h(z,y)\partial_u\ell(z,x,0,0)+h(x,y)\partial_v\ell(z,x,0,0)\Big)d\mu(z,x)\\
&+\int_{X\times X}\Lambda(z,x)d\mu(z,x)\min_{\tilde{\mu}\in\mathfrak{M}_0}\frac{\int_{X\times X}\Big(\partial_u\ell(z,x,0,0)h(z,y)+\partial_v\ell(z,x,0,0)h(x,y)\Big)d\tilde{\mu}(z,x)}{\int_{X\times X}\Lambda(z,x)d\tilde{\mu}(z,x)}\\
&\geq 0.
\end{aligned}$$

It follows that $v_y\leqslant u_0$ and evaluating at y yields $u_0(y)\geq -h(y,y)+\hat{u}_0(y)$. Let $y\in\mathfrak{A}$, we have $h(y,y)=0$, and $u_0(y)\geq\hat{u}_0(y)$. By comparison (Proposition 1.3), as u_0 is a solution and $\hat{u}_0$ a subsolution, we finally get $u_0\geq\hat{u}_0$. □

5 Uniqueness of u_λ

Theorem 6. *The fixed point u_λ of T_λ is unique if λ is small and one of the following holds*

(1) $\ell(z,x,u,v)$ is concave in u and concave in v;

(2) $\partial_{u,v}\ell(z,x,u,v)$ exist and are continuous for (u,v) near $(0,0)$.

Proof. We argue by contradiction. Let u_λ and v_λ be two fixed points. Assume

$$u_\lambda(x_0)-v_\lambda(x_0)=\max_{x\in X}\big(u_\lambda(x)-v_\lambda(x)\big)>0.$$

Let $(x_{-k})_{k\in\mathbb{N}}$ be a minimizing sequence associated to $v_\lambda(x_0)$ as defined in Remark 4.2.

Step 1. We first show that if λ is small enough, we have $u_\lambda(x_{-k}) > v_\lambda(x_{-k})$ for all $k \geq 0$. Assume $u_\lambda(x_{-1}) \leq v_\lambda(x_{-1})$. By (l1) and (l2), we have

$$\begin{aligned} u_\lambda(x_0) - u_\lambda(x_{-1}) &\leq \ell\big(x_{-1}, x, \lambda u_\lambda(x_{-1}), \lambda u_\lambda(x_0)\big) \\ &\leq \ell\big(x_{-1}, x_0, \lambda u_\lambda(x_{-1}), \lambda v_\lambda(x_0)\big) \\ &\leq \ell\big(x_{-1}, x_0, \lambda v_\lambda(x_{-1}), \lambda v_\lambda(x_0)\big) + \lambda\kappa_u(v_\lambda - u_\lambda)(x_{-1}) \\ &= v_\lambda(x_0) - v_\lambda(x_{-1}) + \lambda\kappa_u(v_\lambda - u_\lambda)(x_{-1}), \end{aligned}$$

which implies that

$$(1 - \lambda\kappa_u)(u_\lambda - v_\lambda)(x_{-1}) \geq u_\lambda(x_0) - v_\lambda(x_0) > 0,$$

which leads to a contradiction. Then $u_\lambda(x_{-1}) > v_\lambda(x_{-1})$. We then go on to find, using (l2), that

$$\begin{aligned} u_\lambda(x_{-1}) - u_\lambda(x_{-2}) &\leq \ell\big(x_{-2}, x, \lambda u_\lambda(x_{-2}), \lambda u_\lambda(x_{-1})\big) \\ &\leq \ell\big(x_{-2}, x_{-1}, \lambda u_\lambda(x_{-2}), \lambda v_\lambda(x_{-1})\big) \\ &\leq \ell\big(x_{-2}, x_{-1}, \lambda v_\lambda(x_{-2}), \lambda v_\lambda(x_{-1})\big) \\ &= v_\lambda(x_{-1}) - v_\lambda(x_{-2}), \end{aligned}$$

therefore $u_\lambda(x_{-2}) - v_\lambda(x_{-2}) \geq u_\lambda(x_{-1}) - v_\lambda(x_{-1}) > 0$.

By induction, we have $u_\lambda(x_{-k}) > v_\lambda(x_{-k})$ for all $k \geq 0$.

Step 2. By Step 1 and (l2), we have

$$\begin{aligned} v_\lambda(x_{-k+1}) - v_\lambda(x_{-k}) &= \ell\big(x_{-k}, x_{-k+1}, \lambda v_\lambda(x_{-k}), \lambda v_\lambda(x_{-k+1})\big) \\ &\geq \ell\big(x_{-k}, x_{-k+1}, \lambda u_\lambda(x_{-k}), \lambda u_\lambda(x_{-k+1})\big) \\ &\geq u_\lambda(x_{-k+1}) - u_\lambda(x_{-k}), \end{aligned} \tag{5.1}$$

which implies

$$u_\lambda(x_{-k}) - v_\lambda(x_{-k}) \geq \cdots \geq u_\lambda(x_0) - v_\lambda(x_0).$$

By the definition of x_0, all inequalities above are equalities, that is,

$$u_\lambda(x_{-k}) - v_\lambda(x_{-k}) = \max_{x\in X}\big(u_\lambda(x) - v_\lambda(x)\big) > 0, \quad \forall k \geq 0. \tag{5.2}$$

Step 3. Define the probability measure

$$\mu_N := N^{-1} \sum_{k=-N}^{-1} \delta_{(x_k, x_{k+1})}. \tag{5.3}$$

By compactness of measures, let $(N_n)_n$ be an extraction such that $\mu_{N_n} \to \mu$ as $N \to +\infty$. Since

$$\left|\int_{X\times X} \big(f(x) - f(y)\big) d\mu_N\right| = \left|\frac{f(x_0) - f(x_{-N})}{N}\right| \leq \frac{2\|f\|_\infty}{N} \to 0,$$

the measure μ is closed. We also have

$$\int_{X\times X} \ell\big(z,x,\lambda v_\lambda(z),\lambda v_\lambda(x)\big)d\mu_N = \frac{v_\lambda(x_0)-v_\lambda(x_{-N})}{N} \to 0.$$

As all inequalities in (5.1) are equalities, it follows that

$$\int_{X\times X} \ell\big(z,x,\lambda v_\lambda(z),\lambda v_\lambda(x)\big)d\mu = \int_{X\times X} \ell\big(z,x,\lambda u_\lambda(z),\lambda u_\lambda(x)\big)d\mu = 0. \tag{5.4}$$

Since $\mu_{N_n} \to \mu$, for each $(z,x) \in$ supp(μ), there is a sequence $(z_n,x_n) \in$ supp(μ_{N_n}) with $(z_n,x_n) \to (z,x)$. By Step 2, $u_\lambda(z_n) - v_\lambda(z_n)$ equals a positive constant M. Therefore, $u_\lambda(z) - v_\lambda(z) = M > 0$. Similarly, we have $u_\lambda(x) - v_\lambda(x) = M > 0$. By (l2), we have

$$\ell(z,x,r,s) = \ell\big(z,x,\lambda u_\lambda(z),\lambda u_\lambda(x)\big),$$

for all $(z,x) \in$ supp(μ) and $r \in [\lambda v_\lambda(z),\lambda u_\lambda(z)]$, $s \in [\lambda v_\lambda(x),\lambda u_\lambda(x)]$.

Conclusion under hypothesis (1). By (5.2), the intervals $[\lambda v_\lambda(z),\lambda u_\lambda(z)]$ and $[\lambda v_\lambda(x),\lambda u_\lambda(x)]$ have no-empty interior, by the concavity we have

$$\ell\big(z,x,\lambda u_\lambda(z),\lambda u_\lambda(x)\big) = \max_{r,s} \ell(z,x,r,s), \quad \forall (z,x) \in \text{supp}(\mu).$$

By (l2), we have

$$\ell(z,x,r,s) = \ell\big(z,x,\lambda u_\lambda(z),\lambda u_\lambda(x)\big), \quad \forall r \leq \lambda u_\lambda(z),\ s \leq \lambda u_\lambda(x).$$

Let $u_0 < 0$ and $u_0 \leq \min_{x\in X} \lambda u_\lambda(x)$, we have

$$\int_{X\times X} \ell(z,x,u_0,u_0)d\mu = \int_{X\times X} \ell\big(z,x,\lambda u_\lambda(z),\lambda u_\lambda(x)\big)d\mu = 0.$$

By (l2) we also have

$$0 = \int_{X\times X} \ell(z,x,u_0,u_0)d\mu \geq \int_{X\times X} \ell(z,x,0,0)d\mu.$$

Thus, μ is a Mather measure of $\ell(z,x,0,0)$. By (l2) again we have

$$\ell(z,x,r,s) = \ell(z,x,u_0,u_0), \quad \forall (z,x) \in \text{supp}(\mu),\ \forall r,s \in [u_0,0].$$

Since $u_0 < 0$, we have

$$\partial_{u,v}\ell(z,x,0,0) = 0, \quad \forall (z,x) \in \text{supp}(\mu),$$

which contradicts (l4).

Conclusion under hypothesis (2). Since the set of Mather measures is compact, there is $\epsilon > 0$ such that

$$\int_{X\times X} \Lambda(z,x)d\mu(z,x) < -2\epsilon, \quad \forall \mu \in \mathfrak{M}_0.$$

We first show that there is $r > 0$ and $N_0 > 0$ such that

$$N^{-1} \sum_{k=-N}^{-1} \Lambda(x_k, x_{k+1}) < -2\epsilon, \quad \forall \lambda \in (0, r),\ \forall N \geq N_0.$$

If not, we assume that there is a sequence $\lambda_n \to 0$ and $N_n \to +\infty$ such that

$$N_n^{-1} \sum_{k=-N_n}^{-1} \Lambda(x_k, x_{k+1}) \geq -2\epsilon.$$

Extracting a subsequence if necessary, let μ_{λ_n} be the limit given by (5.3) and $\mu_{\lambda_n} \to \mu$. By (5.4), μ is a Mather measure. We then get a contradiction.

Since $\partial_{u,v}\ell$ is continuous for (u, v) near $(0, 0)$, for λ small and N large, we have

$$N^{-1} \sum_{k=-N}^{-1} \Big[\partial_u \ell\big(x_k, x_{k+1}, \lambda u_\lambda(x_k), \lambda u_\lambda(x_{k+1})\big) + \partial_v \ell\big(x_k, x_{k+1}, \lambda u_\lambda(x_k), \lambda u_\lambda(x_{k+1})\big)\Big] < -\epsilon.$$

Since all inequalities in (5.1) are equalities, we have

$$\partial_u \ell\big(x_k, x_{k+1}, \lambda u_\lambda(x_k), \lambda u_\lambda(x_{k+1})\big) = \partial_v \ell\big(x_k, x_{k+1}, \lambda u_\lambda(x_k), \lambda u_\lambda(x_{k+1})\big) = 0,$$

which leads to a contradiction. □

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