CONVERGENCE OF THE SOLUTIONS OF THE DISCOUNTED EQUATION: THE DISCRETE CASE

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Abstract. We derive a discrete version of the results of [2]. If M is a compact metric space, $c : M \times M \to \mathbb{R}$ a continuous cost function and $\lambda \in (0,1)$, the unique solution to the discrete λ-discounted equation is the only function $u_\lambda : M \to \mathbb{R}$ such that

$$\forall x \in M, \quad u_\lambda(x) = \min_{y \in M} \lambda u_\lambda(y) + c(y,x).$$

We prove that there exists a unique constant $\alpha \in \mathbb{R}$ such that the family of $u_\lambda + \alpha/(1-\lambda)$ is bounded as $\lambda \to 1$ and that for this α, the family uniformly converges to a function $u_0 : M \to \mathbb{R}$ which then verifies

$$\forall x \in X, \quad u_0(x) = \min_{y \in X} u_0(y) + c(y,x) + \alpha.$$

The proofs make use of Discrete Weak KAM theory. We also characterize u_0 in terms of Peierls barrier and projected Mather measures.

1. Introduction

In [2], it was proven that the unique viscosity solution of the λ-discounted Hamilton Jacobi equation converges, as λ tends to zero, to a particular solution of the critical Hamilton–Jacobi equation. In other words, the limit selects one solution among the several possible choices. In this work, we prove the discrete version of the same result. In this discrete setting, minimization of the action of curves is replaced by minimization of costs for sequences, and the Hamilton–Jacobi equation by fixed points of the Lax–Oleinik semigroup. This theory is known as Discrete Aubry-Mather Theory. It was mainly developed in [1] and [4].

Let M be a compact metric space, and $c : M \times M \to \mathbb{R}$ a continuous function, that will be called cost function. The discrete version of the Hamilton–Jacobi equation $H(x, d_x u) = \alpha$ is to find a $u \in C^0(M, \mathbb{R})$ such that

$$u(x) = \mathcal{T}(u)(x) + \alpha \qquad \text{for every } x \in M, \tag{1.1}$$

where $\mathcal{T}$ is the Lax–Oleinik operator, defined on the set $C^0(M, \mathbb{R})$ of continuous functions from M to $\mathbb{R}$ as

$$\mathcal{T}(g)(x) = \inf_{y \in M} g(y) + c(y,x) \qquad \text{for every } x \in M \text{ and } g \in C^0(M, \mathbb{R}).$$

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Due to the compactness of M, there is only one constant α for which we can find solutions of (1.1). This number is called the critical value. As we will see, it has several characterizations.

The discrete version of the λ-discounted Hamilton–Jacobi equation $\lambda u + H(x, d_x u) = \alpha$ is

$$u(x) = \mathcal{T}_\lambda(u)(x) + \alpha \qquad \text{for every } x \in M, \tag{1.2}$$

where λ is a parameter between 0 and 1, and $\mathcal{T}_\lambda$ is an operator defined on $C^0(M, \mathbb{R})$ as

$$\mathcal{T}_\lambda(g)(x) = \inf_{y \in M} \lambda g(y) + c(y, x) \qquad \text{for every } x \in M \text{ and } g \in C^0(M, \mathbb{R}).$$

Equation (1.2) admits a unique solution u_λ. Moreover, the family of solutions $(u_\lambda)_{0<\lambda<1}$ is equicontinuous and equibounded, see Proposition 2.3. Clearly, any accumulation point of the u_λ, as $\lambda \to 1^-$, will be a solution of the discrete Hamilton–Jacobi equation.[1] Yet, equation (1.1) has several possible solutions, therefore it is not a priori clear whether the family u_λ is fully convergent as $\lambda \to 1^-$. The main Theorem of this work is to establish this convergence.

Theorem 1.1. *The solutions u_λ of equation (1.2) converge, as $\lambda < 1$ tends to 1, to a particular solution u_0 of equation (1.1).*

Stephane Gaubert pointed out that the result was already known when M is finite. For example, it could be deduced from [3]

Subsolutions of (1.1) do not need to be continuous, however, according to Proposition A.10, all subsolutions of (1.1) are integrable with respect to any projected Mather measure (see Definition A.4). We denote by $\mathcal{F}_-$ the set of subsolutions such that

$$\int_M u(x)\, d\mu(x) \leq 0$$

for all projected Mather measures. We have the following characterizations for the solution selected in the limit.

Proposition 1.2. *The limit solution u_0 in Theorem 1.1 can be characterized in either of the following two ways:*

(1) $$u_0(x) = \sup_{u \epsilon \mathcal{F}_-} u(x);$$

(2) $$u_0(x) = \min_{\mu} \int_M h(y, x)\, d\mu(y),$$

where h is the Peierls barrier and μ varies in the set of projected Mather measures.

The definitions involved, subsolutions, projected Mather measures and Peierls barrier, will be given in the appendix. For convenience of the reader, we will also state, in the appendix, the results we use from Discrete Aubry–Mather

[1] In the continuous case we study the limit as $\lambda > 0$ tends to 0, and in the discrete case the limit as $\lambda < 1$ tends to 1.

Theory, see [1] or [4] where proofs can be found. The non-expert reader should probably first look at the appendix.

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2. Preliminaries

In this section we shall state and prove some preliminary facts about the discounted equation

$$u(x) = \mathcal{T}_\lambda(u)(x) + \beta = \inf_{z \in M} \lambda u(z) + c(z,x) + \beta, \tag{2.1}$$

for every $x \in M$, where β is a fixed real constant.

Let u be a continuous function on M. We will say that u is a *subsolution* of (3.1) if $u(x) \leq \mathcal{T}_\lambda(u)(x) + \beta$ for every $x \in M$. We will say that u is a *supersolution* of (3.1) if $u(x) \geq \mathcal{T}_\lambda(u)(x) + \beta$ for every $x \in M$.

In the sequel, we shall denote by $S_n(x)$ the set of M-valued sequences of the form $(x_{-n}, x_{-n+1}, \dots, x_{-1}, x_0)$ with $x_0 = x$, and by $S_\infty(x)$ the set of M-valued sequences $\bar{x} = (x_{-n})_{n \geq 0}$ such that $x_0 = x$. It will also be convenient to set $S_n(M) = \cup_{x \in M} S_n(x)$, and $S_\infty(M) = \cup_{x \in M} S_\infty(x)$.

One of our main tools is he following comparison principle:

Proposition 2.1. *Let v and w be a pair of continuous function on M that are, respectively, a sub and a supersolution of (2.1). Then*

$$v(x) \leq \min_{\bar{x} \in S_\infty(x)} \sum_{n=0}^{\infty} \lambda^n \left(c(x_{-n-1}, x_{-n}) + \beta \right) \leq w(x) \tag{2.2}$$

for every $x \in M$.

Proof. Replacing the cost function $c(x,y)$ with $c(x,y) + \beta$, we can always assume that $\beta = 0$. Let us pick a point $x \in M$. By the definition of $\mathcal{T}_\lambda$ and the fact that $v \leq \mathcal{T}_\lambda(v)$ on M we get

$$\begin{aligned} v(x) &\leq \min_{x_{-1}} \left(\lambda v(x_{-1}) + c(x_{-1}, x) \right) \leq \min_{x_{-1}} \left(\lambda \mathcal{T}_\lambda(v)(x_{-1}) + c(x_{-1}, x) \right) \\ &= \min_{x_{-2}, x_{-1}} \left(\lambda^2 v(x_{-2}) + \lambda c(x_{-2}, x_{-1}) + c(x_{-1}, x) \right). \end{aligned}$$

Arguing inductively, we derive

$$v(x) \leq \min_{\bar{x} \in S_\infty(x)} \left(\lambda^n v(x_{-n}) + \sum_{k=0}^{n-1} \lambda^k c(x_{-k-1}, x_{-k}) \right).$$

Since v, c are continuous functions defined on compact spaces, and $\lambda < 1$, the sequence of continuous continuous $\lambda^n v(x_{-n}) + \sum_{k=0}^{n-1} \lambda^k c(x_{-k-1}, x_{-k})$ converges uniformly to $\sum_{k=0}^{\infty} \lambda^k c(x_{-k-1}, x_{-k})$ on the compact space $S_\infty(x)$. Therefore, the left hand side inequality in (2.2) holds. The inequality for w follows arguing analogously. $\square$

The existence of a (unique) solution of equation (2.1) is established in the next proposition.

Proposition 2.2. *For $0 < \lambda < 1$ there is only one solution u_λ of the discounted equation (2.1) and it can be represented by*

$$u_\lambda(x_0) = \min_{\bar{x} \in S_\infty(x_0)} \sum_{n=0}^{\infty} \lambda^n \big(c(x_{-n-1}, x_{-n}) + \beta\big) \quad \textit{for every } x_0 \in M. \tag{2.3}$$

Proof. As before, to simplify notations, replacing the cost function $c(x,y)$ with $c(x,y)+\beta$, we will assume $\beta = 0$. For λ strictly smaller than 1, the operator $u \mapsto \mathcal{T}_\lambda$ is a contraction in the space of continuous functions with the C^0–norm. Indeed, let f and g be two continuous functions. For a given x in M, let y such that $\mathcal{T}_\lambda f(x) = \lambda f(y) + c(y,x)$. By definition we have $\mathcal{T}_\lambda g(x) \le \lambda g(y) + c(y,x)$, so

$$\mathcal{T}_\lambda g(x) - \mathcal{T}_\lambda f(x) \le \lambda(g(y) - f(y)) \le \lambda \|f-g\|_0,$$

where $\|\cdot\|_0$ denotes the C^0 norm. Reversing the roles of f and g we obtain

$$|\mathcal{T}_\lambda g(x) - \mathcal{T}_\lambda f(x)| \le \lambda \|f-g\|_0.$$

Since this is true for every x, we obtain

$$\|\mathcal{T}_\lambda f - \mathcal{T}_\lambda g\|_0 \le \lambda \|f-g\|_0.$$

Therefore, from the Banach fixed point theorem, we obtain that there is a unique fixed point u_λ. The representation formula (2.3) is a direct consequence of Proposition 2.1. Alternatively, since iterates of the contraction map converge to the fixed point, we can obtain the same formula as the limit, as n tends to infinity, of $\mathcal{T}_\lambda^n(0)$. □

Some crucial properties of the solutions of the discounted equation are established in the next proposition. It incidentally entails a characterization for the critical value α (see Theorem A.2).

Proposition 2.3. *For every $\beta \in \mathbb{R}$, the family $\{u_\lambda^\beta : 0 < \lambda < 1\}$ of solutions of (2.1) is equicontinuous. Furthermore, it is equibounded if and only if β is equal to the critical value α.*

Proof. For the first part, let x and y be two points in M and z a point realizing the infimum of the discounted Hamilton–Jacobi equation (2.1) for the point x. We obtain

$$u_\lambda^\beta(y) - u_\lambda^\beta(x) \le c(z,y) - c(z,x),$$

so the solutions u_λ^β have all the same modulus of continuity as the cost function c.

Let us prove that they are equibounded when β equals the critical constant α. Take a solution u of

$$u = \mathcal{T}(u) + \alpha. \tag{2.4}$$

a $\mathcal{T}$, it is then easily seen that

$$\underline{u}(x) = \mathcal{T}(\underline{u})(x) + \alpha \leq \mathcal{T}_\lambda(\underline{u})(x) + \alpha,$$
$$\overline{u}(x) = \mathcal{T}(\overline{u})(x) + \alpha \geq \mathcal{T}_\lambda(\overline{u})(x) + \alpha,$$

for $x \in M$, and $0 < \lambda < 1$. Namely, for every $0 < \lambda < 1$, the continuous functions $\underline{u}$ and $\overline{u}$, respectively, are a subsolution and a supersolution of (2.1), with $\beta = \alpha$. By the comparison principle stated in Proposition 2.1, we conclude that $\underline{u} \leq u_\lambda^\alpha \leq \overline{u}$ on M for every $0 < \lambda < 1$. This implies that the family $(u_\lambda^\alpha)_{0<\lambda<1}$ is equibounded.

To prove the *only if* part, it is enough to observe that $u_\lambda^\beta = u_\lambda^\alpha - \frac{\alpha-\beta}{1-\lambda}$. □

3. Proof of the main Theorem

In this section, we will prove both Theorem 1.1 and the first characterization given in Proposition 1.2.

Again, to simplify notations, we will assume in the sequel that the critical value α is equal to 0. As previously noted, this does not affect the generality. Therefore the discounted equation rereads as

$$u = \mathcal{T}_\lambda(u), \tag{3.1}$$

where λ is a real parameter such that $0 < \lambda < 1$. We will denote by u_λ the unique solution of equation (3.2). Since we are assuming $\alpha = 0$, the discrete version of the critical Hamilton–Jacobi equation is

$$u = \mathcal{T}(u). \tag{3.2}$$

Let us denote by $\mathcal{M}_0$ the set of projected Mather measures (on M) for (3.2) (see the appendix, Definition A.4) and set $u_0(x) := \sup_{u\in\mathcal{F}_-} u(x)$ for every $x \in M$, where $\mathcal{F}_-$ is the set of subsolutions u of (3.2) such that

$$\int_M u(x)\, d\mu(x) \leq 0 \quad \text{for every } \mu \in \mathcal{M}_0$$

The following holds:

Proposition 3.1. *Let u be a limit point of the functions u_λ, as $\lambda \to 1^-$. Then, $u \in \mathcal{F}_-$, i.e. for any measure μ in $\mathcal{M}_0$ we have*

$$\int_M u(x)\, d\mu(x) \leq 0.$$

In particular, $u \leq u_0$.

Proof. Since u_λ is a solution of (3.1) we have

$$u_\lambda(x) - \lambda u_\lambda(y) \leq c(y,x) \qquad \text{for every } (x,y) \in M \times M.$$

Integrating the inequality above with respect to a Mather measure $\tilde{\mu}$ defined on $M \times M$ yields

$$\int_{M\times M} \big(u_\lambda(x) - \lambda u_\lambda(y)\big)\, d\tilde{\mu}(x,y) \leq \int_{M\times M} c(y,x)\, d\tilde{\mu}(x,y).$$

Since $\tilde{\mu}$ is a Mather measure the right hand side of this inequality is zero. Therefore, we have

$$(1-\lambda)\int_M u_\lambda(x)\,d\mu(x) \le 0,$$

where μ is the projection of $\tilde{\mu}$ on either the first or second factors of $M \times M$. Dividing by $1-\lambda > 0$ we conclude that $\int_M u_\lambda(x) d\mu(x) \le 0$. So if u is a uniform limit of u_{λ_i} for some sequence $\lambda_i \to 1^-$, we obtain the first assertion. Since u is a solution, it is in particular a subsolution; therefore $u \le u_0$. □

To prove the other inequality, we need to introduce a special class of measures. Given a sequence $\bar{x}$ in $S_\infty(M)$ and a positive $\lambda < 1$, we denote by $\tilde{\mu}^\lambda_{\bar{x}}$ the probability measure in $M \times M$ defined as

$$\int_{M\times M} f(x,y)\,d\tilde{\mu}^\lambda_{\bar{x}}(x,y) = a_\lambda \sum_{n=0}^{\infty} \lambda^n f(x_{-n-1}, x_{-n})$$

for any continuous function $f : M \times M \to \mathbb{R}$, where $a_\lambda = 1-\lambda$. The choice of the constant a_λ guarantees that $\tilde{\mu}^\lambda_{\bar{x}}$ is a probability measure.

Lemma 3.2. *Let $(\bar{x}^\lambda)$, with $0 < \lambda < 1$, be a family of sequences in $S_\infty(M)$. If $\tilde{\mu}$ is an accumulation point of $\tilde{\mu}^\lambda_{\bar{x}^\lambda}$ as λ tends to 1, then $\tilde{\mu}$ is a closed measure. Moreover, if for every λ the sequence $\bar{x}^\lambda$ realizes the infimum in (2.3), then the measure $\tilde{\mu}$ is a Mather measure.*

Proof. For the first part it is enough to show that for any continuous function $\phi : M \to \mathbb{R}$ we have

$$\int_{M\times M} \big(\phi(x) - \phi(y)\big)\,d\tilde{\mu}(x,y) = 0.$$

We have that

$$\begin{aligned}
\Big|\int_{M\times M} \big(\phi(x)-\phi(y)\big)\,d\tilde{\mu}^\lambda_{\bar{x}^\lambda}\Big| &= \Big|a_\lambda \sum_{n=0}^{\infty} \lambda^n \big(\phi(x^\lambda_{-n-1}) - \phi(x^\lambda_{-n})\big)\Big| \\
&= \Big|a_\lambda \Big[\frac{1}{\lambda}\sum_{n=0}^{\infty} \lambda^{n+1}\phi(x^\lambda_{-n-1}) - \sum_{n=0}^{\infty}\lambda^n \phi(x^\lambda_{-n})\Big]\Big| \\
&= \Big|a_\lambda \Big[\frac{1}{\lambda}\sum_{n=1}^{\infty} \lambda^{n}\phi(x^\lambda_{-n}) - \sum_{n=0}^{\infty}\lambda^n \phi(x^\lambda_{-n})\Big]\Big| \\
&= \Big|a_\lambda \frac{1-\lambda}{\lambda}\sum_{n=1}^{\infty} \lambda^{n}\phi(x^\lambda_{-n}) - a_\lambda \phi(x^\lambda_0)\Big| \\
&\le 2a_\lambda \|\phi\|_0,
\end{aligned}$$

where $\|\phi\|_0$ is, as before, the C^0 norm of ϕ. Since $a_\lambda = (1-\lambda) \to 0$, as $\lambda \to 1$, we obtain the first part of the lemma.

To prove the second part, we note that, for $\lambda < 1$, by (2.3) we have

$$\begin{aligned}\lim_{\lambda\to 1^-}\int_{M\times M} c(x,y)\,d\tilde{\mu}^\lambda_{\bar{x}^\lambda}(x,y) &= \lim_{\lambda\to 1^-} a_\lambda \sum_{n=0}^{\infty}\lambda^n c(x^\lambda_{-n-1},x^\lambda_{-n})\\ &= \lim_{\lambda\to 1^-} a_\lambda u_\lambda(x^\lambda_0).\end{aligned}$$

Since u_λ is equibounded, and $a_\lambda \to 0$ as $\lambda \to 1$, for any limit measure $\tilde{\mu}$ of $\mu^\lambda_{\bar{x}^\lambda}$, we obtain $\int_{M\times M} c(x,y)\,d\tilde{\mu}(x,y) = 0$. Hence $\tilde{\mu}$ is a Mather measure. □

The following is a key lemma for the end of the proof and for the second characterization.

Lemma 3.3. *Suppose w is a continuous subsolution of (3.2). If $\bar{x}^\lambda = (x^\lambda_{-n})_{n\geq 0}$ is a minimizing sequence for (2.3), we have*

$$u_\lambda(x^\lambda_0) \geq \Big(w(x^\lambda_0) - \int_{M\times M} w(x)\,d\tilde{\mu}^\lambda_{\bar{x}^\lambda}(x,y)\Big).$$

Proof. By using the fact that $\bar{x}^\lambda$ realizes the minimum in (2.3), we get

$$\begin{aligned}u_\lambda(x^\lambda_0) &= \sum_{n=0}^{\infty}\lambda^n c(x^\lambda_{-n-1},x^\lambda_{-n})\\ &\geq \sum_{n=0}^{\infty}\lambda^n\big(w^\lambda(x_{-n}) - w^\lambda(x_{-n-1})\big)\\ &= \sum_{n=0}^{\infty}\lambda^n w(x^\lambda_{-n}) - \sum_{n=0}^{\infty}\lambda^{n+1} w(x^\lambda_{-n-1})\\ &= w(x^\lambda_0) + \sum_{n=0}^{\infty}\lambda^{n+1} w(x^\lambda_{-n-1}) - \sum_{n=0}^{\infty}\lambda^n w(x^\lambda_{-n-1})\\ &= w(x^\lambda_0) - (1-\lambda)\sum_{n=0}^{\infty}\lambda^n w(x^\lambda_{-n-1})\\ &= w(x^\lambda_0) - \int_{M\times M} w(x)\,d\tilde{\mu}^\lambda_{\bar{x}^\lambda}(x,y)\end{aligned}$$

□

We now obtain the following key result.

Proposition 3.4. *Suppose u is a uniform limit of a subsequence of u_λ as λ converges to 1. Then for every (possibly discontinuous) subsolution we have*

$$u \geq w - \sup_{\mu\in\mathcal{M}_0}\int_M w(x)\,d\mu(x). \tag{3.3}$$

Therefore $u \geq u_0$

Proof. We first prove (3.3) when w is continuous. We fix $x \in M$, and find, for $0 < \lambda < 1$, a sequence $\bar{x}^\lambda = (x^\lambda_{-n})_{n\geq 0}$ minimizing (2.3), with $x^\lambda_0 = x$. For each such sequence $\bar{x}^\lambda$, we consider the probability measure $\tilde{\mu}^\lambda_{\bar{x}^\lambda}$ on $M \times M$.

Extracting further, we can assume that $\tilde{\mu}^{\lambda_i}_{\bar{x}^{\lambda_i}}$ converges weakly to a measure $\tilde{\mu}$ on $M \times M$. From Lemma 3.2, we know that $\tilde{\mu}$ is a Mather measure. By Lemma 3.3, we have

$$u_{\lambda_i}(x) \geq w(x) - \int_{M\times M} w(y)\, d\tilde{\mu}^{\lambda_i}_{\bar{x}^{\lambda_i}}(x,y).$$

As $i \to +\infty$, the sequence λ_i converges to 1, the functions u_{λ_i} converge uniformly to u, and the probability measures $\tilde{\mu}^{\lambda_i}_{\bar{x}^{\lambda_i}}$ converge to $\tilde{\mu}$. Therefore we can pass to the limit in the inequality above, we obtain

$$u(x) \geq w(x) - \int_M w(y)\, d\mu(y),$$

where $\mu = \pi_2^*\tilde{\mu}$ is a projected Mather measure. This proves (3.3) when w is continuous.

Let us consider the case of a not necessarily continuous subsolution w. By Proposition A.5, we know $\mathcal{T}(w)$ is a continuous subsolution of (A.1) such that $\mathcal{T}(w) \geq w$, and $\mathcal{T}(w) = w$ on $\mathcal{A}$. Therefore, by Proposition A.9, we get $\int_M w(y)\, d\mu(y) = \int_M \mathcal{T}(w)\, d\mu(y)$, for every projected Mather measure μ. Since (3.3) is true for the continuous subsolution $\mathcal{T}(w)$, we obtain

$$\begin{aligned} u &\geq \mathcal{T}(w) - \int_M \mathcal{T}(w)(y)\, d\mu(y), \\ &\geq w - \int_M w(y)\, d\mu(y). \end{aligned}$$

If $w \in \mathcal{F}_-$, we note that $\int_M w(y)\, d\mu(y) \leq 0$, for every projected Mather measure μ. Therefore, in this case, the inequality (3.3) implies $u \geq w$. Taking the sup over all $w \in \mathcal{F}_-$ yields $u \geq u_0 = \sup_{w\in\mathcal{F}_-} w$. □

We now finish the proof of Theorem 1.1.

Proof of Theorem 1.1 . The family $(u_\lambda)_{0<\lambda<1}$ is equicontinuous, hence relatively compact in the C^0 topology by the Arzelà-Ascoli theorem. It therefore suffices to show that any limit of a subsequence of solutions u_λ, as λ converges to 1, is necessarily equal to u_0. Let u be such a limit. From Proposition 3.1, we obtain $u \leq u_0$, and from Proposition 3.4 we obtain $u_0 \geq u$.

□

4. Proof of the second Characterization

In this section, we will prove the second characterization stated in Proposition 1.2, namely that the function u_0 obtained as the limit of the solutions of the discounted equations coincide with the function $\hat{u}_0 : M \to \mathbb{R}$ defined by

$$\hat{u}_0(x) = \min_{\mu\epsilon\mathcal{M}_0} \int h(z,x)\, d\mu(z) \qquad \text{for every } x \in M.$$

We start by proving the following results:

Lemma 4.1. *The function $\hat{u}_0$ is a subsolution.*

Proof. For $\mu \in \mathcal{M}_0$, define the function

$$h_\mu(x) = \int_M h(y,x)\,d\mu(y) \qquad \text{for every } x \in M.$$

Since h_μ is the convex combination of an equibounded family of solutions of (3.2), see (1) of Proposition A.8, by (1) of Proposition A.6, it is itself a subsolution of (3.2). As an infimum of *equibounded* subsolutions, we infer, by the second item of the same proposition, that $\hat{u}_0$ is also a subsolution of (3.2). □

Lemma 4.2. *We have that* $u_0 \leq \hat{u}_0$

Proof. By definition of u_0 and $\hat{u}_0$, we only need to show that $u \leq h_\mu$ on M, where u is a subsolution in $\mathcal{F}_-$ and μ is a projected Mather measure. By proposition A.7 we have

$$u(x) - u(z) \leq h(z,x) \qquad \text{for every } (x,z) \in M \times M.$$

Integrating with respect to $\mu \in \mathcal{M}_0$ we obtain

$$u(x) - \int_M u(z)\,d\mu(z) \leq h_\mu(x).$$

Using that $\int_M u(z)d\mu(z)$ is non-positive, we get the assertion. □

We are now ready to prove the announced equality.

Proof of Proposition 1.2 . By the previous lemma we already know that $u_0 \leq \hat{u}_0$. We have to show the reverse inequality $u_0 \geq \hat{u}_0$. Since we know that u_0 is a solution and that $\hat{u}_0$ is a subsolution, by Proposition A.11 we only have to show the inequality on the projected Aubry set $\mathcal{A}$.

Fix y in $\mathcal{A}$. By Proposition A.8, the function $x \mapsto -h(x,y)$ is a subsolution of (3.2). Hence, adding the constant $\hat{u}_0(y)$ to this function, we obtain that

$$w(x) = -h(x,y) + \hat{u}_0(y),$$

is a subsolution, which is clearly in $\mathcal{F}_-$. So $u_0 \geq w$, in particular, by evaluating at y, we get

$$u_0(y) \geq -h(y,y) + \hat{u}_0(y).$$

Since $h(y,y) = 0$ on the projected Aubry set $\mathcal{A}$, this yields $u_0 \geq \hat{u}_0$ on $\mathcal{A}$ and therefore also on M. □

Appendix A. Discrete Aubry-Mather Theory

A.1. **Lax Oleinik Operator.** Let M be a compact metric space, and

$$c : M \times M \to \mathbb{R}$$

be a continuous cost function. For every function $u : M \to \mathbb{R}$ (not necessarily continuous), we set

$$\mathcal{T}(u)(x) = \inf_{y \in M} u(y) + c(y,x) \qquad \text{for every } x \in M. \tag{A.1}$$

When computed for functions u that are continuous on M, the operator $\mathcal{T}$ is the Lax–Oleinik operator as defined in the introduction. As a direct consequence of the definition, we derive

Proposition A.1. *Let u be a (possibly discontinuous) real function on M such that $\mathcal{T}(u)(x) > -\infty$ for every $x \in M$. Then $\mathcal{T}(u)$ is continuous on M, with the same continuity modulus as c.*

Since c is continuous, and therefore bounded on the compact space $M \times M$, it is not difficult to see that $\mathcal{T}(u)$ is bounded below if and only if u is bounded below, and also if and only if $\mathcal{T}(u)(x) > -\infty$ for some $x \in M$.

The following Theorem is the discrete version of the Weak KAM theorem a proof of which can be found in [4, Theorem 1.2]

Theorem A.2. *There is a unique constant α such that the equation*

$$u = \mathcal{T}(u) + \alpha$$

admits (necessarily continuous) solutions $u : M \to \mathbb{R}$.

Such a constant α is called the *critical value*.

A.2. **Mather measures.** For $i = 1, 2$, let π_i be the projection to each factor of $M \times M$, that is $(x_1, x_2) \mapsto x_i$. For a probability measure $\tilde{\mu}$ in $M \times M$, the projected measures $\pi_i^* \tilde{\mu}$ are given by the formula

$$\int_M f(x)\, d\pi_i^* \tilde{\mu}(x) = \int_{M \times M} (f \circ \pi_i)(x, y)\, d\tilde{\mu}(x, y).$$

A Borel probability measure in $M \times M$ is called *closed* if the projections are the same, i.e. $\pi_1^* \tilde{\mu} = \pi_2^* \tilde{\mu}$. A proof of the following proposition can be found in [1, Theorem 15].

Proposition A.3. *We have*

$$-\alpha = \min_{\nu} \int_{M \times M} c(x, y)\, d\tilde{\nu}(x, y), \tag{A.1}$$

where the minimum is computed for $\tilde{\nu}$ in the class of closed probability measures on $M \times M$.

Definition A.4. A closed probability measure that is a solution of the minimization problem (A.1) is called *Mather measure*. The set of Mather measures on $M \times M$ is denoted by $\mathcal{M}$ and the set of projected Mather measures is denoted by $\mathcal{M}_0$.

Measures on $M \times M$ will be denoted with a tilde, while for the projection we will use the same letter without the tilde.

A.3. **Peierls Barrier and Aubry set.** For each pair of points x and y let $S_n(x,y)$ be the set of M-valued sequences of the form $\bar{x} = (x_{-n}, x_{-n+1}, \dots, x_{-1}, x_0)$ such that $x_{-n} = x$ and $x_0 = y$. Denote by $c(\bar{x})$ the cost of the sequence, that is, $\sum_{i=1}^{n} c(x_{-i}, x_{-i+1})$. For each natural number n define the function

$$c_n(x,y) = \inf_{\bar{x} \in S_n(x,y)} c(\bar{x}).$$

For each real value κ, we define the functions

$$h_n^\kappa(x,y) = c_n(x,y) + n\kappa,$$
$$h^\kappa(x,y) = \liminf_{n\to\infty} h_n^\kappa(x,y).$$

The function h^κ is finite valued if and only if κ equals the critical value α, see [4]. The function h^α is called the Peierls barrier. As noted before, up to adding a constant to the cost, we can always suppose that α is zero. To simplify notations, this will be always assumed in the rest of this appendix. Moreover, we will write h, h_n instead of h^α, h_n^α, respectively.

Since

$$h_{n+m}(x,z) \leq h_n(x,y) + h_m(y,z) \quad \text{for every positive integers } m \text{ and } n, \tag{A.1}$$

it is easy to show that h satisfies the triangle inequality. The symmetrized function $h(x,y) + h(y,x)$ is nonnegative, but, usually, it is not a distance function because in general $h(x,x)$ can be positive and there might be different points such that $h(x,y) + h(y,x) = 0$. Partially motivated by this, we define the projected Aubry $\mathcal{A}$ by

$$\mathcal{A} := \{y \in M \,:\, h(y,y) = 0\,\}.$$

A.4. **Subsolutions and supersolutions.** We consider the following discrete version of the Hamilton–Jacobi equation:

$$u(x) = \mathcal{T}(u)(x) \quad \text{for every } x \in M. \tag{A.1}$$

A function u defined on M is called subsolution of (A.1) if

$$u \leq \mathcal{T}(u).$$

A function u defined on M is called supersolution of (A.1) if

$$u \geq \mathcal{T}(u).$$

A function u defined on M is called solution if it is both a subsolution and supersolution, i.e. if it is a fixed point of the operator $\mathcal{T}$. The following holds:

Proposition A.5. *Let u be a subsolution of (A.1), which is bounded from below. Then $\mathcal{T}(u)$ is a continuous subsolution of (A.1) such that $\mathcal{T}(u) \geq u$ on M, and $\mathcal{T}(u) = u$ on $\mathcal{A}$. In particular, u is continuous on the projected Aubry set $\mathcal{A}$.*

Proof. The first assertion has been proven in [4, Proposition A.10]. This immediately gives the continuity of u on $\mathcal{A}$ in view of Proposition A.1. □

The following is mostly contained in [4, Lemma 2.32 and Proposition A.8.3].

Proposition A.6.

1) *A convex combination of an equibounded family of subsolutions is a subsolution.*
2) *If* $\{u_i\}_{i\in\mathcal{I}}$ *is an equibounded family of subsolutions, then* $v = \inf_{i\in\mathcal{I}} u_i$ *and* $V = \sup_i u_i$ *are subsolutions.*
3) *If* u_i *is an equibounded family of supersolutions, then* $V = \inf_{i\in\mathcal{I}} u_i$ *is a supersolution.*

Proof. For completeness, let us explain the last point. It is a direct consequence of the monotonicity of $\mathcal{T}$:

$$\mathcal{T}V = \mathcal{T}\inf_i u_i \le \inf_i \mathcal{T}u_i \le \inf_i u_i = V.$$

□

The next two propositions are contained in [4, Theorem 2.32], and [4, Theorem 2.29], respectively.

Proposition A.7. *If* u *is a subsolution then we have*

$$u(x) - u(y) \le h(y, x).$$

This is [4, Theorem 2.29].

Proposition A.8. *Let* $y \in M$.

1) *The function* $h(y,\cdot)$ *is a solution* (A.1).
2) *The function* $-h(\cdot, y)$ *is a continuous subsolution of* (A.1).
3) *The family of functions* $(-h(\cdot, y))_{y\in M}$ *is equi-bounded.*

The following proposition is from [1, Theorem 13]. We provide here a different proof using subsolutions.

Proposition A.9. *The support of a projected Mather measure is contained in the projected Aubry set.*

Proof. Let (x, y) be in the support of a Mather measure $\tilde{\mu}$. Then we have

$$u(y) - u(x) = c(x, y). \tag{A.2}$$

for all *continuous subsolutions.* Indeed, since u is a subsolution, if we integrate with respect to $\tilde{\mu}$ we obtain

$$\int_{M\times M} \big(u(y) - u(x)\big)\, d\tilde{\mu}(x, y) \le \int_{M\times M} c(x, y)\, d\tilde{\mu}(x, y).$$

But both sides are equal to zero, the left hand side because the measure is closed and the right hand side because it is a Mather measure. So the inequality $u(y) - u(x) \le c(x, y)$ is an equality $\tilde{\mu}$-almost everywhere, and since they are continuous functions the equality is everywhere on the support of $\tilde{\mu}$. So $u(y) = u(x)+c(x, y) \ge \mathcal{T}(u)(y)$, but u being a subsolution we conclude that $\mathcal{T}(u)(y) = u(y)$. Applying (A.2) to $\mathcal{T}(u)$ we obtain

$$\mathcal{T}(u)(y) - \mathcal{T}(u)(x) = c(x, y), \tag{A.3}$$

yielding that also in x we have $\mathcal{T}(u)(x) = u(x)$.

We have thus proven that, for a given Mather measure $\tilde{\mu}$ and for every continuous subsolution u of (A.1),

$$\mathcal{T}(u)(x) = u(x) \quad \text{and} \quad \mathcal{T}(u)(y) = u(y) \quad \text{for every } (x, y) \in \operatorname{supp}(\tilde{\mu}).$$

We will prove that this implies that the points x and y are in the projected Aubry set $\mathcal{A}$, that is $h(x, x) = h(y, y) = 0$. Indeed, suppose that $\mathcal{T}(u)(z) = u(z)$ for all continuous subsolution u of (A.1). By using $u(\cdot) = -h(\cdot, z)$ and by an immediate induction we infer that for every positive integer n

$$u(z) = \mathcal{T}^n(u)(z) = \inf_{z' \in M} u(z') + c_n(z', z) = u(z_n) + c_n(z_n, z),$$

where z_n is a suitable point that exists by compactness of M and continuity of u and c_n. Up to extracting a subsequence, $(n_k)_k$, we may assume that z_{n_k} converges to some $\xi \in M$ and then get

$$u(z) = \liminf_k u(z_{n_k}) + c_{n_k}(z_{n_k}, z) \geq u(\xi) + h(\xi, z).$$

We just proved that $-h(z, z) \geq -h(\xi, z) + h(\xi, z) = 0$. The other inequality being always true, we get $h(z, z) = 0$ and $z \in \mathcal{A}$. □

As a corollary of Proposition A.5 and Proposition A.9 we obtain

Corollary A.10. *Any subsolution of* (A.1) *is integrable with respect to any projected Mather measure.*

We remark that the main point is that the measurability of the subsolution is not a priori required.

A.5. **Maximum Principle.**

Proposition A.11. *If v is a subsolution and w is a continuous supersolution of* (A.1) *such that $v \leq w$ on the projected Aubry set $\mathcal{A}$, then $v \leq w$ on M.*

Proof. Let us set $u := \mathcal{T}(v)$. Then u is still a subsolution of (A.1) such that $u = v$ on $\mathcal{A}$, see Proposition A.5. Since $v \leq u$ on M, it suffices to prove the statement with u in place of v. The advantage is that u is always a continuous function, while the subsolution v we started with may be discontinuous a priori.

Let us proceed to prove the statement with u in place of v. Let x be an arbitrary point in M. Since w is a supersolution, we can find a point x_{-1} such that

$$w(x_{-1}) + c(x_{-1}, x) \leq w(x)$$

Arguing inductively, we construct a sequence $\bar{x}$ in $S_\infty(x)$ such that

$$w(x_{-i-1}) + c(x_{-i-1}, x_{-i}) \leq w(x_{-i}) \quad \text{for every } i \in \mathbb{N}$$

and

$$w(x_{-n}) + c(x_{-n}, x_{-n+1}) + ... + c(x_{-1}, x) \leq w(x)$$

On the other hand, since u is a subsolution, we have

$$\begin{aligned} u(x) &\leq u(x_{-1}) + c(x_{-1}, x) \\ u(x_{-1}) &\leq u(x_{-2}) + c(x_{-2}, x_{-1}) \\ &\vdots \\ u(x_{-n+1}) &\leq u(x_{-n}) + c(x_{-n}, x_{-n+1}) \end{aligned}$$

for every $n \in \mathbb{N}$, so

$$u(x) \leq u(x_{-n}) + c(x_{-n}, x_{-n+1}) + ... + c(x_{-1}, x).$$

Therefore

$$w(x) - u(x) \geq w(x_{-n}) - u(x_{-n})$$

We claim that any accumulation point of $\bar{x}$ belongs to the projected Aubry set. This is enough to conclude. Indeed, let $(x_{-n_k})_k$ be an appropriate subsequence converging to a point $z \in \mathcal{A}$. Then

$$w(x) - u(x) \geq \lim_{k \to +\infty} w(x_{-n_k}) - u(x_{-n_k}) = w(z) - u(z) \geq 0.$$

Let us then prove the claim. Let z be an accumulation point of $\bar{x}$. Then we can find two diverging sequences $(n_k)_k$ and $(M_k)_k$ such that the points $x_{-n_k-M_k}$ and x_{-n_k} are converging to z as $k \to +\infty$. We have

$$\begin{aligned} h_{M_k}(x_{-n_k-M_k}, x_{-n_k}) &\leq c(x_{-n_k-M_k}, x_{-n_k-M_k+1}) + ... + c(x_{-n_k+1}, x_{-n_k}) \\ &\leq -w(x_{-n_k-M_k}) + w(x_{-n_k-M_k+1}) + ... - w(x_{-n_k+1}) + w(x_{-n_k}) \\ &= -w(x_{-n_k-M_k}) + w(x_{-n_k}), \end{aligned}$$

so sending $k \to +\infty$ we infer that $h(z, z) = 0$, i.e. z belongs to the projected Aubry set $\mathcal{A}$, as it was claimed. □

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