Abstracts

Convergence in the vanishing discount method for Hamilton–Jacobi equations

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(joint work with Andrea Davini, Albert Fathi, Renato Iturriaga)

The results presented build up on previous results [6] and were since extended in various settings [6, 8, 1, 4].

Let M be a closed compact manifold and $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian, convex and coercive with respect to the momentum variable p. It is then known that if $\lambda > 0$ there exists a unique viscosity solution u_λ to the discounted Hamilton–Jacobi equation $\lambda u(x) + H(x, D_x u) = 0$, $x \in M$. We prove the following:

Theorem 1. *There exists a unique constant $c(H)$ for which the functions $u_\lambda + c(H)/\lambda$ uniformly converge to a function u_0 (as $\lambda \to 0$) which then solves the stationary undiscounted equation $H(x, D_x u_0) = c(H)$.*

Such a function u_0 is called a weak KAM solution. In a way, this result closes a loop in the history of weak KAM theory. Let us describe the setting and explain why.

1. History of the problem

In 1987, Lions, Papanicolaou and Varadhan issue a (never published) preprint [7] on the Homogenization of Hamilton–Jacobi equations. They study the following equation with unknown $u^\varepsilon : [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}$:

$$\frac{\partial u^\varepsilon}{\partial t} + H\big(\frac{x}{\varepsilon}, D_x u^\varepsilon\big) = 0$$

with initial condition $u^\varepsilon(0, x) = u_0(x)$. In the above, $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function, which is 1-periodic in the first variable x (meaning it is the lift of a function on $\mathbb{T}^N \times \mathbb{R}^N$) and uniformly coercive with respect to the second variable p. The initial condition $u_0 : \mathbb{R}^N \to \mathbb{R}$ is a bounded uniformly continuous function on $\mathbb{R}^N$. It is then known there exists a unique continuous solution[1] $u^\varepsilon : [0, +\infty) \times \mathbb{R}^N$ to the above equation.

Theorem (Lions, Papanicolaou, Varadhan)**.** *The functions u^ε uniformly converge to a function u^0 which solves a new Hamilton–Jacobi equation $\frac{\partial u^0}{\partial t} + \overline{H}(D_x u^0) = 0$, with same initial condition. The effective Hamiltonian, $\overline{H} : \mathbb{R}^N \to \mathbb{R}$, is characterized as follows: for any $P \in \mathbb{R}^N$, $\overline{H}(P)$ is the only constant for which the following equation admits a 1- periodic solution:*

$$H(x, P + D_x u) = \overline{H}(P). \tag{1}$$

[1] All solutions subsolutions or supersolutions will be implicitly continuous and in the viscosity sense and the terms will be omitted from now on.

Solutions to (1) were later on independently introduced by Fathi as weak KAM solutions. In order to prove such a constant exists, they use an ergodic perturbation and solve $\lambda u + H(x, P + D_x u) = 0$, where $\lambda > 0$ is a parameter that will be sent to 0. It is known that such an equation admits a unique periodic solution u_λ. Moreover, because of the coercivity of H, the family u_λ is equi–Lipschitz. Therefore, the functions $\hat{u}_\lambda = u_\lambda - \min u_\lambda$ admit converging subsequences. They then prove that up to extracting, the λu_λ uniformly converge to a constant $-\overline{H}(P)$ and $\hat{u}_\lambda$ converges to a function u which then solves (1).

Our theorem is that under the extra condition of convexity, no extraction is needed.

2. A formula for the solutions of the discounted equation

Our proof of Theorem 1 relies on an explicit formula for u_λ when H is convex in p. Let us introduce the Lagrangian function: $L : TM \to \mathbb{R} \cup \{+\infty\}$ by

$$\forall (x,v) \in TM, \quad L(x,v) = \sup_{p \in T_x M} p(v) - H(x,p).$$

Then the following holds for all $t > 0$ and $\lambda > 0$ and $x \in M$,

$$\begin{aligned} u_\lambda(x) &= \inf_\gamma e^{-\lambda t} u_\lambda\big(u_\lambda(\gamma(-t)\big) + \int_{-t}^0 e^{\lambda s} L\big(\gamma(s), \dot\gamma(s)\big) \mathrm{d}s \\ &= \inf_\gamma \int_{-\infty}^0 e^{\lambda s} L\big(\gamma(s), \dot\gamma(s)\big) \mathrm{d}s. \end{aligned}$$

Where the infima are taken amongst absolutely continuous curves such that $\gamma(0) = x$.

Note that the function u_0 given by Theorem 1 will then verify a similar relation for all $t > 0$:

$$\forall x \in M, \quad u_0(x) = \inf_\gamma u_0\big(\gamma(-t)\big) + \int_{-t}^0 \big[L\big(\gamma(s), \dot\gamma(s)\big) + c(H)\big] \mathrm{d}s. \tag{2}$$

This is Fathi's original characterization of weak KAM solutions. The $(\inf, +)$ convolution it involves is called Lax-Oleinik semi–group.

3. The discrete setting

At this point, we may introduce a discrete analogue of the previous problem. The philosophy is that the Lagrangian represents a cost to pay to move infinitesimally in a direction v. This is replaced by discretizing the time variable and introducing a cost function which evaluates the cost to go between two points in time 1.

Let (X, d) be a compact metric space, and $c : X \times X \to \mathbb{R}$ be a continuous function. The Lax–Oleinik operator, acting on continuous functions $u : X \to \mathbb{R}$ is defined by $u \mapsto \mathcal{T} u$:

$$\forall x \in X, \quad \mathcal{T}u(x) = \inf_{y \in X} u(y) + c(y, x).$$

It is easily verified that $\mathcal{T}$ has values in (equi–)continuous functions, is 1-Lipschitz for the sup–norm, is order preserving and commutes with addition of constants. Therefore

Proposition 2 (weak KAM)**.** *There exists a unique constant c_0 such that there is a continuous function $u : X \to \mathbb{R}$ verifying $u = \mathcal{T}u + c_0$.*

The discounted operators are defined as follows: given a constant $\mu \in (0,1)$ (which may be seen as $e^{-\lambda}$ in the continuous setting) $\mathcal{T}_\mu$ acts on continuous functions by $\mathcal{T}_\mu u = \mathcal{T}(\mu u)$. This operator is now μ–Lipschitz for the sup–norm, hence it admits a unique fixed point which may be computed taking the limit of iterates starting with any function u, for instance the 0 function. A computation gives that this unique fixed point is given by the formula

$$\forall x \in X, \quad u_\mu(x) = \inf_{(x_n)_{n\leqslant 0}} \sum_{n=-\infty}^{-1} \mu^{n+1} c(x_n, x_{n+1}),$$

where the infimum is taken on all sequences such that $x_0 = x$.

Our second theorem is then:

Theorem 3. *There exists a function u_0 such that $u_\mu + c_0/(1-\mu)$ converges to u_0 as $\mu \to 1$. Moreover, u_0 verifies $u_0 = \mathcal{T}u_0 + c_0$.*

The characterization of u_0 and the proof of the convergence heavily rely on the notion of closed minimizing measures, that is probability measures m on $X \times X$ which have the same projection on both factors and minimize the quantity $\int c(x,y)\mathrm{d}m(x,y)$. They are discrete analogues of Mather measures in the classical Hamiltonian setting. Similar ideas previously appeared in [5] and were recovered independently in the present works..

References

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