INVARIANT FOLIATIONS FOR SYMPLECTIC TWIST MAPS OF THE ANNULUS: A CONSTRUCTION THROUGH GENERATING FUNCTIONS

MARIE-CLAUDE ARNAUD[†,‡], MAXIME ZAVIDOVIQUE[*,**]

ABSTRACT. For an exact symplectic twist diffeomorphism (ESTwD) of the 2-dimensional annulus, we prove that there is a choice of weak K.A.M. solutions $u_c = u(., c)$ that depend in a continuous way on the cohomology class c. Then the graphs of $c + u'_c$ are backward invariant pseudographs and we deduce that the Aubry-Mather sets are contained in pseudographs that are vertically ordered by their rotation numbers. In the C^0-integrable case, we prove that u is C^1 and can be understood as a generating function. For a C^0-integrable ESTwD, we prove the equivalence of (1) the invariant foliation is straightenable via a symplectic homeomorphism, (2) the Dynamics restricted to every leaf of the foliation is C^0-conjugated to a rotation, (3) there exists some global C^0 Arnol'd-Liouville coordinates. We prove that every Lipschitz integrable ESTwD satisfies these properties. We also give a criterion for a foliation to be straightenable via a symplectic homeomorphism. We then provide examples of 'strange' Lipschitz foliations that cannot be straightened by a symplectic homeomorphism, and thus that are not invariant by an ESTwD, but can be invariant by an exact symplectic twist homeomorphism that is a C^1 diffeomorphism.

1. INTRODUCTION AND MAIN RESULTS.

1.1. **Main results.** In this article, we study invariant foliations for exact symplectic twist diffeomorphisms (ESTwD)[1] of the 2-dimensional annulus when different kinds of Dynamics occur:

- any ESTwD;
- the C^0-integrable ESTwDs;
- the Lipschitz integrable ESTwDs.

Because there exist ESTwDs that are not C^0 integrable, there is no hope to find a true invariant foliation in the general case. However, a result of Katznelson and Ornstein [29], provides a family of backward invariant discontinuous graphs for any ESTwD that we call *pseudographs*[2]. Also, weak KAM theory, which was developed

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† Avignon Université, Laboratoire de Mathématiques d'Avignon (EA 2151) F-84018 Avignon, FRANCE .

‡ member of the *Institut universitaire de France.*

∗ IMJ-PRG, UPMC 4 place Jussieu, Case 247 75252 Paris Cedex 5.

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[1] see Definition in section 2.

[2] To be more precise, a pseudograph is the graph $\mathcal{G}(c + u')$ of $c + u'$, where $c \in \mathbb{R}$ and u is a semi-concave function, see subsection 2.3 for the definition of semi-concave function.

in the 90s by A. Fathi [21, 22] in the time continuous case and then extended to the discrete case [8, 9, 15, 24] by others gives some backward invariant pseudographs. More precisely, for a given ESTwD f, weak KAM theory provides for every $c \in \mathbb{R}$ at least one semi-concave function $u_c : \mathbb{T} \to \mathbb{R}$ such that the graph of $c + u'_c$ is backward invariant. However, the question of the regularity with respect to c is open. Our first result states the existence of a continuous choice of u_c and also of the associated pseudograph.

Theorem 1.1. *Let f be a C^1 ESTwD of $\mathbb{T} \times \mathbb{R}$. Then there exists a continuous map $u : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ such that*

- $u(0, c) = 0$*;*
- *the map $(\theta, c) \mapsto \frac{\partial u_c}{\partial \theta}(\theta)$ is continuous on its set of definition;*
- *each $u_c = u(\cdot, c)$ is a weak K.A.M. solution for the cohomology class c, this implies that:*
  - *each $u_c = u(\cdot, c)$ is semi-concave (hence derivable almost everywhere)*[3]*;*
  - *each partial graph $\mathcal{G}(c + u'_c)$ of $c + \frac{\partial u_c}{\partial \theta}$ is backward invariant by f.*

Following [29], we can associate to every pseudograph $\mathcal{G}(c + u')$ its *full pseudograph* $\mathcal{PG}(c + u')$ that is an embedded circle obtained by adding to $\mathcal{G}(c + u')$ some vertical segments. An equivalent and more analytical definition in given in Appendix B. In [29], the authors prove the existence of what they call a pseudo-foliation by full pseudographs whose associated pseudographs are backward invariant. Such a pseudo-foliation is continuous and fills the whole annulus. What we prove in the following theorem implies that our pseudo-foliation by the full pseudographs $\mathcal{PG}(c + du_c)$ satifies the same property and a little more. We will prove in Lemma 2.1 that all the points of $\mathcal{G}(c+du_c)$ have the same rotation number in negative time and the additional result concerns this rotation number.

Theorem 1.2. *With the notations of Theorem 1.1, we have*

(1) *the map $c \mapsto \mathcal{PG}(c + u'_c)$ is continuous for the Hausdorff topology;*
(2) $\bigcup\limits_{c\in\mathbb{R}} \mathcal{PG}(c + u'_c) = \mathbb{A}$;
(3) *if the rotation number*[4] *associated to c is strictly smaller than the one associated to c', then for all $(q, p) \in \mathcal{PG}(c + u'_c)$ and $(q, p') \in \mathcal{PG}(c + u'_{c'})$, we have $p < p'$;*
(4) *we may furthermore construct u in such a way that for all $c \leq c'$, then $c + u'_c(\theta) \leq c' + u'_{c'}(\theta)$ at all $\theta \in \mathbb{T}$ where both derivatives exist;*
(5) *in the later case, the function u is furthermore locally Lipschitz continuous.*

As a result of the proof, we will deduce (see Proposition 2.3) that the Aubry-Mather[5] sets are contained in pseudographs that are vertically ordered by their rotation numbers.

Once we have proved that there always exists a continuous choice $u(\theta, c)$ of weak K.A.M. solutions, we wonder when u can be more regular. We recall that an ESTwD is said to be C^0-integrable if the annulus $\mathbb{T} \times \mathbb{R}$ is C^0-foliated by C^0 invariant graphs.

Theorem 1.3. *With the notations of Theorem 1.1, we have equivalence of*

[3]The definition of a semi-concave function is given in subsection 2.3.
[4]see point (f) in subsection 2.3.
[5]The definition of Aubry-Mather set is given in subsection 2.2.

(1) f is C^0-integrable;
(2) the map u is C^1.

Moreover, in this case, u is unique and we have[6]

- *the graph of $c+u'_c$ is a leaf of the invariant foliation;*
- *$h_c:\theta\mapsto\theta+\frac{\partial u}{\partial c}(\theta,c)$ is a semi-conjugation between the projected Dynamics $g_c:\theta\mapsto\pi_1\circ f\big(\theta,c+\frac{\partial u}{\partial\theta}(\theta,c)\big)$ and a rotation R of $\mathbb{T}$, i.e. $h_c\circ g_c=R\circ h_c$.*

The striking fact is the regularity in c. Indeed, if we have a C^k foliation in graphs for some $k\geq 1$, we can only claim that u and $\frac{\partial u_c}{\partial\theta}$ are C^k. So in the C^0 case, even the derivability with respect to c is surprising, which is a result of the invariance by an ESTwD. Also, the fact that the semi-conjugation h_c continuously depends on c even at a c where the rotation number is rational is very surprising. At an irrational rotation number, this is an easy consequence of the uniqueness of the invariant measure supported on the corresponding leaf, but what happens for a rational rotation number is more subtle.

In the C^0-integrable case, the Dynamics restricted to a leaf with a rational rotation number is completely periodic. It is an open question if it can be a Denjoy counter-example when restricted to a leaf with an irrational rotation number.

We will now give some conditions that imply that the Dynamics restricted to a leaf cannot be Denjoy. We introduce the notion of exact symplectic homeomorphism, which is a particular case of the notion of symplectic homeomorphism that is due to Oh and Müller, [36]. Their notion coincides in this 2-dimensional setting with the one of orientation and Lebesgue measure preserving homeomorphism. Also, driven by the classical Arnol'd-Liouville theorem [18], we introduce a notion of continuous Arnol'd-Liouville coordinates.

Definition.

- An *exact symplectic homeomorphism* is a homeomorphism that is locally a C^0 uniform limit of exact symplectic diffeomorphisms[7].
- If $f:\mathbb{A}\to\mathbb{A}$ is a symplectic homeomorphism, C^0 *Arnol'd-Liouville coordinates* are given by a symplectic homeomorphism $\Phi:\mathbb{A}\to\mathbb{A}$ such that the standard foliation into graphs $\mathbb{T}\times\{c\}$ is invariant by $\Phi\circ f\circ\Phi^{-1}$ and $\Phi\circ f\circ\Phi^{-1}(x,c)=(x+\rho(c),c)$ for some (continuous) function $\rho:\mathbb{R}\to\mathbb{R}$.

A C^1 foliation into continuous graphs can always be straightened via a symplectic homeomorphism (see subsection 6.4). We discover that such a result cannot be adapted to the case of C^0 foliations, as explained by the following statement.

Theorem 1.4. *A C^0-foliation of $\mathbb{A}$: $(\theta,c)\mapsto\big(\theta,\eta_c(\theta)\big)$, where $\int_{\mathbb{T}}\eta_c=c$, is exact symplectically homeomorphic to the standard foliation if and only if there exists a C^1 map $u:\mathbb{A}\to\mathbb{R}$ such that*

- *$u(0,r)=0$ for all $r\in\mathbb{R}$,*
- *$\eta_c(\theta)=c+\frac{\partial u}{\partial\theta}(\theta,c)$ for all $(\theta,c)\in\mathbb{A}$,*
- *for all $c\in\mathbb{R}$, the map $\theta\mapsto\theta+\frac{\partial u}{\partial c}(\theta,c)$ is a homeomorphism of $\mathbb{T}$.*

[6]See the notation π_1 at the beginning of subsection 2.1.

[7]We recall that a diffeomorphism $f:\mathbb{A}\to\mathbb{A}$ is exact symplectic if f is homotopic to Id and the 1-form $f^*(rd\theta)-rd\theta$ is exact.

Remark. In the setting of the previous theorem, the function u is unique and given by the following formula:

$$\forall(\theta,c)\in\mathbb{A},\quad u(\theta,c)=\int_0^\theta \eta_c(s)ds-\theta c.$$

Corollary 1.1. *An ESTwD $f:\mathbb{A}\to\mathbb{A}$ is C^0-integrable with the dynamics on each leaf conjugated to a rotation if and only if it admits global C^0 Arnol'd-Liouville coordinates. In particular, the invariant foliation is exact symplectically homeomorphic to the standard foliation.*

Theorem 1.4 allows us to give an example of a C^0-foliation of $\mathbb{A}$ into smooth graphs that is not symplectically homeomorphic to the standard foliation.

Example. Let $\varepsilon:\mathbb{R}\to\mathbb{R}$ be a non-C^1 function that is $\frac{1}{4\pi}$-Lipschitz. Then the function

$$(\theta,c)\mapsto u_c(\theta)=\frac{\varepsilon(c)}{2\pi}\sin(2\pi\theta)$$

defines a non-C^1 Lipschitz foliation of $\mathbb{A}$ into smooth graphs of $\theta\in\mathbb{T}\mapsto c+\varepsilon(c)\cos(2\pi\theta)$ that is not symplectically homeomorphic to the standard foliation. We will deduce from Theorem 1.5 that this foliation cannot be invariant by an ESTwD. But we will see in subsection 6.5 that it can be invariant by an exact symplectic twist homeomorphism that is a C^1-diffeomorphism.

In fact, we state that any symplectic diffeomorphism that has an invariant foliation that is straightenable via a symplectic homeomorphism has Arnol'd-Liouville coordinates (we don't require the diffeomorphism to be an ESTwD) .

Proposition 1.1. *Let $f:\mathbb{A}\to\mathbb{A}$ be an exact symplectic homeomorphism. Let us assume that f preserves each leaf of a foliation $\mathcal{F}$ into C^0 graphs which is symplectically homeomorphic (by $\Phi:\mathbb{A}\to\mathbb{A}$) to the standard foliation $\mathcal{F}_0=\Phi(\mathcal{F})$. Then there exists a continuous function $\rho:\mathbb{R}\to\mathbb{R}$ such that*

$$\forall(\theta,r)\in\mathbb{A},\quad \Phi\circ f\circ\Phi^{-1}(\theta,r)=(\theta+\rho(r),r).$$

In the case where the invariant foliation by an ESTwD is Lipschitz, we are in the case of Proposition 1.1 and so the Dynamics restricted to every leaf cannot be Denjoy.

Theorem 1.5. *With the notations of Theorem 1.3, we have equivalence of*

(1) *f is Lipschitz integrable[8];*
(2) *the map u is C^1 with*
    - *$\frac{\partial u}{\partial\theta}$ locally Lipschitz continuous;*
    - *$\frac{\partial u}{\partial c}$ uniformly Lipschitz continuous in the variable θ on any compact set of c's;*
    - *for every compact subset $\mathcal{K}\subset\mathbb{A}$, there exists a constant $k>-1$ such that $\frac{\partial^2 u}{\partial\theta\partial c}>k$ Lebesgue almost everywhere in $\mathcal{K}$.*

In this case, there exists $\Phi:\mathbb{T}\times\mathbb{R}\to\mathbb{T}\times\mathbb{R}$ exact symplectic homeomorphism that is C^1 in the θ variable and maps the invariant foliation onto the standard one such that:

$$\forall(x,c)\in\mathbb{T}\times\mathbb{R},\quad \Phi\circ f\circ\Phi^{-1}(x,c)=(x+\rho(c),c);$$

[8]See the definition in subsection 3.1

where $\rho : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism.

Then all the leaves are C^1 and the foliation is a C^1 lamination[9]; moreover the Dynamics restricted to every leaf is C^1 conjugated to a rotation. Theorem 1.5 provides some C^0 Arnol'd-Liouville coordinates. More precisely, the function Φ can be given by the following formula:

$$\Phi\Big(\theta, c + \frac{\partial u}{\partial \theta}(\theta, c)\Big) = \Big(\theta + \frac{\partial u}{\partial c}(\theta, c), c\Big).$$

1.2. **Further comments and related results.**

Theorem 1.1 selects in a continuous way a unique weak KAM solution u_c for every cohomology class $c \in \mathbb{R}$. Let us mention two related results.

- The recent works in [16] for the autonomous case and in [17] and [38] for the discrete case select a unique solution, called discounted solution, for every cohomology class. We give in Appendix A.2 an example of a C^∞ integrable ESTwD (coming from an autonomous Tonelli Hamiltonian) for which the discounted method doesn't select a transversally continuous weak K.A.M. solution. Hence our method is different from the discounted one.
- If we have not a unique choice of a weak K.A.M. solution for every cohomology class $c \in H^1(M, \mathbb{R})$, we cannot speak of C^1 regularity with respect to c for the map $c \mapsto \{u_c\}$ that sends c to the whole set of weak K.A.M. solutions of cohomology class c. Observe nevertheless that a kind of local Lipschitz regularity was studied in [31] (for weak K.A.M. solutions for Tonelli Hamiltonians) with no uniqueness.

Theorem 1.2 compares the cohomology classes of pseudographs that correspond to distinct rotation numbers. In the setting of Hamiltonian flows with two degrees of freedom, an analogous statement is proved in [14] concerning the cyclic order of rotation and cohomology vectors.

In **Theorem 1.3**, the function u can be seen as a C^1 generating function of a continuous map $H : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ that is defined by

$$H(\theta, r) = (x, c) \Longleftrightarrow x = \theta + \frac{\partial u}{\partial c}(\theta, c) \quad \text{and} \quad c = r - \frac{\partial u}{\partial \theta}(\theta, c). \tag{1}$$

and that satisfies $H \circ f(\theta, r) = H(\theta, r) + (\rho(c), 0)$ with $\rho : \mathbb{R} \to \mathbb{R}$ increasing homeomorphism. Observe that if on some curve of the invariant foliation, the Dynamics is not recurrent (i.e. we have a Denjoy counter-example), then H is not an homeomorphism because it is not injective. In [19], the author makes similar remarks concerning the link between the weak K.A.M. solutions and a generating function in the Hamiltonian case.

In the proof we also see that the foliation $\eta_c = c + \frac{\partial u_c}{\partial \theta}$ has a partial derivative with respect to c along any leaf having a rational rotation number.

Theorem 1.4 seems to be global. In fact, one can provide an analogous local statement and an example of a local continuous foliation in C^0 graphs that is not straightenable by a (local) symplectic homeomorphism.

Let us comment on the fact that we seem to restrict ourselves by using *exact* symplectic homeomorphisms.

- Observe that if f is exact symplectic and maps the graph of $c + u'$ onto the graph of $c' + v'$ where $c, c' \in \mathbb{R}$ and $u, v : \mathbb{T} \to \mathbb{R}$ are C^1, then $c = c'$;

[9]See the definition at the beginning of subsection 3.1.

- if a symplectic homeomorphism $\phi : \mathbb{A} \to \mathbb{A}$ gives some Arnol'd-Liouville coordinates for f, then, composing ϕ with $(x, c) \mapsto (\pm x, \pm c + c_0)$ for some $c_0 \in \mathbb{R}$, we may assume that ϕ is exact symplectic.

The same proof as the one of **Proposition 1.1** applies in the slightly more general case where f preserves the foliation $\mathcal{F}$, possibly sending a leaf on a different one. Then the conclusion should be modified by:

$$\exists r_0 \in \mathbb{R}, \exists \lambda \in \{-1, 1\}, \forall (\theta, r) \in \mathbb{A}, \quad \Phi \circ f \circ \Phi^{-1}(\theta, r) = (\theta + \rho(r), \lambda r + r_0).$$

A similar statement to **Theorem 1.5** for Tonelli Hamiltonians is proved in [6].

1.3. **Content of the different sections.** To prove all these results, we will use together Aubry-Mather theory, weak K.A.M. theory in the discrete case and also ergodic theory. Let us detail what will be in the different sections

- Section 2 contains some reminders on ESTwDs, Aubry-Mather theory, on discrete weak K.A.M. theory, some new results on the weak K.A.M. solutions and the proof of Theorems 1.1 and 1.2;
- Section 3 contains the proof of the first implication of Theorem 1.3; after recalling some generalities about ESTwDs, we consider the case of the rational curves by using some ergodic theory, then we build the wanted function u_c and prove its regularity by using also ergodic theory;
- the second implication of Theorem 1.3 is proved in section 4;
- Theorem 1.5 is proved in section 5;
- Section 6 contains proofs of Proposition 1.1 and Theorem 1.4 and gives an example of a C^0 foliation of $\mathbb{A}$ into continuous graphs that is not exact symplectically homeomorphic to the standard foliation. It also contains the proof that any C^1 foliation is symplectically homeomorphic to the standard foliation and the fact that our example of C^0 foliation of $\mathbb{A}$ into continuous graphs that is not exact symplectically homeomorphic to the standard foliation is invariant by a symplectic twist homeomorphism that is a C^1-diffeomorphism.
- Appendix A contains some examples, Appendix B deals with full pseudographs and Appendix C recalls some results about Green bundles.

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2. Aubry-Mather and weak K.A.M. theories for ESTwDs and proof of Theorems 1.1 and 1.2

2.1. **The setting.** The definitions and results that we give here are very classical now. Good references are [23, 25, 34, 35, 9, 33].

Let us introduce some notations

Notations.

- $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the circle and $\mathbb{A} = \mathbb{T} \times \mathbb{R}$ is the annulus ; $\pi : \mathbb{R} \to \mathbb{T}$ is the usual projection;
- the universal covering of the annulus is denoted by $p : \mathbb{R}^2 \to \mathbb{A}$;

- the corresponding projections are $\pi_1 : (\theta, r) \in \mathbb{A} \mapsto \theta \in \mathbb{T}$ and $\pi_2 : (\theta, r) \in \mathbb{A} \mapsto r \in \mathbb{R}$; we denote also the corresponding projections of the universal covering by π_1, $\pi_2 \ : \mathbb{R}^2 \to \mathbb{R}$;
- the Liouville 1-form is defined on $\mathbb{A}$ as being $\lambda = \pi_2 d\pi_1 = r d\theta$; then $\mathbb{A}$ is endowed with the symplectic form $\omega = -d\lambda$.

Let us give the definition of an exact symplectic twist diffeomorphism.

Definition. An *exact symplectic twist diffeomorphism* (in short ESTwD) $f : \mathbb{A} \to \mathbb{A}$ is a C^1 diffeomorphism such that

- f is isotopic to identity;
- f is exact symplectic, i.e. if $f(\theta, r) = (\Theta, R)$, then the 1-form $R d\Theta - r d\theta$ is exact;
- f has the *twist property* i.e. if $F = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is any lift of f, for any $\tilde{\theta} \in \mathbb{R}$, the map $r \in \mathbb{R} \mapsto F_1(\tilde{\theta}, r) \in \mathbb{R}$ is an increasing C^1 diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

A C^2 generating function $S : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that satisfies the following definition can be associated to any lift F of such an ESTwD f.

Definition. The C^2 function $S : \mathbb{R}^2 \to \mathbb{R}$ is a *generating function* of the lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ of an ESTwD if

- $S(\theta + 1, \Theta + 1) = S(\theta, \Theta)$;
- $\displaystyle\lim_{|\Theta - \theta| \to \infty} \frac{S(\theta, \Theta)}{|\Theta - \theta|} = +\infty$; we say that S is *superlinear*;
- for every $\theta_0, \Theta_0 \in \mathbb{R}$, the maps $\theta \mapsto \frac{\partial S}{\partial \Theta}(\theta, \Theta_0)$ and $\Theta \mapsto \frac{\partial S}{\partial \theta}(\theta_0, \Theta)$ are decreasing diffeomorphisms of $\mathbb{R}$;
- for $(\theta, r), (\Theta, R) \in \mathbb{R}^2$, we have the following equivalence

$$F(\theta, r) = (\Theta, R) \Leftrightarrow r = -\frac{\partial S}{\partial \theta}(\theta, \Theta) \quad \text{and} \quad R = \frac{\partial S}{\partial \Theta}(\theta, \Theta).$$

Remark. J. Moser proved in [35] that such an ESTwD is the time 1 map of a C^2 1-periodic in time Hamiltonian $H : \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that is C^2 convex in the fiber direction[10], i.e. such that

$$\frac{\partial^2 H}{\partial r^2}(\theta, r, t) > 0.$$

Then there exists a relation between the Hamiltonian that was built by J. Moser and the generating function. Indeed, if we denote by (Φ_t) the time t map of the Hamiltonian H that is defined on $\mathbb{R}^2$ and by L the associated Lagrangian that is defined by

$$L(\theta, v, t) = \max_{r \in \mathbb{R}} \big(rv - H(\theta, r, t) \big),$$

then we have

- for every $t \in (0, 1]$, Φ_t is an ESTwD and $\Phi_1 = F$;
- there exists a C^1 time-dependent family of C^2 generating functions S_t of Φ_t such $S_1 = S$ and for all $(\theta, r), (\Theta, R) \in \mathbb{R}^2$,

$$\Phi_t(\theta, r) = (\Theta, R) \Rightarrow S_t(\theta, \Theta) = \int_0^t L\big(\pi_1 \circ \Phi_s(\theta, r), \frac{\partial}{\partial s}\big(\pi_1 \circ \Phi_s(\theta, r)\big), s\big) ds.$$

[10] In fact J. Moser assumed that f is smooth.

In other words, the generating function is also the Lagrangian action.

2.2. **Aubry-Mather theory.** Good references for what is in this section are [7], [25] and [4]. Let us recall the definition of some particular invariant sets.

Definition. Let $F:\mathbb{R}^2\to\mathbb{R}^2$ be a lift of an ESTwD f.

- a subset E of $\mathbb{R}^2$ is *well-ordered* if it is invariant under the translation $(\theta, r)\mapsto(\theta+1, r)$ and F and if for every $x_1, x_2\in E$, we have
$$\big[\pi_1(x_1)<\pi_1(x_2)\big]\Rightarrow\big[\pi_1\circ F(x_1)<\pi_1\circ F(x_2)\big];$$
this notion is independent from the lift of f we use;
- a subset E of $\mathbb{A}$ is *well-ordered* if $p^{-1}(E)$ is well-ordered;
- an *Aubry-Mather set* for f is a compact well-ordered set or the lift of such a set;
- a piece of orbit $(\theta_k, r_r)_{k\in[a,b]}$ for F is *minimizing* if for every sequence $(\theta'_k)_{k\in[a,b]}$ with $\theta_a=\theta'_a$ and $\theta_b=\theta'_b$, it holds
$$\sum_{j=a}^{b-1}S(\theta_j,\theta_{j+1})\le\sum_{j=a}^{b-1}S(\theta'_j,\theta'_{j+1});$$
then we say that $(\theta_j)_{j\in[a,b]}$ is a *minimizing sequence* or *segment*;
- an infinite piece of orbit, or a full orbit for F is minimizing if all its finite subsegments are minimizing;
- an invariant set is said to be minimizing if all the orbits it contains are minimizing.

The following properties of the well-ordered sets are well-known

(1) a minimizing orbit and its translated orbits by $(\theta, r)\mapsto(\theta+1, r)$ define a well-ordered set;
(2) the closure of a well-ordered set is a well-ordered set;
(3) any well-ordered set E is contained in the (non-invariant) graph of a Lipschitz map $\eta:\mathbb{T}\to\mathbb{R}$; it follows that the map $N=\big(\cdot,\eta(\cdot)\big):\mathbb{T}\to\mathrm{Graph}(\eta)$ is Lipschitz and so are the maps $\pi_1\circ f\circ N_{|\pi_1(E)}$ and $\pi_1\circ f^{-1}\circ N_{|\pi_1(E)}$. This implies that the projected restricted Dynamics $\pi_1\circ f\big(\cdot,\eta(\cdot)\big)_{|\pi_1(E)}$ to an Aubry-Mather set is the restriction of a biLipschitz orientation preserving circle homeomorphism;
(4) any well-ordered set E in $\mathbb{R}^2$ has a unique rotation number $\rho(E)$ $\big($the one of the circle homeomorphism we mentioned in Point (3)$\big)$, i.e.
$$\forall x\in E,\quad\lim_{k\to\pm\infty}\frac{1}{k}\big(\pi_1\circ F^k(x)-\pi_1(x)\big)=\rho(E);$$
(5) for every $\alpha\in\mathbb{R}$, there exists a minimizing Aubry-Mather set E such that $\rho(E)=\alpha$;
(6) if α is irrational, there is a unique minimizing Aubry-Mather that is minimal (resp. maximal) for the inclusion; the minimal one is then a Cantor set or a complete graph and the maximal one $\mathcal{M}(\alpha)$ is the union of the minimal one and orbits that are homoclinic to the minimal one;
(7) if α is rational, any Aubry-Mather set that is minimal for the inclusion is a periodic orbit;

(8) any essential invariant curve by an ESTwD is in fact a Lipschitz graph (Birkhoff theorem, see [11], [20] and [26]) and a well-ordered set.

We will need more precise properties for minimizing orbits.

Definition. Let $a = (a_k)_{k\in I}$ and $b = (b_k)_{k\in I}$ be two finite or infinite sequences of real numbers. Then

- if $k \in I$, we say that a and b cross at k if $a_k = b_k$;
- if $k, k+1 \in I$, we say that a and b cross between k and $k+1$ if $(a_k - b_k)(a_{k+1} - b_{k+1}) < 0$.

Note that concerning the first item, the traditional terminology also imposes that $(a_{k-1} - b_{k-1})(a_{k+1} - b_{k+1}) < 0$ when k is in the interior of I. However, due to the twist condition, this is automatic for projections of orbits of F as soon as $a_k = b_k$ if the two orbits are distinct.

Proposition 2.1. (Aubry fundamental lemma) *Two distinct minimizing sequences cross at most once except possibly at the two endpoints when the sequence is finite.*

2.3. **Classical results on weak K.A.M. solutions.** Good references are [8], [9] or [24]. We assume that S is a generating function of a lift F of an ESTwD f.

We define on $C^0(\mathbb{T}, \mathbb{R})$ the so-called *negative Lax-Oleinik maps* T^c for $c \in \mathbb{R}$ as follows:

if $u \in C^0(\mathbb{T}, \mathbb{R})$, we denote by $\tilde u : \mathbb{R} \to \mathbb{R}$ its lift and

$$\forall \theta \in \mathbb{R}, \quad \widetilde{T}^c \tilde u(\theta) = \inf_{\theta' \in \mathbb{R}} \big(\tilde u(\theta') + S(\theta', \theta) + c(\theta' - \theta)\big). \tag{2}$$

The function $\widetilde{T}^c \tilde u$ is then 1-periodic and the negative Lax-Oleinik operator is defined as the induced map $T^c u : \mathbb{T} \to \mathbb{R}$.

An alternative but equivalent definition is as follows (see also [39] for similar constructions): define the function

$$\forall (x, x') \in \mathbb{T} \times \mathbb{T}, \quad S^c(x, x') = \inf_{\substack{\pi(\theta) = x \\ \pi(\theta') = x'}} S(\theta, \theta') + c(\theta - \theta'). \tag{3}$$

Then

$$\forall x \in \mathbb{T}, \quad T^c u(x) = \inf_{x' \in \mathbb{T}} u(x') + S^c(x', x).$$

Then it can be proved that there exists a unique function $\alpha : \mathbb{R} \to \mathbb{R}$ such that the map $\widehat{T}^c = T^c + \alpha(c)$ that is defined by

$$\widehat{T}^c(u) = T^c(u) + \alpha(c)$$

has at least one fixed point in $C^0(\mathbb{T}, \mathbb{R})$, i.e. if $u \in C^0(\mathbb{T}, \mathbb{R})$ is such a fixed point, its lift verifies

$$\forall \theta \in \mathbb{R}, \quad \tilde u(\theta) = \inf_{\theta' \in \mathbb{R}} \big(\tilde u(\theta') + S(\theta', \theta) + c(\theta' - \theta) + \alpha(c)\big). \tag{4}$$

Such a fixed point is called a *weak K.A.M. solution*. It is not necessarily unique. For example, if u is a weak K.A.M. solution, so is $u + k$ for every $k \in \mathbb{R}$, but there can also be other solutions. We denote by $\mathcal{S}_c$ the set of these weak K.A.M. solutions. There is no link in general for solutions corresponding to distinct c's. We recall

Definition. Let $u : \mathbb{R} \to \mathbb{R}$ be a function and let $K > 0$ be a constant. Then u is K-semi-concave if for every x in $\mathbb{R}$, there exists some $p \in \mathbb{R}$ so that:

$$\forall y \in \mathbb{R}, \quad u(y) - u(x) - p(y-x) \leq \frac{K}{2}(y-x)^2.$$

A function $v : \mathbb{T} \to \mathbb{R}$ is K-semi-concave if its lift $\tilde{v} : \mathbb{R} \to \mathbb{R}$ is.

A good reference for semi-concave functions is the appendix A of [9] or [12].

Notation. If $u \in C^0(\mathbb{T}, \mathbb{R})$ and $c \in \mathbb{R}$, we will denote by $\mathcal{G}(c + u')$ the partial graph of $c + u'$. This is a graph above the set of derivability of u.
When u is semi-concave, we sometimes say that $\mathcal{G}(c + u')$ is a *pseudograph*.

Let us end with definitions:

Definition. Let $g : \mathbb{T} \to \mathbb{R}$ be a Lipschitz function (hence derivable almost everywhere). We define

$$\forall x \in \mathbb{T}, \quad \partial g(x) = \mathrm{co}\big\{(x,p) \in \mathbb{T} \times \mathbb{R}, \quad (x,p) \in \overline{\mathcal{G}(g')}\big\}.$$

The notation co stands for the convex hull in the fiber direction. The sets $\partial g(x)$ are non empty, (obviously) convex and compact. They are particular instances of the Clarke subdifferential. This set is a good candidate for a generalized derivative because if g is derivable at x then $g'(x) \in \partial g(x)$. Moreover, if $\partial g(x)$ is a singleton, then g is derivable at x. The converse is in general not true, but it is however true for semi-concave functions.

Definition. If $g : \mathbb{T} \to \mathbb{R}$ is Lipschitz and $c \in \mathbb{R}$, we define $\mathcal{PG}(c + g') = \{(0, c) + \partial g(t), \quad t \in \mathbb{T}\}$. If g is semi-concave, we call it the full pseudograph of $c + g'$.

A proof of the following proposition is given in Appendix B.

Proposition 2.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of equi-semi-concave functions from $\mathbb{T}$ to $\mathbb{R}$ that converges (uniformly) to a function f (that is hence also semi-concave). Then $\mathcal{PG}(f_n')$ converges to $\mathcal{PG}(f')$ for the Hausdorff distance.*

The following results can be found in the papers that we quoted

(a) the function α is convex and superlinear;
(b) if $u \in C^0(\mathbb{T}, \mathbb{R})$, then $\widehat{T}^c u$ is semi-concave and then differentiable Lebesgue almost everywhere;
(c) the function $\widehat{T}_c u$ is differentiable at x if and only if there is only one y where the minimum is attained in Equality (4); in this case, if u is semi-concave, then it is differentiable at y and we have

$$f\big(y, c + u'(y)\big) = \big(x, c + (\widehat{T}^c u)'(x)\big);$$

if u is a weak K.A.M. solution for $\widehat{T}^c$ that is differentiable at x then $\Big(f^k\big(x, c + u'(x)\big)\Big)_{k\in\mathbb{Z}_-}$ is a minimizing piece of orbit that is contained in $\mathcal{G}(c + u')$;
(d) moreover, for any compact subset K of $\mathbb{R}$, the weak K.A.M. solutions for T^c with $c \in K$ are uniformly semi-concave (i.e. for a fixed constant of semi-concavity) and then uniformly Lipschitz;

(e) if $u \in C^0(\mathbb{T},\mathbb{R})$, then

$$f^{-1}\big(\overline{\mathcal{G}(c+(\widehat{T}^c u)')}\big) \subset \mathcal{G}(c+u');$$

if u is a weak K.A.M. solution for $\widehat{T}^c$, then $\mathcal{G}(c+u')$ satisfies

$$f^{-1}\big(\overline{\mathcal{G}(c+u')}\big) \subset \mathcal{G}(c+u')$$

and for every $(\theta,r) \in \overline{\mathcal{G}(c+u')}$, then $\big(f^k(\theta,r)\big)_{k\in\mathbb{Z}_-}$ is minimizing; we will give in Appendix A.1 an example of a backward invariant pseudograph that doesn't correspond to any weak K.A.M. solution;

(f) moreover, if u is a weak K.A.M. solution for $\widehat{T}^c$, then the set

$$\bigcap_{n\in\mathbb{N}} f^{-n}\big(\mathcal{G}(c+u')\big)$$

is a f-invariant minimizing compact well-ordered set to which we can associate a unique rotation number. It results from Mather's theory that this rotation number only depends on c and is equal to $\rho(c)=\alpha'(c)$; because of the convexity of α, observe in particular that α is C^1 and ρ is continuous and non-decreasing;

(g) it then follows from the first (a) and the previous (d) and (f) points that, as in (d), for any compact subset K of $\mathbb{R}$, the weak K.A.M. solutions for T^c with $\rho(c) \in K$ are uniformly semi-concave (i.e. for a fixed constant of semi-concavity) and then uniformly Lipschitz;

(h) in the setting of point (f), then for every weak K.A.M. solution for $\widehat{T}^c$, the graph $\mathcal{G}(c+u')$ contains any minimizing Aubry-Mather set with rotation number $\rho(c)$ that is minimal for the inclusion; we denote the union of these Aubry sets by $\mathcal{A}\big(\rho(c)\big)$. If $\rho(c)$ is irrational, then two possibilities may occur:
   - either $\mathcal{A}\big(\rho(c)\big)$ is an invariant Cantor set and $\mathcal{G}(c+u')$ is contained in the unstable set of the Cantor set $\mathcal{A}\big(\rho(c)\big)$;
   - or $\mathcal{A}\big(\rho(c)\big)=\mathcal{G}(c+u')$ and u is C^1.

   If $\rho(c)$ is rational, then $\mathcal{A}\big(\rho(c)\big)$ is the union of some periodic orbits and $\mathcal{G}(c+u')$ is contained in the union of the unstable sets of these periodic orbits.

We noticed that to any $c \in \mathbb{R}$ there corresponds a unique rotation number $\rho(c)$. But it can happen that distinct numbers c correspond to a same rotation number R. In this case, because $\rho(c)=\alpha'(c)$ is non decreasing (because of point (f)), $\rho^{-1}(R)=[c_1,c_2]$ is an interval. It can be proved that this may happen only for rational R's. This is a result of John Mather. A simple proof can be found in [10, Proposition 6.5].

Finally, when c corresponds to an irrational rotation number $\rho(c)$, then there exists only one weak K.A.M. solution up to constants. The argument comes from [10].

Notation. When $\rho(c)$ is irrational, we will denote by u_c the (unique) solution such that $u_c(0)=0$.

2.4. **More results on weak K.A.M. solutions.** We start with a lemma stating that some minimizing sequences admit a rotation number:

Lemma 2.1. *Let u be a weak K.A.M. solution for $\widehat{T}^c$. Let $(\theta_0, r) \in \overline{\mathcal{G}(c+u')}$, and $\tilde\theta_0 \in \mathbb{R}$ a lift of θ_0. Let $(\tilde\theta_k, r_k)_{k\in\mathbb{Z}_-} = \big(F^k(\tilde\theta, r)\big)_{k\in\mathbb{Z}_-}$. Then*

$$\lim_{k\to-\infty} \frac{\tilde\theta_k}{k} = \rho(c).$$

Proof. Let us argue by contradiction. If this is not the case, there exists $\varepsilon > 0$ and a subsequence $n_k \to -\infty$ such that for all k,

$$\left|\frac{\tilde\theta_{n_k}}{n_k} - \rho(c)\right| > \varepsilon.$$

Up to an extra extraction, we may assume that the following sequence of measures converges:

$$\lim_{k\to-\infty} \frac{1}{|n_k|} \sum_{i=-n_k}^{-1} \delta_{(\tilde\theta_i, r_i)} = \mu.$$

We then know that μ is a minimizing Mather measure whose support is made of points having rotation number $\rho(c)$. Consider the function $D(\tilde\theta, r) = \pi_1 \circ F(\tilde\theta, r) - \tilde\theta$. It is a periodic function in $\tilde\theta$ that is the lift of a function on $\mathbb{A}$.

We may compute that

$$\begin{aligned}\left|\int D d\mu - \rho(c)\right| &= \left|\lim_{k\to-\infty} \frac{1}{|n_k|} \sum_{i=-n_k}^{-1} D(\tilde\theta_i, r_i) - \rho(c)\right| \\ &= \left|\lim_{k\to-\infty} \frac{1}{|n_k|}(\tilde\theta_0 - \tilde\theta_{n_k}) - \rho(c)\right| \geq \varepsilon.\end{aligned}$$

This contradicts the fact that μ has rotation number $\rho(c)$. □

Proposition 2.3. *Let u_1, u_2 be two weak K.A.M. solutions corresponding to T^{c_1}, T^{c_2}, such that $\rho(c_1) < \rho(c_2)$. Then we have*

- *$c_1 < c_2$;*
- *for any $t \in \mathbb{T}$, if $(t, p_1) \in \partial u_1(t)$ and $(t, p_2) \in \partial u_2(t)$ then $c_1 + p_1 < c_2 + p_2$;*
- *in particular, at every point of differentiability t of u_1 and u_2: $c_1 + u_1'(t) < c_2 + u_2'(t)$.*

Proof. Let $\tilde u_1$ and $\tilde u_2$ be the lifts of u_1 and u_2. We introduce the notation $v(t) = \tilde u_2(t) - \tilde u_1(t) + (c_2 - c_1)t$. Then v is Lipschitz and thus Lebesgue everywhere differentiable and equal to a primitive of its derivative. Let us assume by contradiction that there exist $(x, c_1 + p_1) \in \overline{\mathcal{G}(c_1 + u_1')}$ and $(x, c_2 + p_2) \in \overline{\mathcal{G}(c_2 + u_2')}$

$$c_2 + p_2 \leq c_1 + p_1. \tag{5}$$

As $\rho(c_1) \neq \rho(c_2)$, the two graphs correspond to distinct rotation numbers. Thanks to (e) their closures have no intersections. The inequality (5) is then strict.

We introduce the notation $(x^1, y^1) = (x, c_1 + p_1)$ and $(x^2, y^2) = (x, c_2 + p_2)$. Then the orbit of (x^i, y^i) is denoted by $(x^i_k, y^i_k)_{k\in\mathbb{Z}}$. We know that the negative orbits $(x^i_k, y^i_k)_{k\in\mathbb{Z}_-}$, that are contained in the corresponding graphs, are minimizing. Hence the sequences $(x^i_k)_{k\in\mathbb{Z}_-}$ are minimizing. By Aubry's fundamental lemma, we know that they can cross at most once (hence only at x). But we have

- because of the twist condition, as $x^1_0 = x^2_0$ and $y^1_0 > y^2_0$, then $x^1_{-1} < x^2_{-1}$;
- as $\rho(c_1) < \rho(c_2)$, and thus for k close enough to $-\infty$, we have: $x^1_k > x^2_k$.

Finally we find two crossings for two minimizing sequences, a contradiction.

We have in particular for any point t of derivability of u_1 and u_2

$$c_1 + u_1'(t) < c_2 + u_2'(t).$$

Integrating this inequality, we deduce that $c_1 < c_2$.

Finally, for any $t \in \mathbb{T}$, as for all $(t, p_1) \in \overline{\mathcal{G}(c_1 + u_1')}$ and $(t, p_2) \in \overline{\mathcal{G}(c_2 + u_2')}$

$$c_2 + p_2 > c_1 + p_1, \tag{6}$$

taking convex hulls, we get the result.

□

As to an irrational rotation number a unique constant c corresponds, we deduce the following corollary.

Corollary 2.1. *With the same notation as in Proposition 2.3, assume that $c_1 < c_2$ are such that at least one of $\rho(c_1)$ and $\rho(c_2)$ is irrational. Then the function $t \in \mathbb{R} \mapsto \tilde{u}_{c_2}(t) - \tilde{u}_{c_1}(t) + t(c_2 - c_1)$ is strictly increasing.*

Remark. A consequence of Proposition 2.3 is that the pseudographs corresponding to the weak K.A.M. solutions having distinct rotation numbers are vertically ordered with the same order as the one between the rotation numbers.

Now we recall some results that are contained in [24] (see especially Theorem 9.3).

Notation. If θ_1, θ_2 are in $\mathbb{R}$, $c \in \mathbb{R}$ and $n \geq 1$, we denote by $\mathcal{S}_n^c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the function that is defined by

$$\mathcal{S}_n^c(\theta, \Theta) = \inf_{\substack{\theta_0 = \theta \\ \theta_n - \Theta \in \mathbb{Z}}} \left\{ \sum_{i=1}^{n} \big(S(\theta_{i-1}, \theta_i) + c(\theta_{i-1} - \theta_i) \big) \right\}.$$

Observe that $\mathcal{S}_n^c$ is $\mathbb{Z}^2$-periodic.

(1) If R is any rotation number, for any $c \in \rho^{-1}(R)$ and any weak K.A.M. solution u for $\widehat{T}^c$, the set of invariant Borel probability measures with support in $\mathcal{G}(c + u')$ is independent from $c \in \rho^{-1}(R)$ and u. Those measures are called Mather measures and the union of the supports of these measures is called the *Mather set* for R and is denoted by $\mathcal{M}(R)$;

(2) We say that a function u defined on a part A of $\mathbb{T}$ is c-dominated if, denoting by $\widetilde{A}$ the lift of A to $\mathbb{R}$, and $\tilde{u}$ a lift of u, we have
$$\forall \theta, \theta' \in \widetilde{A}, \forall n \geq 1, \quad \tilde{u}(\theta) - \tilde{u}(\theta') \leq \mathcal{S}_n^c(\theta', \theta) + n\alpha(c);$$
a weak K.A.M. solution for $\widehat{T}^c$ is always c-dominated; if $A = \mathbb{T}$ a function $u : \mathbb{T} \to \mathbb{R}$ is c-dominated if and only if
$$\forall \theta, \theta' \in \mathbb{R}, \quad \tilde{u}(\theta) - \tilde{u}(\theta') \leq S(\theta', \theta) + c(\theta' - \theta) + \alpha(c);$$

(3) If $u : \mathcal{M}\big(\rho(c)\big) \to \mathbb{R}$ is dominated, then there exists only one extension U of u to $\mathbb{T}$ that is a weak K.A.M. solution for $\widehat{T}^c$. This function is given by
$$\forall x \in \mathbb{T}, \quad U(x) = \inf_{\substack{\pi(\theta) \in \mathcal{M}(\rho(c)) \\ \pi(\theta') = x}} \tilde{u}(\theta) + \mathcal{S}^c(\theta, \theta'),$$

where

$$\mathcal{S}^c(\theta,\Theta)=\inf_{n\in\mathbb{N}}\big(\mathcal{S}^c_n(\theta,\Theta)+n\alpha(c)\big).$$

As we have not found it exactly stated in this way in the literature, we provide a sketch of proof for the reader's convenience in appendix D.

2.5. **Proof of Theorem 1.1.** When there is no ambiguity in the notations, we will put $\sim$ signs to signify that we consider lifts of functions defined on $\mathbb{T}$. We will need the following lemma.

Lemma 2.2. *Let (c_n) be a sequence of real numbers convergent to c and let (u_{c_n}) be a sequence of functions uniformly convergent to v such that u_{c_n} is a weak K.A.M. solution for $\widehat{T}^{c_n}$. Then $\lim\limits_{n\to\infty} u_{c_n}$ is a weak K.A.M. solution for $\widehat{T}^c$.*

Proof. We know from Equation (4) that

$$\tilde u_{c_n}(\theta)=\inf_{\theta'\in\mathbb{R}}\big(\tilde u_{c_n}(\theta')+S(\theta',\theta)+c_n(\theta'-\theta)+\alpha(c_n)\big).$$

Because of the superlinearity of S and the fact that the u_{c_n} and c_n are uniformly bounded, there exists a fixed compact set I in $\mathbb{R}$ such that for every n, we have

$$\tilde u_{c_n}(\theta)=\inf_{\theta'\in I}\big(\tilde u_{c_n}(\theta')+S(\theta',\theta)+c_n(\theta'-\theta)+\alpha(c_n)\big).$$

We deduce from the uniform convergence of (u_{c_n}) to v that

$$\tilde v(\theta)=\inf_{\theta'\in I}\big(\tilde v(\theta')+S(\theta',\theta)+c(\theta'-\theta)+\alpha(c)\big).$$

As we could do the same proof for I as large as wanted, we have in fact

$$\tilde v(\theta)=\inf_{\theta'\in\mathbb{R}}\big(\tilde v(\theta')+S(\theta',\theta)+c(\theta'-\theta)+\alpha(c)\big). \tag{7}$$

□

Let us now prove Theorem 1.1.
We have seen in subsection 2.3 that we have only one possible choice for u_c when $\rho(c)$ is irrational.

Notation. We use the notation $\mathcal{I}=\{c\in\mathbb{R};\quad \rho(c)\in\mathbb{R}\backslash\mathbb{Q}\}$.

Let us prove that any extension $c\mapsto u_c$ that maps c on a weak K.A.M. solution for $\widehat{T}^c$ that vanishes at 0 is continuous at every $c\in\mathcal{I}$. Let us consider a monotone sequence (c_n) that converges to $c\in\mathcal{I}$ and such that the sequence $\rho(c_n)$ is strictly monotone. Then the sequence $(c_n+u'_{c_n})$ is also monotone by Corollary 2.1 and bounded because the u_{c_n} are equi semi-concave and then equiLipschitz so convergent to a function d. We define for every $t\in[0,1]$, $v(t)=\int_0^t d(s)ds-ct$. Then we have $d=c+v'$ Lebesgue almost everywhere. As the sequence $(c_n+u'_{c_n})$ is bounded, the Lebesgue dominated convergence Theorem implies that (u_{c_n}) pointwise converges to v. Because of the Ascoli Theorem, this convergence is uniform. Because of Lemma 2.2, v is a weak K.A.M. solution for $\widehat{T}^c$. As v vanishes at 0 and $\rho(c)$ is irrational, then v is the unique weak K.A.M. solution for $\widehat{T}^c$ that vanishes at 0, i.e. $v=u_c$.
If now (c_n) is any monotone sequence that converges to $c\in\mathcal{I}$, we can choose a monotone sequence (c'_n) that converges to $c\in\mathcal{I}$ and satisfies

- the sequence $\rho(c'_n)$ is a strictly monotone sequence;

- for every $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that $\rho(c_n)$ is strictly between $\rho(c'_{k_n})$ and $\rho(c'_{k_n+1})$.

Then $(c_n + u'_{c_n})$ and $(c'_n + u'_{c'_n})$ have the same limit by Corollary 2.1 and we conclude as before that (u_{c_n}) uniformly converges to u_c.

This gives the wanted continuity at every point of $\mathcal{I}$.

Building a function u, the only problem of continuity we have now to consider is at the points of the set $\rho^{-1}(\mathbb{Q})$.

Observe that if we find a continuous extension to $\mathbb{R}^2$ such that every u_c is a weak K.A.M. solution for $\widehat{T}^c$, replacing u_c by $u_c - u_c(0)$, we obtain an extension as wanted.

Let us now assume that $\frac{p}{q}$ is rational and let us introduce the notations $\rho^{-1}(\frac{p}{q}) = [a_1, a_2]$, $\mathcal{I}_+ = \mathcal{I} \cap [a_2, +\infty)$ and $\mathcal{I}_- = \mathcal{I} \cap (-\infty, a_1]$. Observe that a_1 (resp. a_2) is a limit point of the set $\mathcal{I}_-$ (resp. $\mathcal{I}_+$). Let (c_n) be a decreasing sequence in $\mathcal{I}_+$ that converges to a_2. Then by Proposition 2.3, $(c_n + u'_{c_n})_{n\in\mathbb{N}}$ is a decreasing sequence and then $\big(v_n : \theta \in [0,1] \mapsto c_n\theta + \tilde{u}_{c_n}(\theta)\big)_{n\in\mathbb{N}}$ is also a decreasing sequence, thus convergent and even uniformly convergent by the Ascoli Theorem. By Lemma 2.2, $u_{a_2}(\theta) = \lim\limits_{n\to\infty} v_n(\theta) - c_n\theta$ defines a weak K.A.M. solution for T^{a_2} such that $u_{a_2}(0) = 0$. Observe that we have in fact

$$\lim_{\substack{c\in\mathcal{I}_+\\ c\to a_2}} u_c = u_{a_2}$$

because each such decreasing sequence (c_n) defines a uniformly convergent sequence (u_{c_n}) and so the limit doesn't depend on the considered decreasing sequence.

In a similar way, we define a weak K.A.M. solution for T^{a_1} by taking increasing sequences (c_n)

$$\lim_{\substack{c\in\mathcal{I}_-\\ c\to a_1}} u_c = u_{a_1}.$$

Because u_{a_1} and u_{a_2} are weak K.A.M. solutions, they are dominated and we have

$$\forall x, y \in \mathbb{R}, \forall n \geq 1, \quad u_{a_i}(x) - u_{a_i}(y) \leq \mathcal{S}_n^{a_i}(x,y) + n\alpha(a_i).$$

Let $c = \lambda a_1 + (1-\lambda)a_2 \in [a_1, a_2]$. We use the notation $v_c = \lambda u_{a_1} + (1-\lambda)u_{a_2}$. Observe that $\alpha(c) = \lambda\alpha(a_1) + (1-\lambda)\alpha(a_2)$ because $\alpha' = \frac{p}{q}$ is constant on $[a_1, a_2]$. Then we have

$$\forall x, y \in \widetilde{\mathcal{M}}\Big(\frac{p}{q}\Big), \quad v_c(y) - v_c(x) \leq \mathcal{S}_n^c(x,y) + n\alpha(c);$$

i.e. v_c is c-dominated on $\mathcal{M}\big(\frac{p}{q}\big)$. We deduce from Point (3) of subsection 2.4 that there exists only one extension u_c of v_c restricted to $\widetilde{\mathcal{M}}\big(\frac{p}{q}\big)$ that is a weak K.A.M. solution for $\widehat{T}^c$.

Let us prove that $c \in [a_1, a_2] \mapsto u_c$ is continuous. By definition of u_c, the map $c \mapsto u_{c|\mathcal{M}(\frac{p}{q})}$ is continuous. Let us now consider a sequence (c_n) in $[a_1, a_2]$ that converges to some $c \in [a_1, a_2]$. By Ascoli Theorem the set $\{u_{c_n}, n \in \mathbb{N}\}$ is relatively compact for the uniform convergence distance. Let U be a limit point of the sequence (u_{c_n}). By Lemma 2.2, we know that U is a weak K.A.M. solution for $\widehat{T}^c$. Moreover, we have $U_{|\widetilde{\mathcal{M}}(\frac{p}{q})} = u_{c|\widetilde{\mathcal{M}}(\frac{p}{q})}$. Using Point (3) of subsection 2.4, we deduce that $U = u_c$ and the wanted continuity.

To conclude that our choice is continuous everywhere, we only have to prove that a_1 is a continuity point from the left and that a_2 is a continuity point from the right. If we know that there is only one weak K.A.M. solution for $\widehat{T}^{a_i}$ that vanishes at 0, we will conclude by the same argument we used for any $c \in \mathcal{I}$.
Let us assume that v is another weak K.A.M. solution for $\widehat{T}^{a_1}$ that vanishes at 0. Because of Proposition 2.3, we have

$$\forall c < a_1, \quad c + u_c' < a_1 + v'.$$

Taking the limit in this inequality and using the definition of u_{a_1}, we deduce that $v' \geq u'_{a_1}$. As $0 = \int_{\mathbb{T}} v' = \int_{\mathbb{T}} u'_{a_1}$, we deduce that $u'_{a_1} = v'$ Lebesgue almost everywhere and then $v = u_{a_1}$.

At the end of the previous proof, we have actually established a fact that will be useful later:

Proposition 2.4. *Let $R = \frac{p}{q}$ be a rational number and set $[a_1, a_2] = \rho^{-1}(R)$. Then, up to constants, there exists a unique weak K.A.M. solution for $\widehat{T}^{a_1}$ (resp. $\widehat{T}^{a_2}$).*

In appendix A of [9], it is proved that the uniform convergence of a sequence of equi-semi-concave functions implies their convergence C^1 in some sense. This implies for the function u given in Theorem that if $c_n \to c$, if $\theta_n \to \theta$ and if u_{c_n} is derivable at θ_n and u_c at θ, we have

$$\lim_{n\to\infty} \frac{\partial u}{\partial \theta}(\theta_n, c_n) = \frac{\partial u}{\partial \theta}(\theta, c),$$

i.e. that the map $(\theta, c) \mapsto \frac{\partial u_c}{\partial \theta}(\theta)$ is continuous.

2.6. **More on the constructed function: proof of Theorem 1.2.** In this paragraph, $u : \mathbb{A} \to \mathbb{R}$ is any function given by Theorem 1.1 meaning that

- u is continuous;
- $u(0, c) = 0$;
- each $u_c = u(\cdot, c)$ is a weak K.A.M. solution for the cohomology class c.

We aim to give the range of the map $(\theta, c) \mapsto \big(\theta, c + \frac{\partial u}{\partial \theta}(\theta, c)\big)$. The following proposition asserts that any ESTwD is weakly integrable in the sense that $\mathbb{A}$ is covered by Lipschitz circles arising from weak K.A.M. solutions.

Recall that $\mathcal{PG}(c + u_c') = \{(0, c) + \partial u_c(t), \quad t \in \mathbb{T}\}$ is the full pseudograph of $c + u_c'$.

Proposition 2.5. *The following holds:*

$$\bigcup_{c\in\mathbb{R}} \mathcal{PG}(c + u_c') = \bigcup_{\substack{t\in\mathbb{T}\\ c\in\mathbb{R}}} (0, c) + \partial u_c(t) = \mathbb{A}. \tag{8}$$

Let us define two auxiliary functions with values in $\mathbb{R} \cup \{+\infty, -\infty\}$:

$$\forall \theta \in \mathbb{T}, \quad \eta_+(\theta) = \sup\big\{p \in \mathbb{R}; \quad \exists c \in \mathbb{R}; \quad (\theta, p) \in \overline{\mathcal{G}(c + u_c')}\big\},$$

and

$$\forall \theta \in \mathbb{T}, \quad \eta_-(\theta) = \inf\big\{p \in \mathbb{R}; \quad \exists c \in \mathbb{R}; \quad (\theta, p) \in \overline{\mathcal{G}(c + u_c')}\big\}.$$

Finally define $\mathbb{A}_0 = \big\{(\theta, c) \in \mathbb{A}, \quad \eta_-(\theta) < c < \eta_+(\theta)\big\}$.

The following lemma is proved in Appendix B.2.

Lemma 2.3. *For all $c \in \mathbb{R}$, $\mathcal{PG}(c + u'_c)$ is a Lipschitz one dimensional compact manifold, hence it is an essential circle.*

It follows that the set $\mathbb{A}_0$ is open and connected (we will see at the end that it is in fact $\mathbb{A}$). Indeed, by Jordan's theorem and Proposition 2.3, for $c < c'$ such that $\rho(c) < \rho(c')$, the set $\big\{(t,p) \in \mathbb{A}, \quad c + \partial u_c(t) < p < c' + \partial u_{c'}(t)\big\}$ is open and connected. Now $\mathbb{A}_0$ is an increasing union of such sets.

Proposition 2.6. *The following equality holds:*

$$\mathbb{A}_0 = \bigcup_{c\in\mathbb{R}} \mathcal{PG}(c + u'_c).$$

Proof. We now denote by $\mathcal{B}$ the right hand side of the previous equation. Observe that $\mathcal{B} \subset \mathbb{A}_0$. First we prove that $\mathcal{B}$ is closed in $\mathbb{A}_0$. To this end, let $(t_n, p_n) \in \mathcal{PG}(c_n + u'_{c_n})$ be a sequence converging to $(t,p) \in \mathbb{A}_0$. By definition of $\mathbb{A}_0$, there are $C_0 < C_1$ such that $P_0 < p < P_1$ where P_0 and P_1 are such that $(t, P_0) \in \mathcal{PG}(C_0 + u'_{C_0})$ and $(t, P_1) \in \mathcal{PG}(C_1 + u'_{C_1})$. Now let $c_- < C_0 < C_1 < c_+$ be such that $\rho(c_-) < \rho(C_0)$, $\rho(c_+) > \rho(C_1)$ and $\rho(c_-)$, $\rho(c_+)$ are irrational. As the pseudographs are vertically ordered (Proposition 2.3), (t,p) is trapped in the open sub-annulus between $\mathcal{PG}(c_- + u'_{c_-})$ and $\mathcal{PG}(c_+ + u'_{c_+})$. It follows that for n large enough, so is (t_n, p_n). Hence $\mathcal{PG}(c_n + u'_{c_n})$ is a full pseudograph that contains a point strictly between $\mathcal{PG}(c_- + u'_{c_-})$and $\mathcal{PG}(c_+ + u'_{c_+})$. Proposition 2.3 implies that $\rho(c_-) \le \rho(c_n) \le \rho(c_+)$. As $\rho(c_-)$, $\rho(c_+)$ are irrational, there is a unique weak K.A.M. solution for these rotation numbers and then $\rho(c_n) \notin \{\rho(c_-), \rho(c_+)\}$. We deduce that $\rho(c_-) < \rho(c_n) < \rho(c_+)$ and then that $c_- < c_n < c_+$.

Up to extracting, we may assume that $c_n \to c_\infty$ and by continuity of the pseudographs with respect to c (for the Hausdorff distance, see Proposition 2.2), it follows that $(t,p) \in \mathcal{PG}(c_\infty + u'_{c_\infty}) \subset \mathbb{A}_0$.

Next we prove that $\mathcal{B} = \mathbb{A}_0$. We argue by contradiction, by the first part, if this is not the case, there is an open ball $B = (\theta_0, \theta_1) \times (r_0, r_1)$ such that $\overline{B} \subset \mathbb{A}_0 \setminus \mathcal{B}$.

We say that a topological essential circle $\mathcal{C}$ is above B if B is included in the lower connected component of $\mathbb{A} \setminus \mathcal{C}$[11] and $\mathcal{C}$ is under B if B is included in the upper connected component of $\mathbb{A} \setminus \mathcal{C}$. Therefore, if we set EC_B the set of essential circles of the family $\mathcal{C} \subset \mathbb{A} \setminus B$, EC_B is the union of circles above B: EC_B^+ and those under B: EC_B^-.

We will prove that

Lemma 2.4. *Both EC_B^+ and EC_B^- are open subsets of EC_B for the Hausdorff distance.*

Proof. We prove it for EC_B^+. Let $\mathcal{C}^+$ be a circle above B. As the lower connected component of $\mathbb{A} \setminus \mathcal{C}^+$ is path connected, there is a continuous path $\gamma : [0, +\infty) \to \mathbb{A} \setminus \mathcal{C}^+$ such that $\gamma(0) \in B$ and $\gamma(t) = (0, -t)$ for t large enough. Let $\varepsilon > 0$ be such that $\mathcal{C}^+$ is at distance greater than ε from γ. If $\mathcal{C}^-$ is any circle under B, then it must intersect γ. Hence $d_H(\mathcal{C}^-, \mathcal{C}^+) > \varepsilon$ where d_H stands for the Hausdorff distance. This proves the lemma. □

[11]Recall that by Jordan's theorem, $\mathbb{A} \setminus \mathcal{C}$ has two open connected components, one we call upper that contains $\mathbb{T} \times (k, +\infty)$ and one we call lower, that contains $\mathbb{T} \times (-\infty, -k)$ for k large enough.

We will obtain a contradiction as $\mathbb{R}$ is connected and the map $c \mapsto \mathcal{PG}(c+u_c')$ is continuous for the Hausdorff distance, provided we prove that for c large, $\mathcal{PG}(c+u_c')$ is above B while for c small $\mathcal{PG}(c+u_c')$ is under B.

Lemma 2.5. *For c large, $\mathcal{PG}(c+u_c')$ is above B while for c small $\mathcal{PG}(c+u_c')$ is under B.*

Proof. We establish only the first fact. Let $\theta \in (\theta_0, \theta_1)$. By definition of η_+, there exists C such that for $c > C$, then $p > r_1$ for all p verifying $(\theta, p) \in \mathcal{PG}(c+u_c')$. For $t > 0$ small it follows that $p > r_1$ for all p verifying $(\theta, p) \in \varphi_{-t}\big(\mathcal{PG}(c+u_c')\big)$, where φ denotes here the flow of the pendulum. Moreover, up to taking t smaller, we may require $\varphi_{-t}\big(\mathcal{PG}(c+u_c')\big)$ disjoint from B. But it is proved in [3] that $\varphi_{-t}\big(\mathcal{PG}(c+u_c')\big)$ is the Lipschitz graph for small $t > 0$ of a function $\alpha_t : \mathbb{T} \to \mathbb{R}$. Hence it follows from the intermediate value theorem that $\alpha(\theta) > r_1$ for $\theta \in (\theta_0, \theta_1)$ and it becomes obvious that $B = (\theta_0, \theta_1) \times (r_0, r_1)$ is under $\varphi_{-t}\big(\mathcal{PG}(c+u_c')\big)$ and letting $t \to 0$ and passing to the limit, we obtain that B is under $\mathcal{PG}(c+u_c')$. □

□

In order to conclude, we have to prove that $\mathbb{A} = \mathbb{A}_0$ which is equivalent to proving that η_+ is identically $+\infty$ and η_- is identically $-\infty$. We will establish the result for u_+.

Lemma 2.6. *Let $[a, b]$ be a segment, there exists $C > 0$ depending on $[a, b]$ such that if $|c| > C$ then*

$$\forall \theta \in [0,1], \theta' \in [a,b], \quad S(\theta, \theta') + c(\theta - \theta') > \min_{n \in \mathbb{Z}} S(\theta, \theta' + n) + c(\theta - \theta' - n).$$

Proof. Let us set $\Delta = \max\left\{\left|\frac{\partial S}{\partial \theta'}(\theta, \theta')\right|, \quad \theta \in [0,1], \theta' \in [a-1, b+1]\right\}$ and $C = \Delta + 1$.

If $|c| > C$ two cases may occur:

- either $c > \Delta + 1$. In this case, if $(\theta, \theta') \in [0,1] \times [a,b]$, by Taylor's inequality we find

$$S(\theta,\theta') + c(\theta - \theta') > S(\theta, \theta') + c\big(\theta - (\theta'+1)\big) + \Delta \geq S(\theta, \theta'+1) + c\big(\theta - (\theta'+1)\big);$$

- or $c < -\Delta - 1$, in which case

$$S(\theta,\theta') + c(\theta - \theta') > S(\theta, \theta') + c\big(\theta - (\theta'-1)\big) + \Delta \geq S(\theta, \theta'-1) + c\big(\theta - (\theta'-1)\big).$$

□

Corollary 2.2. *The function η_+ is identically $+\infty$.*

Proof. Let us fix $A > 0$. We assume that for all $(\theta, p) \in \mathcal{PG}(u_0')$, then $|p| \leq A$ (or in other words, u_0 is A-Lipschitz). As every map $\theta \mapsto \frac{\partial S}{\partial \Theta}(\theta, \Theta_0)$ is a decreasing diffeomorphism of $\mathbb{R}$, there exists a constant $B > 0$ such that for every $\Theta_0 \in [0,1]$, we have

$$\theta > B \Rightarrow \frac{\partial S}{\partial \Theta}(\theta, \Theta_0) < -(A+1) \quad \text{and} \quad \theta < -B \Rightarrow \frac{\partial S}{\partial \Theta}(\theta, \Theta_0) > A+1.$$

Let C be the constant given by Lemma 2.6 for the segment $[-B, B]$ and let us choose $c > \sup\{B, C\}$. Let $\theta_0 \in [0,1]$ be any derivability point of u_c. Because of Lemma 2.6, if $\tilde{u}_c$ is a lift of u_c and if $\tilde{\theta}$ verifies

$$\tilde{u}_c(\theta_0) = \inf_{\theta \in \mathbb{R}} \tilde{u}_c(\theta) + S(\theta, \theta_0) + c(\theta - \theta_0) = \tilde{u}_c(\tilde{\theta}) + S(\tilde{\theta}, \theta_0) + c(\tilde{\theta} - \theta_0),$$

then $\tilde\theta \notin [-B, B]$ and then $\left|\frac{\partial S}{\partial \Theta}(\tilde\theta, \theta_0)\right| > A+1$.

We deduce from point (c) of section 2.3 that $f\big(\tilde\theta, c+u_c'(\tilde\theta)\big) = \big(\theta_0, c+u_c'(\theta_0)\big)$ and then

$$c+\tilde u_c'(\theta_0) = \frac{\partial S}{\partial \Theta}(\tilde\theta, \theta_0),$$

and then $|c+\tilde u_c'(\theta_0)| > A+1$.

As $\int_0^1 \big(c+\tilde u_c'(s)\big)ds = c > 0$, we can choose θ_0 such that $c+\tilde u_c'(\theta_0) > 0$ and so $c+\tilde u_c'(\theta_0) > A+1$.

As the pseudographs are vertically ordered (Proposition 2.3), $\mathcal{PG}(c+u_c')$ is above $\mathcal{PG}(u_0')$. We conclude that for all derivability point θ of u_c then $c+\tilde u_c'(\theta) > A+1$. Finally, the whole full pseudograph $\mathcal{PG}(c+u_c')$ lies above the circle $\{(t, A), \quad t \in \mathbb{T}\}$.

We have just established that if $c > B$, then $\mathcal{PG}(c+u_c')$ lies above the circle $\{(t, A), \quad t \in \mathbb{T}\}$, that concludes the proof.

□

Using technics given in [1], we will prove in Proposition B.2 of Appendix B.3 that the map that maps c on the full pseudograph[12] $\mathcal{PG}(c+u_c') = \{(0, c) + \partial u_c(t), \quad t \in \mathbb{T}\}$ of $c+u_c'$ is continuous for the Hausdorff distance.

We end this section with the proof of points (4) and (5) of Theorem 1.2. Let us state a lemma:

Lemma 2.7. *Let $c_1 < c_2$ be two real numbers. Let $v_1, v_2 : \mathbb{T} \to \mathbb{R}$ be continuous functions.*

If the function $\theta \mapsto (\tilde v_2 - \tilde v_1)(\theta) + (c_2 - c_1)\theta$ is non-decreasing, then so is the function $\theta \mapsto (\widetilde T^{c_2}\tilde v_2 - \widetilde T^{c_1}\tilde v_1)(\theta) + (c_2 - c_1)\theta$.

Proof. Let $\theta < \theta'$ be two real numbers. By definition of the operators T_{c_i} there exist θ_2' and θ_1 such that

$$\widetilde T^{c_2}\tilde v_2(\theta') = \tilde v_2(\theta_2') + S(\theta_2', \theta') + c_2(\theta_2' - \theta'),$$
$$\widetilde T^{c_1}\tilde v_1(\theta) = \tilde v_1(\theta_1) + S(\theta_1, \theta) + c_1(\theta_1 - \theta).$$

There are two cases to consider:

- if $\theta_2' < \theta_1$ we use Aubry's fundamental lemma to obtain

$$\begin{aligned}\widetilde T^{c_2}\tilde v_2(\theta') + \widetilde T^{c_1}\tilde v_1(\theta) &= \tilde v_2(\theta_2') + S(\theta_2', \theta') + c_2(\theta_2' - \theta') + \tilde v_1(\theta_1) + S(\theta_1, \theta) + c_1(\theta_1 - \theta)\\ &> \tilde v_2(\theta_2') + S(\theta_2', \theta) + c_2(\theta_2' - \theta') + \tilde v_1(\theta_1) + S(\theta_1, \theta') + c_1(\theta_1 - \theta)\\ &\geq \widetilde T^{c_2}\tilde v_2(\theta) + \widetilde T^{c_1}\tilde v_1(\theta') + (c_2 - c_1)(\theta - \theta').\end{aligned}$$

  After rearranging the terms, this reads

$$\widetilde T^{c_2}\tilde v_2(\theta') - \widetilde T^{c_1}\tilde v_1(\theta') + (c_2 - c_1)\theta' > \widetilde T^{c_2}\tilde v_2(\theta) - \widetilde T^{c_1}\tilde v_1(\theta) + (c_2 - c_1)\theta.$$

- if $\theta_2' \geq \theta_1$ we use the hypothesis on $\theta \mapsto (\tilde v_2 - \tilde v_1)(\theta) + (c_2 - c_1)\theta$ to show that $\tilde v_2(\theta_2') + \tilde v_1(\theta_1) \geq \tilde v_2(\theta_1) + \tilde v_1(\theta_2') + (c_2 - c_1)(\theta_1 - \theta_2')$ and then

$$\begin{aligned}\widetilde T^{c_2}\tilde v_2(\theta') + \widetilde T^{c_1}\tilde v_1(\theta) &= \tilde v_2(\theta_2') + S(\theta_2', \theta') + c_2(\theta_2' - \theta') + \tilde v_1(\theta_1) + S(\theta_1, \theta) + c_1(\theta_1 - \theta)\\ &\geq \tilde v_2(\theta_1) + S(\theta_2', \theta') + c_2(\theta_1 - \theta') + \tilde v_1(\theta_2') + S(\theta_1, \theta) + c_1(\theta_2' - \theta)\\ &\geq \widetilde T^{c_2}\tilde v_2(\theta) + \widetilde T^{c_1}\tilde v_1(\theta') + (c_2 - c_1)(\theta - \theta').\end{aligned}$$

[12] see the definition in subsection 2.3

As before, this gives the result after rearranging terms.

□

Let us now conclude that the function u constructed verifies the requirements of (4) and (5). Let $\frac{p}{q}$ be rational and let us, as previously, introduce the notations $\rho^{-1}(\frac{p}{q}) = [a_1, a_2]$. As seen before, we denote by u_{a_1} and u_{a_2} the unique weak K.A.M. solutions for T^{a_1} and T^{a_2} vanishing at 0. We have proved that $\theta \mapsto (\tilde{u}_{a_2} - \tilde{u}_{a_1})(\theta) + (a_2 - a_1)\theta$ is non-decreasing.

Let $c = \lambda a_1 + (1-\lambda)a_2 \in [a_1, a_2]$. We use again the notation $v_c = \lambda u_{a_1} + (1-\lambda)u_{a_2}$ and recall that $\alpha(c) = \lambda\alpha(a_1) + (1-\lambda)\alpha(a_2)$ because $\alpha' = \frac{p}{q}$ is constant on $[a_1, a_2]$. It follows that $\tilde{v}_c$ is c dominated and that if $a_1 \le c < c' \le a_2$, the function $\theta \mapsto (\tilde{v}_{c'} - \tilde{v}_c)(\theta) + (c' - c)\theta$ is non decreasing.

Finally, as v_c is c-dominated, it can be proved that the function u_c constructed verifies

$$\forall \theta \in \mathbb{R}, \quad \tilde{u}_c(\theta) = \lim_{n\to+\infty} (\widetilde{T}^c)^n \tilde{v}_c(\theta) + n\alpha(c),$$

the limit being that of an increasing sequence. Hence the fact that $\theta \mapsto (\tilde{u}_{c'} - \tilde{u}_c)(\theta) + (c' - c)\theta$ is non decreasing follows from successive applications of the previous lemma.

To prove (5), if $c' \leqslant c$ and $\theta \in [0, 1]$ then

$$0 = (\tilde{u}_{c'} - \tilde{u}_c)(0) \leqslant (\tilde{u}_{c'} - \tilde{u}_c)(\theta) + (c' - c)\theta \leqslant (\tilde{u}_{c'} - \tilde{u}_c)(1) + (c' - c) = c' - c.$$

It follows that

$$(c - c')\theta \leqslant (\tilde{u}_{c'} - \tilde{u}_c)(\theta) \leqslant (c' - c)(1 - \theta).$$

Hence $\tilde{u}$ is uniformly 1-Lipschitz in c and the result follows.

3. Proof of the implication (1) ⇒ (2) in Theorem 1.3

We assume that $f : \mathbb{A} \to \mathbb{A}$ is a C^k ESTwD (with $k \geq 1$) that has a continuous invariant foliation into continuous graphs $a \in \mathbb{R} \mapsto \eta_a \in C^0(\mathbb{T}, \mathbb{R})$ where we choose $\eta_a(0) = a$. Then Birkhoff's theorem (see [11], [20] and [26]) implies that all the η_a are Lipschitz.

Notation. For every $a \in \mathbb{R}$, we will denote by $g_a : \mathbb{T} \to \mathbb{T}$ the restricted-projected Dynamics to the graph of η_a, i.e

$$g_a(\theta) = \pi_1 \circ f\big(\theta, \eta_a(\theta)\big).$$

3.1. **Some generalities.**

Notation.

- In $\mathbb{R}^2$ we denote by $B(x, r)$ the open disc for the usual Euclidean distance with center x and radius r;
- we denote by $R_\alpha : \mathbb{T} \to \mathbb{T}$ the rotation $R_\alpha(\theta) = \theta + \alpha$;
- if E is a finite set, $\sharp(E)$ is the number of elements it contains;
- we denote by $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ the integer part.

Definition.

- We say that $a \mapsto \eta_a$ defines a *Lipschitz foliation* if $(\theta, a) \mapsto \big(\theta, \eta_a(\theta)\big)$ is an homeomorphism that is locally biLipschitz; if f has an invariant Lipschitz foliation, f is *Lipschitz integrable*;
- we say that $a \mapsto \eta_a$ defines a C^k *foliation* if $(\theta, a) \mapsto \big(\theta, \eta_a(\theta)\big)$ is a C^k diffeomorphism; if f has an invariant C^k foliation, f is C^k *integrable*;
- following [28], we say that $a \mapsto \eta_a$ defines a C^k *lamination* if $(\theta, a) \mapsto \big(\theta, \eta_a(\theta)\big)$ is an homeomorphism, every η_a is C^k and the map $a \mapsto \eta_a$ is continuous when $C^k(\mathbb{T}, \mathbb{R})$ is endowed with the C^k topology.

Proposition 3.1. *Assume that the C^1 ESTwD $f : \mathbb{A} \to \mathbb{A}$ has an invariant continuous (resp. locally Lipschitz continuous) foliation into graphs $a \in \mathbb{R} \mapsto \eta_a \in C^0(\mathbb{T}, \mathbb{R})$. Then the map $\mathcal{A} : a \in \mathbb{R} \mapsto \int_{\mathbb{T}} \eta_a(t)dt \in \mathbb{R}$ is an homeomorphism (resp. locally biLipschitz homeomorphism).*

The proof is straightforward. Using this result, we can use $c = \mathcal{A}(a)$ instead of a as a parameter, what we do from now.

Notations. We fix a lift $F : \mathbb{R}^2 \to \mathbb{R}^2$ of f. We denote by $\tilde{\eta}_c : \mathbb{R} \to \mathbb{R}$ the lift of η_c. We denote by ρ the function that maps $c \in \mathbb{R}$ to the rotation number $\rho(c) \in \mathbb{R}$ of the restriction of F to the graph of $\tilde{\eta}_c$.

The map ρ is then an increasing homeomorphism.
When moreover the foliation is biLipschitz, we will prove that ρ is a biLipschitz homeomorphism (see Proposition 5.1).

We recall a well-known result concerning the link between invariant measures and semi-conjugations for orientation preserving homeomorphisms of $\mathbb{T}$.

Proposition 3.2. *Assume that μ_c is a non-atomic Borel invariant probability measure by g_c. Then, if $\rho(c)$ is irrational or g_c is C^0 conjugated to a rotation, the map $h_c : \mathbb{T} \to \mathbb{T}$ defined by $h_c(\theta) = \int_0^\theta d\mu_c$ is a semi-conjugation between g_c and the rotation with angle $\rho(c)$, i.e:*

$$h_c\big(g_c(\theta)\big) = h_c(\theta) + \rho(c).$$

Proof. Let $\tilde{\mu}_c$ be the pull back measure of μ_c to $\mathbb{R}$ and let $\tilde{g}_c : \mathbb{R} \to \mathbb{R}$ be a lift of g_c to $\mathbb{R}$. Then we have for every $\Theta \in [0, 1]$ lift of $\theta \in \mathbb{T}$:

$$\tilde{\mu}_c([0, \Theta]) = \tilde{\mu}_c([\tilde{g}_c(0), \tilde{g}_c(\Theta)]) = \tilde{\mu}_c\big(\big[\lfloor \tilde{g}_c(0)\rfloor, \tilde{g}_c(\Theta)\big]\big) - \tilde{\mu}_c\big(\big[\lfloor \tilde{g}_c(0)\rfloor, \tilde{g}_c(0)\big]\big);$$

where $\lfloor \tilde{g}_c(0)\rfloor$ is the integer part of $\tilde{g}_c(0)$. This implies[13]

$$h_c(\theta) = h_c\big(g_c(\theta)\big) - \tilde{\mu}_c\big(\big[0, g_c(0)\big]\big) = h_c\big(g_c(\theta)\big) - \rho(c).$$

Moreover, as we assumed that μ_c is non-atomic, h_c is continuous. □

Remarks.

[13] Recall that if $f : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism then either $\rho(f)$ is irrational, f is semi-conjugated (by h) to the rotation $R_{\rho(f)}$ and the only invariant measure is the pull back of the Lebesgue measure by h; or $\rho(f)$ is rational and the invariant measures are supported on periodic orbits. When $\rho(f)$ is irrational or when f is C^0 conjugate to a rotation, then for any invariant measure μ and $x \in \mathbb{T}$, $\mu([x, f(x)[) = \rho(f)$.

(1) In the other sense, if h_c is a (non-decreasing) semi-conjugation such that $h_c\circ g_c = h_c+\rho(c)$, then $\mu([0,\theta]) = h_c(\theta)-h_c(0)$ defines a g_c-invariant Borel probability measure;
(2) When $\rho(c)$ is irrational, it is well known that the Borel invariant probability measure μ_c is unique and that the semi-conjugation h_c is unique up to constant.

Notation. When $\rho(c)$ is irrational, we will denote by h_c the semi-conjugation such that $h_c(0) = 0$.

Before entering the core of the proof, let us mention a useful fact about iterates of C^0-integrable ESTwDs:

Proposition 3.3. *Let $f : \mathbb{A} \to \mathbb{A}$ be a C^0-integrable ESTwD, then so is f^n for all $n > 0$.*

This is specific to the integrable case: in general, an iterated ESTwD is not an ESTwD as can be seen in the neighborhood of an elliptic fixed point.

Proof. We argue by induction on $n > 0$. The initialization being trivial, let us assume the result true for some $k > 0$. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of f. For any $c \in \mathbb{R}$ using the notations given at the beginning of section 3, we have

$$\forall\theta \in \mathbb{T}, \forall m > 0, \quad f^m\big(\theta, \eta_c(\theta)\big) = \big(g_c^m(\theta), \eta_c \circ g_c^m(\theta)\big).$$

Observe that if f^m satisfies the twist condition and $c_1 < c_2$ are two real numbers, then we have

$$\tilde g_{c_1}^m(t) = \pi_1 \circ F^m\big(t, \eta_{c_1}(t)\big) < \pi_1 \circ F^m\big(t, \eta_{c_2}(t)\big) = \tilde g_{c_2}^m(t)$$

and $\lim\limits_{t\to\pm\infty} \tilde g_{c_1}(t) = \pm\infty$.

Let us prove this. Let $c_1 < c_2$ and $t \in \mathbb{R}$. Denoting with $\sim$ the lifts of the considered functions we obtain that

$$\pi_1\big(F^{n+1}(t, c_2)\big)-\pi_1\big(F^{n+1}(t, c_1)\big) = \tilde g_{c_2}\circ\tilde g_{c_2}^n(t)-\tilde g_{c_1}\circ\tilde g_{c_1}^n(t) \geq \tilde g_{c_2}\circ\tilde g_{c_1}^n(t)-\tilde g_{c_1}\circ\tilde g_{c_1}^n(t),$$

where we have used the induction hypothesis and the fact that $\tilde g_{c_2}$ is increasing. It follows that $c \mapsto \pi_1\big(F^{n+1}(t, c)\big)$ is an increasing diffeomorphism on its image. Observe also that this inequality implies that $\lim\limits_{c_2\to+\infty} \pi_1\big(F^{n+1}(t, c_2)\big) = +\infty$ because $\lim\limits_{c_2\to+\infty} \tilde g_{c_2}(s) = +\infty$. Moreover

$$\pi_1\big(F^{n+1}(t, c_2)\big)-\pi_1\big(F^{n+1}(t, c_1)\big) = \tilde g_{c_2}\circ\tilde g_{c_2}^n(t)-\tilde g_{c_1}\circ\tilde g_{c_1}^n(t) \leq \tilde g_{c_2}\circ\tilde g_{c_2}^n(t)-\tilde g_{c_1}\circ\tilde g_{c_2}^n(t),$$

implies that $\lim\limits_{c_1\to-\infty} \pi_1\big(F^{n+1}(t, c_1)\big) = -\infty$ because $\lim\limits_{c_1\to-\infty} \tilde g_{c_1}(s) = -\infty$. So finally $c \mapsto \pi_1\big(F^{n+1}(t, c)\big)$ is an increasing diffeomorphism onto $\mathbb{R}$.

□

3.2. **Differentiability and conjugation along the rational curves.** It is proved in [2] that for every $r = \frac{p}{q} \in \mathbb{Q}$, $\eta_c = \eta_{\rho^{-1}(r)}$ is C^k and the restriction of f to the graph Γ_c of η_c is completely periodic: $f^q_{|\Gamma_c} = \mathrm{Id}_{\Gamma_c}$. Moreover, along these particular curves, the two Green bundles (see Appendix C for definition and results) are equal:

$$G_-\big(\theta, \eta_c(\theta)\big) = G_+\big(\theta, \eta_c(\theta)\big).$$

Theorem 3.1.
- *Along every leaf Γ_c such that $\rho(c) \in \mathbb{Q}$, the derivative $\frac{\partial \eta_c(\theta)}{\partial c} = 1 + \frac{\partial^2 u_c}{\partial c \partial \theta} > 0$ exists and C^{k-1} depends on θ;*
- *for any c such that $\rho(c)$ is rational, the measure μ_c on $\mathbb{T}$ with density $\frac{\partial \eta_c}{\partial c}$ is a Borel probability measure invariant by g_c and for $\theta \in [0,1]$, the equality*
$$h_c(\theta) = \mu_c([0,\theta]) = \theta + \frac{\partial u}{\partial c}(\theta, c)$$
defines a conjugation between g_c and the rotation with angle $\rho(c)$;
- *then the map $c \in \mathbb{R} \mapsto \mu_c$ is continuous and also $c \in \mathbb{R} \mapsto h_c$ for the uniform C^0 topology. Thus $(\theta, c) \mapsto h_c(\theta)$ is continuous.*

Remarks.

(1) Observe that because $c \mapsto \eta_c$ is increasing, we know that for Lebesgue almost every $(\theta, c) \in \mathbb{T} \times \mathbb{R}$, the derivative $\frac{\partial \eta_c(\theta)}{\partial c}$ exists (see [30]). But our theorem says something different.
(2) Because of the continuous dependence on θ along the rational curve, we obtain that $\frac{\partial \eta_c(\theta)}{\partial c}$ restricted to every rational curve is bounded (that is clear when we assume that the foliation is Lipschitz but not if the foliation is just continuous).

Proof of the first point. We fix $A \in \mathbb{R}$ such that $\rho(A) = \frac{p}{q} \in \mathbb{Q}$. Replacing f by f^q, we can assume that $\rho(A) \in \mathbb{Z}$. Observe that because of the C^0-integrability of f, f^q is also an (C^0-integrable with the same invariant foliation) ESTwD (Proposition 3.3).

We define $G_A : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ by

$$G_A(\theta, r) = \big(\theta, r + \eta_A(\theta)\big). \tag{9}$$

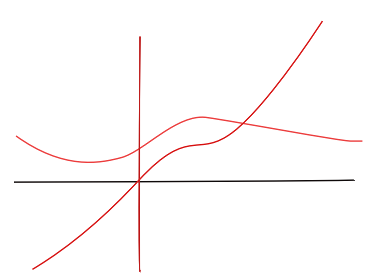
Then $G_A^{-1} \circ f^q \circ G_A$ is also a C^0-integrable C^k ESTwD and $\mathbb{T} \times \{0\}$ is filled with fixed points.

We finally have to prove our theorem in this case and we use the notation f instead of $G_A^{-1} \circ f^q \circ G_A$. We can assume that $A = 0$ instead of $A \in \mathbb{Z}$.

Because of the semi-continuity of the two Green bundles $G_- = \mathbb{R}(1, s_-)$ and $G_+ = \mathbb{R}(1, s_+)$, we have for any point $x = (\theta, r)$ sufficiently close to $\mathbb{T} \times \{0\}$: $\max\{|s_-(x)|, |s_+(x)|\} < \varepsilon$ is small.

Now we fix c small and consider for every $\theta \in \mathbb{T}$ the small triangular domain $\mathcal{T}(\theta)$ that is delimited by the three following red curves

- the graph of η_c;
- the vertical $\mathcal{V}_\theta = \{\theta\} \times \mathbb{R}$;
- the image $f(\mathcal{V}_\theta)$ of the vertical at θ.



To be more precise, $\mathcal{T}(\theta)$ is 'semi-open' in the following sense; it contains its whole boundary except the image $f(\mathcal{V}_\theta)$ of the vertical at θ.
We assume that $c>0$. The case $c<0$ is similar.
As the slope of η_c is almost 0 (because between the slope of the two Green bundles, see Proposition C.1) and the slope of the side of the triangle that is in $f(\mathcal{V}_\theta)$ is almost $\frac{1}{s(\theta)}$ where $s(\theta)>0$ is the torsion that is defined by

$$Df(\theta,0)=\begin{pmatrix}1 & s(\theta)\\ 0 & 1\end{pmatrix}, \tag{10}$$

the area of this triangle is

$$\lambda\big(\mathcal{T}(\theta)\big)=\frac{1}{2}\big(\eta_c(\theta)\big)^2\big(s(\theta)+\varepsilon(\theta,c)\big); \tag{11}$$

where

$$\text{uniformly}\quad\text{for}\quad \theta\in\mathbb{T},\quad \lim_{c\to 0}\varepsilon(\theta,c)=0. \tag{12}$$

Let λ be the Lebesgue measure restricted to the invariant sub-annulus

$$\mathcal{A}_c=\bigcup_{\theta\in\mathbb{T}}\{\theta\}\times[0,\eta_c(\theta)].$$

Being symplectic, f preserves λ. Moreover, every ergodic measure μ for f with support in $\mathcal{A}_c$ is supported on one curve Γ_A with $A\in[0,c]$. But $f_{|\Gamma_A}$ is semi-conjugated to a rotation with an angle $\rho(A)$ that is close to 0. Hence every interval in Γ_A that is between some $\big(\theta,\eta_A(\theta)\big)$ and $f\big(\theta,\eta_A(\theta)\big)$ has the same μ-measure, which is just given by the rotation number $\rho(A)$ on the graph of η_A. This implies that $\theta\mapsto\mu\big(\mathcal{T}(\theta)\big)$ is constant. Hence for every $\theta,\theta'\in\mathbb{T}$ and for every ergodic measure with support in $\mathcal{A}_c$, we have $\mu\big(\mathcal{T}(\theta)\big)=\mu\big(\mathcal{T}(\theta')\big)$. Using the ergodic decomposition of invariant measures (see e.g. [32]) $\lambda=\int\mu_a d\nu(a)$, we deduce that:

$$\forall\theta,\theta'\in\mathbb{T},\quad \lambda\big(\mathcal{T}(\theta)\big)=\lambda\big(\mathcal{T}(\theta')\big)=\int\rho(a)d\nu(a). \tag{13}$$

We deduce from equations (11) and (12) that

$$\text{uniformly}\quad\text{for}\quad \theta,\theta'\in\mathbb{T},\quad \lim_{c\to 0}\frac{\eta_c(\theta')}{\eta_c(\theta)}=\sqrt{\frac{s(\theta)}{s(\theta')}}.$$

Integrating with respect to θ', we deduce that uniformly in θ, we have

$$\lim_{c\to 0}\frac{c}{\eta_c(\theta)}=\sqrt{s(\theta)}\int_{\mathbb{T}}\frac{dt}{\sqrt{s(t)}}.$$

This implies that

$$\frac{\partial\eta_c(\theta)}{\partial c}_{|c=0}=\Big(\int_{\mathbb{T}}\frac{dt}{\sqrt{s(t)}}\Big)^{-1}\frac{1}{\sqrt{s(\theta)}}; \tag{14}$$

and even

$$\eta_c(\theta)=c\Big(\int_{\mathbb{T}}\frac{dt}{\sqrt{s(t)}}\Big)^{-1}\Big(\frac{1}{\sqrt{s(\theta)}}+\varepsilon(\theta,c)\Big) \tag{15}$$

where

$$\text{uniformly}\quad\text{for}\quad \theta\in\mathbb{T},\quad \lim_{c\to 0}\varepsilon(\theta,c)=0. \tag{16}$$

Observe that $\frac{\partial \eta_c}{\partial c} = \left(\int_{\mathbb{T}} \frac{dt}{\sqrt{s(t)}}\right)^{-1} \frac{1}{\sqrt{s(\cdot)}}$ is a C^{k-1} function of θ'. This proves the first point of theorem 3.1.
Proof of the second point. We deduce from the first point that for any c such that $\rho(c)$ is rational, the function $\frac{\partial \eta_c}{\partial c}$ is continuous and positive. Moreover, its integral on $\mathbb{T}$ is 1. Hence $\frac{\partial \eta_c}{\partial c}$ is the density of a Borel probability measure that is equivalent to Lebesgue. We now introduce:

NOTATION. If $c < c'$, we denote by $\Lambda_{c,c'}$ the normalized Lebesgue measure between the graph of η_c and the graph of $\eta_{c'}$.

Then f preserves $\Lambda_{c,c'}$. Observe that for any measurable $I \subset \mathbb{T}$, we have

$$\Lambda_{c,c'}\big(\{(\theta, r); \theta \in I,\ r \in [\eta_c(\theta), \eta_{c'}(\theta)]\}\big) = \frac{1}{c - c'} \int_I \big(\eta_c(\theta) - \eta_{c'}(\theta)\big) d\theta. \tag{17}$$

Lemma 3.1. *If $\rho(c)$ is rational, then $\lim\limits_{c' \to c} \Lambda_{c,c'}$ is a measure supported on the graph of η_c whose projected measure μ_c has density $\frac{\partial \eta_c}{\partial c}$ with respect to Lebesgue of $\mathbb{T}$.*
Hence if $h_c(\theta) = \int_0^\theta \frac{\partial \eta_c}{\partial c}(t)dt$, we have

$$h_c \circ \pi_1 \circ f\big(\theta, \eta_c(\theta)\big) = h_c(\theta) + \rho(c).$$

Proof. Using Equations (15) and (16), we can take the limit in Equation (17) or more precisely for any $\psi \in C^0(\mathbb{A}, \mathbb{R})$ in

$$\int \psi(\theta, r) d\Lambda_{c,c'}(\theta, r) = \int_{\mathbb{T}} \frac{1}{c - c'} \left(\int_{\eta_{c'}(\theta)}^{\eta_c(\theta)} \psi(\theta, r) dr \right) d\theta$$

and obtain that the limit is an invariant measure supported in the graph of η_c whose projected measure μ_c has a density with respect to Lebesgue that is equal to $\frac{\partial \eta_c}{\partial c}$. We then use Proposition 3.2 to conclude that h_c is the wanted conjugation. □

Proof of the third point. We noticed that when $\rho(c)$ is irrational, there is only one invariant Borel probability measure that is supported on the graph of η_c. This implies the continuity of the map $c \mapsto \mu_c$ at such a c. Let us look at what happens when $\rho(c)$ is rational.

Proposition 3.4. *For every $c_0 \in \mathbb{R}$ such that $\rho(c_0)$ is rational, for every $\theta \in [0, 1]$, we have*

$$\lim_{c \to c_0} \mu_c([0, \theta]) = \mu_{c_0}([0, \theta])$$

and the limit is uniform in θ.

This joint with the continuity of h_{c_0} implies the continuity of $(\theta, c) \mapsto h_c(\theta)$ at (θ, c_0).

Proof. In this proof, we will use different functions $\varepsilon_i(\tau, c)$ and all these functions will satisfy uniformly in τ

$$\lim_{c \to 0} \varepsilon_i(\tau, c) = 0.$$

As in the proof of the first point of Theorem 3.1, we can assume that $u_{c_0} = 0$ (and then $c_0 = 0$) and $\rho(0) = 0$.
We fix $\varepsilon > 0$. Because of the continuity of ρ, we can choose α such that if $|c| < \alpha$, then $|\rho(c)| < \varepsilon$.

Let us introduce the notation $N_c = \lfloor \frac{1}{\rho(c)} \rfloor$ for $c \neq 0$. Let us assume that $c > 0$ and $\theta \in (0, 1]$. We also denote by $\tilde{g}_c$ the lift of g_c such that $\tilde{g}_c(0) \in [0, 1)$ and by $M_c(\theta)$

$$M_c(\theta) = \sharp\{j \in \mathbb{N}; \quad \tilde{g}_c^j(0) \in [0, \theta]\}.$$

Hence, $M_c(\theta)$ is the number of points of the orbit of 0 under $\tilde{g}_c$ that belong to $[0, \theta]$. Observe that $M_c(\theta)$ is non-decreasing with respect to θ.
As $\eta_c > 0$, any primitive $\mathcal{N}_c$ of η_c is increasing, hence $M_c(\theta)$ is also the number of $\tilde{g}^k(0)$ such that $\mathcal{N}_c\big(\tilde{g}^k(0)\big)$ belongs to $[\mathcal{N}_c(0), \mathcal{N}_c(\theta)]$, i.e.

$$\begin{aligned} M_c(\theta) \quad &= \sharp\Big\{j \in \mathbb{N}; \quad \textstyle\int_0^{\tilde{g}_c^j(0)} \eta_c(t)dt \leq \int_0^\theta \eta_c(t)dt\Big\} \\ &= \sup\Big\{j \in \mathbb{N}; \quad \textstyle\int_0^{\tilde{g}_c^j(0)} \eta_c(t)dt \leq \int_0^\theta \eta_c(t)dt\Big\}. \end{aligned} \tag{18}$$

Note that $M_c(1) = N_c$ because g_c has rotation number $\rho(c)$ and that we have $\forall \theta \in (0, 1]$, $M_c(\theta) \leq N_c$ as M_c is non decreasing. We have also

$$\mu_c([0, \theta]) = \sum_{j=0}^{M_c(\theta)-1} \mu_c([\tilde{g}_c^j(0), \tilde{g}_c^{j+1}(0)[) + \mu_c([\tilde{g}^{M_c(\theta)}(0), \theta])$$

and thus $\mu_c([0, \theta]) = M_c(\theta)\rho(c) + \Delta\rho(c)$ with $\Delta \in [0, 1]$ because $[\tilde{g}^{M_c(\theta)}(0), \theta] \subset [\tilde{g}^{M_c(\theta)}(0), \tilde{g}^{M_c(\theta)+1}(0)[$.
Hence

$$\mu_c([0, \theta]) \in [M_c(\theta)\rho(c), M_c(\theta)\rho(c) + \rho(c)] \subset \left[\frac{M_c(\theta)}{N_c + 1}, \frac{M_c(\theta) + 1}{N_c}\right]. \tag{19}$$

Hence to estimate the measure $\mu_c([0, \theta])$ we need a good estimate of the number of j such that $\tilde{g}_c^j(0)$ belongs to $[0, \theta]$. We have proved in Equations (15) and (16) that

$$\eta_c(\tau) = \Big(\int_{\mathbb{T}} \frac{dt}{\sqrt{s(t)}}\Big)^{-1} \frac{c\big(1 + \varepsilon_0(\tau, c)\big)}{\sqrt{s(\tau)}}. \tag{20}$$

We deduce from Equation (10) that $\tilde{g}_c(\tau) = \tau + \big(s(\tau) + \varepsilon_1(\tau, c)\big)\eta_c(\tau)$ where uniformly in τ, we have: $\lim\limits_{c \to 0} \varepsilon_1(\tau, c) = 0$ and then by Equation (20):

$$\int_\tau^{\tilde{g}_c(\tau)} \eta_c(t)dt = \eta_c(\tau)^2\big(s(\tau) + \varepsilon_2(\tau, c)\big) = \frac{c^2\big(1 + \varepsilon_3(\tau, c)\big)}{\Big(\int_{\mathbb{T}} \frac{dt}{\sqrt{s(t)}}\Big)^2}. \tag{21}$$

This says that the area between τ and $\tilde{g}_c(\tau)$ that is limited by the zero section and the graph of η_c is almost constant (i.e. doesn't depend a lot on τ).

We deduce from Equation (18) that

$$\int_0^{\tilde{g}_c^{M_c(\theta)}(0)} \eta_c(t)dt \leq \int_0^\theta \eta_c(t)dt < \int_0^{\tilde{g}_c^{M_c(\theta)+1}(0)} \eta_c(t)dt.$$

Hence

$$\sum_{j=0}^{M_c(\theta)-1} \int_{\tilde{g}^j(0)}^{\tilde{g}^{j+1}(0)} \eta_c(t)dt \leq \int_0^\theta \eta_c(t)dt \leq \sum_{j=0}^{M_c(\theta)} \int_{\tilde{g}^j(0)}^{\tilde{g}^{j+1}(0)} \eta_c(t)dt.$$

Using Equation (21), we deduce that

$$M_c(\theta)\frac{c^2\big(1+\varepsilon_4(\theta,c)\big)}{\Big(\int_{\mathbb{T}}\frac{dt}{\sqrt{s(t)}}\Big)^2}\le\frac{c\big(1+\varepsilon_5(\theta,c)\big)}{\int_{\mathbb{T}}\frac{dt}{\sqrt{s(t)}}}\int_0^\theta\frac{dt}{\sqrt{s(t)}}<(M_c(\theta)+1)\frac{c^2\big(1+\varepsilon_6(\theta,c)\big)}{\Big(\int_{\mathbb{T}}\frac{dt}{\sqrt{s(t)}}\Big)^2},$$

and then

$$M_c(\theta)=\left\lfloor\frac{1}{c}\Big(\int_{\mathbb{T}}\frac{dt}{\sqrt{s(t)}}\Big(\int_0^\theta\frac{dt}{\sqrt{s(t)}}+\varepsilon_7(\theta,c)\Big)\Big)\right\rfloor. \tag{22}$$

This implies that

$$N_c=M_c(1)=\left\lfloor\frac{1}{c}\Big(\Big(\int_{\mathbb{T}}\frac{dt}{\sqrt{s(t)}}\Big)^2+\varepsilon_8(\theta,1)\Big)\right\rfloor \tag{23}$$

and by Equations (14), (19), (22) and (23).

$$\mu_c([0,\theta])=\frac{M_c(\theta)}{N_c}+\varepsilon_9(\theta,c)=\frac{\int_0^\theta\frac{dt}{\sqrt{s(t)}}}{\int_{\mathbb{T}}\frac{dt}{\sqrt{s(t)}}}+\varepsilon_{10}(\theta,c)=\mu_0([0,\theta])+\varepsilon_{11}(\theta,c). \tag{24}$$

As none of the measures μ_c has atoms, this implies that $c\mapsto\mu_c$ and all the maps $c\mapsto\mu_c([0,\theta])=h_c(\theta)$ are continuous. As every map h_c is non decreasing in the variable θ, we deduce from the Dini-Polyà Theorem [37, Exercise 13.b page 167] that $c\mapsto h_c$ is continuous for the C^0 uniform topology. □

Remark. If $\rho(c)=\frac{p}{q}$, then we proved that $\frac{\partial\eta_c(\theta)}{\partial c}=\Big(\int_{\mathbb{T}}\frac{dt}{\sqrt{s_q(t,\eta_c(t))}}\Big)^{-1}\frac{1}{\sqrt{s_q(\theta,\eta_c(\theta))}}$ where

$$Df^q(x)=\begin{pmatrix}a_q(x)&s_q(x)\\c_q(x)&d_q(x)\end{pmatrix}.$$

This gives for the conjugation

$$h_c(\theta)=\mu_c([0,\theta])=\Big(\int_{\mathbb{T}}\frac{dt}{\sqrt{s_q\big(t,\eta_c(t)\big)}}\Big)^{-1}\int_0^\theta\frac{1}{\sqrt{s_q\big(t,\eta_c(t)\big)}}dt.$$

Observe that this C^k depends on θ.
Observe too that Equations (15) and (16) can be rewritten as

$$\eta_c(\theta)=c\Big[\Big(\int_{\mathbb{T}}\frac{dt}{\sqrt{s_q\big(t,\eta_c(t)\big)}}\Big)^{-1}\frac{1}{\sqrt{s_q\big(\theta,\eta_c(\theta)\big)}}+\varepsilon(\theta,c)\Big], \tag{25}$$

where

$$\text{uniformly for}\quad\theta\in\mathbb{T},\quad\lim_{c\to0}\varepsilon(\theta,c)=0. \tag{26}$$

3.3. **Generating function and regularity.** Let $u_c : \mathbb{T} \to \mathbb{R}$ be the C^1 function such that $u_c(0) = 0$ and $\eta_c = c + u_c'$. In other words, identifying $\mathbb{T}$ with $[0,1]$, we have

$$u_c(\theta) = \int_0^\theta \eta_c(t)dt - c\theta.$$

Observe that for every θ, the map $c \mapsto u_c(\theta) + c\theta$ is increasing because every $c \mapsto \eta_c(\theta)$ is increasing.

Theorem 3.2. *The map $(\theta, c) \mapsto u_c(\theta)$ is C^1. Moreover, in this case, u is unique and we have*

- *the graph of $c + \frac{\partial u_c}{\partial \theta}$ is a leaf of the invariant foliation;*
- *$\theta \mapsto \theta + \frac{\partial u_c}{\partial c}(\theta)$ is the semi-conjugation h_c between g_c and $R_{\rho(c)}$ given in Theorem 3.1. We have: $h_c \circ g_c = h_c + \rho(c)$.*

Corollary 3.1. *The semi-conjugation h_c continuously depends on c.*

Proof. The first point is a consequence of the definition of u_c.
Then u_c and $\frac{\partial u_c}{\partial \theta} = \eta_c - c$ continuously depend on (θ, c).
Observe that with the notation (17), we have

$$\Lambda_{c,c'}\big(\{(\theta, r);\quad \theta \in [\theta_1, \theta_2], r \in [\eta_c(\theta), \eta_{c'}(\theta)]\}\big) = \\ = \frac{1}{c' - c}\Big(\big(u_{c'}(\theta_2) - u_{c'}(\theta_1)\big) - \big(u_c(\theta_2) - u_c(\theta_1)\big)\Big) + (\theta_2 - \theta_1).$$

Moreover, if $\rho(c_0) \in \mathbb{Q}$, we deduce from Lemma 3.1 that u_c admits a derivative with respect to c at c_0

$$\frac{\partial u_c}{\partial c}_{|c=c_0}(\theta) = \lim_{c \to c_0} \frac{1}{c - c_0}\Big(\big(u_c(\theta) - u_c(0)\big) - \big(u_{c_0}(\theta) - u_{c_0}(0)\big)\Big)$$

that is given by

$$\frac{\partial u_c}{\partial c}_{|c=c_0}(\theta) = \mu_{c_0}([0,\theta]) - \theta = h_{c_0}(\theta) - \theta$$

and this derivative continuously depends on θ.
Assume now that $\rho(c_0)$ is irrational and let c tend to c_0. Every limit point of Λ_{c,c_0} when c tends to c_0 is a Borel probability measure that is invariant by f and supported on the graph of η_{c_0}. As there exists only one such measure, whose projection was denoted by μ_{c_0}, we deduce that

$$\pi_{1*}\Big(\lim_{c \to c_0} \Lambda_{c,c_0}\Big) = \mu_{c_0}.$$

As μ_{c_0} has no atom, we have for all $\theta_0 \in [0,1)$

$$\begin{aligned} h_{c_0}(\theta_0) = \ & \mu_{c_0}([0,\theta_0]) \\ & = \lim_{c \to c_0} \Lambda_{c_0,c}(\{(\theta, r); \theta \in [0,\theta_0], r \in [\eta_{c_0}(\theta), \eta_c(\theta)]\}) \\ & = \lim_{c \to c_0} \frac{1}{c - c_0}\Big(\big(u_c(\theta_0) - u_c(0)\big) - \big(u_{c_0}(\theta_0) - u_{c_0}(0)\big)\Big) + \theta_0 \\ & = \frac{\partial u_c}{\partial c}(\theta_0)_{|c=c_0} + \theta_0, \end{aligned}$$

hence u_c admits a derivative with respect to c and

$$h_{c_0}(\theta) = \mu_{c_0}([0,\theta]) = \theta + \frac{\partial u_c}{\partial c}(\theta)_{|c=c_0}.$$

Because of Theorem 3.1, $(\theta, c) \mapsto \frac{\partial u_c}{\partial c}(\theta) = h_c(\theta) - \theta$ is continuous. As the two partial derivatives $\frac{\partial u_c}{\partial \theta}$ and $\frac{\partial u_c}{\partial c}$ are continuous in (θ, c), we conclude that u is C^1. $\square$

4. Proof of the implication (2) ⇒ (1) in Theorem 1.3

We use the same notations as in Theorem 1.1. We assume that the map u is C^1. Then the graph of every $\eta_c = c + \frac{\partial u_c}{\partial \theta}$ is a continuous graph that is backward invariant, hence invariant. If for $c_1 \neq c_2$ the two graphs of η_{c_1} and η_{c_2} have a non-empty intersection, then their common rotation number is rational because an ESTwD has at most one invariant curve with a fixed irrational rotation number (see [26]). Moreover, for every $c \in [c_1, c_2]$, we have $\rho(c) = \rho(c_1)$.
Using results of [7] (see section 5), we know that above any $\theta \in \mathbb{T}$, there are at most two $r_1, r_2 \in \mathbb{R}$ such that the orbit of (θ, r_i) is minimizing with rotation number $\rho(c_1)$. As $c_1 \neq c_2$, there exists then $\theta \in \mathbb{T}$ such that $r_1 = \eta_{c_1}(\theta) \neq \eta_{c_2}(\theta) = r_2$. But for $c \in [c_1, c_2]$, the orbit of $\big(\theta, \eta_c(\theta)\big)$ is minimizing with rotation number equal to $\rho(c_1)$ and then $\eta_c(\theta) \in \{r_1, r_2\}$. As $c \mapsto \eta_c(\theta)$ is continuous with values in $\{\eta_{c_1}(\theta), \eta_{c_2}(\theta)\}$ and satisfies $\eta_{c_1}(\theta) \neq \eta_{c_2}(\theta)$, we obtain a contradiction.

So finally the graphs of the η_c define a lamination of $\mathbb{A}$ and then f is C^0-integrable.

5. Proof of Theorem 1.5

Proposition 5.1. *Assume that the C^1 ESTwD $f : \mathbb{A} \to \mathbb{A}$ has an invariant locally Lipschitz continuous foliation into graphs $c \in \mathbb{R} \mapsto \eta_c \in C^0(\mathbb{T}, \mathbb{R})$. Then the map $\rho : c \in \mathbb{R} \mapsto \rho(c)$ is a locally biLipschitz homeomorphism.*

This result will not be used in what follows and its proof is postponed to the end of this section.

5.1. **Proof of the first implication.** We assume that the invariant foliation is K-Lipschitz on a compact $\mathcal{K} = \{(\theta, \eta_c(\theta)); \theta \in \mathbb{T}, c \in [a, b]\}$, which means

$$\forall \theta \in \mathbb{T}, \forall c_1, c_2 \in [a, b], \quad \frac{|c_1 - c_2|}{K} \leq |\eta_{c_1}(\theta) - \eta_{c_2}(\theta)| \leq K|c_1 - c_2|. \tag{27}$$

As the Lispchitz constant of the invariant graphs are locally uniform in c, changing $\mathcal{K}$ and K, we also have

$$\forall \theta_1, \theta_2 \in \mathbb{R}, \forall c \in [a, b], \quad |\eta_c(\theta_1) - \eta_c(\theta_2)| \leq K|\theta_1 - \theta_2|.$$

and then

$$\forall \theta_1, \theta_2 \in \mathbb{R}, \forall c_1, c_2 \in [a, b], \quad |\eta_{c_1}(\theta_1) - \eta_{c_2}(\theta_2)| \leq K\left(|\theta_1 - \theta_2| + |c_1 - c_2]\right).$$

Hence the map $(\theta, c) \mapsto \eta_c(\theta)$ is Lipschitz and then Lebesgue almost everywhere differentiable by the Rademacher theorem. We denote the set of its differentiability points in $\mathbb{T} \times [a, b]$ by $\mathcal{N}$. Let us fix some $(\theta_0, c_0) \in \mathbb{T} \times \mathbb{R}$ where $(\theta, c) \mapsto \eta_c(\theta)$ is differentiable. Because of Equation (27), we have $\frac{\partial \eta}{\partial c}(\theta_0, c_0) \geq \frac{1}{K}$.
Along the orbit $\big(\theta_k, \eta_{c_0}(\theta_k)\big)$ of $\big(\theta_0, \eta_{c_0}(\theta_0)\big)$, we use the basis $\big(1, \eta'_{c_0}(\theta_k)\big)$ of the

tangent subspace. We have in the basis $\Big(\big(1,\eta'_{c_0}(\theta_j)\big),(0,1)\Big)_{j\in\mathbb{Z}}$ the following symplectic matrix

$$Df^k\big(\theta_0,\eta_{c_0}(\theta_0)\big)=\begin{pmatrix}a_k & b_k\\ 0 & d_k\end{pmatrix}$$

where $a_k=\frac{\partial g^k_{c_0}}{\partial\theta}(\theta_0)$.
We recall that $g_c(\theta)=\pi_1\circ f\big(\theta,\eta_c(\theta)\big)$ and that

$$\forall c\in\mathbb{R},\ \forall k\in\mathbb{Z},\ \forall\theta\in\mathbb{T},\quad f^k\big(\theta,\eta_c(\theta)\big)=\big(g^k_c(\theta),\eta_c\circ g^k_c(\theta)\big). \tag{28}$$

Observe that this implies that $g^k_c(\theta)=\pi_1\circ f^k\big(\theta,\eta_c(\theta)\big)$. Moreover, using the fact that f^k is an ESTwD (see Proposition 3.3), we also deduce $b_k>0$ and

$$\frac{\partial g^k_{c_0}}{\partial c}(\theta_0)=b_k\frac{\partial\eta_{c_0}}{\partial c}(\theta_0).$$

Equation (28) implies that the functions $(\theta,c)\mapsto g^k_c(\theta)$ are differentiable at (θ_0,c_0) and

$$Df^k\big(\theta_0,\eta_{c_0}(\theta_0)\big)\Big(0,\frac{\partial\eta_{c_0}}{\partial c}(\theta_0)\Big)=\Big(\frac{\partial g^k_{c_0}}{\partial c}(\theta_0),\frac{\partial\eta_{c_0}}{\partial c}\big(g^k_{c_0}(\theta_0)\big)+\eta'_{c_0}\big(g^k_{c_0}(\theta_0)\big)\frac{\partial g^k_{c_0}}{\partial c}(\theta_0)\Big),$$

i.e.

$$Df^k\big(\theta,\eta_{c_0}(\theta_0)\big)\Big(0,\frac{\partial\eta_{c_0}}{\partial c}(\theta_0)\Big)=\frac{\partial g^k_{c_0}}{\partial c}(\theta_0)\Big(1,\eta'_c\big(g^k_{c_0}(\theta_0)\big)\Big)+\frac{\partial\eta_{c_0}}{\partial c}\big(g^k_{c_0}(\theta_0)\big)(0,1),$$

i.e.

$$b_k\frac{\partial\eta_{c_0}}{\partial c}(\theta_0)=\frac{\partial g^k_{c_0}}{\partial c}(\theta_0)\quad\text{and}\quad d_k=\frac{\partial\eta_{c_0}}{\partial c}\big(g^k_{c_0}(\theta_0)\big)\left(\frac{\partial\eta_{c_0}}{\partial c}(\theta_0)\right)^{-1}.$$

The matrix being symplectic, we have $a_kd_k=1$ and then

$$\frac{\partial g^k_{c_0}}{\partial\theta}(\theta_0)=\frac{\partial\eta_{c_0}}{\partial c}(\theta_0)\left(\frac{\partial\eta_{c_0}}{\partial c}\big(g^k_{c_0}(\theta_0)\big)\right)^{-1}\in\Big[\frac{1}{K^2},K^2\Big]$$

is uniformly bounded.
As $\mathcal{N}$ has full Lebesgue measure in $\mathbb{T}\times[a,b]$, there exists a set $C\subset[a,b]$ with full Lebesgue measure such that for every $c_0\in C$, $\mathcal{N}\cap(\mathbb{T}\times\{c_0\})$ has full Lebesgue measure in $\mathbb{T}\times\{c_0\}$. We obtain that for every $c\in C$, the familly $(g^k_c)_{k\in\mathbb{Z}}$ is uniformly K^2-Lipschitz. As C is dense in $[a,b]$, we deduce, by continuity of $c\mapsto g_c$, that the maps $\{g^k_c,\ \ k\in\mathbb{Z},\ c\in[a,b]\}$ are K^2 Lipschitz.
Finally, every g_c is a biLipschitz orientation preserving homeomorphism of $\mathbb{T}$ whose all iterated homeomorphisms are equiLipschitz. We deduce from results of [2] that η_c is in fact C^1 (and the two Green bundles coincide along its graphs) and that g_c is C^1 conjugated to a rotation. Hence all the points are recurrent. Moreover, as the two Green bundles are equal everywhere, they are continuous. Because they coincide with the tangent space to the foliation, the foliation is a C^1 lamination.
As the $(g^k_c)'$ are equibounded by some constant $\widetilde{K}$, we deduce from results that are contained in [27] that the conjugations h_c to a rotation are $\widetilde{K}$-equibiLipschitz.
Finally, we deduce from Theorem 1.3 that u is C^1 with partial derivatives that are

- $\frac{\partial u_c}{\partial\theta}(\theta)=\eta_c(\theta)-c$ which is locally Lipschitz (as η_c is) as a function of (θ,c);

- $\frac{\partial u_c}{\partial c}(\theta) = h_c(\theta) - \theta$ which is uniformly Lipschitz[14] in the variable θ on any compact set of c's.

If we denote by K a local Lipschitz constant for h_c and h_c^{-1}, we have Lebesgue almost everywhere that

$$\frac{\partial h_c}{\partial \theta}(\theta) \in \Big[\frac{1}{K}, K\Big]$$

and then

$$\frac{\partial^2 u_c}{\partial \theta \partial c}(\theta) \in \Big[-1 + \frac{1}{K}, -1 + K\Big],$$

that gives the last point of Theorem 1.5.
Note that this improves the fact that u is C^1.

Let $v : \mathbb{R}^2 \to \mathbb{R}_+$ be the C^∞ function with support in $B(0,1)$ defined by $v(\theta, c) = a\exp\big((1 - \|(\theta, c)\|)^{-2}\big)$ for $(\theta, c) \in B(0,1)$ and where a is such that $\int v = 1$. We denote by v_ε the function $v_\varepsilon(x) = \frac{1}{\varepsilon^2} v(\frac{x}{\varepsilon})$. Then we define for every $\varepsilon > 0$.

$$U_\varepsilon(\theta, c) = (u * v_\varepsilon)(\theta, c),$$

where we recall the formula for the convolution

$$u * v(x) = \int u(x - y)v(y)dy.$$

Then U_ε is 1-periodic in θ and smooth and when ε tends to 0, the functions U_ε tend to U in the C^1 compact-open topology.
Observe that for every θ, the function $c \mapsto c + \frac{\partial u}{\partial \theta}(\theta, c)$ is increasing. We deduce that the convolution $c \mapsto c + \frac{\partial U_\varepsilon}{\partial \theta}(\theta, c)$ is a C^∞ diffeomorphism as it is a mean of C^∞ diffeomorphisms thanks to Lemma 5.1. Finally, the maps $F_\varepsilon : (\theta, c) \mapsto \big(\theta, c + \frac{\partial U_\varepsilon}{\partial \theta}(\theta, c)\big)$ define C^∞ foliations that converge to the initial foliation $F_0 : (\theta, c) \mapsto \big(\theta, c + \frac{\partial u}{\partial \theta}(\theta, c)\big)$ for the C^0 compact-open topology when ε tends to 0.
Observe that the h_c's are assumed to be increasing. We deduce that the maps $G_\varepsilon : (\theta, c) \mapsto (\theta + \frac{\partial U_\varepsilon}{\partial c}(\theta, c), c)$ are C^∞ diffeomorphisms of $\mathbb{T} \times \mathbb{R}$ that converge for the C^0 compact-open topology to $G_0 : (\theta, c) \mapsto (\theta + \frac{\partial u}{\partial c}(\theta, c), c)$.
Finally, the $\mathcal{H}_\varepsilon = G_\varepsilon \circ F_\varepsilon^{-1}$ are C^∞ diffeomorphisms of $\mathbb{T} \times \mathbb{R}$ that converge for the C^0 compact-open topology to $G_0 \circ F_0^{-1} = \Phi$.

This exactly means that Φ is a symplectic homeomorphism. Moreover, we have

$$\Phi \circ f \circ \Phi^{-1}(x, c) = G_0 \circ F_0^{-1} \circ F \circ F_0 \circ G_0^{-1}(x, c) = (x + \rho(c), c).$$

Lemma 5.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative, non-trivial, smooth, integrable and even function such that $f' \leq 0$ on $[0, +\infty)$. Then if $g : \mathbb{R} \to \mathbb{R}$ is increasing, $f * g$ is an increasing C^∞ diffeomorphism.*

Proof. As f is even, f' is odd. Just notice that

$$(f * g)'(x) = \int_{\mathbb{R}} f'(y)g(x - y)dy = \int_0^{+\infty} f'(y)\big(g(x - y) - g(x + y)\big)dy.$$

The result follows as $g(x - y) - g(x + y) < 0$ and $f'(y) \leq 0$ and does not vanish everywhere. □

We conclude this section by returning to the proof of Proposition 5.1. We will use the following

[14]This function is even C^1 at c's such that $\rho(c)$ is irrational.

Lemma 5.2. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be lifts of homeomorphisms of $\mathbb{T}$ that preserve orientation (implying $f(\cdot + 1) = f(\cdot) + 1$ and $g(\cdot + 1) = g(\cdot) + 1$). Assume that*

- *either f or g is conjugated to a translation $t_\alpha : x \mapsto x + \alpha$ by a homeomorphism h lift of a homeomorphism of $\mathbb{T}$ that preserves orientation;*
- *h and h^{-1} are K-Lipschitz.*

(1) If there exists $d > 0$ such that $f < g + d$, then $\rho(f) \leq \rho(g) + Kd$.
(2) If there exists $d > 0$ such that $f + d < g$ then $\rho(f) + \frac{d}{K} \leq \rho(g)$.

Proof. Let us say that $h \circ g \circ h^{-1} = t_\alpha$, hence $\rho(g) = \alpha$ (the proof when f is conjugated to a translation is the same).

(1) By hypothesis, $f \circ h^{-1} < g \circ h^{-1} + d$. Using that h is increasing and K-Lipschitz, it follows that for all $x \in \mathbb{R}$,

$$h \circ f \circ h^{-1}(x) < h(g \circ h^{-1}(x) + d) < h \circ g \circ h^{-1}(x) + Kd = x + \alpha + Kd.$$

Finally, as $\rho(f) = \rho(h \circ f \circ h^{-1})$, we conclude that

$$\rho(f) \leq \alpha + Kd = \rho(g) + Kd.$$

(2) By hypothesis, $f \circ h^{-1} + d < g \circ h^{-1}$. Using that h is increasing, it follows that

$$\forall x \in \mathbb{R}, \quad h(f \circ h^{-1}(x) + d) < h \circ g \circ h^{-1}(x) = x + \alpha.$$

Because h^{-1} is K-Lipschitz and increasing, observe that

$$d = h^{-1}(h(f \circ h^{-1}(x) + d)) - h^{-1}(h \circ f \circ h^{-1}(x)) \\ \leq K\left(h(f \circ h^{-1}(x) + d) - h \circ f \circ h^{-1}(x)\right).$$

Then

$$h \circ f \circ h^{-1}(x) \leq h(f \circ h^{-1}(x) + d) - \frac{d}{K} < x + \alpha - \frac{d}{K};$$

hence $\rho(f) + \frac{d}{K} \leq \rho(g)$.

□

Proof of Proposition 5.1. The proof is now a direct application of the previous Lemma. Indeed, we have seen that when the foliation is K-Lipschitz, if c varies in a compact set $\mathcal{K}$, the dynamics g_c are all conjugated to rotations. We have moreover proven there exists a constant $\widetilde{K}$ such that the conjugating functions h_c may be chosen equi-Lipschitz (for $c \in \mathcal{K}$). Finally, when $\rho(c)$ is irrational, we deduce from results of [27] (see also [6]) that h_c^{-1} is also $\widetilde{K}$-Lipschitz. We therefore conclude that ρ is $K\widetilde{K}$-Lipschitz when restricted to $\rho^{-1}(\mathbb{R} \setminus \mathbb{Q})$. By density, ρ is Lipschitz.

We denote the minimum torsion on $\mathcal{K}$ by

$$b_{\min} = \min_{x \in \mathcal{K}} \frac{\partial f_1}{\partial \theta}(x).$$

For $c_1 < c_2$ in $[a, b]$ such that either $\rho(c_1)$ or $\rho(c_2)$ is irrational, we have

$$\tilde{g}_{c_2}(\theta) - \tilde{g}_{c_1}(\theta) = F_1\big(\theta, \eta_{c_2}(\theta)\big) - F_1\big(\theta, \eta_{c_1}(\theta)\big) \\ \geq b_{\min}\big(\eta_{c_2}(\theta) - \eta_{c_1}(\theta)\big) \geq \frac{b_{\min}}{K}(c_2 - c_1).$$

We deduce from the second point of Lemma 5.2 that

$$\rho(g_{c_2})-\rho(g_{c_1})\geq\frac{b_{\min}}{K^2}(c_2-c_1).$$

As previously, by density, we get that ρ^{-1} is also locally Lipschitz. □

5.2. **Proof of the second implication.** We assume that the map u is C^1 with $\frac{\partial u}{\partial\theta}$ locally Lipschitz continuous and $\frac{\partial u}{\partial c}$ uniformly Lipschitz in the variable θ on any compact set of c's and there exists a constant $k>-1$ such that $\frac{\partial^2 u}{\partial\theta\partial c}(\theta,c)>k$ almost everywhere.
Theorem 1.3 yields that the graphs of the η_c define a lamination of $\mathbb{A}$ into Lipschitz graphs and that the map $h_c:\theta\mapsto\theta+\frac{\partial u_c}{\partial c}(\theta)$ is a semi-conjugation between the projected Dynamics $g_c:\theta\mapsto\pi_1\circ f\big(\theta,c+\frac{\partial u_c}{\partial\theta}(\theta)\big)$ and a rotation R of $\mathbb{T}$, i.e. $h_c\circ g_c=R\circ h_c$.

By assumption, $\eta_c=c+\frac{\partial u_c}{\partial\theta}$ is locally Lipschitz. We want to prove that $(\theta,c)\mapsto\big(\theta,\eta_c(\theta)\big)$ is locally biLipschitz. We only need to prove that locally, we have for Lebesgue almost every (θ,c) a uniform positive lower bound for $\frac{\partial\eta_c}{\partial c}$ (observe that $\frac{\partial\eta_c}{\partial c}$ is always non-negative because every $c\mapsto\eta_c(\theta)$ is increasing).
As $\frac{\partial^2 u}{\partial\theta\partial c}(\theta,c)>k$ almost everywhere, the set

$$C=\Big\{c\in\mathbb{R};\quad\frac{\partial^2 u}{\partial\theta\partial c}(\theta,c)>k\quad\text{for}\quad\text{Lebesgue}\quad\text{almost}\quad\text{every}\quad\theta\in\mathbb{T}\Big\}$$

has full Lebesgue measure.

Then, for every $c\in C$ and $\theta>\theta'$ in $[0,1]$, we have

$$\frac{\partial u}{\partial c}(\theta,c)-\frac{\partial u}{\partial c}(\theta',c)=\int_{\theta'}^{\theta}\frac{\partial^2 u}{\partial\theta\partial c}(a,c)da\geq k(\theta-\theta').$$

Hence, if $c>c'$ in $\mathbb{R}$, we have

$$u(\theta,c)-u(\theta,c')-u(\theta',c)+u(\theta',c')=\int_{c'}^{c}\left(\frac{\partial u}{\partial c}(\theta,t)-\frac{\partial u}{\partial c}(\theta',t)\right)dt\geq k(\theta-\theta')(c-c').$$

If we divide by $\theta-\theta'$ and take the limit $\theta'\to\theta$, we obtain

$$\frac{\partial u}{\partial\theta}(\theta,c)-\frac{\partial u}{\partial\theta}(\theta,c')\geq k(c-c'),$$

that is equivalent to

$$\forall\theta\in\mathbb{T},\quad\eta_c(\theta)-\eta_{c'}(\theta)\geq(1+k)(c-c').$$

As $1+k>0$, we conclude that the foliation is biLipschitz.

6. Foliations by graphs

6.1. **Proof of Proposition 1.1.** Let $f:\mathbb{A}\to\mathbb{A}$ be an exact symplectic homeomorphism. We assume the f invariant foliation $\mathcal{F}$ into C^0 graphs is symplectically homeomorphic (by $\Phi:\mathbb{A}\to\mathbb{A}$) to the standard foliation $\mathcal{F}_0=\Phi(\mathcal{F})$. Then the standard foliation is invariant by the exact symplectic homeomorphism $g=\Phi\circ f\circ\Phi^{-1}$. Hence we have

$$g(\theta,r)=(g_1(\theta,r),r).$$

As g is area preserving, for every $\theta \in [0,1]$ and every $r_1 < r_2$, the area of $[0,\theta] \times [r_1, r_2]$ is equal to the area of $g\big([0,\theta] \times [r_1, r_2]\big)$, i.e.

$$\theta(r_2 - r_1) = \int_{r_1}^{r_2} \big(g_1(\theta, r) - g_1(0, r)\big) dr.$$

Dividing by $r_2 - r_1$ and taking the limit when r_2 tends to r_1, we obtain

$$g_1(\theta, r_1) = \theta + g(0, r_1).$$

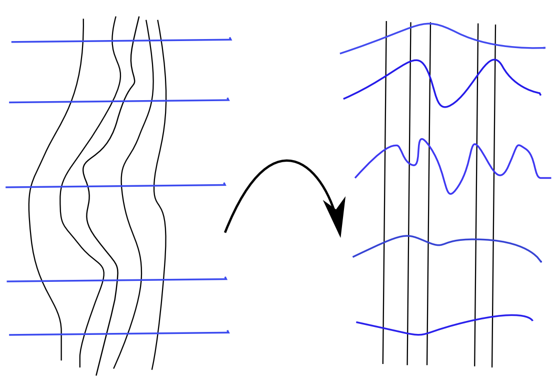
This proves the proposition for $\rho = g_1(0, \cdot)$.

6.2. **Proof of Theorem 1.4.** Let us consider a C^0-foliation $\mathcal{F}$ of $\mathbb{A}$: $(\theta, c) \mapsto \big(\theta, \eta_c(\theta)\big)$, where $\int_{\mathbb{T}} \eta_c = c$. Then there exists a continuous function $u : \mathbb{A} \to \mathbb{R}$ that admits a continuous derivative with respect to θ such that $\eta_c(\theta) = c + \frac{\partial u}{\partial \theta}(\theta, c)$ and $u(0, c) = 0$.

Proof of the first implication.

We assume that this foliation is exact symplectically homeomorphic to the standard foliation $\mathcal{F}_0 = \Phi(\mathcal{F})$ by some exact symplectic homeomorphism Φ.

Observe that the foliation $\mathcal{F}$ is transverse to the "vertical" foliation $\mathcal{G}_0$ into $\{\theta\} \times \mathbb{R}$ for $\theta \in \mathbb{T}$. Hence the foliation $\mathcal{G} = \Phi(\mathcal{G}_0)$ is a foliation that is transverse to the standard ("horizontal") foliation $\mathcal{F}_0 = \Phi(\mathcal{F})$. This exactly means that the foliation $\mathcal{G}$ is a foliation into graphs of maps $\zeta_\theta : \mathbb{R} \to \mathbb{T}$. Hence there exists a continuous function $v : \mathbb{A} \to \mathbb{R}$ that admits a continuous derivative with respect to r such that the foliation $\mathcal{G}$ is the foliation into graphs $\Phi(\{\theta\} \times \mathbb{R})$ of $\zeta_\theta : r \mapsto \theta + \frac{\partial v}{\partial r}(\theta, r)$. Observe that by definition of ζ_θ, we have $\Phi\big(\zeta_\theta(c), c\big) = \big(\theta, \eta_c(\theta)\big)$. As a result, every map $\theta \mapsto \zeta_\theta(c)$ is a homeomorphism of $\mathbb{T}$.



We now use the preservation of the area. We fix $\theta_1 < \theta_2$ in $[0,1]$ and $r_1 < r_2$ in $\mathbb{R}$. Because Φ is a symplectic homeomorphism, Φ preserves the area and so the two following domains have the same area

- the domain delimited by $\mathbb{T} \times \{c_1\}$, $\mathbb{T} \times \{c_2\}$, the graph of $c \in \mathbb{R} \mapsto \zeta_{\theta_1}(c)$ and the graph of $c \in \mathbb{R} \mapsto \zeta_{\theta_2}(c)$;
- the domain delimited by the graphs of η_{c_1}, η_{c_2} and the verticals $\{\theta_1\} \times \mathbb{R}$ and $\{\theta_2\} \times \mathbb{R}$.

This can be written

$$\int_{c_1}^{c_2} \Big(\big(\theta_2 + \frac{\partial v}{\partial c}(\theta_2, c)\big) - \big(\theta_1 + \frac{\partial v}{\partial c}(\theta_1, c)\big)\Big) dc = \int_{\theta_1}^{\theta_2} \Big(\big(c_2 + \frac{\partial u}{\partial \theta}(\theta, c_2)\big) - \big(c_1 + \frac{\partial u}{\partial \theta}(\theta, c_1)\big)\Big) d\theta.$$

It follows that

$$u(\theta_2,c_2)-u(\theta_1,c_2)-u(\theta_2,c_1)+u(\theta_1,c_1)=$$
$$=v(\theta_2,c_2)-v(\theta_2,c_1)-v(\theta_1,c_2)+v(\theta_1,c_1).$$

Evaluating for $\theta_1=0$ we find

$$u(\theta_2,c_2)-u(\theta_2,c_1)=v(\theta_2,c_2)-v(\theta_2,c_1)-v(0,c_2)+v(0,c_1).$$

Finally, as v admits a continuous partial derivative with respect to c, we conclude that $\frac{\partial u}{\partial c}(\theta,c)=\frac{\partial v}{\partial c}(\theta,c)-\frac{\partial v}{\partial c}(0,c)$ exists and is continuous. Hence u is C^1. Moreover, every map $\theta\mapsto\theta+\frac{\partial u}{\partial c}(\theta,c)=\zeta_c(\theta)-\frac{\partial v}{\partial c}(0,c)$ is a homeomorphism of $\mathbb{T}$ and we have established the first implication.

Proof of the second implication.

We assume that there exists a C^1 map $u:\mathbb{A}\to\mathbb{R}$ such that

- $u(0,r)=0$ for all $r\in\mathbb{R}$,
- $\eta_c(\theta)=c+\frac{\partial u}{\partial\theta}(\theta,c)$ for all $(\theta,c)\in\mathbb{A}$,
- for all $c\in\mathbb{R}$, the map $\theta\mapsto\theta+\frac{\partial u}{\partial c}(\theta,c)$ is a homeomorphism of $\mathbb{T}$.

Then we can define a unique homeomorphism Φ of $\mathbb{A}$ by

$$\Phi\Big(\theta+\frac{\partial u}{\partial c}(\theta,c),c\Big)=\Big(\theta,c+\frac{\partial u}{\partial\theta}(\theta,c)\Big).$$

The previous computations (with $v=u$) proves that Φ preserves the area and so is an exact symplectic homeomorphism.

6.3. **Proof of Corollary 1.1.** The *if* part is obvious.

Let us prove the *only if* part, that is we assume f is C^0-integrable with the Dynamics on each leaf conjugated to a rotation. We denote by $u:\mathbb{A}\to\mathbb{R}$ the map given by theorem 1.1 and that enjoys the properties of Theorems 1.3 and 3.1. Hence $h_c:\theta\mapsto\theta+\frac{\partial u_c}{\partial c}(\theta)$ is a semi-conjugation between the projected Dynamics $g_c:\theta\mapsto\pi_1\circ f\big(\theta,c+\frac{\partial u_c}{\partial\theta}(\theta)\big)$ and the rotation $R_{\rho(c)}$ of $\mathbb{T}$ and even is a conjugation when $\rho(c)$ is rational.

If $\rho(c)$ is irrational, it follows from the hypothesis that g_c is conjugated to a rotation. As the dynamics is minimal, there is up to constants a unique (semi)-conjugacy and and then h_c is a true conjugation.

6.4. C^1 **foliations are always straightenable by a symplectic homeomorphism.** Let $r\mapsto f_r$ be a foliation of $\mathbb{A}$ in continuous graphs. Given $r\in\mathbb{R}$, the set $\mathcal{F}_r=\big\{\big(\theta,f_r(\theta)\big),\theta\in\mathbb{T}\big\}$ is a leaf of the foliation. Given an integer $k>0$, we say it is a C^k foliation if the map $F:(\theta,r)\mapsto\big(\theta,f_r(\theta)\big)$ is a C^k-diffeomorphism.

Having in mind Theorem 1.4 and the remark page 4, the following Theorem is quite natural. However, this proof, communicated to us by Philippe Bolle, gives precise formulas in the smooth case:

Theorem 6.1. *Let $k>0$ and $r\mapsto f_r$ be a C^k-foliation in graphs. Then there exists a C^{k-1} exact symplectic homeomorphism $H:(\theta,r)\mapsto\big(h(\theta,r),\eta(h(\theta,r),r)\big)$ such that for each $r\in\mathbb{R}$, the set $\big\{\big(\theta,\eta(\theta,r)\big),\theta\in\mathbb{T}\big\}$ is a leaf of the foliation.*

Proof. As the map $r\mapsto f_r(0)$ is a C^k diffeomorphism, up to composing with its inverse in the vertical direction, we may assume $f_r(0)=r$ for all r.

For $r\in\mathbb{R}$, let us set $A(r)=\int_0^1 f_r(\theta)d\theta$. One computes that $\frac{dA}{dr}(r)=\int_0^1\frac{\partial f}{\partial r}(\theta,r)d\theta>0$. Hence A is a C^k-diffeomorphism of $\mathbb{R}$ and the map

$$G:(\theta,r)\mapsto\big(\theta,f_{A^{-1}(r)}(\theta)\big)=\big(\theta,\eta_r(\theta)\big)$$

is a C^k-diffeomorphism such that for all r, the graph of η_r is the leaf of the foliation of cohomology r.

Let us now look for a map $h:\mathbb{A}\to\mathbb{T}$ such that $H(\theta,r)=\big(h(\theta,r),\eta(h(\theta,r),r)\big)$ is exact symplectic. One computes that

$$\begin{aligned}\det\big(DH(\theta,r)\big)&=dh\wedge\left(\frac{\partial\eta}{\partial\theta}(h(\theta,r),r)dh+\frac{\partial\eta}{\partial r}(h(\theta,r),r)dr\right)\\&=\frac{\partial\eta}{\partial r}(h(\theta,r),r)dh\wedge dr=\frac{\partial\eta}{\partial r}(h(\theta,r),r)\left(\frac{\partial h}{\partial\theta}(\theta,r)d\theta+\frac{\partial h}{\partial r}(\theta,r)dr\right)\wedge dr\\&=\frac{\partial\eta}{\partial r}(h(\theta,r),r)\frac{\partial h}{\partial\theta}(\theta,r)d\theta\wedge dr.\end{aligned}$$

It follows we want to solve

$$\frac{\partial\eta}{\partial r}(h(\theta,r),r)\frac{\partial h}{\partial\theta}(\theta,r)=1.\tag{29}$$

Let us set $g(\theta,r)=\int_0^\theta\frac{\partial\eta}{\partial r}(s,r)ds$. Recall that $\frac{\partial\eta}{\partial r}$ is everywhere positive. Moreover, we have seen that $\int_0^1\frac{\partial\eta}{\partial r}(s,r)ds=1$ so that for each $r\in\mathbb{R}$, the map $g_r=g(\cdot,r):\mathbb{T}\to\mathbb{T}$ is a C^{k-1} orientation preserving diffeomorphism. Hence, by integrating, (29) becomes $g\big(h(\theta,r),r\big)=\theta$, hence

$$\forall(\theta,r)\in\mathbb{A},\quad h(\theta,r)=g_r^{-1}(\theta).$$

If $k\geq 2$, the map H obtained is C^1 and the previous computations apply to show that H is area preserving. For $k=1$, H is only C^0 (more precisely, it varies only continuously in the vertical direction) and we prove below that it preserves the area. Recall that $G^{-1}\circ H(\theta,r)=(h(\theta,r),r)$. Let $0<\theta_1<\theta_2$ and $r_1<r_2$. Let us set $R=H([\theta_1,\theta_2]\times[r_1,r_2])$. We compute

$$\begin{aligned}\int_R drd\theta&=\int_{G^{-1}(R)}\frac{\partial\eta}{\partial r}(\theta,r)d\theta dr\\&=\int_{r_1}^{r_2}\int_{h(\theta_1,r)}^{h(\theta_2,r)}\frac{\partial\eta}{\partial r}(\theta,r)d\theta dr\\&=\int_{r_1}^{r_2}\int_{g_r^{-1}(\theta_1)}^{g_r^{-1}(\theta_2)}\frac{\partial g_r}{\partial\theta}(\theta)d\theta dr\\&=\int_{r_1}^{r_2}(\theta_2-\theta_1)dr=(\theta_2-\theta_1)(r_2-r_1).\end{aligned}$$

Hence H preserves the area. □

6.5. **A strange foliation.** As an application of Theorem1.4, here is a Lipschitz foliation that is not symplectically homeomorphic to the standard foliation. Let $\eta_c(\theta)=c+\varepsilon(c)\cos(2\pi\theta)$. We assume that ε is a contraction (k-Lipschitz with $k<1$) that is not everywhere differentiable. It follows that $(\theta,c)\mapsto\eta_c(\theta)$ is a biLipschitz foliation. Were it symplectically homeomorphic to the standard foliation, the associated function given by Theorem 1.4 would be

$$(\theta, c) \mapsto u_c(\theta) = \frac{\varepsilon(c)}{2\pi}\sin(2\pi\theta).$$

However, by Theorem 1.4, this function should be C^1 which is not the case as it does not admit partial derivatives with respect to c.
Theorem 1.5 implies that this (Lipschitz) foliation cannot be invariant by an ESTwD.
Let us prove however that this foliation, for a simple choice of ε, can be invariant by a certain C^1 exact symplectic twist map.

DEFINITION. An exact symplectic homeomorphism $f : \mathbb{A} \to \mathbb{A}$ has the *weak twist property* if when $F = (F_1, F_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is any lift of f, for any $\tilde\theta \in \mathbb{R}$, the map $r \in \mathbb{R} \mapsto F_1(\tilde\theta, r) \in \mathbb{R}$ is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

Let us now assume that ε is a C^2 function away from $c = 0$ and that at 0 it has a left and a right derivatives up to order 2. For the sake of simplicity, let us assume also that $\varepsilon(0) = 0$ so that $\mathbb{T} \times \{0\}$ is a leaf of the foliation and that ε restricted to $[0, +\infty)$ (resp. $(-\infty, 0]$) is the restriction of a C^2 periodic function.

The proof of Theorem 6.1 gives us two C^1 functions

$$H^\pm : (\theta, r) \mapsto \big(h^\pm(\theta, r), \eta(h^\pm(\theta, r), r)\big)$$

where H^+ is a C^1 exact symplectic diffeomorphism of $\mathbb{A}^+ = \mathbb{T} \times [0, +\infty)$ to itself (up to the boundary) and $H-$ is a C^1 exact symplectic diffeomorphism of $\mathbb{A}^- = \mathbb{T} \times (-\infty, 0]$ to itself (up to the boundary). Note that here H^+ and H^- do not coincide on $\mathbb{T} \times \{0\}$ explaining why the foliation is not straightenable.

Let $\rho : \mathbb{R} \to \mathbb{R}$ be an increasing, C^1 homeomorphism such that $\rho(0) = \rho'(0) = 0$. We denote by $f_\rho : (\theta, r) \mapsto (\theta + \rho(r), r)$. The function $f = H^\pm \circ f_\rho \circ (H^\pm)^{-1}$ is well defined on $\mathbb{A}$, it is the identity on $\mathbb{T} \times \{0\}$. It is clearly an area preserving homeomorphism that is C^1 away from $\mathbb{T} \times \{0\}$.

If $r > 0$ and $\theta \in \mathbb{T}$, let us set $(\Theta, R) = H^+(\theta, r)$. Then one finds that

$$\begin{aligned} Df(\Theta, R) &= DH^+(\theta + \rho(r), r) \cdot Df_\rho(\theta, r) \cdot DH^+(\theta, r)^{-1} \\ &= DH^+(\theta + \rho(r), r) \cdot \begin{pmatrix} 1 & \rho'(r) \\ 0 & 1 \end{pmatrix} \cdot DH^+(\theta, r)^{-1} \end{aligned}$$

It follows from the properties on H^+ and $\rho(0) = \rho'(0) = 0$ that as $R \to 0$, $Df(\Theta, R)$ uniformly converges to the identity. As the same holds for $R < 0$, we deduce that f is in fact C^1 with a differential on $\mathbb{T} \times \{0\}$ being identity.

It is left to chose ρ in such a way that the obtained map is a twist map. We construct it on $[0, +\infty)$. The twist condition we aim at is: for every $\Theta \in \mathbb{R}$, the map $r \mapsto h^+((h_r^+)^{-1}(\Theta) + \rho(r), r)$ is an increasing homeomorphism of $\mathbb{R}$.

After computation, if we denote $h^+(\theta_r, r) = \Theta$, the derivative of the above function is the following for $r > 0$ (the inequality is our goal):

$$\begin{aligned} &\frac{\partial h^+}{\partial r}(\theta_r + \rho(r), r) - \left(\frac{\partial h^+}{\partial \theta}(\theta_r, r)\right)^{-1} \frac{\partial h^+}{\partial \theta}(\theta_r + \rho(r), r)\frac{\partial h^+}{\partial r}(\theta_r, r) \\ &\qquad + \frac{\partial h^+}{\partial \theta}(\theta_r + \rho(r), r)\rho'(r) > 0. \end{aligned}$$

The first line above is smaller in absolute value than $M_1\rho(r)$ where (recall that by hypothesis, all the functions at play are continuous periodic hence bounded)

$$M_1 = \left\|\frac{\partial^2 h^+}{\partial r\partial\theta}\right\|_\infty + \left\|\left(\frac{\partial h^+}{\partial\theta}\right)^{-1}\right\|_\infty \cdot \left\|\frac{\partial^2 h^+}{\partial\theta^2}\right\|_\infty \left\|\frac{\partial h^+}{\partial r}\right\|_\infty .$$

On the other hand, the second line is greater than $M_2\rho'(r)$ where we set $M_2 = \min\frac{\partial h^+}{\partial\theta} > 0$. If $\rho(t) = t^2e^{Mt}$ with $M = 2M_1/M_2$, then we have $\rho(0) = \rho'(0) = 0$ and the previous derivative is at least $M_1\rho(t)$ that implies the twist condition.

Appendix A. Examples

A.1. **An example a semi-concave function that is not a weak K.A.M. solution for $\widehat{T}^c$ and that satisfies $f^{-1}\big(\overline{\mathcal{G}(c+u')}\big) \subset \mathcal{G}(c+u')$.** Let us begin by introducing $g_t : \mathbb{A} \to \mathbb{A}$ as being the time t map of the Hamiltonian flow of the double pendulum Hamiltonian

$$H(\theta, r) = \frac{1}{2}r^2 + \cos(4\pi\theta).$$

If $t > 0$ is small enough, g_t is an ESTwD.
Observe that H is a so-called Tonelli Hamiltonian (see [22] for the definition) with associated Lagrangian $L(\theta, v) = \frac{1}{2}v^2 - \cos(4\pi\theta)$. The global minimum -1 of L is attained in $(0, 0)$ and $(\frac{1}{2}, 0)$.
If G_t is the time t map of the lift of H to $\mathbb{R}^2$, then G_t is a lift of g_t and if $G_s(\theta, r) = (\theta_s, r_s)$, a generating function of G_t is

$$S_t(\theta, \theta_t) = \int_0^t L(\theta_s, \dot\theta_s)ds.$$

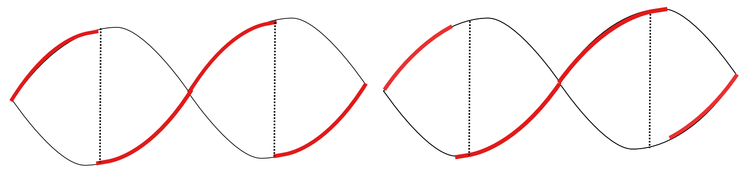
By using this formula, observe that the only ergodic minimizing measures for the cohomology class 0 are the Dirac measure at 0 and $\frac{1}{2}$.

Then we denote by $h : \mathbb{A} \to \mathbb{A}$ the map that is defined by $h(\theta, r) = (\theta + \frac{1}{2}, r)$. Then $f = h \circ g_t = g_t \circ h$ is again an ESTwD and H is an integral for f, which means that $H \circ f = H$.
It is easy to check that a generating function of a lift F of f is given by

$$S(\theta, \Theta) = S_t\big(\theta, \Theta - \frac{1}{2}\big).$$

From this, we deduce that the Mather set corresponding to the cohomology class zero (and the rotation number $\frac{1}{2}$) is the support of a unique ergodic measure, that is the mean of two Dirac measure $\frac{1}{2}(\delta_{(0,0)} + \delta_{(\frac{1}{2},0)})$.
As there is only one such minimizing measure, we know that there is a unique, up to constants, weak K.A.M. solution u with cohomology class 0. But there are a lot of graphs of v' with $v : \mathbb{T} \to \mathbb{R}$ semi-concave that are invariant by f. The first one we draw corresponds to the weak K.A.M. solution whose graph is strictly mapped into itself by f^{-1}. Perturbing slightly the pseudograph in the level $\{H = 1\}$, we obtain another backward invariant pseudograph that doesn't correspond to a weak K.A.M. solution.

In the right drawing, the perturbation of the pseudograph must be small enough so that, in the right eye on the upper manifold, the piece of pseudograph that goes beyond the vertical dotted line is mapped by f in the upper piece of pseudograph of the left eye.



A.2. **Cases where the discounted solution doesn't depend continuously on c.** Let us start this appendix of counterexamples with a positive result. We will show that even if discounted solutions may depend in a discontinuous way on c, the same is not true for their derivative. In what follows we use the notion of Clarke sub-derivative introduced earlier in Definition 2.3.

Let us recall that by Proposition 2.2, if $g_n : \mathbb{T} \to \mathbb{R}$ are equi-semi-concave functions converging to $g : \mathbb{T} \to \mathbb{R}$, then $\mathcal{PG}(g_n')$ converges to $\mathcal{PG}(g')$ for the Hausdorff distance.

Let us now state our result:

Proposition A.1. *Let $f : \mathbb{A} \to \mathbb{A}$ be an ESTwD. For $c \in \mathbb{R}$, we denote by $\mathcal{U}_c$ the weak K.A.M. discounted solution. Then the map $c \mapsto \mathcal{PG}(\mathcal{U}_c')$ is continuous.*

As a straightforward corollary, we deduce for instance that if $c_n \to c$ and $x_n \to x$ and if the $\mathcal{U}_{c_n}'(x_n)$ exist, as well as $\mathcal{U}'(c)(x)$, then $\mathcal{U}_{c_n}'(x_n) \to \mathcal{U}'(c)(x)$.

Proof of Proposition A.1. If $\rho(c_0) \in \mathbb{R} \setminus \mathbb{Q}$, there is a unique weak K.A.M. solution up to constants, hence continuity of $\mathcal{PG}(\mathcal{U}_c')$ at c_0 follows from Proposition 2.2.

If $\rho(c) = r \in \mathbb{Q}$, let us denote $\rho^{-1}(r) = [c_1, c_2]$. Again, continuity at c_1 and c_2 is obvious as there is a unique weak K.A.M solution at these cohomology classes (see Proposition 2.4).

It remains to study what happens inside (c_1, c_2) and we will prove that in this interval, the map $c \mapsto \mathcal{U}_c$ is concave. Let us set $\mathcal{M}_r$ the set of Mather measures corresponding to any cohomology class $c \in (c_1, c_2)$. Recall that as seen in (1) page 13, this set does not depend on c. Moreover, the function α is affine on (c_1, c_2).

From [17], we know that $\mathcal{U}_c(x) = \sup_u u(x)$, where the supremum is taken amongst (continuous) c-dominated functions $u : \mathbb{T} \to \mathbb{R}$ such that $\int u(x)d\mu(x, y) \leq 0$ for all $\mu \in \mathcal{M}_r$. Moreover, it is proven that $\int \mathcal{U}_c(x)d\mu(x, y) \leq 0$ for all $\mu \in \mathcal{M}_r$. Let now $c, c' \in (c_1, c_2)$ and $\lambda \in [0, 1]$. Let us set $v = \lambda\mathcal{U}_c + (1 - \lambda)\mathcal{U}_{c'}$.

As $\int \mathcal{U}_c(x)d\mu(x, y) \leq 0$ and $\int \mathcal{U}_c'(x)d\mu(x, y) \leq 0$ for all $\mu \in \mathcal{M}_r$ we deduce that $\int v(x)d\mu(x, y) \leq 0$ for all $\mu \in \mathcal{M}_r$.

Moreover, passing to lifts (with the same $\sim$ notation as previously), from

$$\forall \theta, \theta' \in \mathbb{R}, \quad \widetilde{\mathcal{U}}_c(\theta) - \widetilde{\mathcal{U}}_c(\theta') \leq S(\theta', \theta) + c(\theta' - \theta) + \alpha(c);$$
$$\forall \theta, \theta' \in \mathbb{R}, \quad \widetilde{\mathcal{U}}_{c'}(\theta) - \widetilde{\mathcal{U}}_{c'}(\theta') \leq S(\theta', \theta) + c'(\theta' - \theta) + \alpha(c');$$

and recalling that $\alpha\big(\lambda c + (1 - \lambda)c'\big) = \lambda\alpha(c) + (1 - \lambda)\alpha(c')$, we get

$$\forall \theta, \theta' \in \mathbb{R}, \quad \tilde{v}(\theta) - \tilde{v}(\theta') \leq S(\theta', \theta) + \big(\lambda c + (1 - \lambda)c'\big)(\theta' - \theta) + \alpha\big(\lambda c + (1 - \lambda)c'\big).$$

Hence v is $\big(\lambda c+(1-\lambda)c'\big)$-dominated. We conclude that $v \le \mathcal{U}_{\lambda c+(1-\lambda)c'}$, proving the claim, and the Proposition.

$\square$

Remark. The previous proof is intimately linked to the 1-dimensional setting we work with. Indeed, it was communicated to us by Patrick Bernard that as soon as we move up to dimension 2, there are examples on $\mathbb{T}^2$ for which it is not possible to construct a function $c \mapsto u_c$ that maps to each cohomology class a weak K.A.M. solution and such that $c \mapsto Du_c$ is continuous (in any possible way).

We obtain as a corollary:

Corollary A.1. *The function $\mathcal{U}(x,c) = \mathcal{U}_c(x) - \mathcal{U}_c(0)$ also satisfies the conclusions of Theorem 1.1.*

We now give a C^∞ integrable example for which the discounted method doesn't select a transversely continuous weak K.A.M. solution.

Example. We use the notation of Theorem 1.3. We define $F_0, H : \mathbb{A} \to \mathbb{A}$ by $F_0(\theta,r) = (\theta + r, r)$ and $H(\theta,r) = (h(\theta), \frac{r}{h'(\theta)})$ where $h : \mathbb{T} \to \mathbb{T}$ is a smooth orientation preserving diffeomorphism of $\mathbb{T}$ such that $h(t) = t + d(t)$ and $d : \mathbb{T} \to \mathbb{R}$ satisfies $d(0) = 0$ and

$$\int_{\mathbb{T}} d(t)dt > \frac{d(\frac12)}{2}. \tag{30}$$

Observe that $h^{-1}(t) = t - d\circ h^{-1}(t)$. As the symplectic diffeomorphism H maps a vertical $\{\theta\}\times\mathbb{R}$ onto a vertical $\{h(\theta)\}\times\mathbb{R}$ and preserves the transversal orientation, the smooth diffeomorphism[15] $F = H\circ F_0 \circ H^{-1}$ is also a symplectic C^∞ integrable ESTwD. The new invariant foliation is the set of the graphs of $\eta_c(\theta) = \frac{c}{h'(h^{-1}(\theta))} = c(h^{-1})'(\theta)$. Hence we have $u_c(\theta) = -cd\circ h^{-1}(\theta)$. Observe that the function u is smooth.

Then $H_c(\theta) = \theta + \frac{\partial u_c}{\partial c}(\theta) = \theta - d\circ h^{-1}(\theta) = h^{-1}(\theta)$. Hence the measure defined on $\mathbb{T}$ by $\mu([0,\theta]) = h^{-1}(\theta)$, i.e. the measure with density $\frac{1}{h'\circ h^{-1}}$, is invariant by the restricted-projected Dynamics g_c. When the rotation number $\rho(c)$ of g_c is irrational, this is the only measure invariant by g_c.

Let us recall that the discounted solution $\mathcal{U}_c$ that is selected in [38] and [17] is the weak K.A.M. solution that is the supremum of the subsolutions that satisfy for every minimizing g_c-invariant measure μ: $\int u_c d\mu \le 0$. When c is irrational, we deduce that

$$\mathcal{U}_c(\theta) = u_c(\theta) - \int u_c(t)d\mu(t) = c\left(\int_{\mathbb{T}} d\circ h^{-1}(t)(h^{-1})'(t)dt - d\circ h^{-1}(\theta)\right);$$

i.e.

$$\mathcal{U}_c(\theta) = c\left(\int_{\mathbb{T}} d(t)dt - d\circ h^{-1}(\theta)\right) = u_c(\theta) + c\int_{\mathbb{T}} d(t)dt. \tag{31}$$

Assume now that $c = \frac12$. Then

$$g_{\frac12}(0) = h\circ R_{\frac12}\circ h^{-1}(0) = h\Big(\frac12\Big) = \frac12 + d\Big(\frac12\Big) \quad\text{and}\quad g_{\frac12}\left(\frac12 + d\Big(\frac12\Big)\right) = 0.$$

[15]Note that F_0 is the time-1 map of the Hamiltonian function $f_0(\theta,r) = \frac12 r^2$. It follows that F, being conjugated to F_0 by a symplectic map, is itself the time-1 map of the Tonelli Hamiltonian $f_0\circ H^{-1}$.

The mean of the two Dirac measures
$$\nu=\frac{1}{2}\left(\delta_0+\delta_{\frac12+d(\frac12)}\right)$$
is a measure that is invariant by $g_{\frac12}$. Hence $\mathcal{U}_{\frac12}(\theta)=u_{\frac12}(\theta)-K$ with $K\geq\int_{\mathbb{T}}u_{\frac12}d\nu$. We deduce that
$$K\geq\frac12\left(u_{\frac12}(0)+u_{\frac12}\left(\frac12+d\Big(\frac12\Big)\right)\right)=-\frac14\left(d\circ h^{-1}(0)+d\circ h^{-1}\left(\frac12+d\Big(\frac12\Big)\right)\right);$$
i.e.
$$K\geq-\frac14 d\Big(\frac12\Big).$$
By Inequality (30), we know that $\varepsilon=\int_{\mathbb{T}}d(t)dt-\frac{d(\frac12)}{2}>0$. We have then
$$\mathcal{U}_{\frac12}(\theta)\leq u_{\frac12}(\theta)+\frac14 d\Big(\frac12\Big)=u_{\frac12}(\theta)+\frac12\int_{\mathbb{T}}d(t)dt-\frac{\varepsilon}{2}$$
Using Equation (31), we deduce that
$$\limsup_{c\to\frac12}\mathcal{U}_c(\theta)\geq\mathcal{U}_{\frac12}(\theta)+\frac{\varepsilon}{2}.$$
Hence $(\theta,c)\mapsto\mathcal{U}_c(\theta)$ is not continuous.
Observe that in the integrable case, there exists a unique weak K.A.M. solution in each cohomology class up to the addition of a constant. Hence selecting a weak K.A.M. solution in every cohomology class is reduced in this case to choosing a constant. Using this remark, it can be proved that for the integrable case, the discounted choice is lower semi-continuous.

A.3. **A foliation by graphs that is the inverse image of the standard foliation by a symplectic map but not by a symplectic homeomorphism.** We will use two special functions

- $\gamma:\mathbb{T}\to\mathbb{R}$ a C^∞ function such that $\gamma'_{[\frac12-\varepsilon,\frac12+\varepsilon]}=-1$ and $\gamma'_{\mathbb{T}\setminus[\frac12-\varepsilon,\frac12+\varepsilon]}>-1$;
- $\zeta:\mathbb{R}\to\mathbb{R}$ a C^∞ function that is increasing, such that $\zeta'(0)=1$ and $\zeta'_{\mathbb{R}\setminus\{0\}}<1$ with $\lim\limits_{\pm\infty}\zeta'=\dfrac12$.

The function $u(\theta,c)=\zeta(c)\gamma(\theta)$ defines the foliation in graphs of
$$\eta_c=c+\frac{\partial u}{\partial\theta}=c+\zeta(c)\gamma'.$$
The derivative with respect to c of $\eta_c(\theta)$ is then $\frac{\partial\eta_c}{\partial c}(\theta)=1+\zeta'(c)\gamma'(\theta)$ that is non negative, vanishes only for $(\theta,c)\in[\frac12-\varepsilon,\frac12+\varepsilon]\times\{0\}$ and is larger that $\frac13$ close to $\pm\infty$. Hence every map $c\in\mathbb{R}\mapsto\eta_c(\theta)\in\mathbb{R}$ is a homeomorphism and we have indeed a C^0 foliation.
Let us introduce $h_c(\theta)=\theta+\frac{\partial u}{\partial c}(\theta)=\theta+\gamma(\theta)\zeta'(c)$. Its derivative is $1+\zeta'(c)\gamma'(\theta)$ that is non negative and vanishes only if $(\theta,c)\in[\frac12-\varepsilon,\frac12+\varepsilon]\times\{0\}$. Hence h_0 is not a homeomorphism but all the other h_c are homeomorphisms.
We deduce from Theorem 1.4 that this foliation is not symplectically homeomorphic to the standard one.

We will now prove that the map defined by $H\big(\theta,\eta_c(\theta)\big)=(h_c(\theta),c)$ is a symplectic map, i.e. the limit (for the C^0 topology) of a sequence of symplectic diffeomorphisms.

Let $\gamma_n : \mathbb{T} \to \mathbb{R}$ be a sequence of C^∞ maps that converges to γ in C^1 topology and satisfies $\gamma_n' > -1$. Let (ζ_n) be a sequence of C^∞ diffeomorphisms of $\mathbb{R}$ that C^1 converges to ζ and satisfies $\zeta_n' < 1$. We introduce $u_n(\theta, c) = \gamma_n(\theta)\zeta_n(c)$. Then $\eta_c^n(\theta) = c + \zeta_n(c)\gamma_n'(\theta)$ defines a smooth foliation, $h_c^n(\theta) = \theta + \gamma_n(\theta)\zeta_n'(c)$ is a smooth diffeomorphism of $\mathbb{T}$ and

$$K_n(\theta, c) = \left(\left(h_c^n\right)^{-1}(\theta), \eta_c^n\left(\left(h_c^n\right)^{-1}(\theta)\right)\right)$$

is a symplectic smooth diffeomorphism that maps the standard foliation to the foliations by the graphs of $(\eta_c^n)_{c\in\mathbb{R}}$.
If $H_n = K_n^{-1}$, observe that $H_n = G_n \circ F_n^{-1}$ where

- $F_n(\theta, c) = \left(\theta, c+\frac{\partial u_n}{\partial \theta}(\theta, c)\right)$ converges uniformly to $F(\theta, c) = \left(\theta, c+\frac{\partial u}{\partial \theta}(\theta, c)\right)$;
- $G_n(\theta, c) = (\theta+\frac{\partial u_n}{\partial c}(\theta, c), c)$ converges uniformly to $G(\theta, c) = (\theta+\frac{\partial u}{\partial c}(\theta, c), c)$.

Finally, $H_n = G_n \circ F_n^{-1}$ converges uniformly to $H = G \circ F^{-1}$

Appendix B. Some results concerning the full pseudographs

Most of the results that follow are standard and even hold in all dimension. One can find them in similar of different formulations in [12]. However, we provide proofs for the reader's convenience.

B.1. An equivalent definition.

Definition. Let $u : \mathbb{R} \to \mathbb{R}$ be a K semi-concave function. Then $p \in \mathbb{R}$ is a *super-derivative* of u at $x \in \mathbb{R}$ if

$$\forall y \in \mathbb{R}, \quad u(y) - u(x) - p(y - x) \leq \frac{K}{2}(y - x)^2.$$

We denote the set of super-derivatives of u at x by $\partial^+ u(x)$. It is a convex set.

Observe that a derivative is always a super-derivative. If $u : \mathbb{R} \to \mathbb{R}$ is K-semi-concave, then $x \mapsto u(x) - \frac{K}{2}x^2$ is concave and thus locally Lipschitz, and $x \mapsto u'(x) - Kx$ is non-increasing. Hence a 1-periodic K-semi-concave function is K-Lipschitz.

Observe also that $\bigcup\limits_{x\in\mathbb{T}} \{x\} \times \partial^+ u(x)$ is compact.

Proposition B.1. *Let $u : \mathbb{R} \to \mathbb{R}$ be a K-semi-concave function. Then, for every $x \in \mathbb{R}$, we have*

$$\partial u(x) = \{x\} \times \partial^+ u(x).$$

Hence the full pseudograph of u is also the subbundle of all the super-derivatives of u.

Proof. Let us prove the inclusion $\partial u(x) \subset \{x\} \times \partial^+ u(x)$. Let us consider $(x, p) \in \partial u(x)$. Then there exist $(x, p_-), (x, p_+) \in \overline{\mathcal{G}(u')}$ such that $p_- \leq p \leq p_+$ and there exist two sequences $(x_n, p_n), (y_n, q_n) \in \mathcal{G}(u')$ that respectively converge to (x, p_-), (x, p_+). Every derivative is a super-derivative and a limit of super-derivatives is a super-derivative. Hence, we have $p_-, p_+ \in \partial^+ u(x)$. By convexity of $\partial^+ u(x)$, we deduce that $p \in \partial^+ u(x)$.

Let us now prove the reverse inclusion. Being K-semi-concave, u is K-Lipschitz, hence the set of all its super-derivatives is bounded (by K). If $x \in \mathbb{R}$, we have then

$\partial^+ u(x) = [p_-, p_+]$ with $-K \le p_- \le p_+ \le K$. We will prove that $(x, p_-), (x, p_+) \in \partial u(x)$. We have

$$\forall y \in \mathbb{R}, \quad u(y) - u(x) - p_-(y-x) \le \frac{K}{2}(y-x)^2$$
$$\text{and} \quad u(y) - u(x) - p_+(y-x) \le \frac{K}{2}(y-x)^2.$$

This implies that

- for $y > x$, we have
$$\frac{u(y)-u(x)}{y-x} \le p_- + \frac{K}{2}(y-x);$$
- for $y < x$, we have
$$\frac{u(y)-u(x)}{y-x} \ge p_+ + \frac{K}{2}(y-x).$$

Recall that $\frac{u(y)-u(x)}{y-x} = \frac{1}{y-x}\int_x^y u'(t)dt$. This gives the existence of two sequences $(x_n) \in (-\infty, x)$ and $(y_n) \in (x, +\infty)$ that converge to x where u is differentiable and

$$\limsup u'(x_n) \ge p_+ \quad \text{and} \quad \liminf u'(y_n) \le p_-.$$

As we know that a derivative is a super-derivative, that the set of super-derivatives is closed and that $\partial^+ u(x) = [p_-, p_+]$, we deduce that

$$\big(x, \lim u'(x_n)\big) = (x, p_+) \in \partial u(x) \quad \text{and} \quad \big(x, \lim u'(y_n)\big) = (x, p_-) \in \partial u(x).$$

□

B.2. **Proof of Lemma 2.3.** We just recall the argument of the proof of

Lemma B.1. *For all $c \in \mathbb{R}$, $\mathcal{PG}(c + u'_c)$ is a Lipschitz one dimensional compact manifold that is an essential circle.*

Proof. It is proved in [3], that for every $c \in \mathbb{R}$ and every K-semi-concave function $u : \mathbb{T} \to \mathbb{R}$, there exists $\tau > 0$ such that $\varphi_{-\tau}\big(\mathcal{PG}(c+u')\big)$ is the graph of a Lipschitz function, where (φ_t) is the flow of the pendulum. This gives the wanted result. □

B.3. **Proof of Proposition 2.2.** Let us now prove the following proposition[16].

Proposition B.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of equi-semi-concave functions from $\mathbb{T}$ to $\mathbb{R}$ that converges (uniformly) to a function f (that is hence also semi-concave).*

Then $\big(\mathcal{PG}(f'_n)\big)$ converges to $\mathcal{PG}(f')$ for the Hausdorff distance.

Proof. Let us prove that the lim sup of the $\mathcal{PG}(f'_n)$ is in $\mathcal{PG}(f')$. Up to a subsequence, we consider $(x_n, p_n) \in \mathcal{PG}(f'_n)$ that converges to some (x, p), and we want to prove that $(x, p) \in \mathcal{PG}(f')$. We have

$$\forall n, \forall y \in \mathbb{R}, \quad f_n(y) - f_n(x_n) - p_n(y - x_n) \le \frac{K}{2}(y - x_n)^2.$$

Taking the limit, we deduce that $(x, p) \in \mathcal{PG}(f')$.

[16]The statement holds in arbitrary dimension and follows from the same result for concave functions. We present here a simple proof relying on the 1-dimensional setting.

Let us now assume that $\big(\mathcal{PG}(f'_n)\big)$ doesn't converge to $\mathcal{PG}(f')$. There exists a point $(x,p)\in\mathcal{PG}(f')$, $r>0$ and $N\geq 1$ such that, up to a subsequence,

$$\forall n\geq N,\quad \mathcal{PG}(f'_n)\cap B\big((x,p),r\big)=\varnothing.$$

Hence, for n large enough, $\mathcal{PG}(f'_n)$ is contained in a small neighbourhood of a simple arc (and not loop). This implies that for n large enough, $\mathcal{PG}(f'_n)$ doesn't separate the annulus into two unbounded connected components, a contradiction.

□

Appendix C. Green bundles

Here we recall the theory of Green bundles. More details or proofs can be found in [4, 2]. We fix a lift F of an ESTwD f.

Notations.

- $V(x)=\{0\}\times\mathbb{R}\subset T_x\mathbb{R}^2$ and for $k\neq 0$, we have $G_k(x)=DF^k(F^{-k}x)V(f^{-k}x)$;
- the slope of G_k (when defined) is denoted by s_k:

$$G_k(x)=\{(\delta\theta, s_k(x)\delta\theta);\quad \delta\theta\in\mathbb{R}\};$$

- if γ is a real Lipschitz function defined on $\mathbb{T}$ or $\mathbb{R}$, then

$$\gamma'_+(x)=\limsup_{\substack{y,z\to x\\ y\neq z}}\frac{\gamma(y)-\gamma(z)}{y-z}\quad\text{and}\quad \gamma'_-(t)=\liminf_{\substack{y,z\to x\\ y\neq z}}\frac{\gamma(y)-\gamma(z)}{y-z}.$$

Then

(1) if the orbit of $x\in\mathbb{R}^2$ is minimizing, we have

$$\forall n\geq 1,\quad s_{-n}(x)<s_{-n-1}(x)<s_{n+1}(x)<s_n(x);$$

(2) in this case, the two *Green bundles* at x are $G_+(x), G_-(x)\subset T_x(\mathbb{R}^2)$ with slopes s_-, s_+ where $s_+(x)=\lim\limits_{n\to+\infty}s_n(x)$ and $s_-(x)=\lim\limits_{n\to+\infty}s_{-n}(x)$;
(3) the two Green bundles are invariant under Df: $Df(G_\pm)=G_\pm\circ f$;
(4) we have $s_+\geq s_-$;
(5) the map s_- is lower semi-continuous and the map s_+ is upper semi-continuous;
(6) hence $\{G_-=G_+\}$ is a G_δ subset of the set of points whose orbit is minimizing (this last set is a closed set) and $s_-=s_+$ is continuous at every point of this set.

Let us focus on the case of an invariant curve that is the graph of γ. Then we have

Proposition C.1. *Assume that the graph of $\gamma\in C^0(\mathbb{T},\mathbb{R})$ is invariant by F. Then the orbit of any point contained in the graph of γ is minimizing and we have*

$$\forall\theta\in\mathbb{T},\quad s_-\big(\theta,\gamma(\theta)\big)\leq\gamma'_-(\theta)\leq\gamma'_+(\theta)\leq s_+\big(\theta,\gamma(\theta)\big).$$

Proposition C.2. **(Dynamical criterion)** *Assume that x has its orbit that is minimizing and that is contained in some strip $\mathbb{R}\times[-K,K]$ (for example x is in some invariant graph) and that $v\in T_x\mathbb{R}^2\backslash\{0\}$. Then*

- *if $\liminf\limits_{n\to+\infty}|D(\pi\circ F^n)(x)v|<+\infty$, then $v\in G_-(x)$;*
- *if $\liminf\limits_{n\to+\infty}|D(\pi\circ F^{-n})(x)v|<+\infty$, then $v\in G_+(x)$.*

In particular, if the Dynamics restricted to some invariant graph is totally periodic, then along this graph we have $G_-=G_+$ and the graph is C^1. The C^1 property can also be proved by using the implicit functions theorem.

Appendix D. Sketch of the proof of point 3 page 13

We wish to explain why if $u:\mathcal{M}\big(\rho(c)\big)\to\mathbb{R}$ is dominated, then there exists only one extension U of u to $\mathbb{T}$ that is a weak K.A.M. solution for $\widehat{T}^c$ that is given by

$$\forall x\in\mathbb{T},\quad U(x)=\inf_{\substack{\pi(\theta)\in\mathcal{M}(\rho(c))\\ \pi(\theta')=x}}\tilde{u}(\theta)+\mathcal{S}^c(\theta,\theta')$$

where $\mathcal{S}^c(\theta,\Theta)=\inf\limits_{n\in\mathbb{N}}\big(\mathcal{S}^c_n(\theta,\Theta)+n\alpha(c)\big)$.

- It is a general fact that if $\pi(\theta)\in\mathcal{M}\big(\rho(c)\big)$ the function $\theta'\mapsto\mathcal{S}^c(\theta,\theta')$ is a weak K.A.M solution that vanishes at $\theta'=\theta$ (see [40, Definition 2.1 and Proposition 2.8] recalling that the function $\mathcal{S}^c$ corresponds to the lift of the Mañé potential φ in the reference and that our Mather set $\mathcal{M}\big(\rho(c)\big)$ is included in the Aubry set). As the set of weak K.A.M. is invariant by addition of constants and an infimum of weak K.A.M. solutions is a weak K.A.M. solution ([40, Lemma 2.33]) it follows that U is a weak K.A.M. solution.
- To prove that $U=u$ on $\mathcal{M}\big(\rho(c)\big)$ just notice that as u is dominated, if $x\in\mathcal{M}\big(\rho(c)\big)$ and $\pi(\theta)=x$

  $$\forall\theta'\in\pi^{-1}\left(\mathcal{M}\big(\rho(c)\big)\right),\quad \tilde{u}(\theta')+\mathcal{S}^c(\theta',\theta)\geq\tilde{u}(\theta)=u(x)+\mathcal{S}^c(\theta,\theta).$$
- It remains to prove that U is unique. This follows from the fact that if two weak K.A.M. solutions U_1 and U_2 coincide on $\mathcal{M}\big(\rho(c)\big)$ they are equal.

  Let $x_0\in\mathbb{T}$. One constructs inductively a sequence $(x_n)_{n\leq 0}$ such that

  $$\forall n<0,\quad U_1(x_0)=U_1(x_n)+\sum_{k=n}^{-1}S^c(x_k,x_{k+1}).$$

  As U_2 is a weak K.A.M. (hence dominated) one also has

  $$\forall n<0,\quad U_2(x_0)\leq U_2(x_n)+\sum_{k=n}^{-1}S^c(x_k,x_{k+1}).$$

  Hence $U_2(x_0)-U_1(x_0)\leq U_2(x_n)-U_1(x_n)$. To conclude, one proves, using a Krylov-Bogoliubov type argument that there exists a subsequence $(x_{\varphi(n)})$ that converges to a point $x\in\mathcal{M}\big(\rho(c)\big)$, hence proving that $U_2(x_0)-U_1(x_0)\leq 0$. Then the result follows by a symmetrical argument.

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Email address: `Marie-Claude.Arnaud@univ-avignon.fr, maxime.zavidovique@upmc.fr`