**Workshop: Dynamische Systeme**

**Table of Contents**

# Abstracts

## Degenerate discounted Hamilton-Jacobi equations

Maxime Zavidovique
(joint work with Q. Chen, A. Fathi and J. Zhang)

If $M$ is a compact, connected smooth manifold without boundary, we consider a Hamiltonian $H : T^*M \to \mathbb{R}$ that is continuous and verifies the following properties:

- convexity: for all $x \in M$, the function $p \mapsto H(x,p)$ is convex,
- superlinearity: the limit $\lim_{\|p\|_x \to +\infty} H(x,p)/\|p\|_x = +\infty$ holds.

The superlinearity condition is stated with the use of an auxiliary riemannian metric, but the property is independent on the metric.

Given this Hamiltonian comes the Lagrangian function $L : TM \to \mathbb{R}$ defined by

$$\forall (x,v) \in TM, \quad L(x,v) = \max_{p \in T_x^*M} p(v) - H(x,p).$$

It is also convex and superlinear in $v$.

### 1. Non-degenerate Hamilton-Jacobi equations

In [1] we obtained the following result:

**Theorem 1.** *For all $\lambda > 0$, there exists a unique viscosity solution $u_\lambda : M \to \mathbb{R}$ to the discounted Hamilton-Jacobi equation;*

$$\lambda u_\lambda(x) + H(x, D_x u_\lambda) = 0. \tag{1}$$

*Moreover, there is a unique constant $c_0$ and function $u_0 : M \to \mathbb{R}$ such that $u_\lambda + c_0/\lambda$ uniformly converges to $u_0$ as $\lambda \to 0$. The function $u_0$ is a weak KAM solution, that is a viscosity solution of $H(x, d_x u_0) = c_0$.*

- The real novelty in the previous result is the convergence one. The rest was known since the 80's and the convergence was known to hold, up to subsequences. Actually, Lions, Papanicolaou and Varadhan introduced this vanishing discount method to prove the existence of weak KAM solutions.
- The set of weak KAM solutions is never reduced to a single function. For example one easily checks that this set is invariant by addition of constant functions.
- All the functions at stake here are automatically Lipschitz, hence differentiable almost everywhere.
- The proof heavily relies on Mather minimizing measures, that are Borel probability measures $\mu$ on $TM$ that are
  (1) closed: for all $f \in C^1(M,\mathbb{R})$, $\int_{TM} D_x f(v) d\mu = 0$,
  (2) minimizing: $\int_{TM} L(x,v) d\mu = -c_0$.
  The limit function $u_0$ is actually expressed in terms of those measures.
- When the Hamiltonian $H$ is Tonelli (smooth and strictly convex in the $C^2$ sense), then all the above objects have dynamical meanings.

  - The Mather measures are invariant by the Euler-Lagrange flow of $L$.
  - If $\lambda > 0$, setting $\mathcal{G}(du_\lambda) = \{(x, D_x u_\lambda), x \in \mathcal{D}(Du_\lambda)\}$ where $\mathcal{D}(Du_\lambda)$ is set of differentiability points of $u_\lambda$, then for $t > 0$ the inclusion $\varphi_{H,\lambda}^{-t}\left(\overline{\mathcal{G}(du_\lambda)}\right) \subset \mathcal{G}(du_\lambda)$ holds, where $\varphi_{H,\lambda}$ is the conformally symplectic flow generated by the equations
$$\begin{cases} \dot{x} = \partial_p H(x,p), \\ \dot{p} = -\partial_x H(x,p) - \lambda p. \end{cases}$$
  - Similarly, for $t > 0$ it holds $\varphi_H^{-t}\left(\overline{\mathcal{G}(du_0)}\right) \subset \mathcal{G}(du_0)$ where $\varphi_H$ is the Hamiltonian flow associated to $H$.
- The functions $u_\lambda$ are given by the following formula (that can be taken as their definition in the present case)
$$\forall x \in M, \quad u_\lambda(x) = \min_{\gamma(0)=x} \int_{-\infty}^{0} e^{\lambda s} L\big(\gamma(s), \dot{\gamma}(s)\big) ds,$$
where the infimum is taken amongst the absolutely continuous curves $\gamma : (-\infty, 0] \to M$ such that $\gamma(0) = x$.
- the function $u_0$ verifies a similar property (that characterizes weak KAM solutions):
$$\forall x \in M, \quad \forall t > 0, \quad u_0(x) = \min_{\gamma(0)=x} u_0\big(\gamma(-t)\big) + \int_{-t}^{0} L\big(\gamma(s), \dot{\gamma}(s)\big) ds + tc_0.$$

## 2. Degenerate Hamilton-Jacobi equations

Having in mind the previous results, one may ask what other kind of perturbations of the stationary Hamilton-Jacobi equation (defining weak KAM solutions) select a unique weak KAM solution. It can be seen from the theory of viscosity solutions that it is important to have an equation with a non-decreasing dependance on the value of the unknown $u_\lambda(x)$. Therefore we will focus here on equations of the form

$$\lambda\alpha(x)u_\lambda(x) + H(x, D_x u_\lambda) = c_0, \tag{2}$$

where $\alpha : M \to [0, +\infty)$ is a given continuous function. If $\alpha$ is identically 0 then there is no perturbation and no reasonable result can be expected. On the countrary, if $\alpha > 0$ everywhere, then, dividing by $\alpha$ one reduces to the results of the previous section. Hence one needs to find an appropriate intermediate condition. We introduce the following:
Non-degeneracy condition: for all Mather measures $\mu$, one has $\int_{TM} \alpha(x) d\mu > 0$.

Note that a Theorem of Mañé asserts that for a generic $H$, there exists a unique Mather measure. Hence for most Hamiltonians, the above condition allows $\alpha$ to vanish on very large sets, hence the equations to be rather degenerate.

Building on the results of [2] and [3], we prove in [4] a generalization of the following:

**Theorem 2.** *Assume $\alpha : M \to [0, +\infty)$ verifies the non-degeneracy condition and $H$ is convex and superlinear as before. For all $\lambda > 0$, there exists a unique viscosity solution $\tilde{u}_\lambda : M \to \mathbb{R}$ to the degenerate discounted Hamilton-Jacobi equation;*

$$\lambda\alpha(x)\tilde{u}_\lambda(x) + H(x, D_x\tilde{u}_\lambda) = c_0. \tag{3}$$

*Moreover, there is a unique constant $c_0$ and function $\tilde{u}_0 : M \to \mathbb{R}$ such that $\tilde{u}_\lambda + c_0/\lambda$ uniformly converges to $\tilde{u}_0$ as $\lambda \to 0$. The function $\tilde{u}_0$ is a weak KAM solution, that is a viscosity solution of $H(x, d_x\tilde{u}_0) = c_0$.*

Most of the previous Theorem in new, including uniqueness of the $\tilde{u}_\lambda$ that requires new, dynamically inspired, methods. The functions $\tilde{u}_\lambda$ no longer verify a nice explicit representation formula as before. However one recovers (with Gronwall's inequality) properties closer to that of weak KAM solutions: for all $x \in M$ and $t > 0$,

$$\tilde{u}_\lambda(x) = \min_{\gamma(0)=x} e^{A_\gamma(-t)}\tilde{u}_\lambda\big(\gamma(-t)\big) + \int_{-t}^{0} e^{A_\gamma(s)}\left[L\big(\gamma(s), \dot{\gamma}(s)\big) + c_0\right] ds,$$

where $A_\gamma(s) = \int_0^s \alpha \circ \gamma(\sigma) d\sigma$. It can be guessed from the above formula that a crucial point will be to ensure that minimizing curves $\gamma$ spend enough time in the regions where $\alpha > 0$ to ensure that $A_\gamma(s)$ goes to $-\infty$ as $s \to -\infty$.

## References


[1] A. Davini, A. Fathi, R. Iturriaga, M. Zavidovique, *Convergence of the solutions of the discounted Hamilton-Jacobi equation*, Invent. Math. **206** (2016), no.1, 2955.

[2] M. Zavidovique, *Convergence of solutions for some degenerate discounted HamiltonJacobi equations*, Analysis & PDE **15** (2022), no. 5, 12871311.

[3] Q. Chen, *Convergence of solutions of HamiltonJacobi equations depending nonlinearly on the unknown function*, Advances in Calculus of Variations **16**, no. 1, (2023), 45-68.

[4] Q. Chen, A. Fathi, M. Zavidovique, J. Zhang, *Convergence of the solutions of the nonlinear discounted Hamilton-Jacobi equation: The central role of Mather measures*, preprint.


*Reporter: Yi Pan*