

Tonelli Hamiltonians without conjugate points and C^0 integrability

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Abstract

We prove that all the Tonelli Hamiltonians defined on the cotangent bundle $T^*\mathbb{T}^n$ of the n -dimensional torus that have no conjugate points are C^0 integrable, i.e. $T^*\mathbb{T}^n$ is C^0 foliated by a family \mathcal{F} of invariant C^0 Lagrangian graphs.

Assuming that the Hamiltonian is C^∞ , we prove that there exists a G_δ subset \mathcal{G} of \mathcal{F} such that the dynamics restricted to every element of \mathcal{G} is strictly ergodic.

Moreover, we prove that the Lyapunov exponents of every C^0 integrable Tonelli Hamiltonian are zero and deduce that the metric and topological entropies vanish.

Key words: Hamiltonian systems. Complete integrability. KAM theorems. Entropy. Weak KAM theory.

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Introduction

This article deals with C^0 integrable Tonelli Hamiltonians¹ and Tonelli Hamiltonians without conjugate points of the cotangent bundle $T^*\mathbb{T}^n$ of the n -dimensional torus.

If the Tonelli Hamiltonian is a Riemannian metric, these properties coincide and have strong implications. Indeed, Heber showed (see [Heb94]) in 1994 that for every Riemannian metric without conjugate points on the torus \mathbb{T}^n , there is a continuous foliation of the unit tangent bundle by tori which are Lipschitz, Lagrangian and invariant by the geodesic flow. The same year, this was improved by Burago and Ivanov who proved (see [BI94]) that such a metric has to be flat; as an immediate consequence, the continuous foliation in Heber's result is in fact smooth.

The notion of Tonelli Lagrangian is a vast extension of the concept of Riemannian metric, but we shall prove in the first section that Heber's result still holds:

Theorem 1. *Let H be a Tonelli Hamiltonian on $T^*\mathbb{T}^n$. Then H has no conjugate points if and only if there is a continuous foliation of $T^*\mathbb{T}^n$ by Lipschitz, Lagrangian, flow-invariant graphs.*

The proof uses ideas coming from weak KAM and Aubry-Mather theory. In fact, we establish that each leaf of the above foliation is the dual Aubry set corresponding

¹All the notions will be defined at the end of this introduction

to some cohomology class. More precisely, the first step of the proof is to see that some of those Aubry sets (later denoted by $\mathcal{G}_{T,r}^*$, with $T > 0$ and $r \in \mathbb{Z}^n$) are covered by periodic orbits of the Hamiltonian flow $(\phi_t^H)_{t \in \mathbb{R}}$, of a given period T (and a given homology class r). In particular, the dynamics on the corresponding leaves is periodic.

The existence of those particular leaves is used again in the second section. Using a KAM theorem, we prove that such sets $\mathcal{G}_{T,r}^*$ are accumulated by KAM tori on which the dynamics is conjugated to an irrational rotation:

Theorem 2. *Assume that H is C^∞ and that $\bar{\omega}$ is strongly Diophantine². There is $m_0 \in \mathbb{N} \setminus \{0\}$ (depending on $\bar{\omega}$) such that, for all $m \geq m_0$, there is a C^∞ Lagrangian embedding $i_m : \mathbb{T}^n \rightarrow T^*\mathbb{T}^n$ such that*

i) $\forall \eta \in \mathbb{T}^n, \quad \phi_{mT}^H(i_m(\eta)) = i_m(\eta + \bar{\omega}).$

ii) *Writing $i_m(\eta) = (\psi_m(\eta), f_m(\eta))$, ψ_m is a C^∞ diffeomorphism of \mathbb{T}^n , isotopic to $id_{\mathbb{T}^n}$ and*

$$\mathcal{T}_m = i_m(\mathbb{T}^n) = \{(\theta, (f_m \circ \psi_m^{-1})(\theta)); \theta \in \mathbb{T}^n\}.$$

is a Lagrangian graph; the sequence (\mathcal{T}_m) converges to $\mathcal{T}_\infty := \mathcal{G}_{T,r}$ in C^∞ topology.

iii) *The sequence (ψ_m) converges in C^∞ topology to a diffeomorphism ψ_∞ of \mathbb{T}^n (independent of $\bar{\omega}$), isotopic to $id_{\mathbb{T}^n}$.*

iv) *The tori \mathcal{T}_m are flow-invariant. More precisely, $i_m^*(X_H) = \frac{r}{T} + \frac{\bar{\omega}}{mT}$, so that*

$$\forall m \geq m_0, \forall t \in \mathbb{R}, \forall \eta \in \mathbb{T}^n, \quad \phi_t^H(i_m(\eta)) = i_m\left(\eta + \frac{t}{T}r + \frac{t}{mT}\bar{\omega}\right).$$

A corollary of this theorem is that the dynamics of the Hamiltonian flow restricted to $\mathcal{G}_{T,r}^*$ is itself conjugated to a rational rotation on \mathbb{T}^n . An important ingredient in the proof of this theorem is to provide a normal form for the flow of H in the neighborhood of $\mathcal{G}_{T,r}^*$, linking it to the geodesic flow of a flat metric on the torus. This is done in proposition 6, with the use of the theorem of Burago and Ivanov, and is of independent interest.

Another consequence of this theorem is that we can deduce some information on the dynamics restricted to a lot of invariant tori. More precisely, let us recall that an invariant set is *strictly ergodic* if there is only one Borel invariant probability measure the support of whose is in this set, and if the support of this measure is the whole set.

Theorem 3. *Let (ϕ_t^H) be a C^∞ Tonelli flow of $T^*\mathbb{T}^n$ with no conjugate points and let \mathfrak{F} be the continuous foliation in invariant Lagrangian tori that is given by theorem 1. Then there is a dense G_δ subset \mathcal{G} of \mathfrak{F} such that, for every $\mathcal{T} \in \mathcal{G}$, then $\phi_{1|\mathcal{T}}^H$ is strictly ergodic.*

²This notion is defined in subsection 2.1

The last section of this article is devoted to studying the entropy of Tonelli Hamiltonians without conjugate points. Indeed, it is not hard to see that a regular completely integrable Hamiltonian system has zero topological entropy. When singularities are allowed, the situation can become more complicated, as shown in the article [BT00] of Bolsinov and Taimanov.

Hence it seems natural to ask what can happen for a C^0 integrable Tonelli Hamiltonian. In this case, we don't know the restricted dynamics to all the invariant tori, hence it is not so obvious to decide if the topological entropy is zero or not. An answer to this question is provided by the following:

Theorem 4. *Let $H : T^*\mathbb{T}^n \rightarrow \mathbb{R}$ be a C^3 Tonelli Hamiltonian that is C^0 integrable. Then for every invariant Borel probability measure, the Lyapunov exponents are zero.*

This implies that both the metric, and topological entropies must also be 0. Observe that the conclusion of theorem 4 is true for a C^0 integrable Tonelli Hamiltonian defined on T^*M for any closed manifold M . We give the statement for $T^*\mathbb{T}^n$ because we define Tonelli Hamiltonians just in this case, but the interested reader can find a definition of Tonelli Hamiltonians on any manifold in [Fat08].

Some interesting questions remain open after this work, as:

Question 1. *Does a C^0 integrable Tonelli Hamiltonian exist that is not C^1 integrable?*

Question 2. *Can an invariant torus of a C^0 integrable Tonelli Hamiltonian flow carry two invariant measures that have not the same rotation number (see the appendix for the definition of the rotation number)?*

Notations and definitions

Tonelli Lagrangians and Hamiltonians

Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ denote the n -dimensional torus, with

$$\text{pr} : \mathbb{R}^n \longrightarrow \mathbb{T}^n, \quad \pi : (x, v) \in T\mathbb{T}^n \longmapsto x \in \mathbb{T}^n, \quad \text{or} \quad \pi : (x, p) \in T^*\mathbb{T}^n \longmapsto x \in \mathbb{T}^n$$

the canonical projections (or their lifts to $T\mathbb{R}^n$ or $T^*\mathbb{R}^n$ when no confusion is possible). For every $(x, v) \in T\mathbb{T}^n$, the vertical space at this point is

$$V(x, v) = \text{Ker}(D\pi|_{T_{(x,v)}T\mathbb{T}^n}),$$

and for every $(x, p) \in T^*\mathbb{T}^n$, the vertical space at this point is $V^*(x, p) = \text{Ker}(D\pi|_{T_{(x,p)}T^*\mathbb{T}^n})$.

A function $L : T\mathbb{T}^n \rightarrow \mathbb{R}$ is named a Tonelli Lagrangian if it verifies the following three conditions:

- it is of regularity at least C^2 ,

- it is super-linear: $\lim_{|v| \rightarrow \infty} \frac{L(x,v)}{|v|} = +\infty$,
- it is strictly convex in the fibers in the sense that $\partial_v^2 L$ is everywhere positive definite as a quadratic form.

Let L be a (C^k , $k \geq 2$) Tonelli Lagrangian on $T\mathbb{T}^n$, and \tilde{L} its ($\mathbb{Z}^n \times \{0\}$ -periodic) lift to $T\mathbb{R}^n$. Its associated Hamiltonian is defined by

$$\forall (x, p) \in T^*\mathbb{T}^n, \quad H(x, p) = \sup_{v \in T_x\mathbb{T}^n} (p \cdot v - L(x, v)).$$

It is also Tonelli and its lift to $T^*\mathbb{R}^n$ is \tilde{H} , the Hamiltonian associated to \tilde{L} . The Lagrangian and Hamiltonian are linked by the Legendre transform:

$$\mathcal{L} : T\mathbb{T}^n \rightarrow T^*\mathbb{T}^n, \quad (x, v) \mapsto (x, \partial_v L(x, v)),$$

as follows:

$$\forall (x, v) \in T\mathbb{T}^n, \quad H \circ \mathcal{L}(x, v) + L(x, v) = \partial_v L(x, v) \cdot v.$$

Note that the Legendre transform is a global C^{k-1} diffeomorphism, but H is of class C^k as L .

The Hamiltonian flow is generated by the vector-field on $T^*\mathbb{T}^n$:

$$(x, p) \mapsto X_H(x, p) = (\partial_p H, -\partial_x H).$$

The flow it generates is denoted $(\phi_t^H)_{t \in \mathbb{R}}$ and it is complete and C^{k-1} . We shall denote by $\varphi_t : T\mathbb{T}^n \rightarrow T\mathbb{T}^n$ the Euler-Lagrange flow of L , which is conjugated to ϕ_t^H by \mathcal{L} . Similarly $\tilde{\varphi}_t$ (resp. $\tilde{\phi}_t^H$) will be the flow of \tilde{L} (resp. \tilde{H}). Recall that H (or equivalently L) is *without conjugate points* if one has

$$\forall (x, v) \in T\mathbb{T}^n, \quad \forall t \in \mathbb{R}^*, \quad V(\varphi_t(x, v)) \cap D\varphi_t(x, v) \cdot V(x, v) = \{0\},$$

or equivalently

$$\forall (x, p) \in T^*\mathbb{T}^n, \quad \forall t \in \mathbb{R}^*, \quad V^*(\phi_t^H(x, p)) \cap D\phi_t^H(x, p) \cdot V^*(x, p) = \{0\}.$$

Extremal curves, Tonelli theorem

We fix a Tonelli Lagrangian L . Let us recall some classical results (a good reference is [Fat08]).

If $\gamma : [a, b] \rightarrow \mathbb{T}^n$ is an absolutely continuous curve, its Lagrangian action is defined by:

$$A_L(\gamma) = \int_a^b L(\gamma(t), \gamma'(t)) dt.$$

A C^2 curve $\gamma : [a, b] \rightarrow \mathbb{T}^n$ is an *extremal curve* for L if for each C^2 variation $\Gamma : [a, b] \times]-\varepsilon, \varepsilon[\rightarrow \mathbb{T}^n$ of γ with $\Gamma(t, s) = \gamma(t)$ in a neighborhood of $(a, 0)$ and $(b, 0)$, we have: $\frac{d}{ds} A_L(\Gamma_s)_{s=0} = 0$.

A curve γ is an extremal curve for L if and only if (γ, γ') is a solution of the Euler-Lagrange equation

$$-\frac{d}{dt} \left(D_v L(\gamma(t), \gamma'(t)) \right) + D_x L(\gamma(t), \gamma'(t)) = 0.$$

Theorem (Tonelli). *Let $L : T\mathbb{T}^n \rightarrow \mathbb{R}$ be a Tonelli Lagrangian. If $C \in \mathbb{R}$, the subset*

$$\Sigma_C = \{ \gamma \in C^{ac}([a, b], \mathbb{T}^n); A_L(\gamma) \leq C \}$$

is a compact subset of the set $C^{ac}([a, b], \mathbb{T}^n)$ of absolutely continuous curves endowed with the topology of uniform convergence.

If $x, y \in \mathbb{T}^n$, there exists a curve $\gamma : [a, b] \rightarrow \mathbb{T}^n$ that minimizes the Lagrangian action among all the absolutely continuous curves joining x to y . Every such curve is then an extremal curve for L .

If $x \in \mathbb{T}^n$ and if $h \in \mathbb{Z}^n$, there exist a loop $\gamma : [a, b] \rightarrow \mathbb{T}^n$ that minimizes the Lagrangian action among all the absolutely curves joining x to x the homotopy class of whose is h . Every such curve is then an extremal curve for L .

A curve $\gamma : [a, b] \rightarrow \mathbb{T}^n$ satisfying the conclusion of the theorem is called a *minimizer*. There is a similar statement (existence of a minimizer with fixed ends) for the lift \tilde{L} .

Palais-Smale condition, coercivity

A good reference for these notions is [Str08].

We consider a C^2 function $E : \mathcal{H} \rightarrow \mathbb{R}$ defined on a Hilbert space \mathcal{H} .

The function E is *coercive* if $\lim_{\|u\| \rightarrow +\infty} E(u) = +\infty$.

The function E satisfies the *Palais-Smale condition* if every sequence $(u_m) \in \mathcal{H}$ that is such that $\lim_{m \rightarrow \infty} \|DE(u_m)\| = 0$ and $(E(u_m))$ is bounded has a subsequence that has a limit.

Mañé, Mather and Fathi theory, Green bundles

See the appendix.

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1 On C^0 integrability

The main theorem of this section is the following:

Theorem 1. *Let L be a Tonelli Lagrangian on $T\mathbb{T}^n$. Then L has no conjugate points if and only if there is a continuous foliation of $T^*\mathbb{T}^n$ by Lipschitz, Lagrangian, flow-invariant graphs.*

Firstly, observe that it is proved in [Arn11] that if there is a continuous foliation of $T^*\mathbb{T}^n$ by Lipschitz, Lagrangian, flow-invariant graphs, then L has no conjugate points. We just have to prove the converse implication.

The proof is not a mere rewriting of Heber's arguments. It is achieved in two steps, each using arguments of very different nature. We first establish (in section 1.2) that it is possible to cover the torus by periodic extremal curves the period and homotopy class of whose may be fixed arbitrarily. To do this, we adapt a technique of metric geometry first used by Busemann (see [Bus55], section 32) while he was investigating G-spaces without conjugated points. The next step (in section 1.3) is to show that it is possible to associate to every $c \in H^1(\mathbb{T}^n, \mathbb{R})$ a Lipschitz graph which is Lagrangian, flow-invariant, and formed by orbits which minimize the action for the Lagrangian $L - \lambda$, where λ is a closed 1-form the cohomology class of whose is c . All this part is inspired by the methods developed in [Arn11], and uses results of weak KAM theory (see [Fat08]). To conclude the proof, we use a topological argument to verify that the union of the previously constructed graphs is the whole of the cotangent space.

1.1 Some properties satisfied by Lagrangians without conjugate points

In this section, we establish some technical results on Lagrangians without conjugate points. They will be used repeatedly in the sequel.

Let $t > 0$ and $x \in \mathbb{R}^n$. Consider the following map, which is reminiscent of the exponential map in Riemannian geometry:

$$F : v \in T_x\mathbb{R}^n \longmapsto \pi \circ \tilde{\varphi}_t(x, v) \in \mathbb{R}^n.$$

Proposition 1. *The map F is injective.*

Before proving this, let us establish some consequences:

Corollary 1.1. *The map F is a C^{k-1} diffeomorphism.*

Proof. This application is of class C^{k-1} . It is injective by proposition 1, and surjective because of the Tonelli theorem. So we only need to check that F is a local diffeomorphism. Let $i : v \in T_x\mathbb{R}^n \mapsto (x, v) \in T\mathbb{R}^n$ be the canonical injection and $(x', v') = \tilde{\varphi}_t(x, v)$. The differential of F at v is

$$DF(v) : w \in T_x\mathbb{R}^n \mapsto D\pi(x', v') \circ D\tilde{\varphi}_t(x, v) \circ Di(v) \cdot w \in T_{x'}\mathbb{R}^n.$$

Let w belong to the kernel of $DF(v)$. Then $D\tilde{\varphi}_t(x, v) \circ Di(v) \cdot w \in V(x', v')$ and $Di(v) \cdot w \in V(x, v)$, so that $Di(v) \cdot w = 0$ (because the Lagrangian has no conjugate points), and hence $w = 0$. This proves that $DF(v)$ is an isomorphism for every $v \in T_x\mathbb{R}^n$ and therefore F is a local diffeomorphism. □

Given two points x and y in \mathbb{R}^n and a positive real number t , let us introduce

$$\mathcal{A}_t(x, y) = \inf\{A_L(c), c : [a, b] \rightarrow \mathbb{R}^n, \text{ with } c(a) = x, c(b) = y, \text{ and } b - a = t\},$$

where $A_L(c) = \int_a^b \tilde{L}(c(t), c'(t)) dt$ is the action of the curve $c : [a, b] \rightarrow \mathbb{R}^n$. Note that the inf (taken over the set of absolutely continuous curves) is actually a min (because of the Tonelli theorem). Moreover, thanks to the hypotheses made on L , this inf is realized by a C^2 curve which is a piece of trajectory of the Euler-Lagrange flow $\tilde{\varphi}$. Hence \mathcal{A}_t is clearly a continuous function, with $\mathcal{A}_t(x + r, y + r) = \mathcal{A}_t(x, y)$ for every $r \in \mathbb{Z}^n$. As a consequence of proposition 1, one has:

Corollary 1.2. *Every extremal curve $c : \mathbb{R} \rightarrow \mathbb{R}^n$ minimizes the action between any two of its points: for every $a, b \in \mathbb{R}$ with $a < b$, $\mathcal{A}_{b-a}(c(a), c(b)) = A_L(c|_{[a,b]})$.*

Let us now come back to the proof of proposition 1.

Proof of proposition 1. To this purpose, we reason by contradiction. Assume therefore that there exist two vectors $v_1, v_2 \in T_x\mathbb{R}^n$ such that $F(v_1) = F(v_2) = y$. Let us moreover denote by $\gamma_i(s) = \pi \circ \tilde{\varphi}_s(x, v_i)$, for $i \in \{1, 2\}$ and $s \in [0, t]$.

Using standard techniques (see for example [AF08, section 5], we modify \tilde{L} and consider a new (periodic) Lagrangian \hat{L} which is still Tonelli, which coincides with \tilde{L} on $\mathbb{R}^n \times B(0, R)$, for some $R > 0$ to be determined later and which is quadratic at infinity (this means there exists $R' > R > 0$ and a smooth \mathbb{Z}^n periodic function V on \mathbb{R}^n such that for all $x \in \mathbb{R}^n$, there are a linear form l_x and a positive definite quadratic form Q_x , such that $\hat{L}(x, v) = V(x) + l_x(v) + Q_x(v)$ as soon as $\|v\| > R'$).

This technical requirement of being quadratic at infinity ensures that the action functional $A_{\hat{L}}$ associated to \hat{L} is C^2 when restricted to the Hilbert space $H^1 = W^{1,2}$ of curves which are absolutely continuous, with L^2 derivative (see for instance [AF08, proposition 4.1]). Moreover, it then verifies the Palais-Smale condition (see [AF08,

proposition 4.2]). However, by construction, any extremal curve for \widehat{L} which remains in $\mathbb{R}^n \times B(0, R)$ does not have conjugate points. It follows from Corollary 4.1 in [CI99] that extremal curves without conjugate points are strict local minima of the action functional (the Hessian of the action functional is positive definite at such a curve). In particular, there exists an $\varepsilon > 0$ and $\alpha > 0$ such that for any non trivial variation $\eta : [0, t] \rightarrow \mathbb{R}^n$ in H^1 verifying $\eta(0) = \eta(t) = 0$ and such that $\|\eta\|_1 = \varepsilon$ then

$$A_{\widehat{L}}(\gamma_i + \eta) > A_{\widehat{L}}(\gamma_i) + \alpha, \quad i \in \{1, 2\}.$$

Let E be the Hilbert space of H^1 curves $\eta : [0, t] \rightarrow \mathbb{R}^n$ verifying $\eta(0) = \eta(t) = 0$ equipped with the norm $\|\eta\|_E$ induced by the norm $\|\cdot\|_1$ on H^1 , and consider the restriction of $A_{\widehat{L}}$ to the affine space $V = \gamma_1 + E = \gamma_2 + E$. As already mentioned, $A_{\widehat{L}|_V}$ is C^2 and verifies the Palais-Smale condition. It also inherits coercivity from the superlinearity of \widehat{L} .

We now apply the Ambrosetti and Rabinowitz mountain pass lemma ([Str08, theorem 6.1 p.109]). The lemma asserts that the value $C = \inf_{\Gamma} \max_{s \in [0, 1]} A_{\widehat{L}}(\Gamma(s))$, where Γ ranges in all the homotopies from γ_1 to γ_2 in V , is a critical value of $A_{\widehat{L}|_V}$. More precisely there exists some curve $\gamma \in V$ realizing the inf max in the sense that γ is an extremal curve of $A_{\widehat{L}|_V}$, $A_{\widehat{L}}(\gamma) = C$ and for any $\varepsilon > 0$, there is a homotopy $H_\varepsilon \in \Gamma$ such that

$$\max_{s \in [0, 1]} A_{\widehat{L}}(H_\varepsilon(s)) \leq C + \varepsilon$$

and $H_\varepsilon([0, 1])$ intersects the ball in V of center γ and radius ε . Obviously, γ must not be a strict local minimum of $A_{\widehat{L}|_V}$ which means it contains conjugate points for \widehat{L} . In order to reach a contradiction, we only need to prove that γ is supported in $\mathbb{T}^n \times B(0, R)$ which we will see is automatic if R is chosen big enough.

Let Γ_0 be the linear homotopy: $s \mapsto (1-s)\gamma_1 + s\gamma_2$. Assume that R is large enough such that Γ_0 is supported in $\mathbb{T}^n \times B(0, R)$. Let

$$C' = \max_{s \in [0, 1]} A_{\widehat{L}}(\Gamma_0(s)) = \max_{s \in [0, 1]} A_L(\Gamma_0(s)) \geq C.$$

Note that C' depends only on L . The contradiction is now a direct consequence of the following lemma:

Lemma 1.1. *Let $T > 0$ and $M > 0$. There exists a constant $R > 0$ such that any critical curve $\delta : [0, T] \rightarrow \mathbb{T}^n$ with action $A_L(\delta) < M$ is R -Lipschitz. Moreover, this holds true for any other Tonelli Lagrangian which coincides with L on $\mathbb{T}^n \times B(0, R)$.*

Proof. Let δ be as in the lemma. By coercivity of L , let $r > 0$ be such that $L(x, v) > M/T$ as soon as $|v| > r$. Since $A_L(\delta) < M$ there exists $s_0 \in [0, 1]$ such that $|\delta'(s_0)| \leq r$.

In particular, δ being an extremal curve, (δ, δ') is a trajectory of the Euler-Lagrange flow of L which yields that

$$\forall s \in [0, 1], \quad (\delta(s), \delta'(s)) = \varphi_{s-s_0}(\delta(s_0), \delta'(s_0)) \in \bigcup_{t \in [-1, 1]} \varphi_t(\mathbb{T}^n \times \overline{B(0, r)}) := K,$$

which is obviously compact. Therefore, it is enough to take R such that $K \subset \mathbb{T}^n \times B(0, R)$.

The second part of the lemma follows from the fact that the previous argument only depends on the restriction of L to $\mathbb{T}^n \times B(0, R)$. □

□

Lemma 1.2. *For any positive real numbers t and t' , for any points x, y , and z in \mathbb{R}^n , the following inequality holds:*

$$\mathcal{A}_{t+t'}(x, z) \leq \mathcal{A}_t(x, y) + \mathcal{A}_{t'}(y, z),$$

It will be referred to as the triangular inequality in the sequel. In this inequality, equality occurs if and only if $y = c(t)$, where $c : \mathbb{R} \rightarrow \mathbb{R}^n$ denotes the extremal curve with $c(0) = x$ and $c(t + t') = z$.

Proof. Let $c_1 : \mathbb{R} \rightarrow \mathbb{R}^n$ be the extremal curve with $c_1(0) = x$ and $c_1(t) = y$; and $c_2 : \mathbb{R} \rightarrow \mathbb{R}^n$ the extremal curve with $c_2(t) = y$ and $c_2(t + t') = z$. If we concatenate $c_1|_{]-\infty, t]}$ and $c_2|_{[t, +\infty[}$, we get a curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ with $\gamma(0) = x$ and $\gamma(t + t') = z$; hence $\mathcal{A}_{t+t'}(x, z) \leq A_L(\gamma|_{[0, t+t']}) = \mathcal{A}_t(x, y) + \mathcal{A}_{t'}(y, z)$. If we have equality, then $\gamma|_{[0, t+t']}$ is an action-minimizing curve and therefore an extremal curve. According to proposition 1, γ is equal to c . In particular $y = \gamma(t) = c(t)$. □

Lemma 1.3. *Let $T > 0$ and $r \in \mathbb{Z}^n$. Define a vector field X on \mathbb{R}^n as such: if $x \in \mathbb{R}^n$, $X(x) = c'(0)$, where c is the extremal curve with $c(0) = x$ and $c(T) = x + r$. Then X is \mathbb{Z}^n -invariant, so it induces a vector field on \mathbb{T}^n , also denoted by X . This vector field is of class C^{k-1} .*

Proof. We shall apply the implicit function theorem to the function

$$\mathcal{F} : (x, v) \in T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \mapsto \pi \circ \tilde{\varphi}_T(x, v) - (x + r).$$

This will ensure that the map sending $x \in \mathbb{R}^n$ to the unique vector $v = X(x)$ for which $\mathcal{F}(x, v) = 0$ is of class C^{k-1} . All we need to do is to check that the differential of \mathcal{F} with respect to v at a point $(x, v) \in T\mathbb{R}^n$ is invertible. This is done in the same way as in the proof of corollary 1.1, using the fact that L has no conjugate points. □

1.2 Construction of totally periodic Lagrangian tori

The goal of this section is to give a proof of the following:

Proposition 2. *Let $T > 0$ and $r \in \mathbb{Z}^n$. There exists a subset $\mathcal{G}_{T,r}$ of $T\mathbb{T}^n$ such that*

- (1) $\mathcal{G}_{T,r}^* = \mathcal{L}(\mathcal{G}_{T,r})$ is a Lagrangian C^{k-1} submanifold of $T^*\mathbb{T}^n$;
- (2) $\mathcal{G}_{T,r}$ is a flow-invariant graph;
- (3) $\forall (x, v) \in \mathcal{G}_{T,r}, \quad \varphi_T(x, v) = (x, v)$;
- (4) For all $(x, v) \in \mathcal{G}_{T,r}$, the extremal curve $c : s \in [0, T] \mapsto \pi \circ \varphi_s(x, v) \in \mathbb{T}^n$ is a smooth loop with homotopy class r and its action does not depend on (x, v) .

Let us fix $T > 0$ and $r \in \mathbb{Z}^n$, and call f the function defined by $f(x) = \mathcal{A}_T(x, x+r)$, where $x \in \mathbb{R}^n$. As $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous and \mathbb{Z}^n -periodic, we have two points a and b in \mathbb{R}^n such that $f(a) = \min_{\mathbb{R}^n} f$ and $f(b) = \max_{\mathbb{R}^n} f$. We first show that there is an extremal curve on the torus with the following properties: it is periodic, T being a period; it contains the point $\text{pr}(b)$; and its homotopy class is r .

Lemma 1.4. *The extremal curve c with $c(0) = b$ and $c(T) = b + r$ is invariant under the translation of vector r . More precisely, we have*

$$\forall s \in \mathbb{R}, \quad c(s+T) = c(s) + r.$$

Therefore $\text{pr} \circ c$ is an extremal curve on the torus that is T -periodic, goes through $\text{pr}(b)$ and the homotopy class of whose is r .

Proof. Let x be any point in \mathbb{R}^n . Using the fact that \tilde{L} is \mathbb{Z}^n -periodic and the triangular inequality, we get

$$\mathcal{A}_{2T}(b, x) = \mathcal{A}_{2T}(b+r, x+r) \leq \mathcal{A}_T(b+r, x) + \mathcal{A}_T(x, x+r) = \mathcal{A}_T(b+r, x) + f(x).$$

We now choose $x = c(2T)$. As c is an extremal curve which contains the points $c(0) = b$, $c(T) = b+r$ and $c(2T) = x$, we have $\mathcal{A}_{2T}(b, x) = \mathcal{A}_T(b, b+r) + \mathcal{A}_T(b+r, x) = f(b) + \mathcal{A}_T(b+r, x)$, and hence $\mathcal{A}_T(b+r, x) = \mathcal{A}_{2T}(b, x) - f(b)$. The last inequality then becomes

$$\mathcal{A}_{2T}(b, x) \leq \mathcal{A}_{2T}(b, x) - f(b) + f(x) \leq \mathcal{A}_{2T}(b, x),$$

because f attains its maximum at the point b . As a consequence, all the above inequalities are in fact equalities. In particular, lemma 1.2 tells us that $x = \gamma(T)$, where γ is the extremal curve with $\gamma(0) = b+r$ and $\gamma(2T) = x+r$.

So we get two extremal curves, namely $s \mapsto \gamma(s)$ and $s \mapsto c(s) + r$, that both go through $b+r$ (at $s=0$) and $x+r$ (at $s=2T$). Proposition 1 implies that they are equal; hence $c(T) + r = \gamma(T)$, i.e. $b+2r = x$. We then apply the same argument to $s \mapsto c(s) + r$ and $s \mapsto c(s+T)$: these two extremal curves coincide at $s=0$ and at $s=T$ (because $b+2r = x$), so they are equal. \square

Lemma 1.5. *The function f is constant.*

Proof. We only need to show that $\max_{\mathbb{R}^n} f = f(b) \leq f(a) = \min_{\mathbb{R}^n} f$. An immediate consequence of lemma 1.4 is that $c(nT) = b + nr$ for all integer n ; therefore if $n \geq 1$, we have

$$\mathcal{A}_{nT}(b, b + nr) = n\mathcal{A}_T(b, b + r) = nf(b).$$

On the other hand, by use of the triangular inequality, we get that for any $n \geq 3$

$$\mathcal{A}_{nT}(b, b + nr) \leq \mathcal{A}_T(b, a + r) + \sum_{i=1}^{n-2} \mathcal{A}_T(a + ir, a + (i+1)r) + \mathcal{A}_T(a + (n-1)r, b + nr).$$

These two lines and the fact that \mathcal{A}_T is \mathbb{Z}^n -invariant imply that

$$nf(b) \leq \mathcal{A}_T(b, a + r) + (n-2)f(a) + \mathcal{A}_T(a, b + r).$$

Dividing this by n and letting n go to infinity yields $f(b) \leq f(a)$. \square

As the function f attains its maximum everywhere, we may apply lemma 1.4 to every point in \mathbb{R}^n . Therefore for every $x \in \mathbb{T}^n$ there exists an extremal curve c_x containing x (we may assume $c_x(0) = x$) which is T -periodic, the homotopy class of whose is r and whose action $A_L(c_x|_{[0,T]})$ does not depend on x . Remark that this curve is unique. For if c and γ are two such curves, we can lift them to \mathbb{R}^n . Hence we obtain two extremal curves C and Γ , chosen in such a way that $C(0) = \Gamma(0)$. As $c(T) = \gamma(T) = x$ and c and γ belong to the same homotopy class, $C(T) = \Gamma(T)$. Using proposition 1 once again, we conclude that $C = \Gamma$, and that $c = \gamma$.

We now define $\mathcal{G}_{T,r}$ as the set for all vectors tangent to the curves c_x . More formally,

$$\mathcal{G}_{T,r} = \{(x, c'_x(0)), \quad x \in \mathbb{T}^n\}.$$

Properties of the curves c_x imply that $\mathcal{G}_{T,r}$ satisfies (2), (3) and (4) of proposition 2. Thanks to lemma 1.3, $\mathcal{G}_{T,r}$ is of class C^{k-1} .

It remains to check that $\mathcal{G}_{T,r}^*$ is Lagrangian. This requires the use of the Green bundles. Recall (see [CI99] and [Arn08a] for details) that if $s \in \mathbb{R} \mapsto (x, p) = \phi_s^H(x_0, p_0) \in T^*\mathbb{T}^n$ is an orbit of the Hamiltonian flow that is free of conjugate points, one may define a bundle G_+ (called the (positive) Green bundle) by

$$G_+(x, p) = \lim_{t \rightarrow +\infty} D\phi_t^H(\phi_{-t}^H(x, p)) \cdot V^*(\phi_{-t}^H(x, p)).$$

Every $G_+(x, p)$ is a Lagrangian subspace of $T_{(x,p)}T^*\mathbb{T}^n$, and this bundle is invariant by the Hamiltonian flow: $D\phi_t^H G_+(x, p) = G_+(\phi_t^H(x, p))$ for all $t \in \mathbb{R}$. We now establish

that at every $(x, p) \in \mathcal{G}_{T,r}^*$, the tangent space $T_{(x,p)}\mathcal{G}_{T,r}^*$ is equal to $G_+(x, p)$, and is therefore Lagrangian.

We will make use of the following criterion (see [CI99] and [Arn08a]): if $w \in T_{(x,p)}(T^*\mathbb{T}^n)$, then

$$w \notin G_+(x, p) \implies \lim_{t \rightarrow +\infty} \|D(\pi \circ \phi_{-t}^H)(x, p) \cdot w\| = +\infty,$$

where $\|\cdot\|$ denotes the Euclidean norm. Assume $w \in T_{(x,p)}\mathcal{G}_{T,r}^*$. As we know that $\phi_T^H|_{\mathcal{G}_{T,r}^*} = \text{Id}|_{\mathcal{G}_{T,r}^*}$, the same equality holds for ϕ_{-nT}^H , for all integer n . Passing to the differential, we get $D\phi_{-nT}^H(x, p) \cdot w = w$, hence $D(\pi \circ \phi_{-nT}^H)(x, p) \cdot w = D\pi(x, p) \cdot w$ has constant norm. The criterion mentioned above implies that $w \in G_+(x, p)$. This proves that $T_{(x,p)}\mathcal{G}_{T,r}^* \subset G_+(x, p)$; but these two spaces have the same dimension, so they coincide.

Remark 1. We do not know if there is an easy way to describe these sets $\mathcal{G}_{T,r}$, except in the case where $r = 0$. In fact, $\mathcal{G}_{T,0} = \{(x, 0), x \in \mathbb{T}^n\}$. To prove this, let $x \in \mathbb{R}^n$ and let c be the extremal curve with $c(0) = c(T) = x$, so that $(x, c'(0)) \in \mathcal{G}_{T,0}$. Denote by γ the extremal curve with $\gamma(0) = x$ and $\gamma(\frac{T}{2}) = x$. As a consequence of lemma 4, we have $\gamma(T) = x$ and hence γ and c are equal. In particular, $c(\frac{T}{2}) = \gamma(\frac{T}{2}) = x$. Repeating the same argument, we get $c(\frac{T}{2^n}) = x$ for every $n \geq 1$; therefore $(x, c'(0)) = (x, 0)$.

Remark 2. A remarkable consequence of proposition 2 is that if $c : \mathbb{R} \rightarrow \mathbb{T}^n$ is an extremal curve, then c is either injective or periodic. For if we can find two real numbers a and b with $a < b$ and $c(a) = c(b)$, we lift c to \mathbb{R}^n and so we have an extremal curve $C : \mathbb{R} \rightarrow \mathbb{R}^n$, and $C(b) = C(a) + r$, with $r \in \mathbb{Z}^n$; lemma 1.4 now tells us that $c = \text{pr} \circ C$ is periodic, and any vector tangent to c belongs to $\mathcal{G}_{b-a,r}$.

1.3 A continuous foliation of $T^*\mathbb{T}^n$

In this section, we will construct a continuous foliation of $T^*\mathbb{T}^n$, with the help of the sets $\mathcal{G}_{T,r}$ introduced above. The method used here is very close to the one introduced in [Arn11], where a similar result is proven under the assumption that the tiered Mané set is the whole cotangent space.

For all $T > 0$ and $r \in \mathbb{Z}^n$, recall that $\mathcal{G}_{T,r}^*$ is $\mathcal{L}(\mathcal{G}_{T,r})$. We first show that each of these sets is in fact an Aubry set associated to a cohomology class.

Proposition 3. *Let $T > 0$ and $r \in \mathbb{Z}^n$. There is a cohomology class $c \in H^1(\mathbb{T}^n, \mathbb{R})$ such that $\mathcal{G}_{T,r}^* = \mathcal{A}_c^* = \mathcal{N}_c^*$.*

Proof. The set $\mathcal{G}_{T,r}^*$ is a C^1 Lagrangian graph in $T^*\mathbb{T}^n$, so it is the graph \mathcal{G}_ω of a C^1 closed 1-form ω . Is it shown in [Fat08] that in this case $\mathcal{A}_c^* \subset \mathcal{G}_\omega \subset \mathcal{N}_c^*$, where c is the cohomology class of ω . We shall prove that these three sets are equal. Let $(x, p) \in \mathcal{G}_\omega = \mathcal{G}_{T,r}^*$. As $\phi_T^H(x, p) = (x, p)$, (x, p) belongs to its omega limit set; and it is known that the omega limit set of every element of \mathcal{N}_c^* is in \mathcal{A}_c^* . So we have $\mathcal{A}_c^* = \mathcal{G}_\omega$. Hence $\pi(\mathcal{A}_c^*) = \pi(\mathcal{G}_\omega) = \mathbb{T}^n$, and this implies $\mathcal{A}_c^* = \mathcal{N}_c^*$ (see the section on weak KAM theory in the appendix). □

We next describe the sets \mathcal{A}_c^* , where c is any cohomology class.

Proposition 4. *For all $c \in H^1(\mathbb{T}^n, \mathbb{R})$, \mathcal{A}_c^* is a graph above \mathbb{T}^n . Moreover, if c and d are two distinct elements in $H^1(\mathbb{T}^n, \mathbb{R})$, then $\mathcal{A}_c^* \cap \mathcal{A}_d^* = \emptyset$.*

Proof. This result is akin to the ones contained in propositions 12 and 13 in [Arn11], and the proof is roughly the same, so we will only give the main lines of the reasoning, and refer to [Arn11] for more details.

Let $c \in H^1(\mathbb{T}^n, \mathbb{R})$. We have to show that \mathcal{A}_c is a graph above \mathbb{T}^n , i.e. that every $y \in \mathbb{T}^n$ is in $\pi(\mathcal{A}_c)$. Let λ be a closed 1-form with cohomology class c , and $(x, v) \in \mathcal{A}_c$. There exist a sequence of real numbers (T_m) with $\lim T_m = +\infty$ and a sequence of extremal curves $\gamma_m : \mathbb{R} \rightarrow \mathbb{T}^n$ such that

$$\gamma_m(0) = \gamma_m(T_m) = x \quad \text{and} \quad \int_0^{T_m} (L(\gamma_m(t), \gamma'_m(t)) - \lambda_{\gamma_m(t)}(\gamma'_m(t)) + \alpha(c)) dt \rightarrow 0,$$

see the part about Aubry sets in the appendix.

According to remark 2 at the end of the last section, each γ_m has to be periodic, and T_m is a period of γ_m , hence $\gamma'_m(0)$ belongs to \mathcal{G}_{T_m, r_m} for a certain $r_m \in \mathbb{Z}^n$. Denote the vector for which $(y, w_m) \in \mathcal{G}_{T_m, r_m}$ by $w_m \in T_y\mathbb{T}^n$, and the associated extremal curve by $\Gamma_m : s \in \mathbb{R} \mapsto \varphi_s(y, w_m) \in \mathbb{T}^n$. We know that each Γ_m is T_m -periodic, homotopic to γ_m , and that $A_L(\Gamma_m|_{[0, T_m]}) = A_L(\gamma_m|_{[0, T_m]})$. These properties imply that

$$\Gamma_m(0) = \Gamma_m(T_m) = y \quad \text{and} \quad \int_0^{T_m} (L(\Gamma_m(t), \Gamma'_m(t)) - \lambda_{\Gamma_m(t)}(\Gamma'_m(t)) + \alpha(c)) dt \rightarrow 0,$$

and therefore $y \in \pi(\mathcal{A}_c)$.

Now suppose that c and d are two cohomology classes with $\mathcal{A}_c^* \cap \mathcal{A}_d^* \neq \emptyset$. Then there exists an action minimizing, flow-invariant, probability measure μ on $T\mathbb{T}^n$, chosen in such a way that its dual measure μ^* has support included in $\mathcal{A}_c^* \cap \mathcal{A}_d^*$. If we express the fact that μ is minimal for both $L - \lambda$ and $L - \eta$, where λ (resp. η) is a closed 1-form with cohomology class c (resp. d), and use the convexity of the α function,

we get $\alpha(\frac{c+d}{2}) = \frac{1}{2}(\alpha(c) + \alpha(d))$. Next, let $(x, p) \in \mathcal{A}_{\frac{c+d}{2}}^*$, (T_m) a sequence of real numbers with $\lim T_m = +\infty$ and $\gamma_m : \mathbb{R} \rightarrow \mathbb{T}^n$ a sequence of extremal curves with $\gamma_m(0) = \gamma_m(T_m) = x$ and

$$\int_0^{T_m} \left(L(\gamma_m(t), \gamma'_m(t)) - \frac{1}{2} \left(\lambda_{\gamma_m(t)}(\gamma'_m(t)) + \eta_{\gamma_m(t)}(\gamma'_m(t)) \right) + \alpha\left(\frac{c+d}{2}\right) \right) dt \rightarrow 0.$$

Using the fact that $\alpha(\frac{c+d}{2}) = \frac{1}{2}(\alpha(c) + \alpha(d))$, we get both limits:

$$\int_0^{T_m} \left(L(\gamma_m(t), \gamma'_m(t)) - \lambda_{\gamma_m(t)}(\gamma'_m(t)) + \alpha(c) \right) dt \rightarrow 0,$$

$$\int_0^{T_m} \left(L(\gamma_m(t), \gamma'_m(t)) - \eta_{\gamma_m(t)}(\gamma'_m(t)) + \alpha(d) \right) dt \rightarrow 0,$$

which imply that the limit of $\gamma'_m(0)$ is in $\mathcal{A}_c \cap \mathcal{A}_d$. This shows that the graph $\mathcal{A}_{\frac{c+d}{2}}^*$ is a subset of $\mathcal{A}_c^* \cap \mathcal{A}_d^*$, so that $\mathcal{A}_c^* = \mathcal{A}_d^*$ and hence $c = d$. \square

We come to the conclusion that the Aubry sets \mathcal{A}_c^* , with c varying in $H^1(\mathbb{T}^n, \mathbb{R})$, are a family of disjoint, flow-invariant, Lipschitz Lagrangian graphs. All we need to show now is that every point of the cotangent space belongs to one of these sets. This is an immediate consequence of the next result.

Proposition 5. *For every $x \in \mathbb{T}^n$, the map*

$$F_x : c \in H^1(\mathbb{T}^n, \mathbb{R}) \mapsto \mathcal{A}_c^* \cap T_x^* \mathbb{T}^n \in T_x^* \mathbb{T}^n$$

is a homeomorphism.

Proof. The map F_x is coercive. Indeed, let K be a compact set in $T_x^* \mathbb{T}^n$ and $X = F_x^{-1}(K)$. We claim that X is bounded: for all $c \in X$, $\alpha(c) = H(F_x(c)) \in H(K)$, hence $\alpha(X)$ is included in the compact set $H(K)$; as α is superlinear, this implies that X is bounded.

We then establish that the map F_x is continuous. Let $\tilde{c} \in H^1(\mathbb{T}^n, \mathbb{R})$ be a cohomology class, and $c_m \rightarrow \tilde{c}$. For all $m > 0$ let λ_m be a C^0 closed 1-form of class c_m such that $\mathcal{A}_{c_m}^*$ is the graph of λ_m , and $\tilde{\lambda}$ a C^0 closed 1-form of class \tilde{c} such that $\mathcal{A}_{\tilde{c}}^*$ is the graph of $\tilde{\lambda}$. We will show that the sequence (λ_m) pointwise converges. It is a general fact that the Mañé sets \mathcal{N}_c^* vary upper semi-continuously with the cohomology class (see [Arn08b, proposition 13]). Let $y \in \mathbb{T}^n$, the sequence $(y, \lambda_{n,y}) \in \mathcal{N}_{c_m}^*$ is bounded by the previous argument. But any converging subsequence must converge to an element of $\mathcal{N}_{\tilde{c}}^*$. Since $\mathcal{N}_{\tilde{c}}^* = \mathcal{A}_{\tilde{c}}^*$ is a graph over the base, necessarily, the limit is $(y, \tilde{\lambda}_y)$. Therefore, the sequence converges to $(y, \tilde{\lambda}_y)$.

Evaluating at x , we have exactly shown that $F_x(c_m) \rightarrow F_x(c)$, hence that F_x is continuous.

It is moreover injective between two vector spaces of the same dimension. The invariance of domain (see [Dol72]) states that F_x is an open map. As we have seen that F_x is coercive, F_x is proper. Thus, $F_x(H^1(\mathbb{T}^n, \mathbb{R}))$ is both open and closed, so it has to be equal to $T_x^*\mathbb{T}^n$. This proves that F_x is surjective. Since F_x is open, it is also a homeomorphism. □

Another consequence of this proposition is that the map

$$\mathcal{F} : (x, c) \in \mathbb{T}^n \times H^1(\mathbb{T}^n, \mathbb{R}) \mapsto F_x(c) \in T^*\mathbb{T}^n$$

is continuous, and therefore the Aubry sets are the leaves of a continuous foliation of $T^*\mathbb{T}^n$.

2 Abundance of KAM tori

2.1 Introduction and statements

In this section, we still study the dynamics of a Tonelli Hamiltonian H on $T^*\mathbb{T}^n = \mathbb{T}^n \times (\mathbb{R}^n)^*$ without conjugate points. We will moreover assume that H is C^∞ (see however Remark 4). The associated Hamiltonian vector-field and Hamiltonian flow will still be denoted by X_H and $(\phi_t^H)_{t \in \mathbb{R}}$. As it was proved in the previous section, for any $T > 0$ and $r \in \mathbb{Z}^n$ there is an invariant Lagrangian graph

$$\mathcal{G}_{T,r}^* := \mathcal{T}_\infty = \{(\theta, I_\infty + Du(\theta)) ; \theta \in \mathbb{T}^n\}$$

such that all the points of $\mathcal{G}_{T,r}^*$ are fixed points of ϕ_T^H , more precisely

$$\forall x \in \mathbb{R}^n, \quad \tilde{\phi}_T^H(x, I_\infty + Du(x)) = (x + r, I_\infty + Du(x)).$$

Here $u \in C^\infty(\mathbb{T}^n, \mathbb{R})$ is identified with a \mathbb{Z}^n -periodic map defined on \mathbb{R}^n and the flow $(\tilde{\phi}_t^H)$ on $T^*\mathbb{R}^n = \mathbb{R}^n \times (\mathbb{R}^n)^*$ is the lift of the flow (ϕ_t^H) .

Our aim is to prove the existence of a rich family of invariant Lagrangian graphs accumulating to \mathcal{T}_∞ , on which the flow is transitive, conjugated to a linear flow of non-resonant vector.

Definition. A vector $\bar{\omega} \in \mathbb{R}^n$ is said strongly Diophantine if there are real numbers $\gamma > 0$ and τ (necessarily $\geq n$) such that

$$\forall k \in \mathbb{Z}^n \setminus \{0\}, \forall l \in \mathbb{Z}, \quad |k \cdot \bar{\omega} + l| \geq \frac{\gamma}{|k|^\tau}. \quad (1)$$

Here is the main result of this section.

Theorem 2. *Assume $\bar{\omega}$ is strongly Diophantine. There is $m_0 \in \mathbb{N} \setminus \{0\}$ (depending on $\bar{\omega}$) such that, for all $m \geq m_0$, there is a C^∞ Lagrangian embedding $i_m : \mathbb{T}^n \rightarrow T^*\mathbb{T}^n$ such that*

$$i) \quad \forall \eta \in \mathbb{T}^n, \quad \phi_{mT}^H(i_m(\eta)) = i_m(\eta + \bar{\omega}).$$

ii) *Writing $i_m(\eta) = (\psi_m(\eta), f_m(\eta))$, ψ_m is a C^∞ diffeomorphism of \mathbb{T}^n , isotopic to $id_{\mathbb{T}^n}$ and*

$$\mathcal{T}_m = i_m(\mathbb{T}^n) = \{(\theta, (f_m \circ \psi_m^{-1})(\theta)); \theta \in \mathbb{T}^n\}.$$

is a Lagrangian graph; the sequence (\mathcal{T}_m) converges to \mathcal{T}_∞ in C^∞ topology.

iii) *The sequence (ψ_m) converges in C^∞ topology to a diffeomorphism ψ_∞ of \mathbb{T}^n (independent of $\bar{\omega}$), isotopic to $id_{\mathbb{T}^n}$.*

iv) *The tori \mathcal{T}_m are flow-invariant. More precisely, $i_m^*(X_H) = \frac{r}{T} + \frac{\bar{\omega}}{mT}$, so that*

$$\forall m \geq m_0, \forall t \in \mathbb{R}, \forall \eta \in \mathbb{T}^n, \quad \phi_t^H(i_m(\eta)) = i_m\left(\eta + \frac{t}{T}r + \frac{t}{mT}\bar{\omega}\right).$$

Remark 3. a) The torus \mathcal{T}_m is the Aubry set $\mathcal{A}_{c_m}^*$, where c_m is the cohomology class of the closed 1-form $f_m \circ \psi_m^{-1}$

b) As a by-result, the diffeomorphism ψ_∞ conjugates the flow on the completely periodic torus \mathcal{T}_∞ to the periodic linear flow on \mathbb{T}^n of vector $\frac{r}{T}$.

The proof of Theorem 2 relies on the following two propositions. The first one provides a normal form for ϕ_T^H in the neighborhood of \mathcal{T}_∞ and uses the theorem of Burago and Ivanov in [BI94]. The second one is a simple application of a KAM theorem for exact symplectic maps.

Remark 4. Using a KAM theorem in finite differentiability ([Pös80], [Dou82], [Sal04]), we could prove our result for H of class C^k , k large enough. More precisely a survey of the proof would show that, $p \geq 1$ being given, if $\bar{\omega}$ satisfies (1) and $k > 2\tau + 7 + p$, then the conclusion of Theorem 2 holds with j_m of class C^p .

Notation. In what follows, any remainder denoted by $O(I^s)$ is a C^∞ map u from $T^*\mathbb{T}^n$ to some finite dimensional vector-space, such that, for any $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^n$ such that $|\beta| < s$, $|\partial_\theta^\alpha \partial_I^\beta u| \leq C_{\alpha,\beta} |I|^{s-|\beta|}$ on some neighborhood W of $0_{\mathbb{T}^n} = \mathbb{T}^n \times \{0\}$. Moreover, if B is an invertible operator, the inverse of its transposed operator is denoted by B^{-T} .

Proposition 6. *Under the assumptions of Theorem 2, there is a C^∞ symplectic diffeomorphism G of $T^*\mathbb{T}^n$ of the form*

$$G(\theta, I) = (\psi(\theta), I_\infty + Du(\psi(\theta)) + D\psi(\theta)^{-T}I),$$

where ψ is a diffeomorphism of \mathbb{T}^n isotopic to $id_{\mathbb{T}^n}$, such that $\mathcal{T}_\infty = G(\mathbb{T}^n \times \{0\})$, and

$$G^{-1} \circ \phi_T^H \circ G(\theta, I) = (\theta + \bar{A}I + O(I^2), I + O(I^3)), \quad (2)$$

where $\bar{A} \in L((\mathbb{R}^n)^*, \mathbb{R}^n)$ is symmetric positive definite.

Proposition 7. *Assume that $H : T^*\mathbb{T}^n \rightarrow \mathbb{R}$ is C^∞ and such that, on a neighborhood of $0_{\mathbb{T}^n}$,*

$$\phi_T^H(\theta, I) = (\theta + \bar{A}I + O(I^2), I + O(I^3)), \quad (3)$$

where $\bar{A} \in L((\mathbb{R}^n)^*, \mathbb{R}^n)$ is symmetric non-degenerate. Let $\bar{\omega} \in \mathbb{R}^n \setminus \{0\}$ be strongly Diophantine. Then there is $m_1 \in \mathbb{N}^*$ such that, for any $m \geq m_1$, there is a C^∞ Lagrangian embedding $j_m : \mathbb{T}^n \rightarrow T^*\mathbb{T}^n$ of the form

$$j_m(\eta) = \left(\eta + u_m(\eta), \bar{A}^{-1} \left(\frac{\bar{\omega}}{m} \right) + v_m(\eta) \right),$$

with $u_m \in C^\infty(\mathbb{T}^n, \mathbb{R}^n)$, $v_m \in C^\infty(\mathbb{T}^n, (\mathbb{R}^n)^*)$, for any k : $\|u_m\|_{C^k(\mathbb{T}^n)} = o(1)$, $\|v_m\|_{C^k(\mathbb{T}^n)} = o(1/m)$ as $m \rightarrow \infty$, such that

$$\forall \eta \in \mathbb{T}^n, \quad \phi_{mT}^H(j_m(\eta)) = j_m(\eta + \bar{\omega}).$$

2.2 KAM meets weak KAM: proof of Theorem 2

In this subsection we prove Theorem 2 from Propositions 6 and 7.

Consider $H_1 = H \circ G$, where G is the symplectic map of Proposition 6. We have $\phi_t^{H_1} = G^{-1} \circ \phi_t^H \circ G$, hence the Hamiltonian H_1 satisfies the assumption of Proposition 7. Consider the sequence $(j_m)_{m \geq m_1}$ of Lagrangian embeddings of \mathbb{T}^n into $T^*\mathbb{T}^n$ provided by Proposition 7, with

$$j_m(\eta) = \left(\eta + u_m(\eta), \bar{A}^{-1} \left(\frac{\bar{\omega}}{m} \right) + v_m(\eta) \right).$$

Since $j_m(\eta + k\bar{\omega}) = \phi_{kmT}^{H_1}(j_m(\eta))$, we have

$$\forall \eta \in \mathbb{T}^n, \quad \forall k \in \mathbb{Z}, \quad H_1(j_m(\eta + k\bar{\omega})) = H_1(j_m(\eta)). \quad (4)$$

Since $\bar{\omega}$ is strongly non-resonant, the orbits of the translation $\tau_{\bar{\omega}}$ are dense in \mathbb{T}^n , hence by (4), $H_1 \circ j_m$ is constant. Since $j_m(\mathbb{T}^n)$ is Lagrangian, this implies that

$$\forall t \in \mathbb{R}, \quad \phi_t^{H_1}(j_m(\mathbb{T}^n)) = j_m(\mathbb{T}^n),$$

and we can define on \mathbb{T}^n a smooth flow $(\alpha_t^{H_1}) = (j_m^{-1} \circ \phi_t^{H_1} \circ j_m)$. Now for any $t \in \mathbb{R}$, $\alpha_t^{H_1}$ commutes with $\tau_{\bar{\omega}}$, hence ($\bar{\omega}$ being non-resonant) $\alpha_t^{H_1}$ itself is a translation of vector $\beta_m(t)$, therefore $j_m^*(X_{H_1}) = \bar{\beta}_m := \beta'_m(0)$ is constant on \mathbb{T}^n and $\beta_m(t) = t\bar{\beta}_m$. We are going to prove that

$$\bar{\beta}_m = \frac{r}{T} + \frac{\bar{\omega}}{Tm}, \quad (5)$$

for m large enough. Since $\alpha_{Tm}^{H_1} = \tau_{\bar{\omega}}$, there is $k_m \in \mathbb{Z}^n$ such that

$$Tm\bar{\beta}_m = \bar{\omega} + k_m. \quad (6)$$

Writing $j_\infty(\eta) = (\eta, 0)$ and $b_\infty(\eta) = (j_\infty^* X_{H_1})(\eta)$, since $\|j_m - j_\infty\|_{C^k(\mathbb{T}^n)} = o(1)$ for any $k \in \mathbb{N}$, we have $\|b_\infty - \bar{\beta}_m\|_{C^k(\mathbb{T}^n)} = o(1)$ for any $k \in \mathbb{N}$. Hence b_∞ is a constant map, equal to $\bar{\beta}_\infty := \lim_{m \rightarrow \infty} \bar{\beta}_m$. Moreover, $\tilde{\phi}_T^{H_1}(x, 0) = (x + r, 0)$ (we use here the fact that ψ is isotopic to $id_{\mathbb{T}^n}$), hence $\bar{\beta}_\infty = \frac{r}{T}$. Thus we obtain

$$\bar{\beta}_m = \frac{r}{T} + o(1). \quad (7)$$

We have

$$j_m(\eta + T\bar{\beta}_m) = \left(\eta + T\bar{\beta}_m + u_m(\eta + T\bar{\beta}_m), \bar{A}^{-1}\left(\frac{\bar{\omega}}{m}\right) + o(1/m) \right). \quad (8)$$

On the other hand,

$$\begin{aligned} j_m(\eta + T\bar{\beta}_m) &= \phi_T^{H_1}(j_m(\eta)) \\ &= \phi_T^{H_1}\left(\eta + u_m(\eta), \bar{A}^{-1}\left(\frac{\bar{\omega}}{m}\right) + v_m(\eta)\right) \\ &= \left(\eta + u_m(\eta) + \frac{\bar{\omega}}{m} + o(1/m), \bar{A}^{-1}\left(\frac{\bar{\omega}}{m}\right) + o(1/m)\right). \end{aligned} \quad (9)$$

Comparing (8) and (9), we derive that there is $l_m \in \mathbb{Z}^n$ such that

$$\forall \eta \in \mathbb{T}^n, T\bar{\beta}_m + u_m(\eta + T\bar{\beta}_m) = u_m(\eta) + \frac{\bar{\omega}}{m} + l_m + o(1/m) \quad (10)$$

Taking the mean-value in (10), we obtain

$$T\bar{\beta}_m = \frac{\bar{\omega}}{m} + l_m + o(1/m). \quad (11)$$

By (11) and (7), $l_m = r + o(1)$, which implies that for m large enough, $l_m = r$. Then (11) gives $Tm\bar{\beta}_m = \bar{\omega} + mr + o(1)$, so that in (6), $k_m = mr + o(1)$. Hence there is $m_0 \geq m_1$ such that for $m \geq m_0$, $k_m = mr$ and (5) is satisfied.

There remains to define $i_m : \mathbb{T}^n \rightarrow T^*\mathbb{T}^n$ as $i_m = G \circ j_m = (\psi_m, f_m)$, with $\psi_m(\eta) = \psi(\eta + u_m(\eta))$; the sequence (ψ_m) clearly converges in C^∞ -topology to $\psi_\infty := \psi$; $j_m(\mathbb{T}^n)$ being a Lagrangian manifold, so is $i_m(\mathbb{T}^n) = G(j_m(\mathbb{T}^n))$, and $i_m(\mathbb{T}^n)$ is a graph because ψ_m is a diffeomorphism of \mathbb{T}^n ; i) and iv) of Theorem 2 are just a consequence of $\phi_t^H \circ G = G \circ \phi_t^{H_1}$.

2.3 The normal form: Proof of Proposition 6

We first introduce a lemma that will be used in the proof of Propositions 6 and 7. In what follows, denoting by pr the canonical projection from $T^*\mathbb{R}^n$ to $T^*\mathbb{T}^n$, we shall identify a map u defined in $T^*\mathbb{T}^n$ with the \mathbb{Z}^n -periodic map $u \circ pr$ defined in $T^*\mathbb{R}^n$.

Lemma 2.1. *Let K be a compact subset of $T^*\mathbb{T}^n$, U be an open bounded neighborhood of K in $T^*\mathbb{T}^n$, and define $\tilde{K} = pr^{-1}(K)$, $\tilde{U} = pr^{-1}(U)$. Let X be a C^{k+1} vector field on $T^*\mathbb{T}^n$, with $k \geq 1$. We shall identify X with the vector field $X \circ pr$ on $T^*\mathbb{R}^n$. We assume that \tilde{K} is preserved by the flow $(\tilde{\varphi}_t)$ of X . For any $c_0 > 0$, there exist $\epsilon_0 > 0$ and $c_1 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ and for any $u \in C^k(T^*\mathbb{T}^n, \mathbb{R}^n \times (\mathbb{R}^n)^*)$ such that $|u - \epsilon X|_{C^k(U)} \leq c_0 \epsilon^2$, for all $m \in \mathbb{N}$ such that $m\epsilon \leq c_0$,*

$$|(id + u)^m - \tilde{\varphi}_{m\epsilon}|_{C^k(\tilde{K})} \leq c_1 \epsilon.$$

Proof. This result is related to the convergence of the Euler method for the differential equations; however it may be convenient to provide some details.

For a C^k map w from \tilde{U} to $\mathbb{R}^n \times (\mathbb{R}^n)^*$, we introduce the notation $|w|_k = |w|_{C^k(\tilde{K})} = \sup_{z \in \tilde{K}, |\alpha| \leq k} |D^\alpha w(z)|$. We shall use the following (rough) estimates, where $C(s)$ stands

for any positive non-decreasing function of the positive variable s .

i) For $\rho \in C^k(T^*\mathbb{T}^n, \mathbb{R}^n \times (\mathbb{R}^n)^*)$ and $a \in C^k(\tilde{U}, T^*\mathbb{R}^n)$ with $a(\tilde{K}) \subset \tilde{U}$,

$$|\rho \circ a|_k \leq C(|a|_k) |\rho|_{C^k(\tilde{U})}.$$

ii) For $w \in C^{k+1}(T^*\mathbb{T}^n, \mathbb{R}^n \times (\mathbb{R}^n)^*)$ and $a, b \in C^k(\tilde{U}, T^*\mathbb{R}^n)$ such that for all $x \in \tilde{K}$, $[a(x), b(x)] \subset \tilde{U}$,

$$|w \circ a - w \circ b|_k \leq C(|a|_k + |b|_k) |a - b|_k |w|_{C^{k+1}(\tilde{U})}$$

iii) For $\epsilon \geq 0$ small enough,

$$|\tilde{\varphi}_\epsilon - id|_{C^{k+1}(\tilde{U})} \leq C\epsilon \quad , \quad |\tilde{\varphi}_\epsilon - id - \epsilon X|_{C^k(\tilde{U})} \leq C\epsilon^2.$$

Let us fix $\delta > 0$ such that $\delta < \text{dist}(\tilde{K}, \partial\tilde{U})$. Under the assumptions of Lemma 2.1, let us introduce, for $m \in \mathbb{N}$, $\alpha_m = |(id + u)^m - \tilde{\varphi}_{m\epsilon}|_k$. As long as $\alpha_m \leq \delta$ and $m\epsilon \leq c_0$, we have

$$\begin{aligned} \alpha_{m+1} &= |(id + u)^{m+1} - \tilde{\varphi}_{(m+1)\epsilon}|_k \\ &\leq |(id + u - \tilde{\varphi}_\epsilon) \circ (id + u)^m|_k + |(id + u)^m - \tilde{\varphi}_{m\epsilon}|_k \\ &\quad + |(\tilde{\varphi}_\epsilon - id) \circ (id + u)^m - (\tilde{\varphi}_\epsilon - id) \circ \tilde{\varphi}_{m\epsilon}|_k \quad (12) \end{aligned}$$

Since $|(id + u)^m - \tilde{\varphi}_{m\epsilon}|_0 \leq \alpha_m \leq \delta$ and $\tilde{\varphi}_{m\epsilon}(\tilde{K}) \subset \tilde{K}$, there holds

$$\forall x \in \tilde{K}, [\tilde{\varphi}_{m\epsilon}(x), (id + u)^m(x)] \subset \tilde{U}.$$

Hence, by i) and iii),

$$\begin{aligned} |(id + u - \tilde{\varphi}_\epsilon) \circ (id + u)^m|_k &\leq C(|(id + u)^m|_k) |id + u - \tilde{\varphi}_\epsilon|_{C^k(\tilde{U})} \\ &\leq C(\alpha_m + |\tilde{\varphi}_{m\epsilon}|_k)(C\epsilon^2 + |u - \epsilon X|_{C^k(\tilde{U})}) \\ &\leq C(\delta, c_0)\epsilon^2, \end{aligned} \tag{13}$$

and by ii),

$$\begin{aligned} |(\tilde{\varphi}_\epsilon - id) \circ (id + u)^m - (\tilde{\varphi}_\epsilon - id) \circ \tilde{\varphi}_{m\epsilon}|_k &\leq C(|(id + u)^m|_k + |\tilde{\varphi}_{m\epsilon}|_k) |(id + u)^m - \tilde{\varphi}_{m\epsilon}|_k \\ &\quad \times |\tilde{\varphi}_\epsilon - id|_{C^{k+1}(\tilde{U})} \\ &\leq C(\delta, c_0)\epsilon\alpha_m. \end{aligned} \tag{14}$$

From (12), (13) and (14), we obtain

$$\alpha_{m+1} \leq C_1(\delta, c_0)\epsilon^2 + (1 + C_2(\delta, c_0)\epsilon)\alpha_m \tag{15}$$

We can assume $C_2(\delta, c_0) \geq 1$ without loss of generality. Setting $\Gamma = \frac{C_1(\delta, c_0)}{C_2(\delta, c_0)}$, using $\alpha_0 = 0$, we derive from (15)

$$\alpha_m + \Gamma\epsilon \leq (1 + C_2(\delta, c_0)\epsilon)^m \Gamma\epsilon$$

as long as $(1 + C_2(\delta, c_0)\epsilon)^m \Gamma\epsilon \leq \delta$ and $m\epsilon \leq c_0$. Hence let us define $c_1 = \sup_{\epsilon \in (0,1]} (1 + C_2(\delta, c_0)\epsilon)^{c_0/\epsilon} \Gamma$ and choose $\epsilon_0 > 0$ such that iii) holds for $\epsilon \in [0, \epsilon_0]$ and $\epsilon_0 \leq \min(1, \delta/c_1)$. Note that c_1 and ϵ_0 depend only on δ, c_0 and the vectorfield X . For $m\epsilon \leq c_0$, and $\epsilon \leq \epsilon_0$ then $\alpha_m \leq c_1\epsilon$. \square

Proof of Proposition 6. We introduce a first symplectic change of variable, with

$$G_0(\theta, I) = (\theta, I + I_\infty + Du(\theta)) \quad , \quad H_0 = H \circ G_0$$

We have $\phi_t^H \circ G_0 = G_0 \circ \phi_t^{H_0}$. Note that, since G_0 preserves the fibers and its restriction to each fiber is a translation, H_0 is a Tonelli Hamiltonian without conjugate points, as H . The Hamiltonian flow associated to H_0 preserves $0_{\mathbb{T}^n} = \mathbb{T}^n \times \{0\}$ and satisfies

$$\forall \theta \in \mathbb{T}^n, \quad \phi_T^{H_0}(\theta, 0) = (\theta, 0).$$

This implies that the differential of $\phi_T^{H_0}$ at $(\theta, 0)$ takes the form

$$D\phi_T^{H_0}(\theta, 0)[\Delta\theta, \Delta I] = (\Delta\theta + A(\theta)\Delta I, Q(\theta)\Delta I)$$

with $A(\theta) = \partial_I \phi_{T,1}^{H_0}(\theta, 0) \in L((\mathbb{R}^n)^*, \mathbb{R}^n)$, and $Q(\theta) = \partial_I \phi_{T,2}^{H_0}(\theta, 0) \in L((\mathbb{R}^n)^*, (\mathbb{R}^n)^*)$. The linear map $D\phi_T^{H_0}(\theta, 0)$ being symplectic, $Q(\theta) = id_{(\mathbb{R}^n)^*}$ and $A(\theta)$ is symmetric.

Let us justify that $A(\theta)$ is positive definite. Given $\theta \in \mathbb{T}^n$, since $\phi_t^{H_0}(0_{\mathbb{T}^n}) \subset 0_{\mathbb{T}^n}$, we can write for $t \in [0, T]$,

$$D\phi_t^{H_0}(\theta, 0)(\Delta\theta, \Delta I) = (P_t\Delta\theta + A_t\Delta I, Q_t\Delta I),$$

with $P_0 = id_{\mathbb{R}^n}$, $Q_0 = id_{(\mathbb{R}^n)^*}$, $A_0 = 0$. Moreover, $D\phi_t^{H_0}(\theta, 0)$ being symplectic, $P_t^T Q_t = id_{(\mathbb{R}^n)^*}$ and $S_t = Q_t^T A_t$ is in the space $L_s((\mathbb{R}^n)^*, \mathbb{R}^n)$ of symmetric linear maps from $(\mathbb{R}^n)^*$ to \mathbb{R}^n . We have

$$\frac{dS_t}{dt} \Big|_{t=0} = \frac{dA_t}{dt} \Big|_{t=0} = \frac{d}{dt} \partial_I \phi_{t,1}^{H_0}(\theta, 0) \Big|_{t=0} = \partial_I X_{H_0,1}(\theta, 0) = \partial_I^2 H_0(\theta, 0).$$

Thus $\frac{dS_t}{dt} \Big|_{t=0}$ is symmetric positive definite. Moreover, since H_0 is without conjugate points, A_t is invertible for all $t \in (0, T]$, and so is $S_t = Q_t^T A_t$. Since S_t is positive definite for small $t > 0$, we can conclude that S_t remains symmetric positive definite for all $t \in (0, T]$. In particular, $A(\theta) = S_T$ is symmetric positive definite.

Let us consider the Taylor expansion of $\phi_T^{H_0}$ with respect to I , in the neighborhood of $0_{\mathbb{T}^n} = \mathbb{T}^n \times \{0\}$,

$$\phi_T^{H_0}(\theta, I) = (\theta + A(\theta)I + O(I^2), I + \frac{1}{2}\langle B(\theta)I, I \rangle + O(I^3)), \quad (16)$$

where $B(\theta) = \partial_I^2 \phi_{T,2}^{H_0}(\theta, 0) \in (\mathbb{R}^n)^* \otimes L_s((\mathbb{R}^n)^*, \mathbb{R}^n)$.

For $\epsilon > 0$, define $\mathcal{R}_\epsilon : T^*\mathbb{T}^n \rightarrow T^*\mathbb{T}^n$ by

$$\mathcal{R}_\epsilon(\theta, I) = (\theta, \epsilon I),$$

and let $\Phi : (0, 1] \times T^*\mathbb{T}^n \rightarrow T^*\mathbb{T}^n$ be defined by $\Phi(\epsilon, \cdot) = \mathcal{R}_\epsilon^{-1} \circ \phi_T^{H_0} \circ \mathcal{R}_\epsilon$. Since $\mathcal{R}_\epsilon^*(I.d\theta) = \epsilon I.d\theta$, the map $\Phi(\epsilon, \cdot)$ is exact symplectic as $\phi_T^{H_0}$ is. By (16), Φ can be smoothly extended to a map on $[0, 1] \times T^*\mathbb{T}^n$ with $\Phi(0, \cdot) = id_{T^*\mathbb{T}^n}$ and, as ϵ tends to 0,

$$\Phi(\epsilon, \theta, I) = (\theta + \epsilon A(\theta)I + O(\epsilon^2), I + \frac{\epsilon}{2}\langle B(\theta)I, I \rangle + O(\epsilon^2)). \quad (17)$$

We have

$$Y(\theta, I) := \frac{d}{d\epsilon} \Phi(\epsilon, \theta, I) \Big|_{\epsilon=0} = (A(\theta)I, \frac{1}{2}\langle B(\theta)I, I \rangle).$$

Now, since $\Phi(\epsilon, \cdot)$ is exact symplectic for all $\epsilon \geq 0$, Y is a Hamiltonian vector-field: there is $F \in C^\infty(T^*\mathbb{T}^n, \mathbb{R})$ such that $Y = X_F$. We have then $\partial_\theta F(\theta, 0) = 0$, hence $F(\theta, 0)$ does not depend on θ and we may impose $F(\theta, 0) = 0$. Since $\partial_I F(\theta, I) = A(\theta)I$, we have $F(\theta, I) = \frac{1}{2}\langle A(\theta)I, I \rangle$, hence $B(\theta) = -(\partial_\theta A)(\theta)$. As a result,

$$\phi_T^{H_0}(\theta, I) = \left(\theta + A(\theta)I + O(I^2), I - \frac{1}{2}\partial_\theta \langle A(\theta)I, I \rangle + O(I^3) \right). \quad (18)$$

Note that the flow (ϕ_t^F) spanned by X_F is the geodesic Hamiltonian flow associated to the Riemannian metric on \mathbb{T}^n

$$g(\theta)[U, V] = \langle A(\theta)^{-1}U, V \rangle$$

We are going to prove that the geodesic flow of g has no conjugate points. Arguing by contradiction, we assume the contrary. Then there are two points x and y of \mathbb{R}^n that are connected in time S by a non-minimizing geodesic path γ_1 . We may assume that γ_1 is non-degenerate, using the fact that along a geodesic path, the points that are conjugate to the starting point are isolated. Now there is also a minimizing geodesic path $\gamma_2 : [0, S] \rightarrow \mathbb{T}^n$ connecting x and y , and we may assume that γ_2 is non-degenerate: if not, we just substitute $S - \delta$ to S and $\gamma_2(S - \delta)$ to y . For $\delta \in (0, S)$, $\gamma_2|_{[0, S-\delta]}$ is a non-degenerate minimizing geodesic path, and if δ is small enough, by the implicit function theorem there is also a non-degenerate geodesic path $\bar{\gamma}_1$ (close to $\gamma_1|_{[0, S-\delta]}$) connecting x and $\gamma_2(S - \delta)$ in time $S - \delta$.

So we have two points x and y of \mathbb{R}^n connected in some time $S > 0$ by two distinct non-degenerate geodesic paths. In other words, there are $I_1 \neq I_2$ such that

$$\tilde{\phi}_{S,1}^F(x, I_1) = \tilde{\phi}_{S,1}^F(x, I_2) = y,$$

with $\partial_I \tilde{\phi}_{S,1}^F(x, I_1)$ and $\partial_I \tilde{\phi}_{S,1}^F(x, I_2)$ invertible.

Let us consider the compact subset of $T^*\mathbb{T}^n$

$$K = \{(\theta, I) \in T^*\mathbb{T}^n : \langle A(\theta)I, I \rangle \leq R\}$$

where R is such that $(x, I_i) \in \text{int}(\tilde{K})$. Note that \tilde{K} is invariant by $(\tilde{\phi}_t^F)$. Let us denote by $\bar{\Phi}_\epsilon : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ the lift of $\Phi(\epsilon, \cdot)$ such that $\bar{\Phi}_\epsilon - id_{T^*\mathbb{R}^n} = O(\epsilon)$. Thus we have

$$\bar{\Phi}_\epsilon(x, I) = \left(x + \epsilon A(x)I + O(\epsilon^2), I - \frac{\epsilon}{2}\partial_x \langle A(x)I, I \rangle + O(\epsilon^2) \right) = (x, I) + \epsilon X_F(x, I) + O(\epsilon^2).$$

By Lemma 2.1, there are $\epsilon_0 > 0$ and $c_1 \geq 0$ such that

$$\forall \epsilon \in (0, \epsilon_0], \forall m \in \mathbb{N} \cap [1, 2S/\epsilon], \quad |\bar{\Phi}_\epsilon^m - \tilde{\phi}_{m\epsilon}^F|_{C^1(\tilde{K})} \leq c_1 \epsilon.$$

Choosing $\epsilon_m = S/m$, for m large enough, $|\overline{\Phi}_{\epsilon_m}^m - \tilde{\phi}_S^F|_{C^1(\tilde{K})} \leq c'/m$. Hence, by the implicit function theorem, there are p and $\overline{I}_1, \overline{I}_2 \in (\mathbb{R}^n)^*$ with $\overline{I}_1 \neq \overline{I}_2$, \overline{I}_i close to I_i , such that

$$\pi \circ \overline{\Phi}_{\epsilon_p}^p(x, \overline{I}_1) = \pi \circ \overline{\Phi}_{\epsilon_p}^p(x, \overline{I}_2) = y. \quad (19)$$

Define $\tau_r : T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n$ by $\tau_r(x, I) = (x + r, I)$; τ_r commutes with $\overline{\Phi}_\epsilon$, and

$$\mathcal{R}_\epsilon^{-1} \circ \tilde{\phi}_T^{H_0} \circ \mathcal{R}_\epsilon = \tau_r \circ \overline{\Phi}_\epsilon.$$

Hence $\mathcal{R}_\epsilon^{-1} \circ \tilde{\phi}_{pT}^{H_0} \circ \mathcal{R}_\epsilon = \tau_{pr} \circ \overline{\Phi}_\epsilon^p$. By (19), we obtain

$$\pi \circ \tilde{\phi}_{pT}^{H_0}(x, \epsilon_p \overline{I}_1) = \pi \circ \tilde{\phi}_{pT}^{H_0}(x, \epsilon_p \overline{I}_2),$$

which contradicts the fact that $(\pi \circ \tilde{\phi}_{pT}^{H_0})|_{\pi^{-1}(x)}$ is injective. As a conclusion, the geodesic flow of g has no conjugate points.

By the theorem of Burago and Ivanov, this implies that g is flat: there exists a C^∞ diffeomorphism ψ of \mathbb{T}^n isotopic to the identity and a positive definite $\overline{B} \in L_s(\mathbb{R}^n, (\mathbb{R}^n)^*)$ such that

$$\forall \theta \in \mathbb{T}^n, \forall (U, V) \in \mathbb{R}^n \times \mathbb{R}^n, \quad \langle A(\psi(\theta))^{-1} D\psi(\theta) \cdot U, D\psi(\theta) \cdot V \rangle = \langle \overline{B}U, V \rangle.$$

Hence we have

$$\forall \theta \in \mathbb{T}^n, \quad D\psi(\theta)^{-1} A(\psi(\theta)) D\psi(\theta)^{-T} = \overline{A} := \overline{B}^{-1}. \quad (20)$$

Let us consider the symplectic diffeomorphism of $T^*\mathbb{T}^n$

$$G_1(\theta, I) = (\psi(\theta), D\psi(\theta)^{-T} I) \quad (21)$$

and the Hamiltonian $H_1 = H_0 \circ G_1$. Then $H_1 = H \circ G$, with

$$G(\theta, I) = G_0 \circ G_1(\theta, I) = (\psi(\theta), I_\infty + Du(\psi(\theta)) + D\psi(\theta)^{-T} I)$$

and

$$G^{-1} \circ \phi_T^H \circ G = G_1^{-1} \circ \phi_T^{H_0} \circ G_1 = \phi_T^{H_1}.$$

We have $\phi_T^{H_1}(\theta, 0) = (\theta, 0)$. By (21) and an easy computation,

$$\partial_I \phi_{T,1}^{H_1}(\theta, 0) = \partial_I (G_1^{-1} \circ \phi_T^{H_0} \circ G_1)_1(\theta, 0) = D\psi(\theta)^{-1} A(\psi(\theta)) D\psi(\theta)^{-T} = \overline{A}$$

With the same arguments that we had used to obtain (18), we can conclude that

$$\phi_T^{H_1}(\theta, I) = (\theta + \overline{A}I + O(I^2), I + O(I^3)).$$

The proof of Proposition 6 is complete. \square

2.4 Proof of Proposition 7

As in the proof of Proposition 6, we introduce $\Phi_\epsilon = \mathcal{R}_\epsilon^{-1} \circ \phi_T^H \circ \mathcal{R}_\epsilon$, where $\mathcal{R}_\epsilon(\theta, I) = (\theta, \epsilon I)$. Then Φ_ϵ is an exact symplectic diffeomorphism of $T^*\mathbb{T}^n$ and by (3), one has

$$\Phi_\epsilon(\theta, I) = (\theta + \epsilon \bar{A}I + O(\epsilon^2), I + O(\epsilon^2)).$$

By Lemma 2.1, the exact symplectic diffeomorphism $\mathcal{U}_m = \Phi_{1/m}^m$ satisfies

$$\mathcal{U}_m(\theta, I) = (\theta + \bar{A}I + O(1/m), I + O(1/m)).$$

Here $O(1/m)$ means that for any $k \in \mathbb{N}$, and any compact neighborhood K of $0_{\mathbb{T}^n}$, the C^k norm of the remainder restricted to K is less or equal to $C_{k,K}/m$.

The matrix \bar{A} being non-degenerate, by a theorem essentially due to J. Moser (see for instance Theorem 1.2.3. in [Bos86], or [Zeh76]), for any strongly Diophantine $\bar{\omega}$, for m large enough (depending on $\bar{\omega}$) there is a Lagrangian embedding $\rho_m : \mathbb{T}^n \rightarrow T^*\mathbb{T}^n$ with $\rho_m(\eta) = (\eta + o(1), \bar{A}^{-1}\bar{\omega} + o(1))$ such that

$$\forall \eta \in \mathbb{T}^n, \quad \mathcal{U}_m(\rho_m(\eta)) = \rho_m(\eta + \bar{\omega}).$$

Here $u_m = o(\alpha_m)$ means that for any $k \geq 0$, $\alpha_m^{-1} \|u_m\|_{C^k(\mathbb{T}^n)}$ tends to 0 as $m \rightarrow \infty$. Now let $j_m(\eta) = \mathcal{R}_{1/m}(\rho_m(\eta))$. Then

$$\phi_{mT}^H(j_m(\eta)) = j_m(\eta + \bar{\omega}).$$

Moreover, we have

$$j_m(\eta) = \left(\eta + o(1), \frac{\bar{A}^{-1}\bar{\omega}}{m} + o(1/m) \right).$$

This completes the proof of Proposition 7.

2.5 Abundance of strictly ergodic invariant tori

Following [FH77], we will say that a set $K \subset T\mathbb{T}^n$ that is invariant by a Tonelli flow (ϕ_t^H) is *strictly ergodic* if:

- the restricted flow $(\phi_t^H|_K)$ has a unique invariant Borel probability measure; this measure is denoted by μ ;
- the support of μ is K .

If K is strictly ergodic, then the restricted flow $(\phi_t^H|_K)$ is minimal (i.e. has no nontrivial invariant closed subset).

A. Fathi and M. Herman proved in [FH77] that if (X, d) is a compact metric space, then the set of its strictly ergodic homeomorphisms is a G_δ -subset of the set of its homeomorphisms endowed with its usual C^0 topology.

If \mathcal{T} is one of the tori given by theorem 2, then the homeomorphism $(\phi_{1|\mathcal{T}}^H)$ is strictly ergodic. As the set of the tori given by theorem 2 is dense in the set of the tori given by theorem 1, we deduce:

Theorem 3. *Let (ϕ_t^H) be a C^∞ Tonelli flow of $T^*\mathbb{T}^n$ with no conjugate points and let \mathfrak{F} be the continuous foliation in invariant Lagrangian tori that is given by theorem 1. Then there is a dense G_δ subset \mathcal{G} of \mathfrak{F} such that, for every $\mathcal{T} \in \mathcal{G}$, then $\phi_{1|\mathcal{T}}^H$ is strictly ergodic.*

3 Zero entropy

Since the arguments of this section are more general, M will denote a closed, smooth manifold. Obviously, the case of the torus \mathbb{T}^n is included in this setting. Recall that in this setting, the notion of C^0 integrability persists. A Hamiltonian H on T^*M is said to be C^0 integrable if T^*M is partitioned by Lipschitz, invariant Lagrangian submanifold which are Hamiltonianly isotopic to the 0-section. As proved in [Arn10, BdS12], in the case of Tonelli Hamiltonians, those submanifolds must be graphs.

Theorem 4. *Let $H : T^*M \rightarrow \mathbb{R}$ a C^3 Tonelli Hamiltonian that is C^0 integrable. Then for every invariant Borel probability measure, the Lyapunov exponents are zero.*

Because of Ruelle's inequality (see [Rue78]), this implies:

Corollary 4.1. *The metric entropy of every Borel probability measure that is invariant by a C^0 integrable Tonelli Hamiltonian is zero.*

Because of the variational principle (see [HK95] p.181 for example), this implies:

Corollary 4.2. *The topological entropy of the Hamiltonian flow of every C^0 integrable Tonelli Hamiltonian is zero.*

Remark 5. Even if the invariant foliation is not C^1 , if all the invariant C^0 Lagrangian graphs are assumed to be everywhere differentiable, the result is straightforward: let us assume that some ergodic Borel probability measure has at least one non-zero Lyapunov exponent. Because the dynamics is symplectic, then the number $d \geq 1$ of positive Lyapunov exponents is equal to the number of negative Lyapunov exponents. Then the support of this ergodic measure is contained in some invariant differentiable C^1 Lagrangian graph Γ . Let $\zeta \in T^*M$ be a regular point for μ . There exist two d -dimensional embedded open disks D^u and D^s that contain ζ , the first one in the unstable set of ζ and the second one in its stable set (see theorem 6.1. in [Rue79]). Moreover, $T_\zeta D^u$ and $T_\zeta D^s$ are transverse and such that $T_\zeta D^u \oplus T_\zeta D^s$ is a symplectic subspace of $T_\zeta(T^*M)$.

Then $D^u \cup D^s \subset \Gamma$. Indeed, let us consider $\zeta' \in D^u \cup D^s$ and let us assume that $\zeta' \notin \Gamma$. Then there exists another invariant Lagrangian graph Γ_1 such that $\zeta' \in \Gamma_1$ and $\Gamma \cap \Gamma_1 = \emptyset$. If for example $\zeta' \in D^s$, we obtain:

- $\lim_{t \rightarrow +\infty} d(\phi_t^H(\zeta), \phi_t^H(\zeta')) = 0$;
- $\forall t \in \mathbb{R}, \quad \phi_t^H(\zeta) \in \Gamma, \quad \phi_t^H(\zeta') \in \Gamma_1$.

Because Γ and Γ_1 are compact, this is impossible.

We deduce that $D^u \cup D^s \subset \Gamma$ and then $T_\zeta D^u \oplus T_\zeta D^s \subset T_\zeta \Gamma$. But $T_\zeta D^u \oplus T_\zeta D^s$ is a $2d$ -dimensional symplectic subspace and $T_\zeta \Gamma$ is Lagrangian, hence this cannot happen.

When a C^0 Lagrangian invariant graph is not assumed to be differentiable, the symplectic product of two of its tangent vectors (in a generalized sense) may be non-zero: consider what happens at the hyperbolic critical point of the Hamiltonian $H : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $H(\theta, r) = \frac{1}{2}r^2 + \cos(2\pi\theta)$: one separatrix is a Lipschitz Lagrangian invariant graph such that the tangent cone at the critical point contains the stable and unstable tangent lines, and the symplectic product of one stable vector with an unstable one is non-zero.

Remark 6. To prove Theorem 4, proposition 8 and subsection 3.2 are useless. But we prefer to give a particular proof in the cases of non-uniform hyperbolicity and of atomic measures because the arguments are simpler in these cases, even if they do not work for the general case.

3.1 Proof of theorem 4 in the case of atomic measures

We assume that H is a C^0 integrable Tonelli Hamiltonian of T^*M . Let us prove:

Proposition 8. *Let $H : T^*M \rightarrow \mathbb{R}$ be a C^k , Tonelli Hamiltonian that is C^0 integrable (with $k \geq 2$). Then the set of all critical points of H is a C^{k-1} Lagrangian graph.*

Remark 7. Proposition 8 is very similar to what was noted in remark 1. However, for completeness, we will recall the main arguments in this setting.

Corollary 4.3. *The Lyapunov exponents of every invariant measure supported by a critical point are zero.*

Proof of proposition 8 and corollary 4.3. As H is C^2 , convex and superlinear in the fiber direction, for every $x \in M$ there exists a unique $p \in T_x^*M$ such that (x, p) is a critical point of $H(x, \cdot)$, i.e. such that $\partial_p H(x, p) = 0$ (we write this equation in charts). Because the Hessian $\partial_p^2 H$ is non-degenerate, we can use the implicit function theorem to deduce that $\Sigma = \{(x, p), \partial_p H(x, p) = 0\}$ is the graph of a C^{k-1} -function.

Let us now prove that all the points of Σ are critical points of H . If not, let $\zeta \in \Sigma$ such that $X_H(\zeta) \neq 0$. Then $X_H(\zeta) = (0, -\partial_x H(\zeta))$ is a non-vanishing vertical vector. But in the section 3.5. of [Arn08a], it is proved that $X_H(\zeta)$ is contained in the two Green bundles at ζ , that are transverse to the vertical. Hence $X_H(\zeta) = 0$ and ζ is a critical point of H .

Let us notice that $(\phi_{t|\Sigma}^H)$ is just the identity map. Hence if $v \in T_\zeta \Sigma$ is any vector tangent to Σ , its orbit $(D\phi_t^H.v)$ is constant and then by the dynamical criterion given in 3.5 of [Arn08a], is in the two Green bundles $G_-(\zeta)$ and $G_+(\zeta)$. As $T_\zeta \Sigma$, $G_-(\zeta)$ and $G_+(\zeta)$ have the same dimension, we deduce that $T_\zeta \Sigma = G_-(\zeta) = G_+(\zeta)$ is Lagrangian and then Σ is Lagrangian.

In a symplectic linear chart where we choose the first coordinates in $T_\zeta \Sigma$, the matrix of $D\phi_t^H(\zeta)$ is a symplectic matrix $A_t = \begin{pmatrix} I_n & B(t) \\ 0_n & D(t) \end{pmatrix}$. Because it is symplectic, we have: $D(t) = I_n$ and $A_t^N = \begin{pmatrix} I_n & NB(t) \\ 0 & I_n \end{pmatrix}$. This implies that all the Lyapunov exponents of the atomic measure supported at ζ are zero. □

Let us now assume that μ is an invariant ergodic measure with at least one non-vanishing Lyapunov exponent the support of which is contained in a certain C^0 Lagrangian invariant graph Γ . By proposition 8, the support of μ contains no critical points of H . As noticed in the previous remark 5, we can choose a point ζ in the support of μ and two embedded disks D^u and D^s containing ζ , the first one in the unstable set of ζ , the second one in the stable set of ζ . Then $D^u \cup D^s \subset \Gamma$.

3.2 Proof of Theorem 4 in the non-uniformly hyperbolic case

Let us assume that there are exactly $2(n-1)$ non vanishing exponents (there is always 2 vanishing Lyapunov exponents, one in the flow direction and the other one in the energy direction). Let us define the weak local stable and unstable manifolds $W^s(\zeta) = \bigcup_{t \in (-\varepsilon, \varepsilon)} \phi_t^H(D^s)$ and $W^u(\zeta) = \bigcup_{t \in (-\varepsilon, \varepsilon)} \phi_t^H(D^u)$. Then they are n -dimensional C^{k-1} submanifolds that are contained in Γ and such that $W^u(\zeta) \cap W^s(\zeta) = \{\phi_t^H(\zeta); x \in (-\varepsilon, \varepsilon)\}$. This cannot happen in the n -dimensional (topological) manifold Γ .

Hence in this case the proof just uses simple topological arguments. But we cannot use the same strategy when there are fewer non-zero Lyapunov exponents.

3.3 Proof of Theorem 4 in the general case

We introduce the notation: $d = \dim(D^u) = \dim(D^s)$. We deduce from theorem 2 of [Arn12] that is recalled in the appendix too that at μ almost every point, we have: $\dim(G_-(\zeta) \cap G_+(\zeta)) = n - d$, hence we can assume this equality for the chosen ζ .

Because $G_-(\zeta)$ and $G_+(\zeta)$ are n -dimensional and transverse to the vertical, their image by $D\pi(\zeta)$ is $T_x M$. Let us now fix a chart U of M at $x = \pi(\zeta)$. We use the notations: $g(x) = D\pi(G_-(\zeta) \cap G_+(\zeta))$. Identifying U with a part of \mathbb{R}^n via the chart, we say that g defines an affine $(n - d)$ -dimensional foliation \mathcal{G} of U .

As $G_-(\zeta) = T_\zeta D^s \oplus (G_-(\zeta) \cap G_+(\zeta))$ and $G_+(\zeta) = T_\zeta D^u \oplus (G_-(\zeta) \cap G_+(\zeta))$ (see [Arn12]), the leaves of the foliation \mathcal{G} are transverse to $d^u = \pi(D^u)$ and $d^s = \pi(D^s)$ if U is small enough.

Let us now choose a non-zero vector $v^s \in T_\zeta D^s \setminus \{0\}$ and let us use the notation: $w^s = D\pi(\zeta) \cdot v^s$. Because $G_+(\zeta) = T_\zeta D^u \oplus (G_-(\zeta) \cap G_+(\zeta))$, we have $T_x M = D\pi(\zeta)(T_\zeta D^u) \oplus g(x)$. Hence there exists $v^u \in T_\zeta D^u$ and $v \in g(x)$ such that $w^s = w^u + v$ if $w^u = D\pi(\zeta) \cdot v^u$. Using the definition of $T_x d^u$, $T_x d^s$ and the transversality of the foliation \mathcal{G} to d^u and d^s , we deduce the existence of $\varepsilon > 0$ and for every $t \in [0, \varepsilon]$ of: $x^s(t) = x + tw^s + o(t) \in d^s$, $x^u(t) = x + tw^u + o(t) \in d^u$ such that $x^s(t) - x^u(t) \in g$.

Then we use the notation: $\zeta^u(t) = \pi_{|\Gamma}^{-1}(x^u(t))$ and $\zeta^s(t) = \pi_{|\Gamma}^{-1}(x^s(t))$. Because Γ is a Lipschitz graph containing D^u and D^s , we deduce that we have in chart:

$$\zeta^u(t) = tv^u + o(t) \in D^u \quad \text{and} \quad \zeta^s(t) = tv^s + o(t).$$

Let us recall (see [Arn12]) that $G_-(\zeta) + G_+(\zeta) = (G_-(\zeta) \cap G_+(\zeta)) \oplus T_\zeta D^u \oplus T_\zeta D^s$. As $v^s \neq 0$, we have then: $v^u - v^s \notin G_-(\zeta) \cap G_+(\zeta)$. Hence (in chart):

$$(*) \quad \lim_{t \rightarrow 0} \frac{1}{t} (\zeta^u(t) - \zeta^s(t)) = v^u - v^s \notin G_-(\zeta) \cap G_+(\zeta).$$

The submanifold Γ is the graph of a Lipschitz map $\gamma : U \rightarrow \mathbb{R}^n$. By Rademacher theorem, γ is Lebesgue almost everywhere differentiable. Let us assume that γ is differentiable almost everywhere along the segment $[\pi \circ \zeta^u(t), \pi \circ \zeta^s(t)]$ for a while. Then $\zeta^u(t) - \zeta^s(t) = (x^u(t) - x^s(t), \gamma(x^u(t)) - \gamma(x^s(t)))$ is equal to:

$$(**) \quad \zeta^u(t) - \zeta^s(t) = \int_0^1 (x^u(t) - x^s(t), D\gamma(x^s(t) + \sigma(x^u(t) - x^s(t)))) \cdot (x^u(t) - x^s(t)) d\sigma.$$

Proposition 4.3. of [Arn08a] states that:

$$\begin{aligned} G_- \circ \pi_{|\Gamma}^{-1}(x^s(t) + \sigma(x^u(t) - x^s(t))) &\leq T_{\pi_{|\Gamma}^{-1}(x^s(t) + \sigma(x^u(t) - x^s(t)))} \Gamma \\ &\leq G_+ \circ \pi_{|\Gamma}^{-1}(x^s(t) + \sigma(x^u(t) - x^s(t))), \end{aligned}$$

i.e. that the tangent Lagrangian subspace to Γ is between the two Green bundles at the points where γ is differentiable. Moreover, $G_- \leq G_+$, G_- is lower semicontinuous and G_+ is upper semicontinuous (see [Arn08a]).

We introduce the following notations: $G_{\pm}(\eta)$ is the graph of the symmetric matrix $S_{\pm}(\eta)$. Because of the semicontinuity of G_- and G_+ , there exist $\Delta S_{\pm}(\eta)$ a semi-positive matrix that depends on η , vanishes at ζ and is continuous at ζ such that: $S_+(\eta) \leq S_+(\zeta) + \Delta S_+(\eta)$ and $S_-(\zeta) - \Delta S_-(\eta) \leq S_-(\eta)$. We obtain then at every point η where Γ is differentiable:

$$S_-(\zeta) - \Delta S_-(\eta) \leq S_-(\eta) \leq D\gamma(\pi(\eta)) \leq S_+(\eta) \leq S_+(\zeta) + \Delta S_+(\eta).$$

Let us consider the restrictions (as quadratic forms) of the previous matrices to $g(x)$ and let us denote them with a “ $\widetilde{}$ ”. We have then $\widetilde{S}_+(\zeta) = \widetilde{S}_-(\zeta)$ and:

$$\widetilde{S}_-(\zeta) - \Delta \widetilde{S}_-(\eta) \leq \widetilde{S}_-(\eta) \leq \widetilde{D\gamma(\pi(\eta))} \leq \widetilde{S}_+(\eta) \leq \widetilde{S}_+(\zeta) + \Delta \widetilde{S}_+(\eta).$$

Moreover, $\lim_{\eta \rightarrow \zeta} \Delta \widetilde{S}_{\pm}(\eta) = 0$. This implies that:

$$(***) \lim_{x' \rightarrow x} \widetilde{D\gamma(x')} = \widetilde{S}_{\pm}(\zeta).$$

Let us prove that this implies that

$$(****) \lim_{\eta \rightarrow \zeta} D\gamma(\pi(\eta))|_g = S_{\pm}(\zeta)|_g.$$

The map γ being Lipschitz, the $D\gamma(x')$ are uniformly bounded. Hence, if (***) is not true, we can find a sequence (x_k) converging to x such that $(D\gamma(x_k))_{k \in \mathbb{N}}$ converges and $\lim_{k \rightarrow \infty} D\gamma(x_k)|_g \neq S_+(\zeta)|_g$. As $D\gamma(x_k) \leq S_+(x_k)$ and as S_+ is upper semicontinuous, we know that $S_+(\zeta) - \lim_{k \rightarrow \infty} D\gamma(x_k)$ is positive semi-definite, hence its kernel coincides with its isotropic cone. And we have proved in (***) that g is in this isotropic cone. This implies $\lim_{k \rightarrow \infty} D\gamma(x_k)|_g = S_+(\zeta)|_g$ and gives a contradiction.

Hence if η is close to ζ and such that γ is differentiable at η : $D\gamma(\pi(\eta))|_g = S + \alpha(\eta)$ where $\lim_{\eta \rightarrow \zeta} \alpha(\eta) = 0$ and $G_-(\zeta) \cap G_+(\zeta)$ is the graph of S above g . Replacing in (**), we obtain: $\zeta^u(t) - \zeta^s(t) =$

$$\left(x^u(t) - x^s(t), \int_0^1 \left[S + \alpha \left(x^s(t) + \sigma(x^u(t) - x^s(t)) \right) \right] d\sigma \cdot (x^u(t) - x^s(t)) \right)$$

i.e:

$$\zeta^u(t) - \zeta^s(t) = (t(w^u - w^s) + o(t), tS(w^u - w^s) + o(t))$$

and we deduce:

$$\lim_{t \rightarrow 0} \frac{1}{t} (\zeta^u(t) - \zeta^s(t)) = (w^u - w^s, S(w^u - w^s)) \in G_-(\zeta) \cap G_+(\zeta),$$

which contradicts (*).

If γ is not differentiable almost everywhere along the segment $[\zeta^u(t), \zeta^s(t)]$, using Fubini theorem, we can replace $\zeta^u(t)$ and $\zeta^s(t)$ by $\eta^u(t) = \zeta^u(t) + o(t)$ and $\eta^s(t) = \zeta^s(t) + o(t)$ in such a way that γ is differentiable almost everywhere along the segment $[\eta^u(t), \eta^s(t)]$, and we use the same argument as before to conclude.

Appendix

Aubry sets

If λ is a C^∞ closed 1-form of \mathbb{T}^n , then the map $T_\lambda : T^*\mathbb{T}^n \rightarrow T^*\mathbb{T}^n$ defined by : $T_\lambda(q, p) = (q, p + \lambda(q))$ is a symplectic C^∞ diffeomorphism; therefore, we have : $(\phi_t^{H \circ T_\lambda}) = (T_\lambda^{-1} \circ \phi_t \circ T_\lambda)$, i.e. the Hamiltonian flow of H and $H \circ T_\lambda$ are conjugated. Moreover, the Tonelli Hamiltonian function $H \circ T_\lambda$ is associated to the Tonelli Lagrangian function $L - \lambda$, and it is well-known that : $(\varphi_t^L) = (\varphi_t^{L-\lambda})$; the two Euler-Lagrange flows are equal. Let us emphasize that these flows are equal, but the Lagrangian functions, and then the Lagrangian actions differ.

For a Tonelli Lagrangian function (L or $L - \lambda$), J. Mather introduced in [Mat93] (see [Mañ96] too) a particular subset $\mathcal{A}(L - \lambda)$ of $T^*\mathbb{T}^n$ which he called the “static set” and which is now usually called the “Aubry set”. There exist different but equivalent definitions of this set (see [CIPP98], [Fat08], [Mañ96]) and it is known that two closed 1-forms that are in the same cohomological class define the same Aubry set:

$$[\lambda_1] = [\lambda_2] \in H^1(\mathbb{T}^n, \mathbb{R}) \Rightarrow \mathcal{A}(L - \lambda_1) = \mathcal{A}(L - \lambda_2).$$

We can then introduce the following notation: if $c \in H^1(\mathbb{T}^n, \mathbb{R})$ is a cohomological class, we have $\mathcal{A}_c = \mathcal{A}_c(L) = \mathcal{A}(L - \lambda)$ where λ is any closed 1-form belonging to c . Then \mathcal{A}_c is compact, non empty and invariant under (φ_t^L) . Moreover, J. Mather proved in [Mat93] that it is a Lipschitz graph above a part of the zero-section (see [Fat08] too).

As we are interested in the Hamiltonian dynamics as well as in the Lagrangian ones, let us define the dual Aubry set:

- if H is the Hamiltonian function associated to the Tonelli Lagrangian function L , its *dual Aubry set* is $\mathcal{A}^*(H) = \mathcal{L}(\mathcal{A}(L))$;
- if $c \in H^1(\mathbb{T}^n, \mathbb{R})$ is a cohomological class, then $\mathcal{A}_c^* = \mathcal{A}_c^*(H) = \mathcal{L}(\mathcal{A}_c(L))$ is the *c-dual Aubry set*; let us notice that for any closed 1-form λ belonging to c , we have: $T_\lambda(\mathcal{A}^*(H \circ T_\lambda)) = \mathcal{A}_c^*(H)$.

These sets are invariant by the Hamiltonian flow (ϕ_t^H) .

Then there exists a real number denoted by $\alpha_H(c)$ such that $\mathcal{A}_c^* \subset H^{-1}(\alpha_H(c))$ (see [Car95] and [Mat91]), i.e. each dual Aubry set is contained in an energy level. The following property is a well-known characterization of the projected Aubry set: $x_0 \in \mathbb{T}^n$ is such that there exists a sequence of absolutely continuous curves $\gamma_k : [0, T_k] \rightarrow \mathbb{T}^n$, with $(T_k) \rightarrow \infty$, such that $\gamma_k(0) = \gamma_k(T_k) = x_0$ and

$$\lim_{k \rightarrow +\infty} \int_0^{T_k} (L(\gamma_k, \gamma_k') - \lambda(\gamma_k') + \alpha_H(c)) = 0,$$

if and only if $x_0 \in \pi(\mathcal{A}_c)$.

The following proposition is proved in [Arn11]:

Proposition. *Let $c \in H^1(\mathbb{T}^n, \mathbb{R})$ and $\lambda \in c$, $\varepsilon > 0$ and let $L : T\mathbb{T}^n \rightarrow \mathbb{R}$ be a Tonelli Lagrangian function. Then there exists $T_0 > 0$ such that:*

$\forall T \geq T_0, \forall (x_0, v_0) \in \mathcal{A}_c, \forall \gamma : [0, T] \rightarrow \mathbb{T}^n$ minimizing for $L - \lambda$ between x_0 and x_0 , i.e.:

$$\forall \eta : [0, T] \rightarrow \mathbb{T}^n,$$

$$\eta(0) = \eta(T) = x_0 \Rightarrow \int_0^T (L(\gamma, \gamma') - \lambda(\gamma') + \alpha_H(c)) \leq \int_0^T (L(\eta, \eta') - \lambda(\eta') + \alpha_H(c))$$

then we have: $d((x_0, v_0), (x_0, \gamma'(0))) \leq \varepsilon$

Mather sets

The general references for this section are [Mat91] and [MS11]. Let $\mathcal{M}(L)$ be the space of compactly supported Borel probability measures that are invariant by the Euler-Lagrange flow (φ_t^L) . To every $\mu \in \mathcal{M}(L)$ we associate its average action $A_L(\mu) = \int_{T\mathbb{T}^n} L d\mu$. It is proved in [Mat91] that for every $f \in C^1(\mathbb{T}^n, \mathbb{R})$, we have:

$$\int df(q) \cdot v d\mu(q, v) = 0.$$

Therefore we can define on $H^1(\mathbb{T}^n, \mathbb{R})$ a linear functional $\ell(\mu)$ by:

$$\ell(\mu)([\lambda]) = \int \lambda(q) \cdot v d\mu(q, v)$$

(here λ designates any closed 1-form). Then there exists a unique element $\rho(\mu) \in H_1(\mathbb{T}^n, \mathbb{R})$ such that:

$$\forall \lambda, \int_{T\mathbb{T}^n} \lambda(q) \cdot v d\mu(q, v) = [\lambda] \cdot \rho(\mu).$$

The homology class $\rho(\mu)$ is called the *rotation vector* of μ . Then the map $\mu \in \mathcal{M}(L) \rightarrow \rho(\mu) \in H_1(\mathbb{T}^n, \mathbb{R})$ is onto. Mather β -function $\beta : H_1(\mathbb{T}^n, \mathbb{R}) \rightarrow \mathbb{R}$ associates to each homology class $h \in H_1(\mathbb{T}^n, \mathbb{R})$ the minimal value of the average action A_L over the set of measures of $\mathcal{M}(L)$ with rotation vector h . We have:

$$\beta(h) = \min_{\substack{\mu \in \mathcal{M}(L) \\ \rho(\mu)=h}} A_L(\mu).$$

A measure $\mu \in \mathcal{M}(L)$ realizing such a minimum, i.e. such that $A_L(\mu) = \beta(\rho(\mu))$ is called a *minimizing measure with rotation vector* $\rho(\mu)$. The β function is convex and superlinear, and its conjugate function (given by Fenchel duality) $\alpha : H^1(\mathbb{T}^n, \mathbb{R}) \rightarrow \mathbb{R}$ is defined by:

$$\alpha([\lambda]) = \max_{h \in H_1(\mathbb{T}^n, \mathbb{R})} ([\lambda] \cdot h - \beta(h)) = - \min_{\mu \in \mathcal{M}(L)} A_{L-\lambda}(\mu).$$

A measure $\mu \in \mathcal{M}(L)$ realizing the minimum of $A_{L-\lambda}$ is called a $[\lambda]$ -*minimizing measure*. Observe that the function α is exactly the same as the function α_H defined in the section on Aubry sets. It is convex and superlinear.

Being convex, Mather's β function has a subderivative at any point $h \in H_1(\mathbb{T}^n, \mathbb{R})$; i.e. there exists $c \in H^1(\mathbb{T}^n, \mathbb{R})$ such that:

$$\forall k \in H_1(\mathbb{T}^n, \mathbb{R}), \quad \beta(h) + c \cdot (k - h) \leq \beta(k).$$

We denote by $\partial\beta(h)$ the set of all the subderivatives of β at h . By Fenchel duality, we have: $c \in \partial\beta(h) \Leftrightarrow c \cdot h = \alpha(c) + \beta(h)$.

Then we introduce the following notations:

- if $h \in H_1(\mathbb{T}^n, \mathbb{R})$, the Mather set for the rotation vector h is:

$$\mathcal{M}^h(L) = \bigcup \{\text{supp}(\mu); \quad \mu \text{ is minimizing with rotation vector } h\};$$

- if $c \in H^1(\mathbb{T}^n, \mathbb{R})$, the Mather set for the cohomology class c is:

$$\mathcal{M}_c(L) = \bigcup \{\text{supp}(\mu); \quad \mu \text{ is } c\text{-minimizing}\};$$

where $\text{supp}(\mu)$ designates the support of the measure μ .

The sets $\mathcal{M}^h(L)$ and $\mathcal{M}_c(L)$ are invariant by φ_t^L .

The following equivalences are proved in [MS11] for any pair $(h, c) \in H_1(M, \mathbb{R}) \times H^1(M, \mathbb{R})$:

$$\mathcal{M}^h(L) \cap \mathcal{M}_c(L) \neq \emptyset \quad \Leftrightarrow \quad \mathcal{M}^h(L) \subset \mathcal{M}_c(L) \quad \Leftrightarrow \quad c \in \partial\beta(h).$$

The dual Mather set for the cohomology class c is defined by: $\mathcal{M}_c^*(H) = \mathcal{L}(\mathcal{M}_c(L))$. If $\mathcal{M}^*(\mathcal{H})$ designates the set of compactly supported Borel probability measures of T^*M that are invariant by the Hamiltonian flow (ϕ_t^H) , then the map $\mathcal{L}_* : \mathcal{M}(L) \rightarrow \mathcal{M}^*(H)$ that push forward the measures by \mathcal{L} is a bijection. We denote $\mathcal{L}_*(\mu)$ by μ^* and say that the measures are dual. We say too that μ^* is minimizing if μ is minimizing in the previous sense.

Moreover, the Mather set $\mathcal{M}_c^*(H)$ is a subset of the Aubry set $\mathcal{A}_c^*(H)$ and every invariant Borel probability measure the support of whose is in $\mathcal{A}_c^*(H)$ is c -minimizing.

Mañé sets

The Mañé set $\mathcal{N}(L)$ of L is the set of $(\gamma(0), \gamma'(0)) \in T\mathbb{T}^n$ such that for all segment $[a, b] \subset \mathbb{R}$, $\gamma|_{[a, b]}$ is a minimizer for L . The *dual Mañé set* is then $\mathcal{N}^*(H) = \mathcal{L}(\mathcal{N}(L))$. For all $c \in H^1(\mathbb{T}^n, \mathbb{R})$ and $\lambda \in c$, then $\mathcal{N}_c = \mathcal{N}_c(L) = \mathcal{N}(L - \lambda)$ is independent of the choice of $\lambda \in c$ and the c -dual Mañé set is $\mathcal{N}_c^*(H) = \mathcal{L}(\mathcal{N}_c(L)) = T_\lambda(\mathcal{N}^*(H \circ T_\lambda))$. It is invariant under (ϕ_t^H) , compact and non empty but is not necessarily a graph.

For every cohomological class $c \in H^1(\mathbb{T}^n)$, we have the inclusion : $\mathcal{M}_c^*(H) \subset \mathcal{A}_c^*(H) \subset \mathcal{N}_c^*(H) \subset H^{-1}(\alpha_H(c))$ (see [Car95] and [Mat91]), i.e. each dual Mañé set is contained in an energy level.

Moreover, the ω and α -limit sets of every point of the Mañé set $\mathcal{N}_c^*(H)$ are contained in the Aubry set $\mathcal{A}_c^*(H)$.

The link with the weak KAM theory

The reference for this section is [Fat08]. We just recall some results that are used in the article; a C^0 Lagrangian graph is the graph of $a + du : \mathbb{T}^n \rightarrow (\mathbb{R}^n)^*$ where $a \in (\mathbb{R}^n)^*$ and $u \in C^1(\mathbb{T}^n, \mathbb{R})$. Then $a \in H^1(\mathbb{T}^n, \mathbb{R})$ is the *cohomology class* of the graph. We have:

if \mathcal{G} is a Lagrangian graph with cohomology class c that is invariant by Φ_t , then

$$\mathcal{A}_c^* \subset \mathcal{G} \subset \mathcal{N}_c^*.$$

Moreover, if \mathcal{A}_c^* (resp. \mathcal{N}_c^*) is a graph above the whole \mathbb{T}^n , then we have $\mathcal{A}_c^* = \mathcal{N}_c^*$ and it is a C^0 Lagrangian graph.

It is proved in [Fat03] that every C^0 Lagrangian graph that is invariant by a Tonelli Hamiltonian is a Lipschitz graph.

Green bundles

Recall (see [CI99] and [Arn08a] for details) that if $s \in \mathbb{R} \mapsto (x, p) = \phi_s^H(x_0, p_0) \in T^*\mathbb{T}^n$ is an orbit of the Hamiltonian flow that is free of conjugate points, one may define two

bundles G_- and G_+ (called the Green bundles) by

$$G_+(x, p) = \lim_{t \rightarrow +\infty} D\phi_t^H(\phi_{-t}^H(x, p)) \cdot V^*(\phi_{-t}^H(x, p))$$

and

$$G_-(x, p) = \lim_{t \rightarrow +\infty} D\phi_{-t}^H(\phi_t^H(x, p)) \cdot V^*(\phi_t^H(x, p)).$$

Then G_- is the negative Green bundle and G_+ is the positive one.

Every $G_\pm(x, p)$ is a Lagrangian subspace of $T_{(x,p)}T^*\mathbb{T}^n$ that is transverse to the vertical space $V^*(x, p)$, and this bundle is invariant by the Hamiltonian flow: $D\phi_t^H G_\pm(x, p) = G_\pm(\phi_t^H(x, p))$ for all $t \in \mathbb{R}$.

We have of the following criteria (see [CI99] and [Arn08a]): if $w \in T_{(x,p)}(T^*\mathbb{T}^n)$, then

$$w \notin G_+(x, p) \implies \lim_{t \rightarrow +\infty} \|D(\pi \circ \phi_{-t}^H)(x, p) \cdot w\| = +\infty,$$

$$w \notin G_-(x, p) \implies \lim_{t \rightarrow +\infty} \|D(\pi \circ \phi_t^H)(x, p) \cdot w\| = +\infty,$$

where $\|\cdot\|$ denotes the Euclidean norm.

Moreover, G_+ is upper semi-continuous and G_- is lower semi-continuous, and we have at every point: $G_- \leq G_+$ (for the usual order relation on the Lagrangian subspaces that are transverse to the vertical, given by the order on symmetric matrices, see [Arn08a] for details). Hence $\{G_- = G_+\}$ is a G_δ subset of $T^*\mathbb{T}^n$.

It is proved in [Arn08a] that if G is any invariant Lagrangian subspace that is transverse to the vertical space (for example the tangent to some invariant Lipschitz Lagrangian graph), then we have: $G_- \leq G \leq G_+$.

There is a strong link between Oseledet's bundle and Green bundles, as explained in [Arn12]:

Theorem. *Let $H : T^*\mathbb{T}^n \rightarrow \mathbb{R}$ be a Tonelli Hamiltonian and let μ be an ergodic minimizing probability measure. Then the two following assertions are equivalent:*

- *at μ almost every point, $\dim(G_-(x) \cap G_+(x)) = p$;*
- *μ has exactly $2p$ zero Lyapunov exponents, $n - p$ positive ones and $n - p$ negative ones.*

Moreover, if the Oseledet's splitting along the support of μ is denoted by $E^s \oplus E^c \oplus E^u$, then we have: $G_- = E^s \oplus (G_- \cap G_+)$ and $G_+ = E^u \oplus (G_- \cap G_+)$.

References

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