Table-turning to Stable Marriage satisfaction and equity

M. Zavidovique and N. Suvonvorn

Abstract—Running a stable marriage algorithm to pairing images in vision, the global satisfaction and sex equality appear as important constraints as the stability itself. In the present paper we outline a novel algorithm based on the rotation of the “marriage table” to align satisfaction and equity with preferences. It turns out that a direct implementation through the lists is doable. Additionally, it shows improved flexibility in balancing constraints. Known algorithms are compared to the present one on 3000 instances of 200 large populations and performances are discussed. Results on real images are displayed in the case of stereo-pairing and motion understanding applications.

I. INTRODUCTION

In many computer vision applications including motion analysis, stereovision or model fitting, matching is a key step that deserves efficient optimization. In this paper we propose a stable marriage algorithm for image matching. It belongs to a bi-partite graph optimization technique based on the so-called marriage table representation [1]. The BZ algorithm proposed in the latter paper achieves an efficient trade-off between the global satisfaction, the fairness (or sex equality) and the stability thanks to this representation. It provides matching results with maximum sex equality and global satisfaction, and with limited instability - about 5% found solutions are unstable. But its complexity is in O(N^{3/2}), to be compared to the basic Gale-Shapley (GS) algorithm in O(N). Recently, the S-procedure was designed [2] in order to obtain stable matching-results by resolving the BZ oscillating and cycling behaviors in the marriage table. The satisfaction and equity are preserved but the complexity grows from O(N^{3/2}) to O(N^2). The present paper describes a process for generating intermediate versions between BZ and GS called RZ or RGS. The stress here is put on the algorithmic complexity decreasing from O(N^{3/2}) to O(N), while controlling the satisfaction or equality as much as possible.

The paper is organized as follows: we first revisit the stable marriage problem, GS and then BZ based on the marriage table, section 2. Then we explain the RZ algorithm motivations and its variations RGS in section 3. New performances are compared with the GS algorithm and the BZ algorithm in section 4.


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II. STABLE MARRIAGE REVISITED

The stable marriage problem was studied by Gale and Shapley [3] who produced the first algorithm and it remains among the popular combinatorial problems [4]–[7]. In this problem, two finite sub-sets M and W of two respective populations, say men and women, have to match. Assume n = \sqrt{N} is the number of elements, M = \{m_i\}_1^N and W = \{w_j\}_1^N. Each element x creates its preference list l(x) i.e. it sorts all members of the opposite sex from most to least preferred (see example in the next section table 1). A matching M is a one to one correspondence between men and women. If (m, w) is a matched pair in M, M(m) = w and M(w) = m, and \rho_m is the rank of m in the list of w (resp. \rho_w the rank of w in the list of m). Man m and woman w form a blocking pair if (m, w) is not in M but m prefers w to M(m) and w prefers m to M(w). The situation where (m, w) is blocking (m, M(m)) and (M(w), w) is called blocking situation. If there is no blocking pair, then the marriage M is stable.

Gale-Shapley proposed the algorithm to find M with complexity O(N). GS has two different versions: men-optimal and women-optimal. GS with men-optimal can be stated as follows: while there is an unpaired man, pick the unpaired man and the first woman on his list. If she is free, they get married. But if she is already engaged, she compares her current fiancée to the new man. If she prefers the new man, she leaves her current partner according to her preferences. The process continues until there is no more unpaired man. However, the stable matching can be such that everybody is unsatisfied. Normally, Men-optimal brings a stable matching in which men have the best possible partner and women may have the worst and conversely.

Let us note that Gusfield and Irving [8] quote open problems in conclusion for their extensive study of the algorithms of stable marriage and the derived models of optimization (15 years ago). One of them, problem 11 is the egalitarian stable marriage and can be solved in O(n^4 \log n).

Feder [9] has claimed a O(n^{2.5} \log n) and O(n^3) [10]. For them "egalitarian" relates to minimal \Sigma(\rho_m + \rho_w), however we preferred to call this property "global satisfaction", more evocative of the properties of solutions in our applications. And "sex-equal" mentioned as open problem 6 is translated by Irving into \Sigma \rho_m = \Sigma \rho_w, here again we preferred "sex equality" to be coded by \min(\Sigma \rho_m - \rho_w).

In [1], the marriage table representation is proposed for the marriage optimization to meet the three objectives of stability, sex equality and global satisfaction. It is a table with (n+1) lines and (n+1) columns. Lines (resp. columns)
frame the preference orders of men, \{1 ... p \ldots N \infty\} (resp. women, \{1 ... q \ldots N \infty\}). The cell \((p, q)\) contains pairs \((m, w)\) such that \(w\) is the \(p\)th choice of \(m\), and \(m\) is the \(q\)th choice of \(w\). Cells can thus contain more than one pair or none. The cell \((p, \infty)\) (resp. \((\infty, q)\)) contains the pairs \((m, w)\), if any, where \(w\) is the \(p\)th choice of \(m\) (resp \(m\) the choice of \(w\) but \(m\) does not exist in her preference list (resp. \(w\) is not in his preference list). Figure 1 shows the marriage table.

The global satisfaction of matching can be measured by
\[
S = \sum_{(m, w) \in M} (\rho_m + \rho_w).
\]
Note that a solution with maximum global satisfaction would get matched pairs around the origin of the table (bottom-left), as shown figure 2. Conversely, sex equality tends to fit the diagonal of the marriage table. It is defined as
\[
E = \sum_{(m, w) \in M} |\rho_m - \rho_w|,
\]
figure 3.

Stability can be translated into the marriage table too, a blocking situation is represented figure 4. Assuming \((x, y)\) and \((m, w)\) were paired, then \((x, w)\) cannot be in the grey rectangle and \((m, y)\) cannot be in the dashed one.

BZ is an algorithm based on that representation. It consists of scanning the marriage table cells in order to first maximize both criteria concurrently. It scans anti-diagonals forward from maximum to minimum global satisfaction while each one is read in swinging from center to sides meaning maximum to minimum sex equality. In each cell, pairs are married if both partners are free. As it is easily proven that no predefined scan warrants stability (see next section), after all cells have been visited the table is scanned again to remove blocking situations: a blocking pair gets married and corresponding blocked pairs are released. The process repeats until there is no more blocking situation or the iteration number is greater than the population size.

Let us stress upon the bounding result in this process. Given a systematic scan i.e. a permutation of \(\mathbb{N}^2, n = s(i, j)\), there exists an instance of population i.e. a set of preference lists \(\{l(m), l(w)\}_{M \times W}\) such that \((m_i, w_j)\), \((m_k, w_l)\) and \((m_k, w_l)\) are met along the scan in that order, then \((m_i, w_j)\), and \((m_k, w_l)\) are married while \((m_k, w_l)\) is blocking them. It is trivially enough that
\[
s(l_j(m_i)), l_i(m_j)) \leq s(l_i(m_k), l_k(m_i)) \leq s(l_j(m_i), l_k(m_j))\]
and \(l_k(w_j) \leq l_j(w_j)\) and \(l_j(w_k) \leq l_i(w_k)\)

With \(l_j(m_i)\) the ranking order of \(w_j\) in \(l(m_i)\). Figure 5 shows an example of such a case for the anti-diagonal regular scan.
In the next section, an alternative algorithm is proposed to get matching results such that everybody is happy (resp. treated fairly) as much as possible while preserving a complexity in O(N).

III. RZ AND RGS ALGORITHMS

Here, we propose an algorithm that takes advantage from the lesser complexity of GS and from the coding of satisfaction and sex equality introduced through the marriage table.

Indeed, if \( w \) is the \( p^{th} \) choice of \( m \) and \( m \) is the \( q^{th} \) choice of \( w \), then \( p + q \) represents the satisfaction and \( |p - q| \) shows sex equality between \( m \) and \( w \). The primary idea is that scanning the marriage table in a diagonal way is (quasi) equivalent to rotate the same 45° before scanning horizontally or vertically. Doing so, at least equivalent results to the first pass in BZ will be obtained. Figure 6(a) shows the rotated version of the marriage table. It is a table with \( 2n+1 \) lines and \( n+1 \) columns. Lines represent the satisfaction and columns the sex equality. The grey area shows the projected area from the original table. It contains in each cell the couples which have the corresponding satisfaction and sex equality. Thanks to the empty cells after rotation in a sampled space, there is room for expanding couples from one cell over several cells if they concern a same person. A complete order is then recovered following additional constraints. By scanning the latter array left-right and bottom-up, and marrying the occurring free pairs (see figure 6(b)), similar results to the OZ-like algorithm [1] are obtained. Let us call RZ (Rotated Zigzag) the algorithm.

![Figure 6](image.png)

Note that OZ is the simplest algorithm based on the marriage table. BZ is improved version of its to target maximum satisfaction and sex equality, and then SBZ [2] reaches the full stability constraint. It means that we can later complete RZ in the \( S - \) fashion to achieve stability, in contradiction with [11].

Actually a horizontal (resp. vertical) scan of the marriage table is also the basis of GS, where the preference list is ordered following \( p \) for men and \( q \) for women. We should then improve the overall result by merely performing GS on lists transformed by \((p \rightarrow p + q; \ q \rightarrow p - q)\): Instead of transforming the preference lists into the marriage table and then finding the matching with maximum satisfaction and sex equality, we introduce the satisfaction and sex equality into the preference lists then finding the matching result by classical GS. Obviously the stability is no more guaranteed, although GS is run, since it would not be according to the original preferences. But one gets additional variants here as flexible stress can be put on satisfaction, equity or stability in departing more or less from the initial lists. The latter flexibility amounts to the above-mentioned additional constraints supporting the cell expansion up to complete order in the marriage table. In the current version of our algorithm, the man’s and woman’s preference lists are first reordered by increasing \( p + q \), then by increasing \(|p - q|\) in case of equal \( p + q \), and then by increasing \( p \) for man (respectively \( q \) for woman) in case of equal \( p + q \) and \(|p - q|\). Table 1 shows an instance of men and women with their preference list ordered by \( p \) and \( q \) respectively.

### TABLE I

<table>
<thead>
<tr>
<th>Man</th>
<th>Woman</th>
</tr>
</thead>
<tbody>
<tr>
<td>p 1 2 3</td>
<td>q 1 2 3</td>
</tr>
<tr>
<td>A B C</td>
<td>A B C</td>
</tr>
<tr>
<td>2 A C B</td>
<td>3 C B A</td>
</tr>
<tr>
<td>3 C B A</td>
<td>3 C B A</td>
</tr>
</tbody>
</table>

Table 2 shows the value of satisfaction \( p + q \) and sex equality \( |p - q| \) for each element in their preference list.

### TABLE II

| Man \((p+q, |p-q|)\) | Woman \((p+q, |p-q|)\) |
|-----------------|-----------------|
| (2,0) (3,1) (6,0) | (3,1) (3,1) (6,0) |
| (3,1) (4,0) (4,2) | (4,2) (4,0) (6,0) |
| (4,2) (4,0) (6,0) | (4,2) (4,0) (4,2) |

Table 3 shows the same instance of men and women with their preference lists reordered by satisfaction, sex equality and initial preference in the order. We can see that the preference list of 3 and B are reordered by \(|p - q|\) since there is the conflict on \( p + q \). Similarly, the preference list of A is reordered by \( q \) since there is a conflict on both \( p + q \) and \(|p - q|\).

### TABLE III

<table>
<thead>
<tr>
<th>Man</th>
<th>Woman</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
</tr>
<tr>
<td>2</td>
<td>B</td>
</tr>
<tr>
<td>3</td>
<td>C</td>
</tr>
</tbody>
</table>

Then the marriages result from executing GS on the preference lists of table 3.

Remark: As both man’s and woman’s lists are shuffled here in stressing the satisfaction first, GS with man-optimal and woman-optimal are likely to give the same solution.
The novel algorithm introduced here is then a
intermediate version between GS and OZ, let us call RGS
this algorithm. Not that as previously mentioned the
S-stability applies to RGS as well.

In the next section, we compare RGS with GS and BZ to
better understand respective performances.

IV. ALGORITHM PERFORMANCE

We study experimentally the global satisfaction, sex
equality and stability obtained by RGS, and we compare
with GS and BZ. About 3,000 instances are built at random
for 200-large populations. Each algorithm is executed and its
results are displayed. Figure 7 and figure 8 zoom on 30
instances out of the 3,000 in order to display more in detail
the global satisfaction and sex equality respectively. Let us
define the number b of instances where RGS is better than
the better GS or than BZ as:

\[ b = \sum_{\text{all instances}} H_{[GS(S),RGS(S)]} \]

With \( H_{[x]} = 1 \) if \( x > 0 \) else \( H_{[x]} = 0 \), then

\[ \beta = \frac{b \times 100}{\text{number of instances}} \]

We can first note that RGS performs totally better than the
better GS for both satisfaction and sex equality, \( \beta = 100 \).
Comparing RGS with BZ, \( \beta = 8.13 \) and \( \beta = 35.77 \)
respectively for the global satisfaction and sex equality. We
note that RGS is comparable to BZ with respect to GS for
the satisfaction and sex equality achieved. This is indicated
by the following distances: \( (S_{RGS} - S_{BS})/S_{BS} = 5.75\% \),
\( (S_{RGS} - S_{GS})/S_{GS} = 22.64\% \), \( (E_{RGS} - E_{BS})/E_{BS} = 7.24\% \),
and \( (E_{RGS} - E_{GS})/E_{GS} = 46.96\% \) in average.

Comparing the stability, the matching results issued by
GS are totally stable while around 5\% and 87\% of instances
are unstable for BZ and RGS respectively. But, more
important, the average number of blocking pairs per unstable
instance is 10.8 for BZ and 3.3 for RGS, meaning that the
expectation of the number of iterations starting from RGS to
overcome oscillations towards complete stability is
significantly lower than with OZ to BZ and then SBZ.

![Fig. 7. Comparing global satisfaction between methods : (a) GS
man-optimal, (b) GS woman-optimal, (c) BZ, (d) RGS.](image)

![Fig. 8. Comparing sex equality between methods : (a) GS man-
onimal, (b) GS woman-optimal, (c) BZ, (d) RGS.](image)

Table 4 shows numeric comparison between RGS and BZ
or GS for the stability (number of blocking pairs), global
satisfaction and sex equality.

| Table 4: COMPARISON OF STABILITY, GLOBAL SATISFACTION AND SEX EQUALITY BETWEEN METHODS |
|-----------------|-----------------|-----------------|-----------------|
| GS | BZ | RGS | RZw |
| INSTABILITY | 10.8 | 3.3 | 100 |
| BLOCCING PARES | 10.8 | 3.3 | 100 |
| DST | 23.3 | 11.7 | 73.1 |
| LOCAL GOODNESS | 3.3 | 11.7 | 73.1 |
| S | 100 | 100 | 100 |
| E | 100 | 100 | 100 |

V. CONCLUSION

For a quick and temporary conclusion let us show two
types of results: (1) comparative results of matching \( M \)
and \( M' \) to compute the transform \( T(\text{affine + projections}) \)
between left \( L \) and right \( R \) images, (2) dominant motion
extraction by our algorithms from an image sequence. In the
comparison of stereo results, images come from
“http://www.gravitram.com/stereoscopic_photography.htm”,
figure 9. Features to be matched are level line junctions [12].
Figures 10 and 11, we display images of the kind
optical flow classification separates the dominant mobile
figure 15 shows the rebuilt vehicle.

In the motion understanding application, extracted
features are again level line junctions. The program runs on
a PowerPC G4 /1.3GHz and delivers a dependable result
every second. We will soon implement it on a Bi-processor
Xeon onboard our autonomous vehicle PICAR [13] and
extraction every 200 ms is expected. Figure 12 shows two
consecutive images in a sequence. Matching results are
displayed as flows in figure 13. Motion identification using
optical flow classification separates the dominant mobile
object, figure 14(a), and the background, figure 14(b). And
figure 15 shows the rebuilt vehicle.
Fig. 9. Stereo images.

Fig. 10. Comparative results.

Fig. 11. Comparative results.

Fig. 12. Original images.
REFERENCES


Fig. 13. Matching results.

Fig. 14. Motion identification: (a) object, (b) background.

Fig. 15. Extracted object.