BEILINSON-BERNSTEIN LOCALIZATION OVER $\mathbb{Q}$ AND PERIODS OF AUTOMORPHIC FORMS

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Introduction

The present paper is the first in a projected series of articles whose purpose is to draw conclusions from the comparison between an arithmetic conjecture of Deligne and an analytic conjecture of Ichino and Ikeda [D2,II]. Let F be a totally real field. The Ichino-Ikeda conjecture for unitary groups over F [II,H*] relates certain expressions in special values of automorphic L-functions with what we call *Gross-Prasad period integrals* of automorphic representations. When these automorphic representations are of discrete series type at archimedean places, the L-functions are attached to motives over F, with coefficients in some number field E, and the special values are critical in Deligne's sense [D2]. Deligne's conjecture then implies that the normalized quotients of the special values by the motivically defined *Deligne periods* are algebraic numbers whose $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates are again normalized special values of L-functions of the same type. In order to compare the Deligne and Ichino-Ikeda conjectures, one needs to consider the Gross-Prasad period integrals attached to all real embeddings of F simultaneously. This leads inevitably to the problem of constructing models over number fields of discrete series $(Lie(G), K)$-modules that are compatible under conjugation by $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

Although the discrete series representations of a reductive Lie group G are initially defined in terms of the decomposition of the regular representation of $G \times G$ on the Hilbert space $L_2(G)$, many of their principal features can be studied by purely algebraic means. The Beilinson-Bernstein localization theory, which realizes these representations in terms of equivariant perverse sheaves or D-modules on the flag variety, provides an efficient way to encode these features in terms of algebraic geometry. In particular, when G is the group of real points of an algebraic group given with a rational structure over a number field, one sees easily that the $(Lie(G), K)$-module attached to a given discrete series representation π also can be defined over $\overline{\mathbb{Q}}$, provided the central character of π is algebraic and the maximal compact subgroup K of G is rational over some number field.

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If G is a reductive algebraic group over $\mathbb{Q}$, it is nevertheless not immediately obvious how to make the Beilinson-Bernstein localization theory work over $\mathbb{Q}$: in general G does not have a maximal compact subgroup defined over $\mathbb{Q}$, and a Galois conjugate of the algebraic subgroup of G underlying K is not necessarily attached to a maximal compact group. The present article explains an approach to Beilinson-Bernstein localization over $\mathbb{Q}$, when G is the group attached to a Shimura variety $S(G,X)$, based on the formula conjectured by Langlands, and proved by Borovoi and Milne, for identifying the Galois conjugates of $S(G,X)$ with the Shimura varieties attached to inner twists of G. When K is defined by a special point x of $S(G,X)$, its conjugate $\alpha(K)$, for $\alpha \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, is isomorphic to the algebraic group ${}^{\alpha,x}K$ attached to a maximal compact subgroup of an inner twist of G denoted ${}^{\alpha,x}G$, locally isomorphic to G at all finite primes. We develop a theory of compatible rational structures, over appropriately chosen number fields, on discrete series $(Lie({}^{\alpha,x}G), {}^{\alpha,x}K)$-modules as α varies but with x fixed. In this way we can endow automorphic representations of discrete series type at infinity with canonical rational structures over natural coefficient fields, and show that these are compatible with Galois conjugation. A complete theory, not developed here, would include a comparison of these structures as the special point x varies.

The first two sections are devoted to the construction of our candidate for Beilinson-Bernstein localization over $\mathbb{Q}$ and its comparison to the theory of automorphic coherent cohomology, as developed in [H90]. The two final sections specialize to the Shimura varieties attached to unitary groups and define the Gross-Prasad period invariants, which are families of complex numbers attached to an automorphic representation of $U(n) \times U(n-1)$, where $U(i)$ here denotes the unitary group of some vector space of dimension i over a CM quadratic extension of the totally real field F. Applications to special values of L-functions will be considered in subsequent articles.

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Notation and conventions

All number fields will be considered as subfields of the field $\overline{\mathbb{Q}}$ of algebraic numbers in $\mathbb{C}$, hence are given with privileged embeddings in $\mathbb{C}$.

Throughout the article, we let $\mathcal{K}$ be a CM quadratic extension of a totally real field F, $c \in Gal(\mathcal{K}/F)$ complex conjugation. Let Σ_F denote the set of real places of F, and let Σ denote a CM type of $\mathcal{K}$, a set of extensions of Σ_F to $\mathcal{K}$, so that $\Sigma \coprod c\dot{\Sigma}$ is the set of archimedean embeddings of $\mathcal{K}$. We let $\eta_{\mathcal{K}/F} : Gal(\bar{F}/F) \rightarrow \{\pm 1\}$ denote the Galois character attached to the quadratic extension $\mathcal{K}/F$.

If E is an archimedean local field and G is an open subgroup of finite index in the group of E-valued points of a reductive algebraic group, we will use the term *irreducible admissible representation* of G to refer to an irreducible admissible smooth Frechet representation of moderate growth, in the sense of Casselman and

Wallach [C, Wh]. Hilbert space completions of discrete series representations are admissible in this sense. Where necessary, we will also be working with Harish-Chandra modules; these will be denoted $(\mathfrak{g}, K)$-modules or $(Lie(G), K)$-modules, where $\mathfrak{g} = Lie(G)$ is the complex (or algebraic) Lie algebra of G and K is a subgroup of $G(E)$ containing an open subgroup of the maximal compact subgroup as well as the center of $G(E)$.

Unless otherwise indicated, a discrete series representation of an algebraic group G over $\mathbb{R}$ will always be assumed to be *algebraic*, in the sense that its infinitesimal character is the same as that of a finite-dimensional representation. This is of course a condition on the central character.

1. Review of conjugation of Shimura varieties and automorphic vector bundles

Let G be a reductive algebraic group over $\mathbb{Q}$. We consider a $G(\mathbb{R})$-homogeneous hermitian symmetric domain X, so that (G, X) is a Shimura datum, in Deligne's sense. Thus X is a set of homomorphisms

$$x : \mathbb{S} := R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$$

satisfying Deligne's axioms [D2, cf. M90]. In particular, given a rational representation $\rho : G \to GL(V)$, each $x \in X$ defines a Hodge structure on V by the formula

$$\rho \circ x(z) \mid_{V^{p,q}} = z^{-p}\bar{z}^{-q}.$$

Let $\mu_x : \mathbb{G}_m \to G$ be the homomorphism with the property that $\mu_x(t)$ acts on $V^{p,q}$ by t^p. We call μ_x the *associated cocharacter* to x.

The Shimura variety $S(G, X)$ is the inverse limit of the finite level Shimura varieties

$$S(G, X) = \varprojlim Sh_U(G, X);\ S_U(G, X) = G(\mathbb{Q})\backslash G(\mathbf{A}^f) \times X/U$$

where U runs over open compact subgroups of $G(\mathbf{A}^f)$. Each $S_U(G, X)$ is quasi-projective and the family $\{S_U(G, X)\}$, together with the natural right action of $G(\mathbf{A}^f)$, has a canonical model over the reflex field $E(G, X)$. For details, see [D2, M90, M05].

1.1. Automorphic vector bundles.

Write $E = E(G, X)$. Let $\check{X}$ denote the compact dual of X, and $X \hookrightarrow \check{X}$ the Borel embedding. Concretely, $\check{X}$ is a flag variety of maximal parabolic subgroups of $G_{\mathbb{C}}$ and the image in $\check{X}$ of $x \in X$ is a maximal parabolic P_x with Levi subgroup K_x. Here K_x is the centralizer of the cocharacter μ_x and the unipotent radical $\mathfrak{p}_x^-$ of P_x is the subspace on which $\mu_x(t)$ acts as t.

As flag variety $\check{X}$ inherits from G a natural rational structure over $\mathbb{Q}$, but this rational structure is not relevant to the theory of Shimura varieties. Instead, we consider the canonical E-rational structure introduced in [H85] (following Deligne). Recall that E is the field of definition of the conjugacy class of the cocharacter μ_x for any $x \in X$; in other words, it is the field of definition of the homogeneous space $M = G/K_x$. We can view M as the moduli space of pairs (L, μ), where L is a Levi factor of a maximal parabolic subgroup P conjugate to one of the form P_x and $\mu : \mathbb{G}_m \to L$ is a cocharacter with image in the center of L with weights $-1, 0, 1$ on

$Lie(G)$, so that $L \oplus Lie(G)_1$ and $L \oplus Lie(G)_{-1}$ are opposite maximal parabolics. The compact dual $\check{X}$ is the flag variety of pairs $(P, [\mu])$, where P is a maximal parabolic conjugate to some P_x and $[\mu]$ is the P-conjugacy class of cocharacters μ as above. The map from M to $\check{X}$ takes (L, μ) to $(P = L \oplus Lie(G)_1, [\mu])$ and endows $\check{X}$ with its canonical G-equivariant E-structure.

Let $\mathcal{C}_{G,\check{X}}$ denote the category of G-equivariant vector bundles on $\check{X}$, $\mathcal{C}_{S(G,X)}$ the category of $G(\mathbf{A}^f)$-equivariant vector bundles on $S(G, X)$. The article [H85] defines a functor $\mathcal{V} \mapsto [\mathcal{V}]$ from $\mathcal{C}_{G,\check{X}}$ to $\mathcal{C}_{S(G,X)}$ whose essential image is the category of *automorphic vector bundles*. The functor is compatible with the E-structure in the sense that, if $\mathcal{V}$ is defined as G-equivariant vector bundle over a field $E(\mathcal{V})$ (which can always be taken to be a number field), then for any $\sigma \in Gal(\bar{E}/E)$, we have

$$\sigma[\mathcal{V}] = [\sigma(\mathcal{V})]. \tag{1.1.1}$$

Fix a point $x \in X$ with stabilizer $P_x \subset G_{\mathbb{C}}$. The following obvious fact is the basis of the relation between automorphic vector bundles and automorphy factors.

Construction 1.1.2. *There is a natural equivalence of categories between $\mathcal{C}_{G,\check{X}}$ and the category of finite-dimensional representations (Λ, W_Λ) of P_x; W_Λ is the fiber of $\mathcal{V}$ at x, and Λ is the isotropy representation. If x is rational (as a point of $\check{X}$ over the subfield E_x of $\overline{\mathbb{Q}}$, then this functor is compatible with the action of $Gal(\overline{\mathbb{Q}}/E_x)$.*

In particular, if $(T, x) \subset (G, X)$ is a CM pair, the equivalence of categories (1.1.2) is rational over the reflex field $E_x = E(T, x)$ of the pair (T, x).

An algebraic representation (Λ, W_Λ) of K_x extends trivially to a representation of P_x and thus defines an automorphic vector bundle $\mathcal{F} = \mathcal{F}_\Lambda$ on $S_U(G, X)$ whose fiber at a point beneath $x \times g$ for any $g \in G(\mathbf{A}^f)$ can be identified with W_Λ. The automorphic vector bundle $\mathcal{F}$ can also be identified with the family of bundles $\mathcal{F}_U$ on $S_U(G, X)$. The notation Λ will also be used to denote a highest weight of an irreducible representation of K_x with respect to a choice of positive roots.

The simplest case is that of the flat bundle attached to a representation $\rho : G \to Aut(\mathbb{V})$ for some finite-dimensional vector space $\mathbb{V}$. This defines a G-equivariant vector bundle $\mathbb{V} \otimes \mathcal{O}_{\check{X}}$, with G acting diagonally, and we write $[\mathbb{V}] = [\mathbb{V} \otimes \mathcal{O}_{\check{X}}]$ for the corresponding automorphic vector bundle on $S(G, X)$. This is the Hodge realization of the local system attached to the representation of G on the $\mathbb{Q}$-rational points of $\mathbb{V}$, and $[\mathbb{V}]$ is endowed with an integrable connection $\nabla : [\mathbb{V}] \to [\mathbb{V}] \otimes \Omega^1_{S(G,X)}$, obtained by functoriality from the trivial connection $1 \otimes d : \mathbb{V} \times Y_{\check{X}} \to \mathbb{V} \times \Omega^1_{\check{X}}$.

1.1.3. Fields of rationality. All fields of rationality of automorphic vector bundles are viewed as subfields of the fieldwill be assumed to contain $E(G, X)$. The field of rationality $E([\mathbb{V}])$ is then the field $E(G, X) \cdot E(\mathbb{V})$, where $E(\mathbb{V})$ is the field of rationality of $\mathbb{V}$ as representation of G. Suppose $\mathbb{V}$ is irreducible. Then $E(\mathbb{V})$ can be determined as follows. As in the construction of the L-group, the reductive group G over $\mathbb{Q}$ is attached to a quadruple $(X_*, \Delta_*, X^*, \Delta^*)$; here X_* and X^* are respectively the groups of cocharacters and characters of the universal maximal torus T in G and Δ_* and Δ^* are the simple coweights and weights; we are working with the universal Borel pair $(T \subset B)$. This quadruple is endowed with an action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ that determines the inner class of the group G. In this way, $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the set of highest weights of irreducible representations of G, and $E(\mathbb{V})$ is the field defined by the stabilizer of the highest weight of $\mathbb{V}$, viewed as an element of the universal character group X^*.

More generally, given a point $x \in X$ as above, a maximal torus of K_x can be identified canonically with the universal maximal torus T. In this way, the highest weight Λ of an irreducible representation (Λ, W_Λ) of K_x can be viewed as an element of X^*, and the field of rationality $E([\mathcal{V}])$, where $\mathcal{V}$ is the equivariant vector bundle attached to Λ, is the field defined by the stabilizer of Λ in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

1.2. Conjugation of Shimura varieties and automorphic vector bundles. The present section is based on Langlands' conjecture on conjugation of Shimura varieties [L79], proved by Borovoi and Milne (see [M88,M90]). We follow the systematic discussion in [MS], but we replace $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ by $Aut(\mathbb{C})$, as in [M88].

Let $\alpha \in Aut(\mathbb{C})$. Choose a CM pair $(H, x) \subset (G, X)$ (for example, the pair (H_0, x) defined in §3). There is an algebraic group ${}^{\alpha,x}G$ over $\mathbb{Q}$, an inner twist of G (see the discussion in (1.2.11) for an explicit construction), and a Shimura datum $({}^{\alpha,x}G, {}^{\alpha,x}X)$, such that

$$ {}^{\alpha,x}G(\mathbf{A}^f) \xrightarrow{\sim} G(\mathbf{A}^f); \tag{1.2.1} $$

there are canonical isomorphisms

$$ \begin{aligned} &\phi_{\alpha,x} : \alpha(S_U(G, X)) \xrightarrow{\sim} S_{{}^{\alpha,x}U}({}^{\alpha,x}G, {}^{\alpha,x}X), \\ &\forall U \subset G(\mathbf{A}^f) \text{ open compact, compatibly with (1.2.1)} \end{aligned} \tag{1.2.2} $$

where ${}^{\alpha,x}U$ is the open compact subgroup of ${}^{\alpha,x}G(\mathbf{A}^f)$ corresponding to U under the isomorphism (1.2.1). If $\alpha \in Aut(\mathbb{C}/E(G, X))$ then there is an isomorphism

$$ f_{1,\alpha} : (G, X) \xrightarrow{\sim} ({}^{\alpha,x}G, {}^{\alpha,x}X) \tag{1.2.3} $$

giving rise to a canonical isomorphism

$$ \phi(\alpha; x) : S(G, X) \xrightarrow{\sim} S({}^{\alpha,x}G, {}^{\alpha,x}X) \tag{1.2.4} $$

compatible with the isomorphism (1.2.3) on points over $\mathbf{A}^f$. This construction is described in more detail below in (1.2.11).

Given CM pairs (H, x) and (H', x') contained in (G, X), there is an isomorphism

$$ f_1 : {}^{\alpha,x}G \to {}^{\alpha,x'}G, $$

well-defined up to conjugation by ${}^{\alpha,x}G(\mathbb{Q})$ and mapping ${}^{\alpha,x}X$ to ${}^{\alpha,x'}X$. There is also an isomorphism

$$ f_2 : {}^{\alpha,x}G(\mathbf{A}^f) \to {}^{\alpha,x'}G(\mathbf{A}^f) $$

with a similar ambiguity. The ambiguities vanish upon passage to the Shimura varieties, and there are therefore well-defined isomorphisms (indexed by U):

$$ \phi(\alpha; x', x) : S_{{}^{\alpha,x}U}({}^{\alpha,x}G, {}^{\alpha,x}X) \to S_{{}^{\alpha,x'}U}({}^{\alpha,x'}G, {}^{\alpha,x'}X) \tag{1.2.5} $$

satisfying the obvious compatibilities with ${}^{\alpha,\bullet}G(\mathbf{A}^f)$-actions, and also satisfying

$$ \phi(\alpha; x', x) \circ \phi_{\alpha,x} = \phi_{\alpha,x'}. \tag{1.2.6} $$

The construction includes an isomorphism of algebraic varieties

$$\phi^{\vee}_{\alpha,x} : \alpha(\check{X}) \xrightarrow{\sim} {}^{\alpha,x}\check{X} \tag{1.2.7}$$

that twists the G action to a ${}^{\alpha,x}G$-action, according to an explicit formula [M88,Prop. 2.7]. There is therefore an equivalence of categories of equivariant vector bundles:

$$T^{\vee}_{\alpha,x} : \alpha(\mathcal{C}_{G,\check{X}}) \xrightarrow{\sim} \mathcal{C}_{{}^{\alpha,x}G,{}^{\alpha,x}\check{X}} \tag{1.2.8}$$

where the left hand side is $\mathcal{C}_{G,\check{X}}$ endowed with the twisted Galois structure (the essential image of $\mathcal{C}_{G,\check{X}}$ under the natural functor from vector bundles on $\check{X}$ to vector bundles on $\alpha(\check{X})$. Writing ${}^{\alpha,x}\mathcal{V} = T^{\vee}_{\alpha,x}(\mathcal{V})$ for $\mathcal{V} \in \mathcal{C}_{G,\check{X}}$, there are canonical isomorphisms **over** $\mathbb{C}$

$$\alpha(\mathcal{V}) \xrightarrow{\sim} {}^{\alpha,x}\mathcal{V} \in \mathcal{C}_{{}^{\alpha,x}G,{}^{\alpha,x}\check{X}}$$

that give rise to canonical isomorphisms of automorphic vector bundles

$$\phi_{\alpha} : \alpha([\mathcal{V}]) \xrightarrow{\sim} [{}^{\alpha,x}\mathcal{V}] \in \mathcal{C}_{S({}^{\alpha,x}G,{}^{\alpha,x}X)} \tag{1.2.9}$$

that are compatible with the canonical models on both sides. The main result of [M88] is that the diagram (1.2.9) satisfies a canonical reciprocity law at the special points, defined in terms of the period torsor for abelian varieties with complex multiplication. Actually, in [M88] Milne only proved the analogue of this theorem for connected Shimura varieties, but there is an optimal statement in [M90].

Remark 1.2.10. *The analogues of (1.2.3-1.2.6) are valid in the setting of automorphic vector bundles.*

These will be stated more precisely when they are needed below.

1.2.11. Functoriality for morphisms of Shimura data.

All the above constructions are compatible with an inclusion $(G', X') \subset (G, X)$ of Shimura data. It is a general principle that $E(G', X') \supset E(G, X)$. Then the isomorphisms in (1.2.1-6) can be chosen so that the diagrams like

$$\begin{array}{ccc} \alpha(S_{U'}(G',X')) & \xrightarrow[\phi_{\alpha,x}]{\sim} & S_{{}^{\alpha,x}U'}({}^{\alpha,x}G', {}^{\alpha,x}X') \\ \downarrow & & \downarrow \\ \alpha(S_U(G,X)) & \xrightarrow[\phi_{\alpha,x}]{\sim} & S_{{}^{\alpha,x}U}({}^{\alpha,x}G, {}^{\alpha,x}X) \end{array} \tag{1.2.11.1}$$

commute (it suffices to choose a CM pair $(H, x) \subset (G', X')$). Similar commutative diagrams will be used explicitly without comment.

In particular, suppose (G', X') is a CM pair (T, x). The following fact is built into the construction (see for example [M90, p. 336]):

Fact 1.2.11.2. *Let $\alpha \in Aut(\mathbb{C})$. If $\mu_x \in Hom(\mathbb{G}_m, T)$ is the associated cocharacter to $x : R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)_{\mathbb{C}} \to T_{\mathbb{R}}$, then $\alpha(\mu_x) \in Hom(\mathbb{G}_m, T)$ is the associated cocharacter to ${}^{\alpha}x := {}^{\alpha,x}x$.*

Taking K_x to be the stabilizer in G of the chosen x, we can define ${}^{\alpha,x}K_x \subset {}^{\alpha,x}P_x \subset {}^{\alpha,x}G$ by the same construction. Since K_x is the centralizer of μ_x and P_x is the parabolic subgroup associated to the decreasing filtration defined by μ_x, it follows from (1.2.11.2) that

$$K_{{}^{\alpha}x} = {}^{\alpha,x}(K_x);\ P_{{}^{\alpha}x} = {}^{\alpha,x}(P_x), \alpha \in Aut(\mathbb{C}) \tag{1.2.11.3}$$

as subgroups of G. More precisely,

Lemma 1.2.11.4. *Let $\alpha \in Aut(\mathbb{C})$, and let $(H,x) \subset (G,X)$ be a CM point. Then the group ${}^{\alpha,x}K_x$ is the stabilizer in ${}^{\alpha,x}G$ of the cocharacter $\alpha(\mu_x)$ corresponding to x in the symmetric space ${}^{\alpha,x}X$. In particular, ${}^{\alpha,x}K_x \xrightarrow{\sim} \alpha(K_x)$ as algebraic groups over $E_{\alpha(x)} = \alpha(E_x) \subset \mathbb{C}$.*

Proof. This follows again from Fact 1.2.11.2.

Remark 1.2.11.5. Since the above construction only depends on the image of $\alpha \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, we will feel free to write ${}^{\alpha,x}G$ when $\alpha \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Apparently one needs to work with $Aut(\mathbb{C})$ in setting up the localization theory of §2, although the final results can be expressed in terms of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

1.2.12. Automorphisms fixing the reflex field.
The isomorphism $\phi(\alpha;x)$ of (1.2.4) is constructed explicitly in [MS], §7. Let $\alpha * x : R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)_{\mathbb{C}} \to G_{\mathbb{R}}$ be the homomorphism with associated cocharacter $\alpha(\mu_x)$.

Lemma 1.2.12.1. *There is an element $v \in G(\overline{\mathbb{Q}})$, well defined up to left multiplication by $\beta \in G(\mathbb{Q})$, and an isomorphism $f_1 : (G,X) \xrightarrow{\sim} ({}^{\alpha,x}G, {}^{\alpha,x}X)$ of Shimura data, such that*

$$f_1 \circ ad(v)(\alpha * x) = {}^{\alpha}x.$$

In particular, there is a CM pair $(T', x' = ad(v)\alpha * x) \subset (G,X)$, well-defined up to conjugation by $G(\mathbb{Q})$, such that

$$(T', x') \xrightarrow{\sim} (T, {}^{\alpha}x).$$

Proof. This is immediate from Lemma 7.7 of [MS] and the definition of v given there.

The descent of automorphic vector bundles to the reflex field is proved (cf. [M99,Va]) using the following cocycle condition that is obvious from the construction of [M88] (but unfortunately does not seem to have been stated anywhere):

Lemma 1.2.12.2. *Let $\alpha, \beta \in Aut(\mathbb{C})$. The isomorphisms (1.2.9) satisfy the cocycle property:*

$$\phi_{\alpha\circ\beta} = \phi_{\alpha}^{\beta} \circ \phi_{\beta},$$

where we write ϕ_{α}^{β} for the isomorphism

$$\alpha([{}^{\beta,x}\mathcal{V}]) \xrightarrow{\sim} [{}^{\alpha,{}^{\beta}x}({}^{\beta,x}\mathcal{V})]$$

over $Sh({}^{\beta,x}G, {}^{\beta,x}X)$.

1.3. Harish-Chandra modules as automorphic vector bundles, and their conjugates.

The functor $\mathcal{V} \mapsto [\mathcal{V}] : \mathcal{C}_{G,\check{X}} \to \mathcal{C}_{S(G,X)}$ defined in §1.1 can be extended to a functor on the ind-categories

$$[\bullet] : ind - (\mathcal{C}_{G,\check{X}}) \to ind - (\mathcal{C}_{S(G,X)})$$

of direct limits of finite-dimensional vector bundles. In particular, the adjoint representation of G on the enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G gives rise to an object $U(\mathfrak{g})\otimes\mathcal{O}_{\check{X}} \in ind-(\mathcal{C}_{G,\check{X}})$, and hence to flat (ind-) automorphic vector bundle

$$[U(\mathfrak{g})] = [U(\mathfrak{g})\otimes\mathcal{O}_{\check{X}}]$$

in $ind-(\mathcal{C}_{S(G,X)})$. The adjoint action is compatible with the multiplicative structure on the enveloping algebra, hence $[U(\mathfrak{g})]$ is a $G(\mathbf{A}^f)$-equivariant $\mathcal{O}_{S(G,X)}$-algebra object in $ind-(\mathcal{C}_{S(G,X)})$, just as $U(\mathfrak{g})\otimes\mathcal{O}_{\check{X}}$ is a G-equivariant $\mathcal{O}_{\check{X}}$-algebra object in $ind-(\mathcal{C}_{G,\check{X}})$.

Similarly, if M is a $U(\mathfrak{g})\otimes\mathcal{O}_{\check{X}}$-module object in $ind-(\mathcal{C}_{G,\check{X}})$, the corresponding (ind)-automorphic vector bundle $[M]$ is a $G(\mathbf{A}^f)$-equivariant $[U(\mathfrak{g})]$-module over $S(G,X)$. We can construct such M by fixing a CM pair $(T,x)\subset(G,X)$ and letting M_x denote an irreducible $(\mathfrak{g},K_x)$-module, or Harish-Chandra module. We view M_x as an ind-object in the category of finite-dimensional representations of K_x, hence of finite-dimensional representations of P_x trivial on the unipotent radical. Construction 1.1.2 extends to ind-objects, hence M_x is the fiber at $x\in X\subset\check{X}$ of an object of $ind-(\mathcal{C}_{G,\check{X}})$. By the definition of Harish-Chandra modules, the natural action

$$U(\mathfrak{g})\otimes M_x \to M_x \tag{1.3.1}$$

is K_x equivariant, hence gives rise to morphisms

$$U(\mathfrak{g})\otimes\mathcal{O}_{\check{X}}\otimes M \to M; [U(\mathfrak{g})]\otimes[M]\to[M] \tag{1.3.2}$$

that make M and $[M]$ into modules over $U(\mathfrak{g})\otimes\mathcal{O}_{\check{X}}$ and $[U(\mathfrak{g})]$, respectively, in the corresponding categories. We let $Mod_G(U(\mathfrak{g})\otimes\mathcal{O}_{\check{X}})$ and $Mod_{aut}([U(\mathfrak{g}])$

The restriction to T of the compatible K_x actions on the algebra $U(\mathfrak{g})$ and its Harish-Chandra module M_x makes (1.3.1) into a diagram of CM motives for absolute Hodge cycles, in the sense of [H85] or [M88]. The representation of T on $U(\mathfrak{g})$ is rational over $\mathbb{Q}$, but the representation on M_x is in general only rational over an appropriate extension $E(M_x)$ of the field E_x. The rationality of Harish-Chandra modules will be considered in detail in §2.

In the meantime, we observe that, as CM motives, both $U(\mathfrak{g})\otimes_{\mathbb{Q}}\mathbb{C}$ and $M_x\otimes_{E(M_x)}\mathbb{C}$ are given two rational structures. The *Betti* rational structures, denoted by the subscript $_B$, are the natural $\mathbb{Q}$-structure on $U(\mathfrak{g})$ and the $E(M_x)$-structure on M_x mentioned in the previous paragraph; the former is independent of x, whereas the latter is derived from a rational structure of the equivariant ind-vector bundle M over $\check{X}$ over some extension $E(M)$ of $E(G,X)$. We will see in §2.2 that

Fact 1.3.3. *M has a rational structure over the field of definition of its isomorphism class.*

As K_x-module, the Betti rational structure on $U(\mathfrak{g})$ is only defined over K_x. The *de Rham* rational structures $U(\mathfrak{g})_{DR,x}$ and $M_{DR,x}$ are defined over the same extensions of E_x. (Actually, since $U(\mathfrak{g})$ is generated by the subalgebras $Lie(K_x)$, the unipotent radical $\mathfrak{p}_x^-$ of P_x, and the unipotent radical $\mathfrak{p}_x^+$ of the opposite parabolic, it is obvious that $U(\mathfrak{g})_{DR,x}$ is defined as K_x-module over E_x.) The comparison isomorphisms

$$U(\mathfrak{g})_B\otimes_{E_x}\mathbb{C}\xrightarrow{\sim}U(\mathfrak{g})_{DR}\otimes_{E_x}\mathbb{C};\ M_B\otimes_{E(M_x)}\mathbb{C}\xrightarrow{\sim}M_{DR}\otimes_{E(M_x)}\mathbb{C} \tag{1.3.4}$$

are given by the period torsor, as in [M88] (or in less precise form, in [H85]). What this means explicitly is that $U(\mathfrak{g})_{DR,x}$ is the associative E_x-subalgebra of $U(\mathfrak{g})_B \otimes_{E_x} \mathbb{C}$ generated by $Lie(T)$ and the multiples of the standard root vectors X_a for T in $\mathfrak{g}_{\overline{\mathbb{Q}}}$ by the (generally transcendental) CM periods corresponding to the characters a of T. With respect to their natural rational structures, the fibers of $U(\mathfrak{g})\otimes\mathcal{O}_{\check{X}}$ and M at $x \in \check{X}$ are endowed with the Betti rational structures, whereas the fibers of $[U(\mathfrak{g})]$ and $[M]$ at a point of $S(T,x) \subset S(G,X)$ are isomorphic over $\mathbb{C}$ to $U(\mathfrak{g})_B \otimes_{E_x} \mathbb{C}$ and $M_B \otimes_{E(M_x)} \mathbb{C}$, respectively, but are endowed with their de Rham rational structures.

An irreducible $(\mathfrak{g}, K_x)$-module M_x has an *infinitesimal character* $[\eta] : Z(\mathfrak{g}) \to \mathbb{C}$, where $Z(\mathfrak{g})$ denotes the center of $U(\mathfrak{g})$, and the notation $[\eta]$ is explained in §2.1. The kernel of $[\eta]$ is denoted $I_{[\eta]}$. We write $U(\mathfrak{g})_{[\eta]} = U(\mathfrak{g}/I_{[\eta]}U(\mathfrak{g})$. Then we have algebra objects (equivariant sheaves of algebras) $U(\mathfrak{g})_{[\eta]} \otimes \mathcal{O}_{\check{X}}$ and $[U(\mathfrak{g})_{[\eta]}]$ in $ind-(\mathcal{C}_{G,\check{X}})$ and $ind-(\mathcal{C}_{S(G,X)})$, respectively. Let $Mod_G(U(\mathfrak{g}) \otimes \mathcal{O}_{\check{X}})$ and $Mod_G(U(\mathfrak{g})_{[\eta]} \otimes \mathcal{O}_{\check{X}})$ be the subcategories of $ind-(\mathcal{C}_{G,\check{X}})$ of G-equivariant modules over the corresponding equivariant sheaves of algebras, and $Mod_{aut}([U(\mathfrak{g})])$ the analogous subcategory of $[U(\mathfrak{g})]$-modules in $ind-(\mathcal{C}_{S(G,X)})$

The following is obvious.

Lemma 1.3.5. *If $M \in Mod_G(U(\mathfrak{g}) \otimes \mathcal{O}_{\check{X}})$ and if for one (and therefore every) point M_x has infinitesimal character $[\eta]$, then the actions (1.3.2) factor through actions of $U(\mathfrak{g})_{[\eta]} \otimes \mathcal{O}_{\check{X}}$ and $[U(\mathfrak{g})_{[\eta]}]$ respectively.*

Now suppose $\alpha \in Aut(\mathbb{C})$. Then diagrams (1.2.8) and (1.2.9) apply in particular to the objects $[U(\mathfrak{g})]$ $[U(\mathfrak{g})_{[\eta]}]$, and $[M]$. We have canonical isomorphisms
(1.3.6)
$$\alpha([U(\mathfrak{g})]) \xrightarrow{\sim} [U({}^{\alpha,x}\mathfrak{g})];\ \alpha([U(\mathfrak{g})_{[\eta]}]]) \xrightarrow{\sim} [U({}^{\alpha,x}\mathfrak{g})_{[{}^{\alpha,x}\eta]}]; \alpha([M]) \xrightarrow{\sim} [{}^{\alpha,x}M].$$

It is obvious that ${}^{\alpha,x}\mathfrak{g}$ is canonically isomorphic to $Lie({}^{\alpha,x}G)$.

More precise formulas for $[{}^{\alpha,x}\eta]$ and $[{}^{\alpha,x}M]$ will be derived in §2.2. We note the following corollary to Lemma 1.2.11.4.

Proposition 1.3.7. *Let $M \in Mod_G(U(\mathfrak{g}) \otimes \mathcal{O}_{\check{X}})$. Assume that the representation of K_x on M_x is the direct sum of irreducible representations*

$$M_x = \oplus_{i \in I} M_{x,i}$$

for some index set I. Let $M_i \in \mathcal{C}_{G,\check{X}}$ be the equivariant vector bundle with fiber $M_{x,i}$ at x. Then for any $\alpha \in Aut(\mathbb{C})$,

$${}^{\alpha,x}M_{{}^{\alpha}x} \xrightarrow{\sim} \oplus_{i \in I}\, \alpha(M_{x,i})$$

as representation of ${}^{\alpha,x}K_x \xrightarrow{\sim} \alpha(K_x)$.

1.4. Review of coherent cohomology of automorphic vector bundles.

In this section we consider automorphic representations $\pi = \otimes'_v \pi_v$ of the group G whose archimedean component π_∞ belongs to the discrete series. I reformulate a well-known theorem of Schmid. We fix $x \in X$ as above. Notation for relative Lie algebra cohomology is as in [H90].

Theorem 1.4.1 [Sc, cf. BHR, Prop. 2.4.5, Theorem 3.2.1]. *Let π_∞ be a discrete series representation of G_∞ with algebraic central character. There is a unique index $q(\pi_\infty)$ and a unique irreducible algebraic representation $(\Lambda(\pi_\infty), W_{\Lambda(\pi_\infty)})$ of K_x such that, letting $\mathbb{W}$ denote a variable representation of K_x,*

$$H^i(Lie(P_x)_{\mathbb{C}}, K_x; \pi_\infty \otimes \mathbb{W}) \neq 0 \Leftrightarrow i = q(\pi_\infty) \text{ and } \mathbb{W} = W_{\Lambda(\pi_\infty)}.$$

Moreover, π_∞ is the unique discrete series representation of G_∞ for which

$$H^\bullet(Lie(P_x)_{\mathbb{C}}, K_x; \pi_\infty \otimes \mathbb{W}) \neq 0,$$

and

$$\dim H^{q(\pi_\infty)}(Lie(P_x)_{\mathbb{C}}, K_x; \pi_\infty \otimes W_{\Lambda(\pi_\infty)}) = 1.$$

We let $\mathcal{F}(\pi_\infty)$ be the automorphic vector bundle attached to the representation $(\Lambda(\pi_\infty), W_{\Lambda(\pi_\infty)})$ of K_x.

The global relation between π_∞ and $(\Lambda(\pi_\infty), W_{\Lambda(\pi_\infty)})$ is mediated by the coherent cohomology of $\mathcal{F}(\pi_\infty)$. When $S(G, X)$ is not compact (when G is not anisotropic modulo center) the relation is slightly indirect, and is constructed in [H90]. We recall the constructions briefly. For each level subgroup $U \subset G(\mathbf{A}^f)$ there is a collection of toroidal compactifications $S_U(G, X)_{\mathcal{S}}$, where $\mathcal{S}$ runs through families of fans in certain rational vector spaces and the compactifiations are partially ordered by the relation of refinement. The following theorem is proved in [H89, BHR].

Theorem 1.4.2. *Suppose U is neat in the sense of [H89] or [Pink]. Then*

(1) *There is a well-defined subcollection of families of fans $\mathcal{S}$ such that $S_U(G, X)_{\mathcal{S}}$ is a smooth projective variety over $E(G, X)$.*
(2) *There is a functor taking automorphic vector bundles $\mathcal{F}$ on $S_U(G, X)$ to their canonical extensions $\mathcal{F}^{can} = \mathcal{F}^{can}_{\mathcal{S}}$ over $S_U(G, X)_{\mathcal{S}}$.*
(3) *The action (1.1.1) of $Gal(\overline{\mathbb{Q}}/E(G, X))$ on the collection of automorphic vector bundles respects the canonical extension functor. More generally, the actions of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on the categories of automorphic vector bundles over Shimura varieties (defined in §1.2) extend to an action on families of fans, and for any choice of CM point $x \in X$ we have (in the obvious notation)*

$$\alpha(\mathcal{F}^{can}_{\mathcal{S}}) \xrightarrow{\sim} ({}^{\alpha,x}\mathcal{F}_{\alpha,x\mathcal{S}})^{can}$$

as canonically extended automorphic vector bundles on $S_U({}^{\alpha,x}G, {}^{\alpha,x}X)_{\alpha,x\mathcal{S}}$.

Proof. The first assertion is essentially the theory of toroidal compactifications, due to Ash, Mumford, Rapoport, and Tai; the statement here is in [H89], which also proves (2). The final assertion is Proposition 1.4.5 of [BHR].

For our purposes, the main property of the canonical extension is that there is a canonical isomorphism

$$H^\bullet(S_U(G, X)_{\mathcal{S}}, \mathcal{F}^{can}_{\mathcal{S}}) \xrightarrow{\sim} H^\bullet(S_U(G, X)_{\mathcal{S}'}, \mathcal{F}^{can}_{\mathcal{S}'}) \tag{1.4.3}$$

whenever $\mathcal{S}$ and $\mathcal{S}'$ are families of fans as above. This allows us to define

$$\tilde{H}^\bullet(\mathcal{F}^{can}) = \tilde{H}^\bullet(S(G, X), \mathcal{F}^{can}) = \varinjlim_U \varinjlim_{\mathcal{S}} H^\bullet(S_U(G, X)_{\mathcal{S}}, \mathcal{F}^{can}_{\mathcal{S}}) \tag{1.4.4}$$

where the transition morphisms for the limits over $\mathcal{S}$ are the canonical isomorphisms of (1.4.3). The action of $G(\mathbf{A}^f)$ on the family $\{S_U(G,X)\}$, as U varies, extends to an action on their toroidal compactifications and canonical extensions, giving rise to a canonical action of $G(\mathbf{A}^f)$ on $\tilde{H}^\bullet(\mathcal{F}^{can})$ making the latter an admissible representation.

As explained in (1.1.3), the automorphic vector bundle $\mathcal{F}(\pi_\infty)$ has a canonical model over a finite extension $E(\pi_\infty) := E(\mathcal{F}(\pi_\infty))$ of $E(G,X)$. Thus $\tilde{H}^\bullet(\mathcal{F}(\pi_\infty)^{can})$ has a canonical model over $E(\pi_\infty)$. Now let $\pi = \pi_\infty \otimes \pi_f$ be an irreducible automorphic representation. Here π_∞ is a $(Lie(G)_\mathbb{C}, K_x)$-module for a choice of CM point $x \in X$. Let

$$\tilde{H}[\pi_f] = \tilde{H}^\bullet(\mathcal{F}(\pi_\infty)^{can})[\pi_f] = \tilde{H}^\bullet(S(G,X), \mathcal{F}(\pi_\infty)^{can})[\pi_f]$$

(the longer notation is used when necessary to avoid ambiguity) denote the π_f-isotypic component of $\tilde{H}^\bullet(\mathcal{F}(\pi_\infty)^{can})$. by which we mean the maximal subrepresentation of $\tilde{H}^\bullet(\mathcal{F}(\pi_\infty)^{can})$ all of whose subquotients are isomorphic to π_f (in the applications, $\tilde{H}[\pi_f]$ will be a semisimple representation, and in most cases will be isomorphic to π_f itself). In any case, if π_f has a model over an extension $E(\pi_f)$ of $E(\pi_\infty)$, then so does $\tilde{H}[\pi_f]$.

If π_f is not known to be the finite part of an automorphic representation, we can still define $\tilde{H}^\bullet(\mathcal{F}^{can})[\pi_f] = \tilde{H}^\bullet(S(G,X), \mathcal{F}^{can})[\pi_f]$, for any automorphic vector bundle $\mathcal{F}$ as above.

Theorem 1.4.5 [H90, Proposition 3.6; BHR, Theorem 4.4.1]. *Suppose π is cuspidal. Then $\tilde{H}[\pi_f]$ is non-trivial and semisimple, and the field of rationality $\mathbb{Q}(\pi_f)$ of π_f is a subfield of a CM field.*

An explicit formula for $\tilde{H}[\pi_f]$ in terms of relative Lie algebra cohomology is given in §1.5.

Corollary 1.4.6. *Under the hypotheses of Theorem 1.4.5, π_f has a model over a CM field $E(\pi_f)$ containing $E(\pi_\infty)$.*

Proof. The Brauer obstruction to realizing π_f over its field of rationality $\mathbb{Q}(\pi_f)$ splits over a cyclotomic field $\mathbb{Q}(\zeta_m)$ for some m. We can take $E(\pi_f) = \mathbb{Q}(\pi_f)(\zeta_m) \cdot E(\pi_\infty)$.

We have included $E(\pi_\infty)$ in $E(\pi_f)$ in order to subordinate the representations that contribute to $\tilde{H}^\bullet(\mathcal{F}^{can})$ to the automorphic vector bundle. Here is the relevant list of inclusions:

$$\overline{\mathbb{Q}} \supset E(\pi_f) \supset E(\pi_\infty) \supset E(G,X) \supset \mathbb{Q}.$$

All these fields, as well as the field $\mathbb{Q}(\pi_f)$ of Theorem 1.4.5, have been constructed as subfields of the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ $\mathbb{C}$. For purposes of comparison with Deligne's conjecture, we need to work with all complex embeddings of its extension $E(\pi_f)$ simultaneously. This can be understood geometrically but at the cost of considering the Galois conjugates $\alpha(S(G,X))$, for $\alpha \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, as well as the Galois conjugates of automorphic vector bundles, as discussed in §1.2.

Using the correspondence of 1.4.1 between discrete series representations of G_∞ and (certain)automorphic vector bundles we can define the discrete series representation ${}^{\alpha,x}\pi_\infty$ of ${}^\alpha G(\mathbb{R})$ by the formula

$$\mathcal{F}({}^{\alpha,x}\pi_\infty) = {}^{\alpha,x}\mathcal{F}(\pi_\infty). \tag{1.4.7}$$

Corollary 1.4.8. *In the situation of Theorem 1.4.5, let $\alpha \in Aut(\mathbb{C})$ (or $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$) and let ${}^{\alpha,x}\pi_f$ denote the representation of $G(\mathbf{A}^f) \xrightarrow{\sim} {}^{\alpha,x}G(\mathbf{A}^f)$ obtained by conjugating π_f by α. Let $\tilde{H}[{}^{\alpha,x}\pi]$ denote the ${}^{\alpha}\pi_f$-isotypic component of $\tilde{H}^{\bullet}({}^{\alpha,x}\mathcal{F}(\pi_\infty)^{can})$, with notation as in (1.4.7). Then $\tilde{H}[{}^{\alpha,x}\pi]$ is non-trivial and is canonically isomorphic to $\alpha(\tilde{H}[\pi])$. In particular, there is a representation π' of ${}^{\alpha,x}G_\infty$ such that $\pi' \otimes {}^{\alpha,x}\pi_f$ is an automorphic representation of ${}^{\alpha,x}G$.*

Proof. We adopt the notation of [H90] without comment. Since π is assumed to be cuspidal, π_f actually contributes to the *interior cohomology* $H_!^{\bullet}(\mathcal{F}(\pi_\infty))$, the image of $\tilde{H}^{\bullet}(\mathcal{F}(\pi_\infty)^{sub})$ in $\tilde{H}^{\bullet}(\mathcal{F}(\pi_\infty)^{can})$, (in [H90] this is denoted $\bar{H}$ rather than $H_!$). It follows from (1.2.10) and Theorem 1.4.2 (which applies to the subcanonical extensions as well as the canonical extensions) that ${}^{\alpha,x}\pi_f$ contributes to $H_!^{\bullet}({}^{\alpha,x}\mathcal{F}(\pi_\infty))$. By Theorem 5.3 of [H90], this space is represented by discrete automorphic representations of ${}^{\alpha,x}G$.

We would like to be able to make the stronger assertion that we can take $\pi' = {}^{\alpha,x}\pi_\infty$, and that

$$ {}^{\alpha,x}\pi := {}^{\alpha,x}\pi_\infty \otimes {}^{\alpha,x}\pi_f \tag{1.4.9} $$

is a cuspidal automorphic representation of ${}^{\alpha,x}G$. Much of [H90] and [BHR] are devoted to finding hypotheses on $\mathcal{F}(\pi_\infty)$ that guarantee this to be the case. In the applications we have in mind to special values of L-functions, (1.4.9) should be automatic. In the next section we recall the explicit relation between coherent cohomology and automorphic forms and introduce an ad hoc property corresponding to (1.4.9).

1.4.10. Conjugating π_f without conjugating G. Discrete series representations contribute to topological (Betti) cohomology as well as to coherent cohomology of the Shimura variety $S(G,X)$. Suppose $\rho : G \to Aut(R)$ is an irreducible finite-dimensional representation of G defined over $\mathbb{Q}$, and let $\tilde{R}$ denote the corresponding local system over $S(G,X)$ in $\mathbb{Q}$-vector spaces. Then the Betti cohomology

$$ H^{\bullet}(S(G,X),\tilde{R}) := H^{\bullet}(S(G,X)(\mathbb{C}),\tilde{R}) $$

is naturally an admissible $\mathbb{Q}[G(\mathbf{A}^f)]$-module, so that

$$ Hom_{G(\mathbf{A}^f)}(\pi_f, H^{\bullet}(S(G,X),\tilde{R})) \neq 0 \\ \Rightarrow Hom_{G(\mathbf{A}^f)}(\alpha(\pi_f), H^{\bullet}(S(G,X),\tilde{L})) \neq 0 \ \forall \alpha \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}). $$

Here $\alpha(\pi_f)$ is just obtained by applying α to the (locally constant) matrix coefficients of π_f in a $\overline{\mathbb{Q}}$-model; this was denoted ${}^{\alpha,x}\pi_f$ in the statement of Corollary 1.4.8 when it was considered a representation of the isomorphic group ${}^{\alpha,x}G(\mathbf{A}^f)$, but the result is the same up to equivalence. On the other hand, we have the mixed Hodge decomposition

$$ H^n(S(G,X),\tilde{R}) \otimes \xrightarrow{\sim} \oplus_w H^{n-\ell(w)}(S(G,X),\mathcal{E}_w(L)^{can}) \otimes \mathbb{C} \tag{1.4.11} $$

(cf. [H94]), where w runs through a subset of the absolute Weyl group of G, $\mathcal{E}_w(R)$ are certain irreducible automorphic vector bundles, and the right hand side is coherent cohomology. Each $\mathcal{F}(\pi_\infty)$ occurs as a $\mathcal{E}_w(R)$ for the R whose contragredient

$R^\vee$ has the same infinitesimal character as π_∞. Since (1.4.11) determines an isomorphism of semisimplified $\mathbb{C}[G(\mathbf{A}^f)]$-modules, it follows from the above discussion that if the π_f-isotypic component of $\tilde{H}^\bullet(\mathcal{F}(\pi_\infty)^{can})$ is non-trivial, then the $\alpha(\pi_f)$-isotypic component of the corresponding $H^n(S(G,X),\tilde{R})$ is also non-trivial, for any $\alpha \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Applying (1.4.11) again, we see that $\alpha(\pi_f)$ occurs in the semisimplification[1] $H^\bullet(S(G,X),\mathcal{E}_w(R)^{can})$ for appropriate w, again for any $\alpha \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

Even if L is not assumed rational over $\mathbb{Q}$, one again finds that $\alpha(\pi_f)$ occurs in coherent cohomology of $S(G,X)$, for any $\alpha \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, by replacing L by the sum of its $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ conjugates. Since the resulting local system is not irreducible, the statement is a bit more complicated, but the same argument works.

The conclusion is that $\alpha(\pi_f)$ occurs in the coherent cohomology of $S(G,X)$ as well as in that of $S({}^{\alpha,x}G, {}^{\alpha,x}X)$. That there is nothing remarkable about this is clear if you consider the Betti cohomology of the point $Spec(\mathcal{K})$ – it is a one-dimensional space with a natural $\mathbb{Q}$-structure, but the corresponding de Rham (coherent) cohomology is only rational over $\mathcal{K}$. The Galois action on Betti cohomology should be considered more elementary than that on coherent or de Rham cohomology, and is not considered in the present article.

1.5. Coherent cohomology and relative Lie algebra cohomology.

We fix a CM pair $(T,x) \subset (G,X)$. Let $\mathcal{A}(G) = \mathcal{A}(G,x)$ denote the space of K_x-finite automorphic forms on $G(\mathbb{Q})\backslash G(\mathbf{A})$, $\mathcal{A}_0(G)$ the subspace of cusp forms; the dependence on x will be omitted when possible. Some years ago, Jens Franke announced (privately) the following theorem for coherent cohomology, parallel to his celebrated theorem [Fr] on the representability of the cohomology of locally symmetric spaces by automorphic forms. Notation is as above.

Theorem 1.5.1. *Let $\mathcal{V}$ be an equivariant vector bundle on $\check{X}$, $[\mathcal{V}]$ the corresponding automorphic vector bundle. There is a canonical isomorphism*

$$H^\bullet(Lie(P_x), K_x; \mathcal{A}(G) \otimes \mathcal{V}_x) \xrightarrow{\sim} \tilde{H}^\bullet([\mathcal{V}]^{can}).$$

Since the proof has never been published, I prefer not to refer to this theorem, though when $\mathcal{V} = \mathcal{F}(\pi_\infty)$ for a discrete series Harish-Chandra module π_∞ I suspect it can be derived in a straightforward manner from the results of [Fr] using mixed Hodge theory. In any case, Theorem 1.5.1 is not precise enough to imply (1.4.9). Instead, we use the weaker Proposition 3.6 of [H90], which we restate in the present notation:

Proposition 1.5.2. *Under the hypotheses of Theorem 1.5.1, there is a canonical inclusion*

$$H^\bullet(Lie(P_x), K_x; \mathcal{A}_0(G) \otimes \mathcal{V}_x) \hookrightarrow \tilde{H}^\bullet([\mathcal{V}]^{can}).$$

1.5.3. Remark. It is important to bear in mind that the presence of $\mathcal{V}_x$ makes the map of 1.5.2 is canonical. This would not be the case for a statement purely in terms of the space of cusp forms, defined in a purely transcendental manner, with

[1] This semisimplification is a red herring. If G is anisotropic, then $H^n(S(G,X),\tilde{R})$ carries a pure Hodge structure and Theorem 1.4.5 implies that all the $G(\mathbf{A}^f)$-modules occurring in (1.4.11) are semisimple. This is almost always true for cuspidal cohomology, as we see in the following section.

no reference to Shimura varieties. The fiber $\mathcal{V}_x$ at x of $\mathcal{V}$ reappears in connection with Beilinson-Bernstein localization and with the Ichino-Ikeda conjecture.

Let $\tilde{H}^\bullet_{cusp}([\mathcal{V}]^{can}) \subset \tilde{H}^\bullet([\mathcal{V}]^{can})$ denote the image of the injection of Proposition 1.5.2. Let π_f be an irreducible admissible representation of $G(\mathbf{A}^f)$, and let

$$\tilde{H}^\bullet([\mathcal{V}]^{can})[\pi_f];\ \tilde{H}^\bullet_{cusp}([\mathcal{V}]^{can})[\pi_f]$$

denote the π_f-isotypic components of $\tilde{H}^\bullet([\mathcal{V}]^{can})$ and $\tilde{H}^\bullet_{cusp}([\mathcal{V}]^{can})$, respectively.

Definition 1.5.3. *An admissible irreducible representation π_f of $G(\mathbf{A}^f)$, defined over a number field E, is called* **automorphically cuspidal** *if for all $\beta \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, all inner twists G' of G such that $G'(\mathbf{A}^f) \xrightarrow{\sim} G(\mathbf{A}^f)$, all Shimura data (G', X'), and all automorphic vector bundles $[\mathcal{V}]$ over $S(G', X')$, the inclusion of coherent cohomology of $S(G', X')$*

$$\tilde{H}^\bullet_{cusp}(S(G', X'), [\mathcal{V}]^{can})[\beta(\pi_f)] \hookrightarrow \tilde{H}^\bullet(S(G', X'), [\mathcal{V}]^{can})[\beta(\pi_f)]$$

is an equality.

If in addition every automorphic representation π' of G, with $\pi'_f \xrightarrow{\sim} \pi_f$ has the property that π'_∞ belongs to the discrete series, then we say π_f is **of discrete series type** *(for G).*

Lemma 1.5.4. *Suppose π_f is automorphically cuspidal and of discrete series type. Then for all $\beta \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ and all G' as in Definition 1.5.3, $\beta(\pi_f)$ is of discrete series type for G'.*

Proof. This is an easy consequence of Theorem 1.4.2 and the definitions.

Conjecture 1.5.5. *If π_f is tempered, then it is automorphically cuspidal.*

Lemma 1.5.6. *Suppose π_f is automorphically cuspidal and of discrete series type. Fix $\beta \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Then, for any irreducible automorphic vector bundle $\mathcal{F}$, in the notation of §1.4, there is at most one automorphic representation π' of G, with $\pi'_f \xrightarrow{\sim} \beta(\pi_f)$, and*

$$\tilde{H}(\mathcal{F}^{can})[\beta(\pi)] = Hom_{(Lie(G)_\mathbb{C}, K_x)\times G(\mathbf{A}^f)}(\pi', \mathcal{A}_0(G)) \otimes H^{q(\pi)}(Lie(G)_\mathbb{C}, K_x; \pi' \otimes \Lambda(\pi'))$$
$$= Hom_{(Lie(G)_\mathbb{C}, K_x)\times G(\mathbf{A}^f)}(\pi', \mathcal{A}_0(G)) \otimes H^{q(\pi)}(Lie(G)_\mathbb{C}, K_x; \pi'_\infty \otimes \Lambda(\pi)) \otimes \beta(\pi_f).$$

Proof. By the hypotheses, if π' is as in the statement of the lemma then π'_∞ belongs to the discrete series. Then π'_∞ is uniquely determined by $\mathcal{F}$ by Schmid's Theorem 1.4.1.

In §3 we show that, when G is a unitary similitude group, the π_f that contribute to the coherent cohomology of sufficiently regular $[\mathcal{V}]$ are (usually) automorphically cuspidal.

Definition 1.5.7. *Let π_f be an admissible irreducible representation of $G(\mathbf{A}^f)$, defined over a number field $E'(\pi_f)$. The* **automorphic Hodge-de Rham structure** *attached to π_f is the collection*

$$\{\tilde{H}^\bullet(S({}^{\alpha,x}G, {}^{\alpha,x}X), {}^{\alpha,x}\mathcal{F}^{can})[\beta(\pi_f)]\}$$

where α and β run (independently) over $Aut(\mathbb{C})$ (or $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$) and $\mathcal{F}$ runs over all automorphic vector bundles over $S(G, X)$. We use the existence of isomorphisms ${}^{\alpha,x}G(\mathbf{A}^f) \xrightarrow{\sim} G(\mathbf{A}^f)$ to define the isotypic spaces.

2. Beilinson-Bernstein localization over $\mathbb{Q}$

2.1. The complex theory.

Let $Flag_G$ denote the flag variety of Borel subgroups of G, endowed with its natural $\mathbb{Q}$-structure, and let $\mathcal{B}$ denote the tautological G-homogeneous bundle over $Flag_G$ whose fiber at x is the Borel subgroup corresponding to x. The equivariant bundle of Cartan subgroups $\mathcal{B}/[\mathcal{B},\mathcal{B}]$ is trivial (cf. [V], Lecture 1) and therefore its bundle of Lie algebras $Lie(\mathcal{B}/[\mathcal{B},\mathcal{B}])$ has a canonical fiber, the *canonical Cartan subalgebra* $\mathfrak{h}$, endowed moreover with a no less canonical negative Weyl chamber in the dual $\mathfrak{h}^*$; here and in what follows, we follow the sign conventions of [HMSW1, HMSW2], which is opposite to that of [BB]. As in any of a number of sources (for example [HMSW1, V]), an element $\eta \in \mathfrak{h}^*$ defines a sheaf $\mathcal{D}_\eta$ of twisted differential operators on $Flag_G$.

The Harish-Chandra isomorphism can be viewed as a bijection between Weyl group orbits $[\eta]$ of $\eta \in \mathfrak{h}^*$ and maximal ideals $I_{[\eta]}$ in the center $Z(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$. In the present article we will always assume

Hypothesis 2.1.1. *The maximal ideal $I_{[\eta]}$ is the annihilator in $Z(\mathfrak{g})$ of a finite-dimensional irreducible algebraic representation $W_\eta = W_{[\eta]}$ of G. In other words, $\eta \in \mathfrak{h}^*$ is regular (and integral).*

Under this hypothesis, the orbit $[\eta]$ has a unique *antidominant* representative η – that is, η belongs to the canonical negative chamber – and therefore the natural inclusion of fields of rationality $\mathbb{Q}([\eta]) \subset \mathbb{Q}(\eta)$ is an equality.

We fix a point $x \in X$ and let K_x be the centralizer of its associated cocharacter μ_x, as in §1.1. Let $Mod(\mathcal{D}_\eta, K_x)$ denote the category of K_x-equivariant $\mathcal{D}_\eta$-modules on $Flag_G$; these are automatically regular holonomic, cf. [BB]. Let $Mod(\mathfrak{g}, K_x)_{[\eta]}$ denote the category of finitely generated $(\mathfrak{g}, K_x)$-modules on which $I_{[\eta]}$ acts trivially. If $i : Y \hookrightarrow Flag_G$ is a K_x-orbit in $Flag_G$, the pullback of $\mathcal{D}_\eta$ to Y induces a natural K_x-equivariant twisted sheaf of differential operators $\mathcal{D}^i_\eta$ on Y (cf. [HMSW1], §A.1]), and there is a derived direct image functor

$$Ri_+ : D^b - Mod(\mathcal{D}^i_\eta, K_x) \to D^b - Mod(\mathcal{D}_\eta, K_x) \tag{2.1.2}$$

where D^b denotes the bounded derived category; we denote $R^0 i_+$ the 0-th direct image (cf. [HMSW1], §A.3.3]). All we need to know about these constructions is that they are purely algebraic; in particular, if η, x, and Y are defined over some number field E, then these constructions are $Gal(\overline{\mathbb{Q}}/E)$-equivariant.

The basic facts that we will need about Beilinson-Bernstein localization [BB] are summarized in the following theorem.

Theorem 2.1.3. *Let $[\eta]$ denote the Weyl group orbit of $\eta \in \mathfrak{h}^*$.*

1. *There is a canonical isomorphism $U(\mathfrak{g})/I_{[\eta]}U(\mathfrak{g}) \xrightarrow{\sim} \Gamma(Flag_G, \mathcal{D}_\eta)$.*
2. *Suppose η is antidominant (with respect to the canonical negative chamber). Then the global sections functor*
$$\Gamma : Mod(\mathcal{D}_\eta, K_x) \longrightarrow Mod(\mathfrak{g}, K_x)_{[\eta]}$$
is an equivalence of categories.
3. *The irreducible objects of $Mod(\mathcal{D}_\eta, K_x)$ are in one-to-one correspondence with pairs $(Y, \mathcal{T})$, where $i : Y \hookrightarrow Flag_G$ is a K_x-orbit in $Flag_G$ and $\mathcal{T}$ is*

an irreducible $(\mathcal{D}^i_\eta, K_x)$*-connection on* Y*, in the sense of [HMSW1]; i.e., an irreducible* K_x*-equivariant* $\mathcal{D}^i_\eta$*-module that is coherent as* $\mathcal{O}_Y$*-module. We let* $M(Y,\mathcal{T})$ *denote the object of* $Mod(\mathcal{D}_\eta, K_x)$ *corresponding to* $(Y,\mathcal{T})$ *(in [HMSW1] it is denoted* $\mathcal{L}(Q,\tau)$*, with* $Q = Y$*,* $\tau = \mathcal{T}$*)).*

(4) *Let* $i : Y \hookrightarrow Flag_G$ *be a closed* K_x *orbit. Then there is a unique irreducible* $(\mathcal{D}^i_\eta, K_x)$*-connection* $\mathcal{T}_{Y,\eta}$ *on* Y*. We let* $M(Y) = i_+M(Y,\mathcal{T}_{Y,\eta}) := R^0 i_+ M(Y,\mathcal{T}_{Y,\eta}) \subset Mod(\mathcal{D}_\eta, K_x)$*. Then* $\Gamma(M(Y))$ *is a discrete series* $(\mathfrak{g}, K_x)$*-module with infinitesimal character* $[\eta]$*. In this way,* Γ *defines a one-to-one correspondence between closed* K_x *orbits in* $Flag_G$ *and the discrete series L-packet with infinitesimal character* $[\eta]$*.*

Remarks 2.1.4.

(i) Theorem 2.1.3 (4) is a restatement of Theorem 12.5 of [HMSW2] and the discussion preceding the statement of the theorem.

(ii) Beilinson-Bernstein localization is generally formulated for $(\mathfrak{g}, K_\infty)$-modules, where K_∞ is a maximal compact subgroup of $G(\mathbb{R})$. In our situation, K_x contains the center Z_G of G and the image in $G^{ad}(\mathbb{R})$ of its group of real points is open of finite index in a maximal compact subgroup. In any case, the set of K_x orbits in $Flag_G$ is finite, and the localization construction works just as well in this situation.

2.2. The action of $Aut(\mathbb{C})$ **on the categories** $Mod(\mathcal{D}_\eta, K_x)$**.**

The objects described in the previous section inherit a canonical rational structure from the $\mathbb{Q}$-structure on G. When we incorporate the adjoint action of the CM pair (T,x), we obtain two canonical rational structures: the Betti and de Rham structures, discussed in §1.3. We begin by considering the Betti structure, which is consistent with the natural $\mathbb{Q}$-structure on the pair $(G, Flag_G)$. If η is $\overline{\mathbb{Q}}$-rational (with respect to the Betti structure), then so is $\mathcal{D}_\eta$, and we have

Lemma 2.2.1. *For any* $\alpha \in Aut(\mathbb{C})$*,*

$$\alpha(\mathcal{D}_\eta) \xrightarrow{\sim} \mathcal{D}_{\alpha(\eta)} \xrightarrow{\sim} \mathcal{D}_{^\alpha\eta}.$$

Proof. The first assertion is obvious. Let $B \in Flag_G$ be any Borel subgroup containing T. We have isomorphisms

$$\alpha(U(\mathfrak{g})_\eta) \xrightarrow{\sim} \Gamma(Flag_G, \mathcal{D}_{\alpha(\eta)});\ U({}^{\alpha,x}\mathfrak{g})_{\alpha(\eta)} \xrightarrow{\sim} \Gamma(Flag_{^{\alpha,x}G,\mathcal{D}_{^\alpha\eta}}).$$

So it suffices to show that

$$\alpha(U(\mathfrak{g})_\eta) \xrightarrow{\sim} U({}^{\alpha,x}\mathfrak{g})_{\alpha(\eta)} \tag{2.2.1.1}$$

and this can be done in terms of the de Rham rational structure. Thus it suffices to show that

$$i^*(\alpha([U(\mathfrak{g})_\eta])) \xrightarrow{\sim} i^*(U({}^{\alpha,x}\mathfrak{g})_{\alpha(\eta)}) \tag{2.2.1.3}$$

where i is the inclusion of

$$\alpha(S(T,x)) \xrightarrow{\sim} S({}^{\alpha,x}T, {}^\alpha x) \xrightarrow{\sim} S(T, {}^\alpha x)$$

in $\alpha(S(G,X)) \xrightarrow{\sim} S({}^{\alpha,x}G, {}^{\alpha,x}X)$.

We may identify the Lie algebra $\mathfrak{t}$ of the CM torus T with the canonical Cartan subalgebra $\mathfrak{h}$ on $Flag_G$ of the previous section. Since $Z(\mathfrak{g}) \subset U(\mathfrak{t})$, the problem is thus reduced to case where $G = T$ is a torus, in which case it follows from Fact 1.2.11.2.

The following is obvious.

Corollary 2.2.2. *In the situation of Theorem 2.1.3, suppose K_x is defined over a subfield E_x of $\mathbb{C}$. Then the equivalence of categories in Theorem 2.1.3(2) is rational over $E_x(\eta)$.*

The set of closed K_x-orbits in $Flag_G$ is a finite union of smooth subvarieties; each orbit is rational over a finite extension of E_x, and the union is obviously defined over E_x. Let $Y \subset Flag_G$ be a closed K_x-orbit, defined over a field E_Y. The module $M(Y)$ introduced in Theorem 2.1.3 (4) can be identified with $i_+M(Y, \mathcal{T}_{Y,\eta})$. Since $\mathcal{T}_{Y,\eta}$ is the unique irreducible $(\mathcal{D}^i_\eta, K_x)$-connection on Y, it is rational over the common field of definition $E_Y(\eta)$ of Y and $\mathcal{D}_\eta$. It follows that $i_+M(Y, \mathcal{T}_{Y,\eta})$ is defined over $E_Y(\eta)$. Hence

Lemma 2.2.3. *There is a natural $E_Y(\eta)$-structure on the discrete series $(\mathfrak{g}, K_x)$-module*

$$\pi_{Y,\eta} = \pi_{Y,\eta,x} := \Gamma(M(Y)) = \Gamma(i_{Y,*}(\mathcal{O}_Y)) \in Mod(\mathfrak{g}, K_x)_{[\eta]},$$

associated to the pair (Y, η) by the construction in Theorem 2.1.3.

Moreover, if $\alpha \in Gal(\overline{\mathbb{Q}}/E_x)$, then there is a **canonical** *isomorphism*

$$\alpha(\pi_{Y,\eta}) \xrightarrow{\sim} \pi_{\alpha(Y),\alpha(\eta)}.$$

Proof. The first assertion is obvious; in view of (2.2.1), so is the existence of the isomorphism stated at the end of the theorem. The fact that the isomorphism can be chosen canonically follows from the existence of a canonical isomorphism

$$\alpha(\mathcal{O}_Y) \xrightarrow{\sim} \mathcal{O}_{\alpha(Y)}.$$

Characterization by Blattner K_x-types. Before we can continue, we need to recall the basic properties of $\bar{\partial}$-cohomology of discrete series. We return to the notation of §1. Thus π_∞ is a $(\mathfrak{g}, K_x)$-module for a fixed choice of x. Schmid's Theorem 1.4.1 can be stated more precisely in terms of the representation theory of K_x (cf. [H90, Proposition 4.5]).

Proposition 2.2.4. *With the notation of Theorem 1.4.1, there are canonical isomorphisms of one-dimensional spaces*

$$\begin{aligned} H^{q(\pi_\infty)}(Lie(P_x)_{\mathbb{C}}, K_x; \pi_\infty \otimes W_{\Lambda(\pi_\infty)}) &\xrightarrow{\sim} Hom_{K_x}(\wedge^{q(\pi_\infty)}(\mathfrak{p}_x^-), \pi_\infty \otimes W_{\Lambda(\pi_\infty)}) \\ &\xrightarrow{\sim} Hom_{K_x}(\wedge^{q(\pi_\infty)}(\mathfrak{p}_x^-) \otimes W^*_{\Lambda(\pi_\infty)}, \pi_\infty) \end{aligned}$$

Let λ be any non-trivial element of the last of these spaces. There is a unique irreducible representation $(\tau(\pi_\infty), Bl(\pi_\infty)) = (\tau(\pi_\infty, x), Bl(\pi_\infty, x))$ of K_x (the Blattner type*) such that*

$$\dim Hom_{K_x}(Bl(\pi_\infty, x), \pi_\infty) = 1$$

and if $\lambda' \in Hom_{K_x}(Bl(\pi_\infty, x), \pi_\infty)$ is any non-trivial homomorphism, then $Im(\lambda') = Im(\lambda)$.

Moreover, any unitary $(\mathfrak{g}, K_x)$-module with infinitesimal character $[\eta]$ containing $Bl(\pi_\infty)$ is isomorphic to π_∞.

The uniqueness assertions of Theorem 1.4.1 and Proposition 2.2.4 have the following consequences. We write $\tau(Y, \eta) = \tau(\pi_{Y,\eta})$, $q(Y, \eta) = q(\pi_{Y,\eta})$, etc.

Corollary 2.2.5. *Suppose* $\pi_\infty = \pi_{Y,\eta}$ *in the notation of Lemma 2.2.3. The representations* $(\tau(Y,\eta), B_{\tau(Y,\eta)})$ *and* $W_{\Lambda(Y,\eta)}$ *of* K_x *and the* $(Lie(G), K_x)$*-module* $\pi_{Y,\eta}$ *are all defined over the extension* $E_Y(\eta)$ *of* E_x*, and the one-dimensional* $\bar{\partial}$*-cohomology space* $H^{q(Y,\eta)}(Lie(P_x)_{\mathbb{C}}, K_x; \pi_{Y,\eta} \otimes W_{\Lambda(Y,\eta)})$ *is defined over* $E_Y(\eta)$*. Moreover,* $\alpha \in Aut(\mathbb{C}/E_x(\eta))$ *fixes the isomorphism class of* $\tau(Y,\eta)$ *if and only if* $\alpha(Y) = Y$*.*

More generally, if $\alpha \in Gal(\overline{\mathbb{Q}}/E_x)$*, then*

$$\alpha(W_{\Lambda(Y,\eta)}) = W_{\Lambda(\alpha(Y),\alpha(\eta))};\ \ \alpha(\tau(Y,\eta)) = \tau(\alpha(Y),\alpha(\eta));\ \ q(\alpha(Y),\alpha(\eta)) = q(Y,\eta)$$

and there is an isomorphism of 1*-dimensional* $\alpha(E_Y(\eta))$*-vector spaces*

$$\alpha(H^{q(Y,\eta)}(Lie(P_x), K_x; \pi_{Y,\eta} \otimes W_{\Lambda(Y,\eta)})) \\ \xrightarrow{\sim} H^{q(Y,\eta)}(Lie(P_{^\alpha x}), K_{^\alpha x}; \pi_{\alpha(Y),\alpha(\eta)} \otimes W_{\Lambda(\alpha(Y),\alpha(\eta))}))$$

Proof. For all $\alpha \in Gal(\overline{\mathbb{Q}}/E_Y(\eta))$,

$$\begin{aligned} 0 &\neq \alpha(H^{q(Y,\eta)}(Lie(P_x)_{\mathbb{C}}, K_x; \pi_{Y,\eta} \otimes W_{\Lambda(Y,\eta)}) \\ &= H^{q(Y,\eta)}(Lie(P_x)_{\mathbb{C}}, K_x; \alpha(\pi_{Y,\eta}) \otimes \alpha(W_{\Lambda(Y,\eta)})) \\ &= H^{q(Y,\eta)}(Lie(P_x)_{\mathbb{C}}, K_x; \pi_{Y,\eta} \otimes \alpha(W_{\Lambda(Y,\eta)})). \end{aligned}$$

By Theorem 1.4.1, this implies that $\alpha(W_{\Lambda(Y,\eta)})) = W_{\Lambda(Y,\eta)}$. Since $\mathfrak{p}_x^-$ is defined over E_x, it follows that the space $Hom_{K_x}(\wedge^{q(Y,\eta)}(\mathfrak{p}_x^-) \otimes W^*_{\Lambda(Y,\eta)}, \pi_{Y,\eta})$ is also defined over $E_Y(\eta)$, hence the image of any non-trivial homomorphism in this space is as well. Thus the first assertion follows from Proposition 2.2.4.

Now suppose $\alpha \in Aut(\mathbb{C}/E_x)$ fixes $\tau(Y,\eta)$. Then the above argument shows that $Hom_{K_x}(\tau(Y,\eta), \pi_{\alpha(Y),\alpha(\eta)}) =\neq 0$. Since $\pi_{\alpha(Y),\alpha(\eta)} = \pi_{\alpha(Y),\eta}$ is a discrete series module, it follows from the last statement of Proposition 2.2.4 that $\pi_{\alpha(Y),\eta} \xrightarrow{\sim} \pi_{Y,\eta}$.

The final assertion is proved in the same way.

Remark 2.2.6. The infinitesimal character ξ_π of π_∞ is a homomorphism $Z(\mathfrak{g}) \to \mathbb{G}_a$; by our running hypothesis, it is equal to the infinitesimal character of an algebraic representation $R(\pi)$ and is therefore defined over the field of definition $E(R)$ of $R(\pi)$. Let W_G (resp. W_{K_x}) denote the absolute Weyl group of G (resp. K_x) relative to the torus H_0, and let W^x be the set of Kostant representatives in W_G of the coset space $W_{K_x}\backslash W_G$. It is known (cf. [H94], §2) that there is a unique element $w \in W^x$ such that, letting the notation $R(\pi)$ stand for the highest weight of the representation $R(\pi_\infty)$,

$$\Lambda(\pi_\infty) = w(\rho - R(\pi)) - \rho$$

where ρ is the half-sum of the positive roots with respect to a Borel subgroup contained in P_x. Similarly, the highest weight of $\tau(\pi_\infty)$ has a well-known explicit expression (the Blattner parameter) in terms of $\Lambda(\pi_\infty)$ (cf. the discussion in [BHR, §3.3]). One can also prove Corollary 2.2.5 using these formulas.

Corollary 2.2.7. *The Harish-Chandra module $\pi_{Y,\eta}$ has a model over the field of definition of its isomorphism class, which is the field $E_x(Y,\eta)$.*

Proof. This follows because a finite-dimensional irreducible representation of K_x such as $\tau(Y,\eta)$ has a model over the field of definition of its isomorphism class. It can also be proved directly using its construction as $\Gamma(M(Y))$.

Now return to the situation of §1.3. There is a unique G-equivariant $U(\mathfrak{g})_{[\eta]}\otimes\mathcal{O}_{\check{X}}$-module $M_{\mathcal{Y},\eta}$ on $\check{X}$ whose fiber at x is the discrete series $(\mathfrak{g},K_x)$-module $\pi_{Y,\eta,x}$. (The notation $\mathcal{Y}$ is supposed to suggest a continuous family of closed orbits on $Flag_G$ under the stabilizers of points of $\check{X}$, whose fiber at x is the chosen orbit Y.) Consider the corresponding ind-automorphic vector bundle $[M_{\mathcal{Y},\eta}]$ on $S(G,X)$, with its canonical model over an appropriate subfield $E(\mathcal{Y},\eta)\subset E_x(Y,\eta)$, whose stalk at a point on the Shimura subvariety $S(T,x)$ is the space $\pi_{Y,\eta,x}$ with its de Rham rational structure over $E_x(Y,\eta)$. It follows from Corollary 2.2.7 that $E(\mathcal{Y},\eta)$ is in fact the field of definition of the isomorphism class of $[M_{\mathcal{Y},\eta}]$ (or of $M_{\mathcal{Y},\eta}$); this was asserted as Fact 1.3.3.

Let $x\in X$ be a CM point as above, and suppose $\alpha\in Aut(\mathbb{C})$ does not necessarily fix E_x. By the construction outlined in (1.2.11), we can construct the $\mathbb{Q}$-group ${}^{\alpha,x}G$ and its flag variety $Flag_{{}^{\alpha,x}G}$, as well as the hermitian symmetric domain ${}^{\alpha,x}X$ and its compact dual ${}^{\alpha,x}\check{X}$. By Lemmas 1.3.5 and 2.2.1, the image under the functor $T^{\vee}_{\alpha,x}$ of (1.2.8) of the discrete series module $\pi_{Y,\eta}$ is a $U({}^{\alpha,x}\mathfrak{g})_{[\alpha(\eta)]}$-module. We will show it is in fact a discrete series module of the form $\pi_{{}^{\alpha,x}Y,\alpha(\eta)}$ for some ${}^{\alpha,x}K_x$-orbit ${}^{\alpha,x}Y\in Flag_{{}^{\alpha,x}G}$.

Proposition 2.2.8. *There is a canonical isomorphism of complex algebraic groups*

$$\psi_{\alpha,x}:G(\mathbb{C})\xrightarrow{\sim}{}^{\alpha,x}G(\mathbb{C})$$

taking $\alpha(K_x)$ to ${}^{\alpha,x}K_x$ and covering a unique isomorphism of complex homogeneous algebraic varieties

$$\alpha(\check{X})\xrightarrow{\sim}{}^{\alpha,x}\check{X}. \tag{2.2.8.1}$$

The $(U({}^{\alpha,x}\mathfrak{g})_{[\alpha(\eta)]},{}^{\alpha,x}K_x)$-module $T^{\vee}_{\alpha,x}\pi_{Y,\eta}$ is isomorphic, with respect to the corresponding isomorphism of enveloping algebras, to the $U(\mathfrak{g},\alpha(K)_x)$-module $\pi_{\alpha(Y),\alpha(\eta)}$.

Proof. The construction of $\psi_{\alpha,x}$ is given by Milne on p. 104 of [M88], where it is labeled

$$g\mapsto{}^{\alpha,x}g.$$

Proposition 2.7 of [M88] implies that

$$\psi_{\alpha,x}(\alpha(K_x))={}^{\alpha,x}K_x\subset{}^{\alpha,x}G,$$

as claimed. By functoriality, $\psi_{\alpha,x}$ determines an isomorphism

$$Flag_G(\mathbb{C})\to Flag_{{}^{\alpha,x}G}(\mathbb{C}). \tag{2.2.8.2}$$

By Lemma 2.2.1, (2.2.8.2) gives rise to a canonical equivalence of twisted equivariant D-module categories

$$Mod(\mathcal{D}_{\alpha(\eta)},\alpha(K_x))\xrightarrow{\sim}Mod(\mathcal{D}_{{}^{\alpha,x}\eta},{}^{\alpha,x}K_x). \tag{2.2.8.3}$$

Similarly, and compatibly with (2.2.8.3), if $i : Y \hookrightarrow Flag_G$ is a K_x-orbit in $Flag_G$, then (2.2.8.2) defines a canonical equivalence of equivariant bounded derived categories of $\alpha(\mathcal{D}^i_{(\eta)}) = \mathcal{D}^i_{\alpha,x\eta}$-modules on $\alpha(Y) \xrightarrow{\sim} {}^{\alpha,x}Y \subset Flag_{\alpha,xG}(\mathbb{C})$:

$$D^b - Mod(\alpha(\mathcal{D}^i_{(\eta)}), \alpha(K_x)) \xrightarrow{\sim} D^b - Mod(\mathcal{D}^i_{\alpha,x\eta}, K_{\alpha x}). \tag{2.2.8.4}$$

Say $\pi_{Y,\eta}$ is the fiber at $x \in \check{X}$ of $M \in Mod_G(U(\mathfrak{g}) \otimes \mathcal{O}_{\check{X}})$, in the notation of §1.3. Then the module $T^\vee_{\alpha,x}(\pi_{Y,\eta})$, viewed as an object of the category on the left-hand side of (2.2.8.3), is by definition the fiber at the point ${}^\alpha x \in {}^{\alpha,x}\check{X}$ of ${}^{\alpha,x}M \in Mod_{\alpha,xG}(U({}^{\alpha,x}\mathfrak{g}) \otimes \mathcal{O}_{\alpha,x\check{X}})$. Since $\psi_{\alpha,x}$ covers the isomorphism (2.2.8.1) of homogeneous varieties, it follows that, with respect to the map (2.2.8.4) induced by $\psi_{\alpha,x}$, $T^\vee_{\alpha,x}(\pi_{Y,\eta})$ is isomorphic via $\psi_{\alpha,x}$ to the fiber at $\alpha(x) \in \alpha(\check{X})$ of $\alpha(M) \in Mod_G(U(\mathfrak{g})\otimes\mathcal{O}_{\check{X}})$. This in turn is isomorphic to the $(U(\mathfrak{g}), \alpha(K_x))$-module $\alpha(\pi_{Y,\eta})$, and since the results of Theorem 2.1.3 are algebraic, this is just $\pi_{\alpha(Y),\alpha(\eta)}$, as claimed.

2.2.8.5. Remark. Suppose G is of adjoint type and of the form $R_{F/\mathbb{Q}}G_1$ for some absolutely simple group G_1 over a totally real field F. It then follows from Theorem 1.3 of [MSu] that ${}^{\alpha,x}G(\mathbb{R})$ is obtained from $G(\mathbb{R}) \xrightarrow{\sim} \prod_{\sigma:F\hookrightarrow\mathbb{R}} G_1(F_\sigma)$ by letting α permute the σ. The map $\psi_{\alpha,x}$ is then given, up to a canonical inner automorphism coming from $T(\mathbb{C})$, by the same permutation. On the other hand, the action of α on $Z(\mathfrak{g})$ is given by the same permutation. So the equivalence of categories in the proof of Proposition 2.2.8 is just given by permuting the simple factors of $G(\mathbb{R})$.

In the proof of Proposition 2.2.8, we saw that the functor $T^\vee_{\alpha,x}$ is given canonically by

$$T^\vee_{\alpha,x}\pi_{Y,\eta} = \alpha(\pi_{Y,\eta}). \tag{2.2.8.6}$$

Corollary 2.2.9. *Hypotheses are as in Proposition 2.2.8. If $Y \subset Flag_G$ is a closed K_x-orbit, we let ${}^{\alpha,x}Y \subset Flag_{\alpha,xG}$ denote the closed ${}^{\alpha,x}K = K_{\alpha x}$-orbit obtained as the image of Y under (2.2.8.2). Then there is an isomorphism*

$$\phi_{\alpha,x} : T^\vee_{\alpha,x}\pi_{Y,\eta} \xrightarrow{\sim} \pi_{\alpha,xY,\alpha(\eta)}$$

of $(U({}^{\alpha,x}\mathfrak{g})_{[\alpha(\eta)]}, {}^{\alpha,x}K_x)$-modules.

This isomorphism is compatible with de Rham rational structures. For any $\alpha \in Aut(\mathbb{C})$, we can identify $T^\vee_{\alpha,x}\pi_{Y,\eta} \xrightarrow{\sim} \alpha(\pi_{Y,\eta})$ as in (2.2.8.6). For any $\alpha, \beta \in Aut(\mathbb{C})$, we have with respect to this identification

$$\phi_{\alpha\circ\beta,x} = \phi_{\alpha,{}^\beta x} \circ \phi_{\beta,x} : (\pi_{Y,\eta})_{DR,x} \xrightarrow{\sim} (\pi_{\alpha\circ\beta,xY,\alpha\circ\beta(\eta)})_{DR,{}^{\alpha\circ\beta}x}$$

Proof. The final claim is a consequence of Lemma 1.2.12.2.

2.2.10. Remark. Corollary 2.2.9 affirms the existence of isomorphisms of de Rham rational structures, satisfying the natural cocycle condition; of course these structures are rational relative to the de Rham rational structures on the enveloping algebra, as described in §1.3. Because the discrete series modules are irreducible, one sees that, with respect to appropriate algebra isomorphisms between the de Rham and Betti $E_x(Y,\eta)$-rational structures on $U(\mathfrak{g})$, the de Rham and Betti rational structures on $\pi_{Y,\eta}$ differ by a scalar (period) factor. If we normalize the isomorphisms $\phi_{\alpha,x}$ by the corresponding period factors, we can construct isomorphisms for the Betti rational structures

$$\phi_{\alpha,x,B} : \alpha(\pi_{Y,\eta,B}) \xrightarrow{\sim} \pi_{{}^{\alpha,x}Y,\alpha(\eta)),B}. \tag{2.2.10.1}$$

This is apparently more natural, since the Betti rational structures of $U({}^{\alpha,x}\mathfrak{g})$ is defined by the $\mathbb{Q}$-rational structures of the groups ${}^{\alpha,x}G$, and therefore do not depend on the choice of CM point x. However, it is not at all clear whether or not the $\phi_{\alpha,x,B}$ satisfy the analogue of the cocycle condition of Corollary 2.2.9. In other words, the isomorphisms (2.2.10.1) define a 2-cocycle on $Aut(\mathbb{C})$ (or on $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$) with coefficients in $Aut(\pi_{Y,\eta,B}) \xrightarrow{\sim} GL(1)$ that is not necessarily trivial. As Y and η vary, and as the pair (G,X) varies among the collection $({}^{\alpha,x}G, {}^{\alpha,x}X)$, we obtain a family of 2 cocycles, an example of a Galois gerbe, about which I can say nothing.

2.3. Conjugation by $\alpha \in Aut(\mathbb{C}/E(G,X))$.

We first note a simple invariance property of the rational structures. Let (T,x) be a CM pair as above. The following lemma is obvious.

Lemma 2.3.1. *Let η be a dominant integral weight of the canonical Cartan subalgebra of $Lie(G)$. Let $\beta \in G(\mathbb{Q})$, so that $ad(\beta)K_x = K_{\beta(x)}$ is the centralizer of the cocharacter attached to the CM pair $(ad(\beta)T, \beta(x)) \subset (G,X)$. Then the action of β on $Flag_G$ defines equivalences of derived categories*

$$T(\beta) : D_{K_x}(Flag_G)_{-\eta} \xrightarrow{\sim} D_{K_{\beta(x)}}(Flag_G)_{-\eta}$$

$$T(\beta) : D - Mod(\mathcal{D}_\eta, K_x) \xrightarrow{\sim} D - Mod(\mathcal{D}_\eta, K_{\beta(x)})$$

compatible with the Riemann-Hilbert correspondence, and an equivalence of categories

$$T(\beta) : Mod(\mathfrak{g}, K_x)_{[\eta]} \xrightarrow{\sim} Mod(\mathfrak{g}, K_{\beta(x)})_{[\eta]}$$

compatible with both the global sections functor (between the last two displayed formulas), and with the canonical $E_x(\eta)$-rational structure (on the last two lines).

Now let $\alpha \in Aut(\mathbb{C}/E(G,X))$. Let $v \in G(\overline{\mathbb{Q}})$, f_1, and (T',x') be as in Lemma 1.2.12.1. The isomorphism $f_1 : (G,X) \xrightarrow{\sim} ({}^{\alpha,x}G, {}^{\alpha,x}X)$ of Shimura data induces a canonical equivalence of categories

$$f_{1,*} : Mod(\mathcal{D}_{\alpha(\eta)}, K_{x'}) \xrightarrow{\sim} Mod(\mathcal{D}_{{}^{\alpha,x}\eta}, K_{{}^{\alpha}x}).$$

Here $f_{1,*}$ is just (the restriction to equivariant D_η-modules of) the usual direct image functor on D_η-modules for the isomorphism $f_1 : Flag_G \xrightarrow{\sim} Flag_{G'}$. We identify $\alpha(\eta)$ with a dominant integral weight ${}^{\alpha,x}\eta$ for ${}^{\alpha,x}G$ by means of f_1. Although f_1 is only well-defined up to $G(\mathbb{Q})$-conjugation, this latter identification is canonical.

Lemma 2.3.2. *Let $\alpha \in Aut(\mathbb{C}/E(G,X))$. For any pair (v, f_1) as in Lemma 1.2.12.1, there is a commutative diagram of equivalences of categories*

$$\begin{array}{ccc}
\alpha(Mod(\mathcal{D}_\eta, K_x)) & \xrightarrow{\alpha(\Gamma)} & \alpha(Mod(\mathfrak{g}, K_x)_{[\eta]}) \\
\downarrow & & \downarrow \\
Mod(\mathcal{D}_{\alpha(\eta)}, K_{x'}) & \xrightarrow{\Gamma} & Mod(\mathfrak{g}, K_{x'})_{[\eta]} \\
\downarrow & & \downarrow \\
Mod(\mathcal{D}_{{}^{\alpha,x}\eta}, K_{{}^{\alpha}x}) & \xrightarrow{{}^{\alpha,x}\Gamma} & Mod({}^{\alpha,x}\mathfrak{g}, K_{{}^{\alpha}x})_{{}^{\alpha,x}[\eta]}.
\end{array}$$

The functors respect the $\alpha(E_x(\eta)) = E_{\alpha x}({}^{\alpha,x}\eta)$-rational structures on all categories.

The diagrams corresponding to (v, f_1) and to $(\beta v, f_1 \circ ad(\beta^{-1}))$, with $\beta \in G(\mathbb{Q})$, are related by the equivalences of categories of Lemma 2.3.1.

2.4. Rational models of automorphic representations and Petersson norms. We return to the situation of §2.2. Let $\pi_{Y,\eta}$ be a discrete series $(\mathfrak{g}, K)$-module as in Lemma 2.2.3. Recall the inclusion from Proposition 1.5.2, where we include the dependence on x in the notation:

$$H^\bullet(Lie(P_x), K_x; \mathcal{A}_0(G,x) \otimes \mathcal{V}_x) \xrightarrow{\sim} \bigoplus_{\pi=\pi_\infty\otimes\pi_f\subset\mathcal{A}_0(G,x)} H^\bullet(Lie(P_x), K_x; \pi_\infty \otimes \mathcal{V}_x) \otimes \pi_f \hookrightarrow \tilde{H}^\bullet([\mathcal{V}]^{can}).$$

We write $\tilde{H}^\bullet(S(G,X), [\mathcal{V}]^{can})$ for $\tilde{H}^\bullet([\mathcal{V}]^{can})$ to allow for conjugation of the Shimura variety (this is an abuse of notation inasmuch as the coherent cohomology is attached to the toroidal compactifications of $S(G,X)$). Fix $\pi_\infty = \pi_{Y,\eta}$, let $\mathcal{F}_{Y,\eta}$ be the bundle denoted $\mathcal{F}(\pi_\infty)$ in §1.4, and let $\pi_{0,f}$ be an automorphically cuspidal representation of $G(\mathbf{A}^f)$, as in Definition 1.5.3. Then for any $\beta \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ and any inner twist of G of the form ${}^{\alpha,x}G$, there are canonical isomorphisms

(2.4.1)

$$\bigoplus_{\pi=\pi_\infty\otimes\beta(\pi_{0,f})\subset\mathcal{A}_0(G,x)} Hom_{K_{\alpha x}}(B_{\alpha(Y),\alpha(\eta)}, \pi_\infty) \otimes \beta(\pi_{0,f})$$
$$\xrightarrow{\sim} H^\bullet(Lie(P_{\alpha x}), K_{\alpha x}; \mathcal{A}_0({}^{\alpha,x}G, {}^{\alpha}x) \otimes {}^{\alpha,x}\mathcal{F}_{\alpha(Y),\alpha(\eta),{}^{\alpha}x})[\beta(\pi_{0,f})]$$
$$\xrightarrow{\sim} \tilde{H}^\bullet(S({}^{\alpha,x}G, {}^{\alpha,x}X), [\mathcal{F}_{\alpha(Y),\alpha(\eta)}]^{can})[\beta(\pi_{0,f})]$$

Let α and β be trivial, and suppose $\pi_{0,f}$ is of discrete series type. Then the left-hand side of (2.4.1) simplifies and we have

(2.4.2)

$$\bigoplus_{\pi=\pi_{Y,\eta}\otimes\pi_{0,f}\subset\mathcal{A}_0(G,x)} Hom_{K_x}(B_{Y,\eta}, \pi_{Y,\eta})\otimes\pi_{0,f} \xrightarrow{\sim} \tilde{H}^{q(Y,\eta)}(S(G,X), [\mathcal{F}_{Y,\eta}]^{can})[\pi_{0,f}].$$

As in Proposition 1.4.6, the right hand side of (2.4.2) has a canonical rational structure over $E_x(\pi_{0,f}) \supset E_x(Y,\eta)$, the one-dimensional space $Hom_{K_x}(B_{Y,\eta}, \pi_{Y,\eta})$ has two canonical rational structures – the Betti rational structure and the de Rham rational structure – over $E_x(Y,\eta)$, and $\tilde{H}^{q(Y,\eta)}(S(G,X), [\mathcal{F}_{Y,\eta}]^{can})$ has a canonical rational structure over the reflex field $E(\mathcal{Y},\eta)$ of the automorphic vector bundle $[\mathcal{F}_{Y,\eta}]$. *

Define $U(\mathfrak{g})_{DR,x}$ as in §1.3.

Proposition 2.4.3. *There is a unique $U(\mathfrak{g})_{DR,x} \times G(\mathbf{A}^f)$-stable $E_x(\pi_{0,f})$-rational structure $\pi_{DR,x}$ on the automorphic representation $\pi = \pi_{Y,\eta} \otimes \pi_{0,f}$, so that the isomorphism (2.4.2) is $E_x(\pi_{0,f})$-rational with respect to the de Rham rational structure on $Hom_{K_x}(B_{Y,\eta}, \pi_{Y,\eta})$, the given $E_x(\pi_{0,f})$-rational structure on $\pi_{0,f}$, and the canonical rational structure on the right-hand side of (2.4.2).*

Proof. It follows from Corollary 2.2.5 that the kernel of the natural map

$$U(\mathfrak{g}) \otimes B_{Y,\eta} \to \pi_{Y,\eta}$$

is $E_x(Y,\eta)$-rational with respect to either the Betti or the de Rham rational structure on both sides. We can therefore take the $U(\mathfrak{g})_{DR,x} \times G(\mathbf{A}^f)$-module generated by the image of $B_{Y,\eta}$ under any non-zero de Rham rational element of $Hom_{K_x}(B_{Y,\eta}, \pi_{Y,\eta})$. Uniqueness follows from Corollary 2.2.5.

We consider the $Aut(\mathbb{C})$-translates of $\pi_{DR,x}$. Define ${}^{\alpha,x}\pi_{DR,x}$ to be the unique $U({}^{\alpha,x}\mathfrak{g})_{DR,{}^{\alpha}x} \times {}^{\alpha,x}G(\mathbf{A}^f)$-rational structure on the automorphic representation ${}^{\alpha,x}\pi_{Y,\eta} \otimes \alpha(\pi_{0,f})$ of ${}^{\alpha,x}G$ such that the isomorphism (2.4.2) (with G replaced by ${}^{\alpha,x}G$ and so on) satisfies the conditions of Proposition 2.4.3. It follows from (1.3.6) and Corollary 1.4.8 that, for any $\alpha \in Aut(\mathbb{C})$, $\alpha(\pi_{DR,x})$ is canonically isomorphic to ${}^{\alpha,x}\pi_{DR,x}$.

In the following Lemma, the action of α is acting on the coefficients of the$\pi_{0,f}$, rather than on the field of definition of $S(G,X)$.

Lemma 2.4.4. *Suppose $\alpha \in Aut(\mathbb{C}/E_x)$. Suppose $\pi = \pi_{Y,\eta} \otimes \pi_{0,f}$; let*

$$\alpha[(\mathcal{F})_{Y,\eta}] = [\alpha(\mathcal{F}_{Y,\eta})] = [\mathcal{F}_{\alpha(Y),\alpha(\eta)}]$$

be the automorphic vector bundle on $S(G,X)$ obtained by conjugating $[\mathcal{F}_{Y,\eta}]$ by α, and define

$$\alpha \star \pi = \pi_{\alpha(Y),\alpha(\eta)} \otimes \alpha(\pi_{0,f})$$

Then ${}^{\alpha,x}\pi_{DR,x}$ is canonically isomorphic to $(\alpha \star \pi)_{DR,x}$.

Proof. Any $\alpha \in Aut(\mathbb{C}/E_x)$ stabilizes the inclusion $S(T,x) \subset S(G,X)$. The construction 1.1.2 defines a natural bijection between irreducible automorphic vector bundles on $S(T,x)$ and characters of T, and $Aut(\mathbb{C}/E_x)$ permutes the automorphic vector bundles by the obvious action on characters. Since an irreducible automorphic vector bundle on $S(G,X)$ is determined (by the theory of the highest weight) by its restriction to $S(T,x)$, the action on characters of T also determines the action of $Aut(\mathbb{C}/E_x)$ on automorphic vector bundles on $S(G,X)$. Recalling from Corollary 2.2.5 that $q(\alpha(Y),\alpha(\eta)) = q(Y,\eta)$, we see that the canonical isomorphism of the Lemma is induced by the canonical isomorphism

$$\alpha(H^{q(Y,\eta)}(S(G,X),[(\mathcal{F})_{Y,\eta}]) \rightarrow H^{q(Y,\eta)}(S(G,X),\alpha([(\mathcal{F})_{Y,\eta}])).$$

Indeed, since $\pi_{0,f}$ is assumed automorphically cuspidal and of discrete series type, this follows from Lemma 1.5.6.

Since $E_x(\pi_{0,f})$ is defined as a subfield of $\mathbb{C}$, we can identify $Aut(\mathbb{C})/Aut(\mathbb{C}/E_x(\pi_{0,f}))$ with the set Σ_π of complex embeddings of $E_x(\pi_{0,f})$. We thus have a natural collection

$$\{\alpha(\pi_{DR,x}) = {}^{\alpha,x}\pi_{DR,x}, \alpha \in \Sigma_\pi\} \tag{2.4.5}$$

and for any non-zero vector $v \in \pi_{DR,x}$, the collection

$$\{\alpha(v), \alpha \in \Sigma_\pi\} \tag{2.4.6}$$

is a generator of the $\oplus_{\alpha\in\Sigma_\pi} U({}^{\alpha,x}\mathfrak{g})_{DR,{}^{\alpha}x} \times {}^{\alpha,x}G(\mathbf{A}^f)$-module $\oplus_{\alpha\in\Sigma_\pi}\alpha(\pi_{DR,x})$. This is the automorphic counterpart of the restriction of scalars to $\mathbb{Q}$ of the motive conjecturally attached to π.

2.4.7. Changing x. The analogues of formulas (1.2.5) and (1.2.6) are valid for automorphic vector bundles, and in principle provide means to compare the arithmetic normalizations (2.4.5) and (2.4.6) of $(\mathfrak{g}, K_x)$- and $(\mathfrak{g}, K_{x'^{\iota}})$-modules. Since it is not clear how these can be used, the formulas are omitted. It may be possible to formulate comparisons in such a way as to obtain an automorphic version of a motive over $E(G,X)$ and not over E_x.

Normalized inner products. We fix $x \in X$, and consider an irreducible automorphic representation $\pi \subset \mathcal{A}_0(G,x)$ with central character

$$\xi_\pi : Z_G(\mathbb{Q})\backslash Z_G(\mathbf{A}) \to \mathbb{C}.$$

Let $\nu : G(\mathbb{Q})\backslash G(\mathbf{A}) \to \mathbb{R}_+^\times$ be the unique positive character such that $\nu \mid_{Z_G(\mathbf{A})} = |\xi_\pi|^{-2}$. Let $v_1, v_2 \in \pi$, and define

$$< v_1, v_2 >= \int_{Z_G(\mathbf{A})\cdot G(\mathbb{Q})\backslash G(\mathbf{A})} v_1(g)\cdot \bar{v}_2(g)\cdot \nu(g)dg \tag{2.4.8}$$

This is a $(\mathfrak{g}, K_x) \times G(\mathbf{A}^f)$-invariant positive-definite hermitian form on π, the restriction of the L_2 inner product.

Proposition 2.4.9. *Let $v \in \pi_{DR,x}$ be any non-zero vector, and let $Q(v) =< v, v >$. Let $E_x(\pi_{0,f})^{++}$ be the subset of totally positive elements of the maximal totally real subfield $E_x(\pi_{0,f})^+$ of the CM field $E_x(\pi_{0,f})$. Then for all $v_1, v_2 \in \pi_{DR,x}$*

$$Q(v)^{-1} < v_1, v_2 > \in E_x(\pi_{0,f});\ Q(v)^{-1} < v_1, v_1 > \in\ E_x(\pi_{0,f})^{++}.$$

Proof. Schur's Lemma applies to irreducible admissible $(\mathfrak{g}, K_x) \times G(\mathbf{A}^f)$-modules and implies that the space of invariant hermitian forms on π, resp. on $\pi_{DR,x}$ is of dimension at most 1 over $\mathbb{R}$, resp. over $E_x(\pi_{0,f})^+$, and the two dimensions are equal. The action of $E_x(\pi_{0,f})$ is hermitian with respect to the L_2 inner product on π, because the coefficients are realized as a subfield of $\mathbb{C}$. Thus, up to a positive real scalar multiple, the L_2-inner product restricts to a positive-definite hermitian form on $\pi_{DR,x}$ with values in $E_x(\pi_{0,f})$.

In particular, the representation π admits a non-trivial hermitian inner product taking values in $E_x(\pi_{0,f})$ on the $E_x(\pi_{0,f})$-rational form $\pi_{DR,x}$.

The following lemma is then obvious:

Lemma 2.4.10. *For any non-zero vector $v \in \pi_{DR,x}$, the collection*

$$\{Q(\alpha(v)), \alpha \in \Sigma_\pi\}$$

depends on v only up to multiplication by $E_x(\pi_{0,f})) \subset (E_x(\pi_{0,f})) \otimes \mathbb{C})^\times$.

3. Arithmetic automorphic forms on unitary groups

3.1. Unitary group Shimura varieties.

Let V denote an n-dimensional vector space over $\mathcal{K}$ with non-degenerate hermitian form $\langle , \rangle$, relative to F. Let $V' \subset V$ be a subspace of dimension $n-1$; we assume $\langle , \rangle$ restricts to a non-degenerate hermitian form on V', and let $V_0 = (V')^\perp$ so that $V = V' \oplus V_0$ as a sum of hermitian spaces. Let $U = U(V)$, $U' = U(V') \times U(V_0)$.

These are algebraic groups over the totally real field F and $U' \subset U$. Let G be the $\mathbb{Q}$-group scheme whose group of R-valued points, for any $\mathbb{Q}$-algebra R, is given by

$$G(R) = \{g \in GL(V \otimes R) \mid \langle g(v), g(w)\rangle = \nu(g)\langle v, w\rangle, \text{ for some } \nu(g) \in R^{\times}\}. \tag{3.1.1}$$

Let $G' = G \cap GL(V') \times GL(V_0)$, and let ν' denote the restriction of ν to G'.

For each $\sigma \in \Sigma$, we let (r_σ, s_σ), resp. (r'_σ, s'_σ), denote the signature of the complex hermitian space $V_\sigma := V \otimes_\sigma \mathbb{C}$, resp. $V'_\sigma := V' \otimes_\sigma \mathbb{C}$.

We choose as basepoint the homomorphism

$$h_0 : \mathbb{S}(\mathbb{C}) = \mathbb{C}^{\times} \to GL(V \otimes_{\mathbb{Q}} \mathbb{C}) = \prod_{\sigma \in \Sigma} GL(V_\sigma) \times GL(V_{c\sigma})$$

sending $z \in \mathbb{C}^{\times}$ to the matrix whose $(\sigma, c\sigma)$ component is

$$\left(\begin{pmatrix} zI_{r_\sigma} & 0 \\ 0 & \bar{z}I_{s_\sigma} \end{pmatrix}, \begin{pmatrix} zI_{s_\sigma} & 0 \\ 0 & \bar{z}I_{r_\sigma} \end{pmatrix}\right). \tag{3.1.2}$$

The image of h_0 lies in $G(\mathbb{C})$ and we let X be its $G(\mathbb{R})$-orbit. We may choose a basis of V so that $Im(h_0) \subset G'(\mathbb{C})$; then the $G'(\mathbb{R})$-orbit of h_0 is a hermitian symmetric domain X' contained in X, so that the inclusion $(G', X') \subset (G, X)$ is a morphism of Shimura data.

Write $\mathbb{S}_{\mathbb{C}} = (\mathbb{G}_m)_{\mathbb{C}} \times (\mathbb{G}_m)_{\mathbb{C}}$, as in [D1], with z the first coordinate, and let $\mu : \mathbb{G}_m \to \mathbb{S}$ denote the inclusion of the first coordinate. The reflex field $E(G, X)$ is the field of definition of the conjugacy class of the cocharacter $\mu_{h_0} = h_0 \circ \mu : \mathbb{G}_m \to G$. This is determined with respect to the CM type Σ by Shimura's recipe [Sh], as follows. Let $T_{\mathcal{K}} = N^{-1}_{\mathcal{K}/F}(\mathbb{G}_m)_{\mathbb{Q}} \subset R_{\mathcal{K}/\mathbb{Q}}(\mathbb{G}_m)_{\mathcal{K}}$ be the Serre torus of $\mathcal{K}$. The complex embeddings of $\mathcal{K}$ form a $\mathbb{Z}$-basis of the character group $X^*(R_{\mathcal{K}/\mathbb{Q}}(\mathbb{G}_m)_{\mathcal{K}})$ and define characters of $T_{\mathcal{K}}$ by restriction; if σ is a complex embedding we let $[\sigma] \in X^*(T_{\mathcal{K}})$ denote the corresponding character. Then $E(G, X)$ is the field of rationality of the character

$$\xi_X = \prod_{\sigma \in \Sigma} [\sigma]^{r_\sigma} [c\sigma]^{s_\sigma}. \tag{3.1.3}$$

It follows from Theorem 1.3 of [MSu] that if $G = GU(W)$ as above, $\alpha \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, and $(H, x) \subset (G, X)$ is a CM pair, then for ${}^{\alpha,x}G$ we can take $GU({}^{\alpha}W)$, where ${}^{\alpha}W$ is a hermitian space over L with the same local invariants at all finite places and whose signatures at archimedean places σ are obtained from those of W by the formula

$$(r_\sigma({}^{\alpha}W), s_\sigma({}^{\alpha}W)) = (r_{\alpha\sigma}(W), s_{\alpha\sigma}(W)); \tag{3.1.4}$$

the existence of such a hermitian space is guaranteed by the reciprocity law for the classification of global hermitian spaces by local invariants. The datum ${}^{\alpha,x}X$ is then determined by (3.1.4) and (3.1.2).

Let π_∞ be a discrete series representation of G_∞. We define the discrete series representation ${}^{\alpha,x}\pi_\infty$ of ${}^{\alpha,x}G_\infty$ by (1.4.7). The restriction of ${}^{\alpha,x}\pi_\infty$ to $U({}^{\alpha}W, \mathbb{R}) = \prod_{v \mid \infty} U({}^{\alpha}W)_v$ is obtained by permuting the archimedean local factors as in (3.1.4).

3.2 Rational structures on archimedean and adelic representations.

We work with G in this section for convenience; one can substitute G' for G in all theorems.

The hermitian space V can be written as a direct sum of non-degenerate one-dimensional hermitian spaces

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n. \tag{3.2.1}$$

The basepoint $x \in X$ can be chosen so that $Im(x)$ fixes the decomposition (3.2.1). Moreover, for each V_i the formula (2.0.2) determines a homomorphism $h_i : \mathbb{S} \to GL(V_i)$, with signatures $(r_\sigma(V_i), s_\sigma(V_i))$ either $(1,0)$ or $(0,1)$ at each place. We define x to be the diagonal homomorphism $(h_1, \ldots, h_n) : \mathbb{S} \to \prod GL(V_i)$. It takes values in

$$H_0 = \prod_{i=1}^{n} GL(V_i) \cap G$$

and (H_0, x) is a Shimura datum contained in (G, X), so $E(x) := E(H_0, x) \supset E(G, X)$. The group K_x is the centralizer in G of the homomorphism μ_x (denoted μ_{h_0} earlier in this section, when the point in X was denoted h_0); since $E(x)$ is the field of rationality of μ_x, K_x is a subgroup of G defined over $E(x)$.

Recall that $E(G, X)$ is the field of definition of the character ξ_X of (3.1.3). Consider all (ordered) factorizations

$$\phi : \quad \xi_X = \xi_1 \cdot \xi_2 \cdot \ldots \cdot \xi_n \tag{3.2.2}$$

where each ξ_j is a character of the form $\prod_{\sigma \in \Sigma} [\sigma]^{a_\sigma} [c\sigma]^{b_\sigma}$ with $0 \leq a_\sigma, b_\sigma \leq 1$ and $a_\sigma + b_\sigma = 1$ for all σ. The decomposition (3.2.1) determines one such factorization ϕ_0, and one sees easily that $E(x)$ is the field of rationality $E(\phi_0)$ of ϕ_0.

Lemma 3.2.3. *Any factorization of ξ_X can be obtained from some decomposition of the form (3.2.1).*

Proof. By induction on n, it suffices to show the following. Let $a : \Sigma \to \{0, 1\}$ be a function such that $a(\sigma) \leq r_\sigma$ for all σ. Then there is a non-zero vector $v_1 \in V(\mathcal{K})$ such that $\sigma(\langle v_1, v_1 \rangle) > 0$ if and only if $a(\sigma) = 1$. Indeed, we let $V_1 = \mathcal{K} \cdot v_1$ and apply induction to $V_1^\perp$. But the existence of such a v_1 is obvious because $V(\mathcal{K})$ is dense in $V \otimes_{\mathcal{K}} \mathbb{R}$.

Corollary 3.2.4. *The group $Gal(\overline{\mathbb{Q}}/E(G, X))$ acts on the set of factorizations (3.2.1), and the reflex field $E(G, X)$ is the intersection of the $E(x)$ as x runs over the points of X corresponding to factorizations (3.2.1).*

Proof. Indeed, $E(G, X)$ is clearly the intersection of the $E(\phi)$ as ϕ varies over factorizations (3.2.2).

This is a special case of the well-known theorem, due to Shimura, that the reflex field of $S(G, X)$ is the intersection of the reflex fields of its CM points. Obviously it suffices to take a finite set of x in Corollary 3.2.4. In fact, we can take a representative ϕ_α for each $Gal(\overline{\mathbb{Q}}/E(G, X))$ orbit α of factorizations, and realize ϕ_α as the factorization of a fixed x_α. Then the $Gal(\overline{\mathbb{Q}}/E(G, X))$ orbits of the x_α thus defined suffice.

We now apply the constructions of §2.4 with x and K_x attached to the decomposition (3.2.1). This choice will stay with us for most of the subsequent discussion, but bearing in mind Corollary 3.2.4, we can remove the choice in the study of special values of L-functions.

3.3. Automorphic cuspidality for unitary groups.

Let π_∞ be a discrete series representation of $G(\mathbb{R})$ and let $\mathcal{F} = \mathcal{F}(\pi_\infty)$ be the automorphic vector bundle over $S(G,X)$ defined in Theorem 1.4.1. In the course of the proof of Corollary 1.4.8 we briefly referred to the *subcanonical extension* of $\mathcal{F}$ and the *interior cohomology* $H^\bullet_!(\mathcal{F})$, the image of the coherent cohomology of the subcanonical extension (over any smooth toroidal compactification) in the coherent cohomology of the canonical extension. We have the inclusions

$$H^\bullet_{cusp}(\mathcal{F}) \subset H^\bullet_!(\mathcal{F}) \subset H^\bullet_{(2)}(\mathcal{F}) \tag{3.3.1}$$

where $H^\bullet_{(2)} \subset \tilde{H}^\bullet(S(G,X), \mathcal{F}^{can})$ is the subspace represented by square-integrable automorphic forms.

The following proposition is well known (see [Sch], §2, for the proof of a more difficult version).

Proposition 3.3.2. *Suppose the infinitesimal character of π_∞ is the infinitesimal character of a finite-dimensional representation of G with regular highest weight (i.e., whose highest weight is not only dominant but is in the interior of the positive chamber). Then the inclusions of (3.3.1) are all equalities.*

It follows that if π_∞ has regular infinitesimal character, in the sense of Proposition 3.3.2, then $H^\bullet_{cusp}(\mathcal{F})$ is naturally an $E(\pi_\infty)$ rational subspace of $\tilde{H}^\bullet(S(G,X), \mathcal{F}^{can})$. Moreover, for any $\alpha \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, there is a natural isomorphism of

$$G(\mathbf{A}^f) \xrightarrow{\sim} {}^{\alpha,x}G(\mathbf{A}^f)$$

-modules

$$\alpha(H^\bullet_{cusp}(S(G,X),\mathcal{F})) \xrightarrow{\sim} H^\bullet_{cusp}(S({}^{\alpha,x}G, {}^{\alpha,x}X), {}^{\alpha,x}\mathcal{F}). \tag{3.3.3}$$

Suppose π_f is an irreducible admissible representation of $G(\mathbf{A}^f)$ such that

$$H^\bullet_{cusp}(S(G,X),\mathcal{F})[\pi_f] \neq 0.$$

It follows from (3.3.3) that, for all $\alpha \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$H^\bullet_{cusp}(S({}^{\alpha,x}G, {}^{\alpha,x}X), {}^{\alpha,x}\mathcal{F})[\alpha(\pi_f)] \neq 0.$$

Thus $\alpha(\pi_f)$ occurs as the finite part of a cuspidal automorphic representation for some inner twist of G. If π_f is not automorphically cuspidal, then for some β, some inner twist G' of G, and some automorphic vector bundle $\mathcal{F}'$ over $S(G',X')$, we must have that $\beta(\pi_f)$ also contributes to the quotient of $\tilde{H}^\bullet(S(G',X'), \mathcal{F}'^{,can})$ by the cuspidal cohomology. It is known that this quotient can be expressed in terms of non-cuspidal automorphic representations, in particular in terms of representations parabolically induced from Levi subgroups of G. (This seems to depend on Franke's unpublished theorem 1.5.1, but in fact the boundary cohomology of the coherent automorphic vector bundle $\mathcal{F}$ can be related by the Hodge-de Rham spectral sequence to the boundary cohomology of $S(G,X)$ with coefficients in a local system, so this claim can be derived from the results of [Fr].) Thus as long as

$\beta(\pi_f)$ is not (the finite part of) a CAP representation for any β, we see that π_f is automorphically cuspidal.

Now we have seen in 1.4.10 that $\beta(\pi_f)$ also occurs in the cuspidal cohomology of some $\mathcal{F}^{\prime,can}$ over the original $S(G,X)$, and by a similar argument, we see that if $\beta(f\pi_f)$ is CAP for G', then it is also CAP for the original $S(G,X)$. Applying β^{-1}, we see that π_f is automorphically cuspidal provided the infinitesimal character of π_∞ is regular and $\pi = \pi_\infty \otimes \pi_f$ is not a CAP representation. It is expected that CAP representations of unitary groups are non-tempered, so the regularity of the infinitesimal character of π_∞ should suffice to guarantee automorphic cuspidality.

4. Gross-Prasad period invariants

We return to the situation of §3.1, with $G' \subset G$ an inclusion of unitary similitude groups.

4.1. Review of the Gross-Prasad conjecture for unitary and similitude groups.

Let L/E be a quadratic extension of local fields of characteristic zero, and let $\alpha \in Gal(L/E)$ be the non-trivial element. Let W be an n-dimensional non-degenerate hermitian space over L, relative to E, $W' \subset W$ a subspace on which the restriction of the hermitian form is non-degenerate, so that $W = W' \oplus W_0$ with $W_0 = W^\perp$. The unitary groups of W, W', and W_0 are reductive algebraic groups over E; we write $H' = U(W')(E)$, $H'' = U(W')(E) \times U(W_0)(E)$ and $H = U(W)(E)$ for their groups of E-rational points. As in the global situation, we have $H' \subset H'' \subset H$. The Gross-Prasad conjecture, most recently reformulated in [GGP], describes restrictions of irreducible representations for H to H'', and for more general pairs of classical groups.

Thanks to Waldspurger and Moeglin, the full Gross-Prasad conjecture is now known for pairs of special orthogonal groups over non-archimedean local fields, and a similar argument seems likely to work for pairs of unitary groups as well. In the situation of the present paper, however, only the following weaker statement has been proved: in the non-archimedean case this is in [AGRS] and in the archimedean case in [SZ] (recall that in this case we are working with smooth Frechet representations of moderate growth).

Theorem 4.1.1 ([AGRS], [SZ]). *Let π and π' be irreducible admissible representations of H and H', respectively. Let*

$$L(\pi,\pi') = Hom_{H'}(\pi \otimes \pi', \mathbb{C})$$

be the space of H'-invariant linear forms on the representation $\pi \otimes \pi'$ of $H \times H'$, where H' acts through its diagonal embedding in $H \times H'$. Then

$$\dim L(\pi,\pi') \leq 1.$$

The complete Gross-Prasad conjecture predicts precisely when $\dim L(\pi,\pi') = 1$, at least when π and π' belong to *generic* L-packets. We recall an intermediate version of the conjecture. Let Π and Π' be generic L-packets of H and H', respectively. These are defined by Moeglin, at least in the tempered case, as the fibers of base change to $GL(n,L)$ and $GL(n-1,L) \times GL(1,L)$. Then the Gross-Prasad conjecture includes the following statement:

Conjecture 4.1.2. *The sum*

$$\sum_{\pi\in\Pi,\pi'\in\Pi'} L(\pi,\pi') = 1.$$

In the literature, one usually works with H' rather than H'', but for the local theory there is an equivalent version involving H''. Let $Z \subset H$ denote the center of H. Then $Z \subset H''$. Let π be as above, and let $\xi_\pi : Z \to \mathbb{C}^\times$ denote its central character. Any representation π' of H' can be extended uniquely to a representation $\pi'' = \pi' \times \pi_0$, with central character $\xi_{\pi''}$, such that

$$\xi_\pi^{-1} = \xi_{\pi''}|_Z. \tag{Ξ}$$

Indeed, the group Z is isomorphic to $U(W_0)(E)$, and one takes $\pi_0 = \xi_\pi^{-1}\cdot\xi_{\pi'}^{-1}$. Then Conjecture 4.1.2 is equivalent to the version for the inclusion of $H'' \subset H$, with π' replaced by π'' satisfying (Ξ).

We let G and G' be as above. In the following lemma, we assume E is a completion of $\mathbb{Q}$ at a prime w of F dividing the rational place v, and let $J = G(\mathbb{Q}_v)$ or $G'(\mathbb{Q}_v)$. Thus $J \supset J_0$, where either $J_0 = \prod_{w|v} U(W)(F_w)$ or $J_0 \supset \prod_{w|v} U(W')(F_w)$ and w runs over primes of F dividing v.

Lemma 4.1.3. *Let π_1 be an irreducible admissible representation of J and assume*

$$\pi_1 \mid_{J_0} = \oplus_{\alpha\in A}\pi_\alpha$$

where each π_α is an irreducible representation of J_0. Write $\pi_\alpha = \otimes_{w \mid v}\pi_{\alpha,w}$, with $\pi_{\alpha,w}$ an irreducible representation of $U(W)(F_w)$ or $U(W')(F_w)$. Then as α varies over A, the $\pi_{\alpha,w}$ with w fixed belong to the same L-packet. Moreover, each π_α occurs with multiplicity one.

Because the definition of L-packets for unitary groups is in some flux, we will be satisfied with the following version of Lemma 4.1.3, including an obviously artificial condition on globalization:

Lemma 4.1.3 bis. *In the situation of Lemma 4.1.3, assume further that π_1 is the local component at v of an automorphic representation τ of G or G', as the case may be, such that τ_∞ belongs to the discrete series. Then the conclusion of Lemma 4.1.3 holds.*

Proof. Assume the additional hypothesis. For definiteness, we assume $J = G(\mathbb{Q}_v)$. Let $\alpha, \alpha' \subset A$, and let τ_α and $\tau_{\alpha'}$ be two irreducible constituents of the restriction of τ to $U(W)(\mathbf{A})$. Under the additional hypothesis, we know thanks to Labesse [L] or Morel [Mo] that τ_α and $\tau_{\alpha'}$ admit base change to automorphic representations σ and σ', respectively, of $GL(n,\mathcal{K})$. By the Jacquet-Shalika version of strong multiplicity one, it suffices to show that $\sigma_u = \sigma'_u$ for all places u of $\mathcal{K}$ dividing places of F at which τ is unramified. In other words, it suffices to prove Lemma 4.1.3 when π_1 is an unramified representation. In that case, it has been proved by Clozel that π_1 is in fact *tempered* [C2].[2]

[2] This is also true at ramified places, a fact proved over the past years with increasing generality and completed recently by Caraiani [Ca]. But for our purposes a result at almost all primes is sufficient.

We thus assume π_1 is unramified. Let $G^\sharp$ be the group of all unitary similitudes of W – in the definition (3.1.1) one replaces the condition $\nu(g) \in R^\times$ by $\nu(g) \in (F \otimes_{\mathbb{Q}} R)^\times$. Then $G^\sharp$ is an algebraic group over F, and if we let $J^\sharp = G^\sharp(F \otimes \mathbb{Q}_v)$, we find that $J^\sharp \supset J$. Now π_1 can be extended to an irreducible unramified representation $\pi^\sharp$ of $J^\sharp$. We may thus replace J by $J^\sharp$ and π_1 by $\pi^\sharp$, and since $J^\sharp$ factors over divisors of v in F, we may assume without loss of generality that $F \otimes \mathbb{Q}_v = F_v$ is a field. Thus we have to show that, when U is an unramified unitary group over F_v of rank n attached to a non-trivial quadratic extension $\mathcal{K}_v$ of F_v and GU is the corresponding similitude group, then every constituent of the restriction to $U(F_v)$ of an (irreducible) unramified tempered representation $\pi^\sharp$ of $GU(F_v)$ has the same base change to $GL(n, F_v)$, and occurs with multiplicity one.

But in that case $\pi^\sharp$ is an unramified tempered principal series representation, and base change is given explicitly in terms of the Satake parametrization, and the Lemma is verified by a simple calculation. More precisely, if v splits in $\mathcal{K}$, then the L-packet is a singleton and there is nothing to prove. Next, suppose v is inert and n is odd. Since $\pi^\sharp$ is tempered, it follows from Theorem 4 of [C1] (where the result is attributed to Keys) that it restricts to an irreducible representation of $U(F_v)$, and again there is nothing to prove. Finally, suppose v is inert and n is even. Then $\pi^\sharp$ may restrict to the sum of two irreducible representations of $U(F_v)$, each one with a vector fixed by one of the two hyperspecial maximal compact subgroups (permuted by the action of the adjoint group of $U(F_v)$; see [C1] again). In that case, the base change is calculated as in [Mi, Theorem 4.1] and in particular depends only on the characters defining the induced representation and not the choice of component.

This completes the proof, except for the assertion of multiplicity one. Let $T = GL(1)_{\mathcal{K}_v}$, which we view as the center of the algebraic group GU, and consider the short exact sequence of algebraic groups

$$1 \to T^1 \to T \times U \to GU \to 1.$$

Taking $Gal(\bar{F}_v/F_v)$-invariants, we see that the index of $T(F_v) \cdot U(F_v)$ in $GU(F_v)$ is bounded by $|H^1(F_v, T^1)| = 2$ (cf. [CHL, §1.1]). Let $\gamma \in GU(F_v)$, $\gamma \notin T(F_v) \cdot U(F_v)$. If $|A| = 2$ then $\pi_1 \mid_{J_0} = \pi_\alpha \oplus \pi_\alpha^\gamma$, with π_α irreducible. But if $\pi_\alpha^\gamma \xrightarrow{\sim} \pi_\alpha$ then π_α extends to a representation of $GU(F_v)$, which is impossible because π_1 is irreducible.

Corollary 4.1.4. *Assume Conjecture 4.1.2. Let π_1 and π_1' be irreducible admissible representations of $G(\mathbb{Q}_v)$ and $G'(\mathbb{Q}_v)$, respectively. Assume both π_1 and π_1' can be extended to automorphic representations of G and G', as in Lemma 4.1.3 bis. Then*

$$L(\pi_1, \pi_1') = Hom_{G'(\mathbb{Q}_v)}(\pi_1 \otimes \pi_1', \mathbb{C})$$

is of dimension at most 1.

Let $L_0(\pi_1, \pi_1') = Hom_{J_0'}(\pi_1 \otimes \pi_1', \mathbb{C})$. Then $L_0(\pi_1, \pi_1') \neq 0$ if and only if there is a unique irreducible J_0-subrepresentation π_α of π_1 and a unique irreducible J_0'-subrepresentation π_α' of π_1' such that the second map below

$$L(\pi_1, \pi_1') \to L_0(\pi_1, \pi_1') \to L(\pi_\alpha, \pi_\alpha')$$

is an isomorphism of one-dimensional vector spaces. If $L(\pi_1, \pi_1') \neq 0$, the first map is also an isomorphism.

Proof. Write $\pi_1 \mid_{J_0} = \oplus_{\alpha\in A}\pi_\alpha$ and $\pi'_1 \mid_{J'_0} = \oplus_{\alpha'\in A'}\pi'_{\alpha'}$, in the obvious notation. Restriction defines an injective map

$$L(\pi_1, \pi'_1) \hookrightarrow \oplus_{\alpha\in A,\alpha'\in A'} L(\pi_\alpha, \pi'_{\alpha'}).$$

The Corollary now follows immediately from Lemma 4.1.3 bis and Conjecture 4.1.2.

For the purposes of the following corollary, an "automorphic representation" has a factorization over primes of $\mathbb{Q}$ whose archimedean component is a smooth Frechet representation of moderate growth. Thus we can apply

Corollary 4.1.5. *Let τ and τ' be automorphic representations of G and G', respectively. Then*

$$L(\tau, \tau') = Hom_{G'(\mathbf{A})}(\tau \otimes \tau', \mathbb{C})$$

is of dimension at most 1.

Let $L_0(\tau, \tau') = Hom_{H'(\mathbf{A})}(\tau \otimes \tau', \mathbb{C})$. Then $L_0(\tau, \tau') \neq 0$ if and only if there is a unique irreducible H-subrepresentation $\pi \subset \tau$ and a unique irreducible H'- (or H''-)subrepresentation π' of τ' such that the second map below

$$L(\tau, \tau') \to L_0(\tau, \tau') \to L(\pi, \pi')$$

is an isomorphism of one-dimensional vector spaces. If $L(\pi_1, \pi'_1) \neq 0$, the first map is also an isomorphism.

4.2. Statement of the Ichino-Ikeda conjecture for unitary groups.

The definition of Gross-Prasad periods in the next section, and the conjectures regarding their properties, are motivated by the Ichino-Ikeda conjecture relating periods to special values of L-functions. The conjectures in [II] are stated for pairs of orthogonal groups, and presumably one can define interesting periods in this setting as well, but here we are concerned with the analogue of the Ichino-Ikeda conjecture for unitary groups, due to Neal Harris [NH]. The provisional statement presented here is based on the one explained to me by W. T. Gan in a private communication.

Let π, π', H, and H' be as above. Measures dh and dh' on H and H' will be Tamagawa measures, and for any factorizations $dh = \prod_v dh_v$, $dh' = \prod_v dh'_v$ over the places of F, the local measure at a finite place v is always assumed to take rational values; if H or H' is unramified at v then the corresponding local measure is always assumed to give volume 1 to a hyperspecial maximal compact subgroup.

We let $\sigma = BC(\pi)$, $\sigma' = BC(\pi')$ denote the automorphic representations of $GL(n)_{\mathcal{K}}$ and $GL(n-1)_{\mathcal{K}}$, obtained by stable base change from H and H', respectively (we assume base change has been established; this is usually known to be the case when π_∞ and π'_∞ are in the discrete series [Lab]). The Rankin-Selberg L-functions $L(s, \sigma \otimes \sigma')$ are well defined and are known to have no poles along the critical line $Re(s) = \frac{1}{2}$. The two representations of the Langlands L-group of H on $Lie(H)$, extending the natural adjoint action of the dual group $\hat{H}$ and differing by a quadratic character, are denoted As^+ and As^-, in honor of Asai (see [GGP1, §7] for precise definitions). Partial L-functions including only factors outside a finite set S of plaes of F are denoted L^S. Let

$$\mathcal{L}^S(\pi, \pi') = \frac{L^S(\frac{1}{2}, \sigma \otimes \sigma')}{L^S(1, \pi, As^{\pm}) L^S(1, \pi', As^{\mp})} \tag{4.2.1}$$

where the sign of $As^{\pm}$ (resp. $As^{\mp}$) is $(-1)^n$ (resp. $(-1)^{n-1}$).

Choose $f \in \pi$, $f' \in \pi'$, and assume they are factorizable as $f = \otimes f_v, f' = \otimes f'_v$ with respect to tensor product factorizations

$$\pi \xrightarrow{\sim} \otimes'_v \pi_v;\ \pi' \xrightarrow{\sim} \otimes'_v \pi'_v. \tag{4.2.2}$$

We assume π_v and π'_v are unramified, and f_v and f'_v are the normalized spherical vectors, outside a finite set S including all archimedean places (and which we allow to include some unramified places). The L_2 norm on the adele class groups of H and H', with respect to Tamagawa measure, We define inner products $<,>_\pi$, $<,>_{\pi'}$ on π and π' by restriction of the L_2 norm on $L_2(H(F)\backslash H(\mathbf{A}))$, resp., $L_2(H'(F)\backslash H'(\mathbf{A}))$, and we choose inner products $<,>_{\pi_v}, <,>_{\pi'_v}$ on each of the π_v and π'_v such that at an unramified place v, the local spherical vector in π_v or π'_v taking value 1 at the identity element has norm 1. (In the case of representations of discrete series type, the rationality at archimedean places depends on the choice of maximal compact subgroup; this is discussed below.) Let

(4.2.3)

$$I^{can}(f,f') = \int_{H'(F)\backslash H'(\mathbf{A})} f(h')f'(h')dh';\ \mathcal{P}(f,f') = \frac{|I^{can}(f,f')|^2}{< f,f >_\pi^2 < f',f' >_{\pi'}^2}$$

For each $v \in S$, let

$$c_{f_v}(h_v) =< \pi_v(h_v)f_v, f_v >_{\pi_v};\ c_{f'_v}(h'_v) =< \pi'_v(h_v)f'_v, f'_v >_{\pi'_v},\ h_v \in H_v, h'_v \in H'_v,$$

and define

$$I_v(f_v,f'_v) = \int_{H'_v} c_{f_v}(h'_v)c_{f'_v}(h'_v)dh'_v;\ I^*_v(f_v,f'_v) = \frac{I_v(f_v,f'_v)}{c_{f_v}(1)c_{f'_v}(1)};$$

N. Harris proves that these integrals converge.

Let $\Delta_{U(n)}$ be the value at L-function of the *Gross motive* of $U(n)$:

$$\Delta_{U(n)} = L(1,\eta_{\mathcal{K}/F})\zeta(2)L(3,\eta_{\mathcal{K}/F})....L(n,\eta^{n-1}_{\mathcal{K}/F}).$$

The Ichino-Ikeda conjecture for unitary groups is then [NH]

Conjecture 4.2.4. *Let $f \in \pi$, $f' \in \pi'$ be factorizable vectors as above. Then there is an integer β, depending on the L-packets containing π and π', such that*

$$\mathcal{P}(f,f') = 2^\beta \Delta_{U(n)} \prod_{v\in S} I^*_v(f_v,f'_v)\mathcal{L}^S(\pi,\pi').$$

Return for the moment to the notation of the previous section, with $G \supset H$ and $G' \supset H'$ unitary similitude groups. We let $G'' \supset H''$ be the subgroup of $G' \times GU(W_0)$ on which the two similitude factors coincide. Let Z^+ denote the center of G. We extend π and π' to automorphic representations τ and τ'' of G and G'', respectively, with central characters ξ and ξ'', and assume they verify the analogue of (Ξ) of the previous section:

$$\xi \cdot \xi'' \mid_{Z^+(\mathbf{A})} = 1.$$

More precisely, $\tau = \otimes'_v \tau_v$, where τ_v is an irreducible representation of G_v if v is finite but is an irreducible $(Lie(G_v), K_x)$-module if $v = \infty$. In the latter case, the group G_∞ may be disconnected, in which case the connected algebraic group K_x does not contain the points of a maximal compact subgroup of G_∞; τ_∞ does not then correspond to a representation of the full group G_∞, and its restriction to $H_\infty = \prod_{v|\infty} H_v$ remains irreducible.

There is a unique Shimura datum X'' such that the inclusion $G'' \subset G$ extends to a morphism of Shimura data $(G'', X'') \subset (G, X)$. In what follows, one chooses a CM point x in X'' as in §3.2. Suppose τ and τ'' are of discrete series type at infinity, so that τ_f and τ''_f contribute to coherent cohomology of $S(G, X)$ and $S(G'', X'')$, respectively, as in §§1.4, 1.5.

Lemma 4.2.5. *Assume Conjecture 4.1.2 and the Ichino-Ikeda Conjecture 4.2.4, and assume*

$$0 \neq I^{can} \in Hom_{H'(\mathbf{A})}(\pi \otimes \pi', \mathbb{C}) = L(\pi, \pi').$$

Then $L(\frac{1}{2}, \sigma \otimes \sigma') \neq 0$, with $\sigma = BC(\pi)$, $\sigma' = BC(\pi')$. Moreover, in the notation of Corollary 4.1.5, the restriction map

$$L_0(\tau, \tau') \to L(\pi, \pi')$$

is an isomorphism of one-dimensional vector spaces.

Proof. The hypothesis implies that the left-hand side of the Ichino-Ikeda identity does not vanish for some choice of $f \in \pi$, $f' \in \pi'$. Then each factor right-hand side is non-trivial, and in particular $L^S(\frac{1}{2}, \sigma \otimes \sigma') \neq 0$, and therefore $L(\frac{1}{2}, \sigma \otimes \sigma') \neq 0$. The second assertion is a consequence of Corollary 4.1.5.

4.3. Gross-Prasad periods for the pair (G, G').

In the setting of §4.1, let $E = \mathbb{R}$ and $L = \mathbb{C}$, and consider $H' = U(W') \times U(W_0)$ and $H = U(W)$ as groups over $\mathbb{Q}$ by restriction of scalars. Choose maximal compact subgroups K and K' of $H(\mathbb{R})$ and $H'(\mathbb{R})$, respectively, so that $K' = K \cap H'$. Write $\mathfrak{h} = Lie(H)$, $\mathfrak{h}' = Lie(H')$. The only sensible versions of the main results and conjectures of the present article are predicated on the truth of the following strengthening of the result of Sun and Zhu quoted in Theorem 4.1.1.

Conjecture 4.3.1. *Let π and π' be irreducible $(Lie(H(\mathbb{R})), K)$ and $(Lie(H'(\mathbb{R})), K')$-modules, respectively. Let*

$$L_0(\pi, \pi') = Hom_{(\mathfrak{h}', K')}(\pi \otimes \pi', \mathbb{C})$$

where $\mathfrak{h}'$ and K' act through their diagonal embeddings in $\mathfrak{h} \times \mathfrak{h}'$ and $K' \times K$. Then

$$\dim L_0(\pi, \pi') \leq 1.$$

As the authors of [SZ] explain, this would follow from their main theorem if it were known that every element of $L_0(\pi, \pi')$ extended continuity to the associated canonical smooth Frechet representation of moderate growth of H and H'. This continuity is expected and is the subject of a conjecture, but the continuity conjecture is known in very few cases (mainly restrictions of holomorphic discrete series).

We admit Conjecture 4.3.1 in what follows and draw some global consequences. Henceforward, π and π' denote cuspidal automorphic representations of the groups G and G', respectively. Let $\mathfrak{g} = Lie(G(\mathbb{R}))$, $\mathfrak{g}' = Lie(G'(\mathbb{R}))$. Recall that we have an inclusion $(G', X') \subset (G, X)$ of Shimura data. Choose a base point $x \in X'$, and let K'_x and K_x denote its stabilizers in $G'(\mathbb{R})$ and $G(\mathbb{R})$, respectively, so that $K' = K'_x \cap H'(\mathbb{R})$, $K = K_x \cap H(\mathbb{R})$.

In what follows, we revert to the notation of the previous sections, so that an automorphic representation of G is denoted π rather than τ.

Corollary 4.3.2. *We admit Conjectures 4.1.2 and 4.3.1. Let $\pi = \pi_\infty \otimes \pi_f$ and $\pi' = \pi'_\infty \otimes \pi_f$ be irreducible cuspidal automorphic representations of G and G', respectively, with the convention that π_∞ and π'_∞ are respectively irreducible $(\mathfrak{g}, K_x)$ and $(\mathfrak{g}', K'_x)$-modules. Define*

$$L_0(\pi, \pi') = Hom_{(\mathfrak{h}', K') \times H'(\mathbf{A}^f)}(\pi \otimes \pi', \mathbb{C}).$$

Then $\dim L(\pi, \pi') \leq 1$.

Assume henceforward that π_∞ and π'_∞ are (the Harish-Chandra modules of) discrete series representations. It follows from Corollary 1.4.6, Proposition 2.4.3, and the discussion in 4.2.5, that π and π' have rational models over CM subfields $E(\pi)$ and $E(\pi')$ of $\mathbb{C}$, respectively. We denote these models $\pi_{E(\pi)}$ and $\pi'_{E(\pi')}$. The following corollary is obvious.

Corollary 4.3.3. *Admit Conjectures 4.1.2 and 4.3.1, and assume $L_0(\pi, \pi') \neq 0$. Then the one-dimensional space $L_0(\pi, \pi')$ can be generated by a vector $I(\pi, \pi')$ with the property that*

$$I(\pi, \pi')(\pi_{E(\pi)} \otimes \pi'_{E(\pi')}) = E(\pi) \cdot E(\pi'),$$

the subfield of $\mathbb{C}$ generated by $E(\pi)$ and $E(\pi')$.

In what follows, $Aut(\mathbb{C})$ acts on representations of $(\mathfrak{g}, K_x) \times G(\mathbf{A}^f)$ and $(\mathfrak{g}', K'_x) \times G'(\mathbf{A}^f)$ by acting on their automorphism classes. The groups are allowed to vary in their inner classes, as in the previous sections. Thus, for $\alpha \in Aut(\mathbb{C})$, $\alpha(\pi)$ is a $({}^{\alpha,x}(g), K_{{}^\alpha x}) \times G(\mathbf{A}^f) \xrightarrow{\sim} ({}^{\alpha,x}(g), K_{{}^\alpha x}) \times {}^{\alpha,x}G(\mathbf{A}^f)$-module, and $\alpha(\pi')$ is understood similarly. With these conventions in mind, the next proposition is practically obvious:

Proposition 4.3.4. *For any $\alpha \in Aut(\mathbb{C})$*

$$\dim(L_0(\pi, \pi')) = \dim L_0(\alpha(\pi), \alpha(\pi')).$$

Proof. Indeed, we can identify $\alpha(L_0(\pi, \pi'))$ with $L_0(\alpha(\pi), \alpha(\pi'))$, using Lemma 2.4.4.

For comparison with Deligne's conjecture, it is useful to define a version of π with coefficients in the abstract number field $E(\pi)$, rather than in the subfield $E(\pi) \subset \mathbb{C}$. Let

$$\pi^{mot} = \pi_{E(\pi)} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \bigoplus_{\beta: E(\pi) \to \mathbb{C}} \pi_{E(\pi)} \otimes_{E(\pi), \beta} \mathbb{C}. \tag{4.3.4}$$

Write $E(\pi)\otimes E(\pi')$ for $E(\pi)\otimes_{\mathbb{Q}} E(\pi')$. We define $\pi'^{,mot}$ analogously and define the $E(\pi)\otimes E(\pi')$-module

$$L_0^{mot}(\pi,\pi') = Hom_{(\mathfrak{h}',K')\times G'(\mathbf{A}^f)}(\pi^{mot}\otimes\pi'^{,mot}, E(\pi)\otimes E(\pi')\otimes\mathbb{C}) \tag{4.3.5}$$

in the obvious way.

Since (4.3.4) can be rewritten

$$\pi^{mot} \xrightarrow{\sim} \bigoplus_{\alpha\in Aut(\mathbb{C})/Aut(\mathbb{C}/E(\pi))} \alpha(\pi_{E,\pi})\otimes_{\alpha(E(\pi))}\mathbb{C} \tag{4.3.6}$$

Corollary 4.3.7. *Suppose $L_0^{mot}(\pi,\pi')\neq 0$. Then $L_0^{mot}(\pi,\pi')$ is a free $E(\pi)\otimes E(\pi')$-module of rank* 1.

In other words, if $L_0(\pi,\pi')\neq 0$ then for all embeddings $\beta: E(\pi)\to\mathbb{C}$, $\beta': E(\pi')\to\mathbb{C}$, $L(\pi_{E(\pi)}\otimes_{E(\pi),\beta}\mathbb{C},\pi'_{E(\pi')}\otimes_{E(\pi'),\beta'}\mathbb{C})\neq 0$. Equivalently, for all $\beta\in Aut(\mathbb{C})$, $L_0(\beta(\pi_{E(\pi)}\otimes_{E(\pi)}\mathbb{C}),\beta(\pi'_{E(\pi')}\otimes_{E(\pi')}\mathbb{C}))\neq 0$. But

$$L_0(\beta(\pi_{E(\pi)}\otimes_{E(\pi)}\mathbb{C},\beta(\pi'_{E(\pi')}\otimes_{E(\pi')}\mathbb{C}) = \beta(L_0(\pi_{E(\pi)}\otimes_{E(\pi)}\mathbb{C},(\pi'_{E(\pi')}\otimes_{E(\pi')}\mathbb{C})$$

so the claim is obvious. Note however that if β does not fix $E(G,X)$ the representation $\beta(\pi_{E(\pi)}\otimes_{E(\pi)}\mathbb{C}$ is in general not an automorphic representation of G; if $\beta=\alpha\circ\iota$, where $\iota: E(\pi)\hookrightarrow\mathbb{C}$ is the tautological inclusion, then $\pi_{E(\pi)}\otimes_{E(\pi),\beta}\mathbb{C}$ is an automorphic representation of ${}^{\alpha,x}G$, as in Corollary 1.4.8.

By analogy with Corollary 4.3.3, it follows that the module has a generator $I^{mot}(\pi,\pi')$ such that

$$I^{mot}(\pi,\pi')(\pi^{mot}\otimes\pi'^{,mot}) = E(\pi)\otimes E(\pi').$$

We let dg' denote Tamagawa measure on $G'(\mathbf{A})$, and define the canonical pairing $I^{can}(\pi,\pi')$ by (4.2.3). Since π and π' are cuspidal, the integral converges absolutely and defines an element of $L_0(\pi,\pi')$. Similarly, integration defines a canonical pairing

$$I^{mot}(\pi,\pi'):\pi^{mot}\otimes\pi'^{,mot}\to E(\pi)\otimes E(\pi')\otimes\mathbb{C}. \tag{4.3.8}$$

As in the discussion following Corollary 4.3.7, $\pi_{E(\pi)}\otimes_{E(\pi),\beta}\mathbb{C}$ is in general **not an automorphic representation of the group** G. We let $\sigma=BC(\pi)$, $\sigma'=BC(\pi')$ as in the previous section.

The following is a reformulation of a conjecture of Deligne [D2, 2.7 (ii)].

Conjecture 4.3.9 [Deligne]. *Let M be a pure motive over a number field $\mathcal{K}$ of weight w with coefficients in the number field E. For any $\sigma: E\to\mathbb{C}$ let $L(\sigma,M,s)$ be the corresponding L-function.*

(i) If m is a critical value for M (if the Tate twist $M(m)$ is critical in the sense of [D2]) then $L(\sigma,M,m)\neq 0$ unless $m=\frac{w+1}{2}$; in particular, w must be an odd number.

(ii) Suppose w is odd. The multiplicity of the zero of $L(\sigma,M,\frac{w+1}{2})$ is independent of σ.

Parallel to Corollary 4.3.7 is the following global conjecture, motivated by comparing the Ichino-Ikeda conjecture, recalled in §4.2, to Deligne's Conjecture 4.2.9:

Conjecture 4.3.10. *Suppose $I^{can}(\pi,\pi') \neq 0$. Then the map (4.3.8) is surjective.*

In other words, if the integral is non-vanishing for the pair (π,π'), then it is non-vanishing for all conjugates of π and π' relative to their rational structures.

Definition 4.3.11. *Suppose $I^{mot}(\pi,\pi') \neq 0$. We define the Gross-Prasad period invariant of the pair (π,π') to be the constant*

$$P(\pi,\pi') \in (E(\pi)\otimes E(\pi')\otimes \mathbb{C})^{\times}$$

such that

$$I^{can}(\pi,\pi') = P(\pi,\pi')\cdot I^{mot}(\pi,\pi').$$

This depends on the choice of basis $I^{mot}(\pi,\pi')$ of $L(\pi,\pi')$, but $P(\pi,\pi')$ is well-defined up to a factor in $(E(\pi)\otimes E(\pi'))^{\times}$.

If $I^{can}(\pi,\pi') = 0$, we set $P(\pi,\pi') = 0$.

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