AUTOMORPHIC MOTIVES AND THE ICHINO-IKEDA CONJECTURE

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Conference in honor of Jean-Marc Fontaine

(thanks to Blasius, Gan, Gross, N. Harris)

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Main Formula

(This lecture): (a real lecture):: (CDO): (Real money)

CDO = Collateralized Debt Obligation = bundle of more or less toxic speculations on future production (just like this lecture).

Goal: to determine compatibility of two conjectures on special values of $L$-functions:

**Deligne** (motivic)

(in some indefinite future Bloch-Kato/Fontaine-Perrin-Riou);

and **Ichino-Ikeda** (automorphic).
0. Linear Algebra

Consider a polarized regular motive $M$ over $K$, pure of weight $w = n - 1$, of rank $n$ over its coefficient field $E$.

$\mathcal{K}/F$ a CM quadratic extension of a totally real field.

$c \in Gal(\mathcal{K}/F)$ complex conjugation.

Absolutely everything is true in this generality.

But to simplify notation (usually) $F = \mathbb{Q}$, $\mathcal{K}$ an imaginary quadratic subfield of $\mathbb{C}$. 
This is a collection of realizations

\[ M = (\{M_\lambda\}, M_B, M_{DR}, M_{Hodge}, M_{cris}) : \]

just vector spaces with associated structures.

Here \( \lambda \) runs over finite places of \( E \),

\[ M_{Hodge} = \text{gr}_F^\bullet M_{DR} = \bigoplus M^{p_i,q_i} \]

a \( k \)-vector space (really: free rank \( n \) \( k \otimes E \)-module)

In practice we need the \( \lambda \)-adic Galois representation \( M_\lambda \) (for \( L \)-function)

and \( M_{Hodge} \) (for periods).
**Regularity hypothesis:** $\dim M^{p, w^p} \leq 1$ for all $i$.

Thus we have a decreasing set of $p_i$:

$$p_1 > p_2 > \cdots > p_n \quad \dim H^{p_i, q_i} = 1$$

For each $i$ choose a rational vector $0 \neq \omega_i \in M^{p_i, q_i}$.

The conjugate motive $M^c$ has Hodge types $(p^c_i, q^c_i)$.

Let $RM = M \oplus M^c$ over $F (= \mathbb{Q})$.

**Polarization hypothesis:**

$$<, >_M: M \otimes M^c \to E(1 - n).$$
Finally, $F_\infty$ is an $E$-linear isomorphism of Betti realizations

$$F_\infty : M_B \simto M_B^c$$

(the king of Frobenii, according to Hida).

Induces a map

$$F_\infty : M_{Hodge} \otimes \mathbb{C} \simto M_{Hodge}^c \otimes \mathbb{C};$$

$$F_\infty (M^{p_i,q_i}) = (M^c)^{p_{n+1-i},q_{n+1-i}}.$$

does not respect the $\mathcal{K}$-rational structure.

$$Q_i = Q_i(M) = \langle \omega_i, F_\infty (\omega_i) \rangle_M \in (\mathbb{C} \otimes \mathcal{K} \otimes E)^\times / (\mathcal{K} \otimes E)^\times$$

(we will work only mod $\overline{\mathbb{Q}}^\times$ in this talk.)
The polarization induces one relation:

$$\prod_{i=1}^{n} Q_i(M) = (2\pi i)^{n(n-1)} \delta(M)^{-2}.$$ 

Here $\delta(M)$ is the determinant of

$$I_\infty : M_B \otimes \mathbb{C} \sim \rightarrow M_{DR} \otimes \mathbb{C}.$$ 

This is the same as the comparison map

$$\det(M)_B \otimes \mathbb{C} \sim \rightarrow \det(M)_{DR} \otimes \mathbb{C}.$$ 

det($M$) is the (abelian) motive, of a CM Hecke character.
Thus

\[ \prod_{i=1}^{n} Q_i(M) = 1 \pmod{A^\times} \]

where \( A^\times \subset \mathbb{C}^\times \) is the subgroup generated by peri-
ods of CM motives.

To avoid having to keep track of CM periods we work in \( \mathbb{C}^\times / A^\times \).

(Anyway, I haven’t had time to finish the necessary calculations.)
Deligne’s conjecture. If $m$ is a critical integer for $M$, there is a Deligne period

$$c^+(M(m)) \in \mathbb{C}^\times /\mathbb{Q}^\times$$

defined using $M_{Hodge}$ and $F_{\infty}$, such that

$$L(M, m)/c^+(M(m)) \in \overline{\mathbb{Q}}.$$

**Principle:** Many $c^+(M(m))$ can be expressed in terms of the $Q_i(M)$. 
1. Automorphic motives

Π a cuspidal cohomological automorphic representation of $GL(n, \mathcal{K})$.

*Polarization condition*

\[ \Pi^\vee \sim \Pi^c. \]

$E = E(\Pi)$ the field of definition of $\Pi_f$, a CM field.

$M_{\Pi, \lambda}$.

$\Pi$ gives rise to a compatible system of $\lambda$-adic representations

\[ \rho_{\Pi, \lambda} \to GL(n, E_\lambda), \]

where $\lambda$ runs over places of $E$, non-degenerate pairing

\[ \rho_{\Pi, \lambda} \otimes \rho_{\Pi, \lambda}^c \to E_\lambda(1 - n). \]

(cf. Eichler-Shimura)
References:

$n = 2$: essentially the theory of Galois representations attached to Hilbert modular forms (Carayol, Wiles, Blasius-Rogawski, and Taylor).

$n = 3$: Blasius-Rogawski.


2005+ (After the proof of the Fundamental Lemma by Laumon and Ngô): Book project (mainly Clozel, Labesse, Chenevier, and MH), S.W. Shin.

Complete statement in Chenevier-MH.

$\rho_{\Pi,\lambda}$ are pure of weight $n - 1$. (proved at most bad places by Shin except in limit cases; in limit cases results of Clozel, Taylor, etc.)
Hypothesis: there is a polarized regular motive $M = M_\Pi$ over $\mathcal{K}$ of rank $n$, pure of weight $w = n - 1$, coefficients in $E$, with

$$M_{\Pi, \lambda} = \rho_{\Pi, \lambda}.$$ 

Relation between $M$ and $\Pi$:

$$L(s, M) = L\left(s + \frac{1 - n}{2}, \Pi\right) = L\left(s, \Pi \otimes (|\bullet| \circ \det)^{\frac{1-n}{2}}\right)$$

Actually true! except for some cases (the “limit cases”) treated by $\lambda$-adic congruences.

The **Fontaine-Mazur conjecture** in this setting: every such motive (even a single $M_\lambda$) is automorphic.
In most cases (the “motivic cases”) $\rho_{\Pi,\lambda}$ realized in the cohomology of certain Shimura varieties, defined over $\mathcal{K}$ with coefficients in certain local systems.

Similarly, $M_{\Pi}$ can be realized as a Grothendieck motive using Hecke correspondences on generalized Kuga-Sato varieties (mixed Shimura varieties).

In the motivic cases, $M_{\Pi, Hodge} (+ F_\infty)$ can be related directly to automorphic forms (see below).

The limit case is important: it includes symmetric powers of modular forms as in proofs of various cases of the Sato-Tate conjecture and arises in all potential automorphy arguments.
$M_{\Pi,B}$. Purely hypothetical (but see Taylor on descent of Shimura varieties)

Interpret $F_\infty$ as complex conjugation of automorphic forms, $Q_i(M_\Pi)$: Petersson (square) norm of automorphic form $\omega_i$.

$M_{\Pi,DR}$.

Since $\Pi$ is cohomological, $M_\Pi$ is regular, which means that

$$p_i > p_{i+1}, \quad p_i^c > p_{i+1}^c$$

There is a simple expression for the Hodge types $p_i, p_i^c$ in terms of the component $\Pi_\infty$ of $\Pi$. 
Problem 1.  Study $M_{Π, \text{cris}}$ in the automorphic setting.

The comparison isomorphisms of $p$-adic Hodge theory are used, e.g. in Clozel-MH-Taylor (to compute Zariski tangent space of crystalline deformation functor).

$M_{Π, \text{cris}}$ plays no direct role (but see Mokrane-Tilouine as well as earlier work of Coleman, Gross, Buzzard-Taylor on $p$-adic cohomology and modular forms).
2. Periods

Shimura varieties $Sh(V)$ attached to unitary groups $U(V)$,

$V$ an $n$-dimensional hermitian space over $\mathcal{K}$.

For each $V$, base change map

(cohomological automorphic representations of $U(V)$)

$\updownarrow$

(cohomological automorphic reps. of $GL(n)_\mathcal{K}$)

Studied in detail by Labesse, (S. Morel for $F = \mathbb{Q}$).

Inverse image of $\Pi$: the $L$-packet $\Pi_V = \{\pi_\alpha\}$ of $U(V)$ attached to $\Pi$.

Depending on local conditions, $\Pi_V$ may be empty, but often is rich enough to define variants of the motive $M$, as follows.
First case: $V = V_1$ has signature $(1, n - 1)$. Pick $\pi = \pi_\infty \otimes \pi_f \in \Pi_{V_1}$, fix $\pi_f$.

$$\dim Sh(V_1) = n - 1$$

Local system $L(\Pi_\infty)/Sh(V_1)$ ($\ell$-adic or VHS) defined by $\Pi_\infty$

Of geometric origin (in direct image of cohomology of a family of abelian varieties over $Sh(V_1)$).

The group $U(V_1)(A^f)$ (actually $GU(V_1)(A^f)$) acts on $L(\Pi_\infty)$ and $Sh(V_1)$. 
Define

\[ M(V_1, \Pi) = \text{Hom}_{U(V_1)(\mathbb{A}_f)}(\pi_f, H^{n-1}(\text{Sh}(V_1), L(\Pi_\infty))). \]

Usually \( \dim M(V_1, \Pi) = n \): this is the motive over \( \mathcal{K} \) we want to call \( M_\Pi \), up to an explicit abelian twist.

\( n \)-dimensional because the archimedean (discrete series) \( L \)-packet \( \Pi_{V_1,\infty} \) has \( n \)-members – basis of Hodge realization indexed by \( \Pi_{V_1,\infty} \) (see below).
More generally, if $U(V) = U(r, s)$ with $r + s = n$ the signature, then

$$|\Pi_{V,\infty}| = d(V) := \binom{n}{r}$$

so $M(V, \Pi)$, defined by the formula analogous to that used with $V = V_1$, should be a motive of rank $d(V)$. In fact we have (Kottwitz + stable trace formula)

$$M(V, \Pi)_\lambda \sim \wedge^r M_{\Pi,\lambda} \otimes M(\xi)$$

for an explicit CM Hecke character $\xi$. 
The point to keep in mind:

Cohomology of Shimura varieties not only gives us a large collection of $n$-dimensional polarized regular motives. It also gives us motivic realizations of all exterior powers of the $\lambda$-adic realizations.

Problem 2. The Tate conjecture implies the corresponding relations at the level of motives. Prove them independently, at least for the Hodge realization (see below); i.e., prove the corresponding period relations.

(All motives have coefficients in some Hecke field $E = E(\Pi)$, but I ignore this.)
We actually know more than this. The automorphic theory naturally gives us a \( \mathcal{K} \)-rational basis (actually \( \mathcal{K} \otimes E \)-rational) \( (\omega_1, \ldots, \omega_n) \) of the Hodge realization

\[
M_{Hodge} := \bigoplus_{i=1}^{n} H^{p_i,q_i}(M_\Pi),
\]

which is naturally a \( \mathcal{K} \)-vector space via the identification of the \( H^{p_i,q_i} \) with coherent cohomology.

Thus the pure tensors

\[
\omega_I = \omega_I(\Pi) := \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_r}; \quad I = (i_1 < i_2 < \cdots < i_r)
\]

form a canonical basis for \( \wedge^r M_{Hodge} \).

(For general \( F \), we have to tensor over the \( \sigma \in \text{Hom}(F, \mathbb{R}) \), but the idea is the same, and we get rational bases of pure tensors for each \( M(V, \Pi)_{Hodge} \) in terms of the \( \omega_{i,\sigma} \).)
The *periods* of the $\omega_i$—integrals over rational homology classes—are completely inaccessible by automorphic means.

But $\omega_i$ is a concrete automorphic form on $U(V_1)$, generating a specific $\pi_i \in \Pi_{V_1}$, and its Petersson (square) norm $Q_i = Q_i(\Pi)$ is a well-defined invariant in $\mathbb{C}^\times/\mathbb{Q}^\times$, up to algebraic factors.

In particular, $\omega_1$ is a *holomorphic* automorphic form.

Similarly, if $V$ is fixed, for each $\pi_I \in \Pi_V$, there is a concrete automorphic form $\omega_{I,V}$ on $U(V)$, generating an automorphic representation of $U(V)$ isomorphic to $\pi_I \otimes \pi_f$ when $|I| = r$; these form a rational basis for the Hodge realization of the motive $M(V, \Pi)$, canonical up to (rational) scalar multiples.

(Here we are making the assumption that $U(V)(\mathbb{A}^f) \simto U(V)$ so their local representations can be identified. This is legitimate when $n$ is odd but not when $n$ is even, in which case the construction needs to be modified.)

For example, $\omega_{I,V}$ is holomorphic, as is the formal wedge product $\omega_I$, if and only if $I = \{1, \ldots, r\}$. 
Let $Q_{I,V}$ be the Petersson square norm of $\omega_{I,V}$. Under the hypothetical isomorphism

$$M(V, \Pi) \xrightarrow{\sim} \wedge^r M_{\Pi} \otimes M(\xi)$$

we want to identify $\omega_{I,V}$ with the wedge product $\omega_I$.

**Problem 3.** Prove that $Q_{I,V} = \prod_{i \in I} Q_i \pmod{A^\times}$, more precisely up to the CM period corresponding to the abelian twist), as predicted by the Tate conjecture.

When $F = \mathbb{Q}$ and $\Pi \xrightarrow{\sim} \Pi^c$, known (MH) for the holomorphic $Q_I$, up to an undetermined constant that depends only on $\Pi_\infty$.

The proof is based on the theta correspondence.

MH-Li-Sun (2009-2010): the theta correspondence alone cannot solve Problem 3 for general $I$!

Can it be solved without the Tate conjecture?
The interest of the $Q$-invariants: for certain motives $\mathcal{M}$ constructed from polarized regular $M$, the Deligne periods $c^+(\mathcal{M})$ (of Deligne’s conjecture on special values) can be expressed formally in terms of the $Q_i$ and some abelian (CM) periods.

In the heuristic discussion that follows, assume the period relations hold for the $Q$-invariants. In particular, we identify the formal wedge product $\omega_I$ with the automorphic form $\omega_{I,V}$ on $U(V)$. 
3. The Gross-Prasad conjectures

Let now $\Pi$ and $\Pi'$ be cohomological automorphic representations of $GL(n)_\mathcal{K}$ and $GL(n - 1)_\mathcal{K}$.

Choose hermitian spaces $V$ and $V'$, $V = V' \oplus E$ for a non-isotropic line $E$, $\dim V = n$.

Let $\pi \in \Pi_V$, $\pi' \in \Pi'_{V'}$, $G = U(V)$, $G' = U(V')$. 
Consider the abstract space of linear forms

\[ L(\pi, \pi') = Hom_{G'}(\pi \otimes \pi', \mathbb{C}) \]

where \( G' \subset G \times G' \) is embedded diagonally.

The following theorem is due to Aizenbud-Gurevich-Rallis-Schiffman (for the non-archimedean factors) and Sun-Zhu (for the archimedean factors):

**Theorem 3.** \( \dim L(\pi, \pi') = \prod_v \dim L(\pi_v, \pi'_v) \leq 1 \), where the factorization is defined in the obvious way.
Waldspurger has proved much more (so far only for orthogonal groups).

The full Gross-Prasad conjecture in this setting is roughly

$$\sum_{\pi \in \Pi_V, \pi' \in \Pi_{V'}} \dim L(\pi, \pi') \leq 1$$

with equality if the $L$-function considered by Ichino-Ikeda (see below) does not vanish at $s = \frac{1}{2}$.

Waldspurger has proved the non-archimedean part of this statement (for tempered $L$-packets, which should include all the local terms of $\Pi_V$ and $\Pi_{V'}$).

The archimedean statement is not yet proved but will be assumed in what follows.
Consequences of Theorem 3

Global consequences. Suppose $\pi = \pi_I$ for some subset $I \subset \{1, \ldots, n\}$ as above, with $|I| = r$, $G_\infty = U(r,s)$. Assume $\pi' = \pi_{I'}$ similarly.

Hypotheses.

(a) $\dim L(\pi, \pi') = 1$

(b) A generator $L \in L(\pi, \pi')$ is given by integration

$$L_{can}(f, f') = \int_{G'(F) \backslash G'(A)} f(g')f'(g')dg'$$

where $dg'$ is Tamagawa measure.

The original global Gross-Prasad conjecture is roughly the following:

Problem 4. Assuming (a), prove that (b) is true if and only if $L(\frac{1}{2}, \Pi \otimes \Pi') \neq 0$.

The Ichino-Ikeda conjecture is a refined version of Problem 4, expressing the $L$-value in terms of the integral $L_{can}$ and other terms.
**Proposition.** The spaces $\pi$ and $\pi'$ have $\overline{Q}$-rational structures generated by $\omega_I(\Pi)$ and $\omega_I'(\Pi')$. In particular, $L(\pi, \pi')$ has a generator $L_0$ rational with respect to these structures.

One can replace $\overline{Q}$ by $\mathcal{K} \otimes E$. For $\pi_f$ and $\pi'_f$, standard considerations about automorphic cohomology.

For the archimedean part, use algebraic realization of $\pi_{I,\infty}, \pi'_{I',\infty}$ as Zuckerman derived functor modules.

(For this need to choose max. compact, max torus $K_\infty \supset T_\infty$; OK over a CM point attached to $\mathcal{K}$.)
Corollary. There is a constant $P(\pi, \pi') \in \mathbb{C}^\times / \overline{\mathbb{Q}}^\times$ (an occult period) such that, for any $f \in \pi$, $f' \in \pi'$, rational over $\overline{\mathbb{Q}}$, $L_0(f, f')$ is a $\overline{\mathbb{Q}}$-multiple of $P(\pi, \pi')$.

Proposition. If $\pi_I$ is holomorphic, $P(\pi, \pi') = 1$.

Indeed, $\pi_{I, \infty} |_{G'(\mathbb{R})}$ is a discrete (countable) direct sum of holomorphic representations. Thus if $L(\pi, \pi') \neq 0$, $\pi'$ must be antiholomorphic, and the integral $L_{\text{can}}$ is just cup-product in coherent cohomology.

With the rational structures I (haven’t) defined, we can thus take $L_0 = L_{\text{can}}$. 
More generally, for certain pairs \((I, I')\), \(L_{can}\) can be interpreted as a cup product in coherent cohomology, and then one again has \(L_0 = L_{can}\). We then say the pair \((\pi_{I,\infty}, \pi_{I',\infty})\) is coherent.

One has \(\omega_i \in H^{p_i, q_i} \xrightarrow{\sim} H^{i-1}(\text{Sh}(V_1), \mathcal{E}_i)\) for some automorphic vector bundle \(\mathcal{E}_i\) on \(\text{Sh}(V_1)\).

(Don’t always have \(q_i = i - 1\): \(\mathcal{E}_i\) has a built-in twist by (the infinity type of) a CM Hecke character.)

Similarly, \(\omega_I\) belongs to \(H^j(\text{Sh}(V), \mathcal{E}_I)\) where \(j\) is the number of elements in \(I\) not in the set \(\{1, \ldots, r\}\).

**Problem 5.** Determine the set of coherent pairs \((\pi_{I,\infty}, \pi_{I',\infty})\); \(L_0 = L_{can}\) (possibly up to an element of \(A^\times\), to account for the built-in twist).
In some cases, for example, when $\pi_I$ is holomorphic, one can get not only direct cup product pairings but also cup products after application of “nearly holomorphic” differential operators (Maass operators). So a special case of Problem 5 is

**Problem 6.** *Classify nearly holomorphic differential operators on higher coherent cohomology.*

For holomorphic cohomology these have been classified long ago.

When $L_0 = L_{can}$, we say the pair $(\pi_I, \infty, \pi'_I, \infty)$ is *coherent.*
Four pages relevant to this conference

When $V_\mathbb{R}$ is a definite hermitian space – so $G(\mathbb{R})$ and $G'(\mathbb{R})$ are necessarily compact –

$$\dim \pi_\infty, \dim \pi'_\infty < \infty.$$ 

The classification of pairs such that $L(\pi, \pi') \neq 0$ given by classical branching laws for algebraic representations of $U(n)$ (or $GL(n)$).

At a finite prime $p$

$$G_p = G(\mathbb{Q}_p) \overset{\sim}{\longrightarrow} \{ GL(n, \mathbb{Q}_p) \text{ some unitary group} \}$$
Let $\pi_p \otimes \pi'_p$ (resp. $W \otimes W'$) be a smooth (resp. algebraic) irreducible representations of $G_p \times G'_p$, with coefficients in $\mathbb{C}_p$. Suppose

$L(\pi_p, \pi'_p) \neq 0; \quad L(W, W') := \text{Hom}_{G'_p}(W \otimes W', \mathbb{C}_p) \neq 0$.

Let $\sigma, \sigma'$ be irreducible Banach space completions of the locally algebraic representations $\pi_p \otimes W, \pi'_p \otimes W'$. Define

$L(\sigma, \sigma') = \text{Hom}_{G'_p}^{cont}(\sigma \otimes \sigma', \mathbb{C}_p)$.

Locally algebraic vectors dense

$\Rightarrow \text{dim}_{\mathbb{C}_p}(L(\sigma, \sigma')) \leq 1$. 
Problem 7. For which Banach space completions of $\pi_p \otimes W$ and $\pi'_p \otimes W'$ do the non-trivial elements of $L(\pi_p, \pi'_p) \otimes L(W, W')$ extend continuously?

$GL(2, \mathbb{Q}_p) \ (n = 2)$ is already interesting.

$L(\pi_p, \pi'_p)$ determined by Waldspurger, Tunnell, H. Saito.

Non-vanishing depends on sign of the local $\varepsilon$ factor of $\pi_p \times \pi'_p$ (by Waldspurger, this is true more generally).

Specialists expect to classify Banach space completions by admissible filtrations on the local $(\phi, N)$-module attached to $\pi_p \otimes W$.

Can the $\varepsilon$ factor be extended to allow for the admissible filtration?
$U(2) \times U(1)$ at a non-split prime $\leftrightarrow G = GL(2, \mathbb{Q}_p)$,
$G' = L^\times$, $L$ is a quadratic extension of $\mathbb{Q}_p$

What can be said about the Banach representations of $G$ as $G'$-modules?

As for modular representations:

**Problem 8.** $G = GL(2, K)$ for some $p$-adic field $K$,
$[L : K] = 2$, $\pi$ an irreducible smooth representation of $G$ over $k = \overline{\mathbb{F}}_p$.

*What is the structure of $\pi$ as $L^\times$-module?*

*Breuil and Paskunas have constructed uncountable families of such $\pi$; do they all have the same $L^\times$-module structure?*
4. The Ichino-Ikeda conjectures

Ichino and Ikeda formulated an explicit conjectural expression for the central value $L(\frac{1}{2}, \pi \otimes \pi')$ when $\pi$ and $\pi'$ are representations of $SO(n)$ and $SO(n - 1)$, respectively.

Neal Harris (Ph.D. thesis under the direction of W. T. Gan) is working out the analogue for pairs $(G, G')$ as above:
Let $f \in \pi$, $f' \in \pi'$.

$$\frac{|L_{\text{can}}(f, f')|^2}{|f, f|^2|f', f'|^2} = 2^\beta C_0 \Delta_{U(n)}^S \prod_{v \in S} Z_v(f_v, f'_v) P^S\left(\frac{1}{2}, \pi, \pi'\right)$$

where $L_{\text{can}}(\bullet, \bullet')$ is the period integral and

$$P^S(s, \pi, \pi') = \frac{L^S(s, \pi \times \pi')}{L^S(s + \frac{1}{2}, \pi, \text{Ad})L^S(s + \frac{1}{2}, \pi', \text{Ad})}$$

is the main term.
Here

(1) $S$ is a finite set of places including archimedean places

(2) the superscript $S$ refers to partial $L$-functions

(3) $C_0$ is an elementary constant (quotient of volumes, can be taken in $\mathbb{Q}^\times$)

(4) $\Delta^S_U(n) = \alpha \pi^m$, $\alpha \in \overline{\mathbb{Q}}, m \in \mathbb{Z}$ (a product of special values of abelian $L$-functions),

(5) each $Z_v(f_v, f'_v) = \frac{I(f_v, f'_v)}{|f_v|^2 |f'_v|^2}$ can be taken algebraic except for $v | \infty$;

(6) $L(\ast, \ast, Ad)$ is the Langlands $L$-function attached to the adjoint representation of the $L$-group of $G$ (or $G'$).
Modulo $A^\times$, and being hopeful about the term $Z_\infty$, we have

$$\left| L_{\text{can}}(f, f') \right|^2 \equiv \frac{L(\frac{1}{2}, \pi \times \pi')}{L(1, \pi, \text{Ad})L(1, \pi', \text{Ad})},$$

Problem 9. Formulate a $p$-adic Ichino-Ikeda conjecture (for completed cohomology?)

(cf. Villegas-Zagier, Stevens, A. Mori, MH-Tilouine.)

All special values are theoretically covered by definite case (Chenevier, Emerton).

But this must be very difficult to prove!
Remarks.

The terms $Z_v(f_v, f'_v)$ are explicit integrals of matrix coefficients of representations.

Problem 10. *Compare these with local terms in Bloch-Kato-Fontaine-Perrin-Riou conjecture.*
For $p$-adic families, the $\mathbb{Q}$-invariants are not sufficiently precise. For $n = 2$, to get the correct periods, have to divide by congruence ideal (Hida).

**Problem 11.** *(Very hard.)* What about higher dimensions?

Background motivation for construction of $p$-adic $L$-functions:

Generalization of Skinner-Wiles, irreducibility of automorphic Galois representations.
5. Results

Let

\[ M = M_\Pi, M' = M_{\Pi'}, f \in \pi_I(\mathbb{Q}), f' \in \pi'_{I'}(\mathbb{Q}). \]

Then the left hand side is

\[ \frac{|L_{can}(f, f')|^2}{Q_I(M)Q_{I'}(M')}. \]

Meanwhile, the right-hand side can be rewritten

\[ \frac{L \left( \frac{w+1}{2}, M \otimes M' \right)}{L(1, M, Ad)L(1, M', Ad)} \]

\[ w = (n - 1) + (n - 2) = 2n - 3. \]

All values in this quotient are critical in Deligne’s sense.
Thus, if $L(\pi, \pi') \neq 0$, Deligne’s conjecture and the Ichino-Ikeda conjecture express the occult $P(\pi, \pi')$ in terms of Deligne periods and the $Q$-invariants.

In fact, one can express the Deligne periods in terms of $Q$-invariants and (explicit) abelian periods, as I explain below.

**Problem 12.** When the pair $(\pi_I, \infty, \pi'_I, \infty)$ is coherent, verify that this expression is compatible with the formula $P(\pi, \pi') = 1$.

*Warning:* the Gross-Prasad conjecture implies that the central value is often calculated by an occult period.
So far I have only checked the simplest cases.

**Proposition.** *In the coherent case, the expressions are compatible, up to factors in $A^\times$, when $\pi_I$ is holomorphic or when $V = V_1$.*

The proof is based on explicit expressions for the Deligne periods of the motives that appear in the right-hand side.
The formulas for the adjoint motives are uniform:

\[ c^+(\text{Ad}(M)(1)) \equiv \prod_{i=1}^{n} Q_i(M)^{-i} \pmod{A^\times}. \]

Since \( \prod_{i=1}^{n} Q_i(M) \in A^\times \), a more suggestive expression for this is

\[ \prod_{i=1}^{n} Q_i(M)^{\frac{n+1}{2}-i}. \]

Likewise for \( M' \).

The formula for \( c^+(M \otimes M'(\frac{w+1}{2})) \) depends on the relative positions of the Hodge types of \( M \) and \( M' \). So does the pair \((I, I')\) such that \( L(\pi_I, \pi'_{I'}) \neq 0 \). The main point is that \( c^+(M \otimes M'(\frac{w+1}{2})) \) can be expressed explicitly in terms of the \( Q_i(M) \) and \( Q_j(M') \), and of CM periods.