AUTOMORPHIC MOTIVES AND THE ICHINO-IKEDA CONJECTURE

Michael Harris

Conference in honor of Jean-Marc Fontaine

(thanks to Blasius, Gan, Gross, N. Harris)

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Main Formula

(This lecture): (a real lecture):: (CDO): (Real money)

CDO = Collateralized Debt Obligation = bundle of more or less toxic speculations on future production (just like this lecture).

Goal: to determine compatibility of two conjectures on special values of L-functions:

Deligne (motivic)

(in some indefinite future Bloch-Kato/Fontaine-Perrin-Riou) ;

and **Ichino-Ikeda** (automorphic).

0. Linear algebra

Consider a *polarized regular motive* M over $\mathcal{K}$, pure of weight $w = n - 1$, of rank n over its coefficient field E.

$\mathcal{K}/F$ a CM quadratic extension of a totally real field.

$c \in Gal(\mathcal{K}/F)$ complex conjugation.

Absolutely everything is true in this generality.

But to simplify notation (usually) $F = \mathbb{Q}$, $\mathcal{K}$ an imaginary quadratic subfield of $\mathbb{C}$.

This is a collection of realizations

$$M = (\{M_\lambda\}, M_B, M_{DR}, M_{Hodge}, M_{cris}) :$$

just vector spaces with associated structures.

Here λ runs over finite places of E,

$$M_{Hodge} = gr^\bullet_F M_{DR} = \oplus M^{p_i,q_i}$$

a $\mathcal{K}$-vector space (really: free rank n $\mathcal{K} \otimes E$-module)

In practice we need the λ-adic Galois representation M_λ (for L-function)

and M_{Hodge} (for periods).

Regularity hypothesis: $\dim M^{p,w-p} \leq 1$ for all i.

Thus we have a decreasing set of p_i:

$$p_1 > p_2 > \cdots > p_n \ \dim H^{p_i,q_i} = 1$$

For each i choose a rational vector $0 \neq \omega_i \in M^{p_i,q_i}$.

The conjugate motive M^c has Hodge types (p_i^c, q_i^c).

Let $RM = M \oplus M^c$ over F $(= \mathbb{Q})$.

Polarization hypothesis:

$$<,>_M \colon M \otimes M^c \to E(1-n).$$

Finally, F_∞ is an E-linear isomorphism of Betti realizations

$$F_\infty : M_B \xrightarrow{\sim} M_B^c$$

(the *king of Frobenii*, according to Hida).

Induces a map

$$F_\infty : M_{Hodge} \otimes \mathbb{C} \xrightarrow{\sim} M^c_{Hodge} \otimes \mathbb{C};$$

$$F_\infty(M^{p_i,q_i}) = (M^c)^{p^c_{n+1-i},q^c_{n+1-i}}.$$

does not respect the $\mathcal{K}$-rational structure.

$$Q_i = Q_i(M) = < \omega_i, F_\infty(\omega_i) >_M \in (\mathbb{C} \otimes \mathcal{K} \otimes E)^\times / (\mathcal{K} \otimes E)^\times$$

(we will work only mod $\overline{\mathbb{Q}}^\times$ in this talk.)

The polarization induces one relation:

$$\prod_{i=1}^{n} Q_i(M) = (2\pi i)^{n(n-1)} \delta(M)^{-2}.$$

Here $\delta(M)$ is the determinant of

$$I_\infty : M_B \otimes \mathbb{C} \xrightarrow{\sim} M_{DR} \otimes \mathbb{C}.$$

This is the same as the comparison map

$$\det(M)_B \otimes \mathbb{C} \xrightarrow{\sim} \det(M)_{DR} \otimes \mathbb{C}.$$

$\det(M)$ is the (abelian) motive, of a CM Hecke character.

Thus

$$\prod_{i=1}^{n} Q_i(M) = 1 \pmod{A^\times}$$

where $A^\times \subset \mathbb{C}^\times$ is the subgroup generated by periods of CM motives.

To avoid having to keep track of CM periods we work in $\mathbb{C}^\times / A^\times$.

(Anyway, I haven't had time to finish the necessary calculations.)

Deligne's conjecture. If m is a *critical* integer for M, there is a *Deligne period*

$$c^+(M(m)) \in \mathbb{C}^\times/\overline{\mathbb{Q}}^\times$$

defined using M_{Hodge} and F_∞, such that

$$L(M,m)/c^+(M(m)) \in \overline{\mathbb{Q}}.$$

Principle: Many $c^+(M(m))$ can be expressed in terms of the $Q_i(M)$.

1. Automorphic motives

Π a cuspidal cohomological automorphic representation of $GL(n, \mathcal{K})$.

Polarization condition

$$\Pi^\vee \xrightarrow{\sim} \Pi^c.$$

$E = E(\Pi)$ the field of definition of Π_f, a CM field.

$M_{\Pi,\lambda}$.

Π gives rise to a compatible system of λ-adic representations

$$\rho_{\Pi,\lambda} \to GL(n, E_\lambda),$$

where λ runs over places of E, non-degenerate pairing

$$\rho_{\Pi,\lambda} \otimes \rho^c_{\Pi,\lambda} \to E_\lambda(1-n).$$

(cf. Eichler-Shimura)

References :

$n = 2$: essentially the theory of Galois representations attached to Hilbert modular forms (Carayol, Wiles, Blasius-Rogawski, and Taylor).

$n = 3$: Blasius-Rogawski.

$n > 3$, 1988-2005: Clozel, Kottwitz, MH-Taylor, Taylor-Yoshida.

2005+ (After the proof of the Fundamental Lemma by Laumon and Ngô): Book project (mainly Clozel, Labesse, Chenevier, and MH), S.W. Shin.

Complete statement in Chenevier-MH.

$\rho_{\Pi,\lambda}$ are pure of weight $n - 1$. (proved at most bad places by Shin except in limit cases; in limit cases results of Clozel, Taylor, etc.)

Hypothesis: there is a polarized regular motive $M = M_\Pi$ over $\mathcal{K}$ of rank n, pure of weight $w = n-1$, coefficients in E, with

$$M_{\Pi,\lambda} = \rho_{\Pi,\lambda}.$$

Relation between M and Π:

$$L(s, M) = L(s + \frac{1-n}{2}, \Pi) = L(s, \Pi \otimes (|\bullet| \circ \det)^{\frac{1-n}{2}})$$

Actually true! except for some cases (the "limit cases") treated by λ-adic congruences.

The **Fontaine-Mazur conjecture** in this setting: every such motive (even a single M_λ) is automorphic.

In most cases (the "motivic cases") $\rho_{\Pi,\lambda}$ realized in the cohomology of certain Shimura varieties, defined over $\mathcal{K}$ with coefficients in certain local systems.

Similarly, M_Π can be realized as a Grothendieck motive using Hecke correspondences on generalized Kuga-Sato varieties (mixed Shimura varieties) .

In the motivic cases, $M_{\Pi,Hodge}$ (+ F_∞) can be related directly to automorphic forms (see below).

The limit case is important: it includes symmetric powers of modular forms as in proofs of various cases of the Sato-Tate conjecture and arises in all potential automorphy arguments.

$M_{\Pi,B}$. Purely hypothetical (but see Taylor on descent of Shimura varieties)

Interpret F_∞ as complex conjugation of automorphic forms, $Q_i(M_\Pi)$: Petersson (square) norm of automorphic form ω_i.

$M_{\Pi,DR}$.

Since Π is cohomological, M_Π is *regular*, which means that

$$p_i > p_{i+1},\ p_i^c > p_{i+1}^c$$

There is a simple expression for the Hodge types p_i, p_i^c in terms of the component Π_∞ of Π.

Problem 1. *Study $M_{\Pi,cris}$ in the automorphic setting.*

The comparison isomorphisms of p-adic Hodge theory are used, e.g. in Clozel-MH-Taylor (to compute Zariski tangent space of crystalline deformation functor).

$M_{\Pi,cris}$ plays no direct role (but see Mokrane-Tilouine as well as earlier work of Coleman, Gross, Buzzard-Taylor on p-adic cohomology and modular forms).

2. Periods

Shimura varieties $Sh(V)$ attached to unitary groups $U(V)$,

V an n-dimensional hermitian space over $\mathcal{K}$.

For each V, base change map

(cohomological automorphic representations of $U(V)$)

$$\updownarrow$$

(cohomological automorphic reps. of $GL(n)_{\mathcal{K}}$)

Studied in detail by Labesse, (S. Morel for $F = \mathbb{Q}$).

Inverse image of Π: the L-packet $\Pi_V = \{\pi_\alpha\}$ of $U(V)$ attached to Π.

Depending on local conditions, Π_V may be empty, but often is rich enough to define variants of the motive M, as follows.

First case: $V = V_1$ has signature $(1, n-1)$. Pick $\pi = \pi_\infty \otimes \pi_f \in \Pi_{V_1}$, fix π_f.

$$\dim Sh(V_1) = n - 1$$

Local system $L(\Pi_\infty)/Sh(V_1)$ (ℓ-adic or VHS) defined by Π_∞

Of geometric origin (in direct image of cohomology of a family of abelian varieties over $Sh(V_1)$).

The group $U(V_1)(\mathbf{A}^f)$ (actually $GU(V_1)(\mathbf{A}^f)$) acts on $L(\Pi_\infty)$ and $Sh(V_1)$.

Define

$$M(V_1,\Pi) = Hom_{U(V_1)(\mathbf{A}^f)}(\pi_f, H^{n-1}(Sh(V_1), L(\Pi_\infty))).$$

Usually $\dim M(V_1,\Pi) = n$: this is the motive over $\mathcal{K}$ we want to call M_Π, up to an explicit abelian twist.

n-dimensional because the archimedean (discrete series) L-packet $\Pi_{V_1,\infty}$ has n-members – basis of Hodge realization indexed by $\Pi_{V_1,\infty}$ (see below).

More generally, if $U(V) = U(r,s)$ with $r+s=n$ the signature, then

$$|\Pi_{V,\infty}| = d(V) := \binom{n}{r}$$

so $M(V,\Pi)$, defined by the formula analogous to that used with $V = V_1$, should be a motive of rank $d(V)$. In fact we have (Kottwitz + stable trace formula)

$$M(V,\Pi)_\lambda \xrightarrow{\sim} \wedge^r M_{\Pi,\lambda} \otimes M(\xi)$$

for an explicit CM Hecke character ξ.

The point to keep in mind:

Cohomology of Shimura varieties not only gives us a large collection of n-dimensional polarized regular motives. It also gives us motivic realizations of all exterior powers of the λ-adic realizations.

Problem 2. *The Tate conjecture implies the corresponding relations at the level of motives. Prove them independently, at least for the Hodge realization (see below); i.e., prove the corresponding period relations.*

(All motives have coefficients in some Hecke field $E = E(\Pi)$, but I ignore this.)

We actually know more than this. The automorphic theory naturally gives us a $\mathcal{K}$-*rational basis* (actually $\mathcal{K}\otimes E$-rational) $(\omega_1, \ldots, \omega_n)$ of the Hodge realization

$$M_{Hodge} := \oplus_{i=i}^{n} H^{p_i,q_i}(M_\Pi),$$

which is naturally a $\mathcal{K}$-vector space via the identification of the H^{p_i,q_i} with coherent cohomology.

Thus the pure tensors

$$\omega_I = \omega_I(\Pi) := \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_r};\ I = (i_1 < i_2 \cdots < i_r)$$

form a canonical basis for $\wedge^r M_{Hodge}$.

(For general F, we have to tensor over the $\sigma \in Hom(F, \mathbb{R})$, but the idea is the same, and we get rational bases of pure tensors for each $M(V, \Pi)_{Hodge}$ in terms of the $\omega_{i,\sigma}$.)

The *periods* of the ω_i – integrals over rational homology classes – are completely inaccessible by automorphic means.

But ω_i is a concrete automorphic form on $U(V_1)$, generating a specific $\pi_i \in \Pi_{V_1}$, and its Petersson (square) norm $Q_i = Q_i(\Pi)$ is a well-defined invariant in $\mathbb{C}^\times/\overline{\mathbb{Q}}^\times$, up to algebraic factors.

In particular, ω_1 is a *holomorphic* automorphic form.

Similarly, if V is fixed, for each $\pi_I \in \Pi_V$, there is a concrete automorphic form $\omega_{I,V}$ on $U(V)$, generating an automorphic representation of $U(V)$ isomorphic to $\pi_I \otimes \pi_f$ when $|I| = r$; these form a rational basis for the Hodge realization of the motive $M(V,\Pi)$, canonical up to (rational) scalar multiples.

(Here we are making the assumption that $U(V)(\mathbf{A}^f) \xrightarrow{\sim} U($
so their local representations can be identified. This is legitimate when n is odd but not when n is even, in which case the construction needs to be modified.)

For example, $\omega_{I,V}$ is holomorphic, as is the formal wedge product ω_I, if and only if $I = \{1, \ldots, r\}$.

Let $Q_{I,V}$ be the Petersson square norm of $\omega_{I,V}$. Under the hypothetical isomorphism

$$M(V,\Pi) \xrightarrow{\sim} \wedge^r M_\Pi \otimes M(\xi)$$

we want to identify $\omega_{I,V}$ with the wedge product ω_I.

Problem 3. *Prove that* $Q_{I,V} = \prod_{i \in I} Q_i$ *(mod* $A^\times$*, more precisely up to the CM period corresponding to the abelian twist), as predicted by the Tate conjecture.*

When $F = \mathbb{Q}$ and $\Pi \xrightarrow{\sim} \Pi^c$, known (MH) for the *holomorphic* Q_I, up to an undetermined constant that depends only on Π_∞.

The proof is based on the theta correspondence.

MH-Li-Sun (2009-2010): the theta correspondence alone cannot solve Problem 3 for general I!

Can it be solved without the Tate conjecture?

The interest of the Q-invariants: for certain motives $\mathcal{M}$ constructed from polarized regular M, the *Deligne periods* $c^+(\mathcal{M})$ (of Deligne's conjecture on special values) can be expressed formally in terms of the Q_i and some abelian (CM) periods.

In the heuristic discussion that follows, assume the period relations hold for the Q-invariants. In particular, we identify the formal wedge product ω_I with the automorphic form $\omega_{I,V}$ on $U(V)$.

3. The Gross-Prasad conjectures

Let now Π and Π' be cohomological automorphic representations of $GL(n)_{\mathcal{K}}$ and $GL(n-1)_{\mathcal{K}}$.

Choose hermitian spaces V and V', $V = V' \oplus E$ for a non-isotropic line E, $\dim V = n$.

Let $\pi \in \Pi_V$, $\pi' \in \Pi'_{V'}$, $G = U(V)$, $G' = U(V')$.

Consider the abstract space of linear forms

$$L(\pi, \pi') = Hom_{G'}(\pi \otimes \pi', \mathbb{C})$$

where $G' \subset G \times G'$ is embedded diagonally.

The following theorem is due to Aizenbud-Gurevich-Rallis-Schiffman (for the non-archimedean factors) and Sun-Zhu (for the archimedean factors):

Theorem 3. $\dim L(\pi, \pi') = \prod_v \dim L(\pi_v, \pi'_v) \leq 1$, *where the factorization is defined in the obvious way.*

Waldspurger has proved much more (so far only for orthogonal groups).

The full Gross-Prasad conjecture in this setting is roughly

$$\sum_{\pi \in \Pi_V, \pi' \in \Pi_{V'}} \dim L(\pi, \pi') \leq 1$$

with equality if the L-function considered by Ichino-Ikeda (see below) does not vanish at $s = \frac{1}{2}$.

Waldspurger has proved the non-archimedean part of this statement (for tempered L-packets, which should include all the local terms of Π_V and $\Pi_{V'}$).

The archimedean statement is not yet proved but will be assumed in what follows.

Consequences of Theorem 3

Global consequences. Suppose $\pi = \pi_I$ for some subset $I \subset \{1, \ldots, n\}$ as above, with $|I| = r$, $G_\infty = U(r, s)$. Assume $\pi' = \pi_{I'}$ similarly.

Hypotheses.

(a) $\dim L(\pi, \pi') = 1$

(b) A generator $L \in L(\pi, \pi')$ is given by integration

$$L_{can}(f, f') = \int_{G'(F)\backslash G'(\mathbf{A})} f(g')f'(g')dg'$$

where dg' is Tamagawa measure.

The original global Gross-Prasad conjecture is roughly the following:

Problem 4. *Assuming (a), prove that (b) is true if and only if $L(\frac{1}{2}, \Pi \otimes \Pi') \neq 0$.*

The Ichino-Ikeda conjecture is a refined version of Problem 4, expressing the L-value in terms of the integral L_{can} and other terms.

Proposition. *The spaces* π *and* π' *have* $\overline{\mathbb{Q}}$*-rational structures generated by* $\omega_I(\Pi)$ *and* $\omega_{I'}(\Pi')$*. In particular,* $L(\pi, \pi')$ *has a generator* L_0 *rational with respect to these structures.*

One can replace $\overline{\mathbb{Q}}$ by $\mathcal{K} \otimes E$. For π_f and π'_f, standard considerations about automorphic cohomology.

For the archimedean part, use algebraic realization of $\pi_{I,\infty}, \pi_{I',\infty}$ as Zuckerman derived functor modules.

(For this need to choose max. compact, max torus $K_\infty \supset T_\infty$; OK over a CM point attached to $\mathcal{K}$.)

Corollary. *There is a constant* $P(\pi, \pi') \in \mathbb{C}^\times/\overline{\mathbb{Q}}^\times$ *(an* **occult period***) such that, for any* $f \in \pi$, $f' \in \pi'$*, rational over* $\overline{\mathbb{Q}}$*,* $L_0(f, f')$ *is a* $\overline{\mathbb{Q}}$*-multiple of* $P(\pi, \pi')$*.*

Proposition. *If* π_I *is holomorphic,* $P(\pi, \pi') = 1$*.*

Indeed, $\pi_{I,\infty} \mid_{G'(\mathbb{R})}$ is a discrete (countable) direct sum of holomorphic representations. Thus if $L(\pi, \pi') \neq 0$, π' must be *antiholomorphic*, and the integral L_{can} is just cup-product in coherent cohomology.

With the rational structures I (haven't) defined, we can thus take $L_0 = L_{can}$.

More generally, for certain pairs (I, I'), L_{can} can be interpreted as a cup product in coherent cohomology, and then one again has $L_0 = L_{can}$. We then say the pair $(\pi_{I,\infty}, \pi'_{I',\infty})$ is *coherent.*

One has $\omega_i \in H^{p_i,q_i} \xrightarrow{\sim} H^{i-1}(Sh(V_1), \mathcal{E}_i)$ for some automorphic vector bundle $\mathcal{E}_i$ on $Sh(V_1)$.

(Don't always have $q_i = i-1$: $\mathcal{E}_i$ has a built-in twist by (the infinity type of) a CM Hecke character.)

Similarly, ω_I belongs to $H^j(Sh(V), \mathcal{E}_I)$ where j is the number of elements in I *not* in the set $\{1, \ldots, r\}$.

Problem 5. *Determine the set of coherent pairs* $(\pi_{I,\infty}, \pi'_{I',\infty})$*;* $L_0 = L_{can}$ *(possibly up to an element of* $A^\times$*, to account for the built-in twist).*

In some cases, for example, when π_I is holomorphic, one can get not only direct cup product pairings but also cup prodcuts after application of "nearly holomorphic" differential operators (Maass operators). So a special case of Problem 5 is

Problem 6. *Classify nearly holomorphic differential operators on higher coherent cohomology.*

For holomorphic cohomology these have been classified long ago.

When $L_0 = L_{can}$, we say the pair $(\pi_{I,\infty}, \pi'_{I',\infty})$ is *coherent*.

Four pages relevant to this conference

When $V_{\mathbb{R}}$ is a definite hermitian space – so $G(\mathbb{R})$ and $G'(\mathbb{R})$ are necessarily compact –

$$\dim \pi_\infty,\ dim \pi'_\infty < \infty.$$

The classification of pairs such that $L(\pi, \pi') \neq 0$ given by classical branching laws for algebraic representations of $U(n)$ (or $GL(n)$).

At a finite prime p

$$G_p = G(\mathbb{Q}_p) \xrightarrow{\sim} \left\{ \begin{array}{c} GL(n, \mathbb{Q}_p) \\ \text{some unitary group} \end{array} \right.$$

Let $\pi_p \otimes \pi'_p$ (resp. $W \otimes W'$) be a smooth (resp. algebraic) irreducible representations of $G_p \times G'_p$, with coefficients in $\mathbb{C}_p$. Suppose

$$L(\pi_p, \pi'_p) \neq 0; L(W, W') := Hom_{G'_p}(W \otimes W', \mathbb{C}_p) \neq 0.$$

Let σ, σ' be irreducible Banach space completions of the locally algebraic representations $\pi_p \otimes W$, $\pi'_p \otimes W'$.

Define

$$L(\sigma, \sigma') = Hom^{cont}_{G'_p}(\sigma \otimes \sigma', \mathbb{C}_p).$$

Locally algebraic vectors dense

$$\Rightarrow \dim_{\mathbb{C}_p}(L(\sigma, \sigma')) \leq 1.$$

Problem 7. *For which Banach space completions of* $\pi_p \otimes W$ *and* $\pi'_p \otimes W'$ *do the non-trivial elements of* $L(\pi_p, \pi'_p) \otimes L(W, W')$ *extend continuously?*

$GL(2, \mathbb{Q}_p)$ $(n = 2)$ is already interesting.

$L(\pi_p, \pi'_p)$ determined by Waldspurger, Tunnell, H. Saito.

Non-vanishing depends on sign of the local ε factor of $\pi_p \times \pi'_p$ (by Waldspurger, this is true more generally).

Specialists expect to classify Banach space completions by admissible filtrations on the local (ϕ, N)-module attached to $\pi_p \otimes W$.

Can the ε factor be extended to allow for the admissible filtration?

$U(2) \times U(1)$ at a non-split prime $\leftrightarrow G = GL(2, \mathbb{Q}_p)$, $G' = L^\times$, L is a quadratic extension of $\mathbb{Q}_p$

What can be said about the Banach representations of G as G'-modules?

As for modular representations:

Problem 8. *$G = GL(2, K)$ for some p-adic field K, $[L : K] = 2$, π an irreducible smooth representation of G over $k = \bar{\mathbb{F}}_p$.*

What is the structure of π as $L^\times$-module?

Breuil and Paskunas have constructed uncountable families of such π; do they all have the same $L^\times$-module structure?

4. The Ichino-Ikeda conjectures

Ichino and Ikeda formulated an explicit conjectural expression for the central value $L(\frac{1}{2}, \pi \otimes \pi')$ when π and π' are representations of $SO(n)$ and $SO(n-1)$, respectively.

Neal Harris (Ph.D. thesis under the direction of W. T. Gan) is working out the analogue for pairs (G, G') as above:

Let $f \in \pi$, $f' \in \pi'$.

$$\frac{|L_{can}(f,f')|^2}{|f,f|^2|f',f'|^2} = 2^{\beta} C_0 \Delta^S_{U(n)} \prod_{v \in S} Z_v(f_v, f'_v) P^S(\frac{1}{2}, \pi, \pi')$$

where $L_{can}(\bullet, \bullet')$ is the period integral and

$$P^S(s, \pi, \pi') = \frac{L^S(s, \pi \times \pi')}{L^S(s+\frac{1}{2}, \pi, Ad) L^S(s+\frac{1}{2}, \pi', Ad)}$$

is the main term.

Here

1. S is a finite set of places including archimedean places
2. the superscript ^{S} refers to partial L-functions
3. C_0 is an elementary constant (quotient of volumes, can be taken in $\mathbb{Q}^\times$)
4. $\Delta^S_{U(n)} = \alpha\pi^m$, $\alpha \in \overline{\mathbb{Q}}, m \in \mathbb{Z}$ (a product of special values of abelian L-functions),
5. each $Z_v(f_v, f'_v) = \frac{I(f_v, f'_v)}{|f_v|^2|f'_v|^2}$ can be taken algebraic except for $v \mid \infty$;
6. $L(*, *, Ad)$ is the Langlands L-function attached to the adjoint representation of the L-group of G (or G').

Modulo $A^\times$, and being hopeful about the term Z_∞, we have

$$\frac{|L_{can}(f,f')|^2}{|f,f|^2|f',f'|^2} \equiv \frac{L(\frac{1}{2},\pi\times\pi')}{L(1,\pi,Ad)L(1,\pi',Ad)}.$$

Problem 9. *Formulate a p-adic Ichino-Ikeda conjecture (for completed cohomology?)*

(cf. Villegas-Zagier, Stevens, A. Mori, MH-Tilouine.)

All special values are theoretically covered by definite case (Chenevier, Emerton).

But this must be very difficult to prove!

Two more pages relevant to this conference

Remarks.

The terms $Z_v(f_v, f'_v)$ are explicit integrals of matrix coefficients of representations.

Problem 10. *Compare these with local terms in Bloch-Kato-Fontaine-Perrin-Riou conjecture.*

For p-adic families, the Q-invariants are not sufficiently precise. For $n = 2$, to get the correct periods, have to divide by congruence ideal (Hida).

Problem 11. *(Very hard.) What about higher dimensions?*

Background motivation for construction of p-adic L-functions:

Generalization of Skinner-Wiles, irreducibility of automorphic Galois representations.

5. Results

Let

$$M = M_\Pi, M' = M_{\Pi'}, f \in \pi_I(\overline{\mathbb{Q}}), f' \in \pi'_{I'}(\overline{\mathbb{Q}}).$$

Then the left hand side is

$$\frac{|L_{can}(f, f')|^2}{Q_I(M)Q_{I'}(M'}.$$

Meanwhile, the right-hand side can be rewritten

$$\frac{L(\frac{w+1}{2}, M \otimes M')}{L(1, M, Ad)L(1, M', Ad)}$$

$w = (n-1) + (n-2) = 2n - 3.$

All values in this quotient are critical in Deligne's sense.

Thus, if $L(\pi, \pi') \neq 0$, Deligne's conjecture and the Ichino-Ikeda conjecture express the occult $P(\pi, \pi')$ in terms of Deligne periods and the Q-invariants.

In fact, one can express the Deligne periods in terms of Q-invariants and (explicit) abelian periods, as I explain below.

Problem 12. *When the pair* $(\pi_{I,\infty}, \pi'_{I',\infty})$ *is coherent, verify that this expression is compatible with the formula* $P(\pi, \pi') = 1$.

Warning: the Gross-Prasad conjecture implies that the central value is often calculated by an occult period.

So far I have only checked the simplest cases.

Proposition. *In the coherent case, the expressions are compatible, up to factors in* $A^\times$*, when* π_I *is holomorphic or when* $V = V_1$*.*

The proof is based on explicit expressions for the Deligne periods of the motives that appear in the right-hand side.

The formulas for the adjoint motives are uniform:

$$c^+(Ad(M)(1)) \equiv \prod_{i=1}^{n} Q_i(M)^{-i} \pmod{A^\times}.$$

Since $\prod_{i=1}^{n} Q_i(M) \in A^\times$, a more suggestive expression for this is

$$\prod_{i=1}^{n} Q_i(M)^{\frac{n+1}{2}-i}.$$

Likewise for M'.

The formula for $c^+(M \otimes M'(\frac{w+1}{2}))$ depends on the relative positions of the Hodge types of M and M'. So does the pair (I, I') such that $L(\pi_I, \pi'_{I'}) \neq 0$. The main point is that $c^+(M \otimes M'(\frac{w+1}{2}))$ can be expressed explicitly in terms of the $Q_i(M)$ and $Q_j(M')$, and of CM periods.