IV.1. UNITARY GROUPS AND BASE CHANGE

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1. Classification of unitary groups over number fields

Let F/F^+ be a (separable) quadratic extension of fields with Galois group $\{1, c\}$, where c is the non-trivial automorphism. Let V be an n-dimensional vector space over F. A *hermitian form* on V is a pairing

$$h : V \times V \to F$$

that is linear in the first variable, c-linear in the second variable:

$$h(\alpha v, \beta w) = \alpha \cdot c(\beta) h(v, w)$$

and satisfies $h(v, w) = c(h(w, v))$. We always assume h to be non-degenerate; i.e. for $v \in V$, $v \neq 0$, there exists $w \in V$ such that $h(v, w) \neq 0$. The unitary group $U(V)$ is the subgroup of $g \in GL(V)$ that preserve h:

$$h(g(v), g(w)) = h(v, w).$$

This relation defines an algebraic group over F^+. Two hermitian vector spaces are *equivalent* if there is an F linear isomorphism between them that identifies the hermitian forms. If V and V' are two hermitian vector spaces, one constructs the hermitian vector space $V \oplus V'$ in the obvious way.

If $F = \mathbb{C}$, $F^+ = \mathbb{R}$, and $n = 1$, then any hermitian form on $\mathbb{C}$ is equivalent to either $h^+(z_1, z_2) = z_1 \cdot \bar{z}_2$ or $h^-(z_1, z_2) = -z_1 \cdot \bar{z}_2$. Denote the corresponding one-dimensional hermitian spaces V^+ and V^-, and note that $U(V^+) \simeq U(V^-) = U(1)$, the unit circle in $\mathbb{C}^\times$. Any hermitian form on $\mathbb{C}^n$ is isomorphic to one of the form $V^{p,q} = (V^+)^p \oplus (V^-)^q$, where $p + q = n$; the signature (p, q) determines the hermitian space up to isomorphism. The unitary groups $U(p, q) = U(V^{p,q})$ and $U(q, p)$ are isomorphic, but there are no other isomorphisms. The unitary groups $U(n, 0) = U(0, n)$ are the familiar compact unitary groups; the others are non-compact Lie groups.

Suppose F/F^+ is a quadratic extension of p-adic fields, for some prime p. For each n there are exactly two n-dimensional hermitian spaces over F, up to isomorphism, say V^+ and V^-, If n is even then $U(V^+)$ and $U(V^-)$ are not isomorphic; if G is the unitary group of an n-dimensional hermitian space V over F, and we assign the Hasse invariant $\epsilon(G) = \pm 1$ as $V \simeq V^\pm$, in such a way that V^+ has a maximal isotropic subspace of dimension $\frac{n}{2}$; then $U(V^+)$ is *quasi-split* in the sense that it contains an F^+-rational Borel subgroup. If n is odd then $U(V^+) \simeq U(V^-)$ is always quasi-split and we set $\epsilon(G) = 1$.

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Returning to the real case, we set $\epsilon(U(p,q)) = 1$ if $p+q$ is odd. If $p+q = 2m$ is even, we set $\epsilon(U(p,q)) = (-1)^{m-p}$.

Now suppose F/F^+ is an extension of number fields. We will always assume F^+ is totally real and F is a totally imaginary quadratic extension. Let V be an n-dimensional hermitian vector space over F, $G = U(V)$. For all places v of F^+, let $G_v = U(V_v)$ over F_v^+. If v splits as $w \cdot w^c$ in F then $V_v = V \otimes_{F^+} (F_w \oplus F_{w^c}) = V_w \oplus V_{w^c}$, c exchanges V_w and V_{w^c}. A simple calculation shows that any element $G(V_v)$ fixes each of the two summands and projection on the former defines an isomorphism

$$G(V)_v \xrightarrow{\sim} GL(n, F_w) \xrightarrow{\sim} GL(n, F_v^+).$$

If v is not split, then we obtain a local Hasse invariant $\epsilon(G_v) = \pm 1$. Classification of unitary groups over number fields is contained in the following theorem:

Theorem. *For each place v of F^+ that is not split in F, choose a sign ϵ_v and suppose $\epsilon_v = 1$ for all but finitely many v. Then there exists a unitary group $G = U(V)$ with $\epsilon(G_v) = \epsilon_v$ if and only if $\prod_v \epsilon_v = 1$. In particular, if n is odd, any collection of local unitary groups can be realized as a global unitary group.*

The distinct F^+ forms of a given unitary group G are classified by $H^1(\Gamma, ad(G)(F)) = H^1(\Gamma, PGL(n,F))$, where $c \in \Gamma$ acts by the outer automorphism $c^*(g) = {}^t c(g)^{-1}$. The short exact sequence of Γ-modules

$$1 \to GL(1,F) \to GL(n,F) \to PGL(n,F) \to 1$$

gives rise to an exact sequence

$$H^1(\Gamma, F^\times) \to H^1(\Gamma, GL(n,F)) \to H^1(\Gamma, PGL(n,F)) \to H^2(\Gamma, F^\times)$$

which can be calculated, and the theorem is ultimately a consequence of class field theory. (Note that $H^1(\Gamma, GL(n,F))$ is not trivial for the outer action; Hilbert's Theorem 90 does not apply.)

In [CHT] and [T] a more general kind of unitary group is considered. Let B be a central division algebra over F of dimension n^2 whose opposite algebra isomorphic to its c-conjugate. Assume it is split outside a finite set of primes of F, all of which split over F^+, and let $S(B)$ be the corresponding set of primes of F^+. We assume the divisors of $S(B)$ include $v_0, c(v_0), v_1, c(v_1)$ and at each prime of F dividing a prime of $S(B)$ B is a division algebra. The hypothesis on B^{op} implies that the local Hasse invariants of B at the two primes of F above any $v \in S(B)$, so in particular such a B exists. Now let $\ddagger$ be an involution of B of the *second kind*, i.e. $\ddagger$ restricts to complex conjugation c on F. Let G_0 (denoted G in [CHT] and [T]) be the reductive algebraic group over F^+ such that, for any F^+-algebra R,

$$G_0(R) = \{g \in (B \otimes_{F^+} R)^\times \mid g^{\ddagger \otimes 1} \cdot g = 1\}. \tag{2.3}$$

At non-split primes v of F^+ (or F) B_v is a matrix algebra, $\ddagger$ is a c-antilinear automorphism of B_v, and by the classification of such anti-automorphisms $G_0(F_v^+)$ can be identified to the unitary group of some hermitian form on F_v^n. The general classification of unitary groups recalled above extends. We have arranged things so that the division algebra structure does not interfere with the local Hasse invariants. In particular, we have

Fact. *Suppose $d = [F^+ : \mathbb{Q}]$ is even. Then ‡ can be chosen so that $G_0(F_v^+)$ is quasi-split at every finite prime of F^+ not in $S(B)$, and such that $G_0(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ is compact, and isomorphic to the product of d copies of the compact unitary group $U(n)$.*

The same conclusion holds if $S(B)$ is empty, i.e. if $B \simeq M(n, F)$.

In the present course I will always assume $B = M(n, F)$. Some of the results will therefore be conditional.

2. Unitary groups and base change

Henceforward, we always assume F totally imaginary, so $[F : F^+] = 2$ and the Galois conjugation c is non-trivial. It's also convenient henceforward to assume n to be even, since this is the only case to which our potential modularity methods apply, although the modular lifting theorems work as well for odd n. We let $d = [F^+ : \mathbb{Q}]$.

The construction of the arrow from left to right:

$$\begin{Bmatrix} \text{automorphic} \\ \text{representations} \\ \text{of } GL(n) \end{Bmatrix} \to \begin{Bmatrix} \text{Compatible systems of} \\ n\text{-dimensional } \ell\text{-adic} \\ \text{representations} \end{Bmatrix} \tag{2.1}$$

proceeds by replacing the automorphic representation Π of $GL(n)$ by a packet $\{\pi\}$ of representations of some unitary group $G(V)$ that contribute to the cohomology in middle degree $n-1$ of a Shimura variety we denoted $Sh(G(V))$. There is no known way to construct the packet $\{\pi\}$ directly, starting with Π. The existence of $\{\pi\}$, which we can call the $G(V)$-avatar of Π, is deduced from the comparison of a (stable) twisted trace formula attached to $GL(n)$ to the stable trace formula for $G(V)$, or more generally to the stable trace formula for an inner twist G' of $G(V)$ that also defines a Shimura variety of dimension $n-1$. The automorphic representation Π factors over the places of F:

$$\Pi = \otimes'_v \Pi_v$$

where almost all Π_v are unramified, and in particular are determined by the eigenvalues of the corresponding Hecke operators. There are n basic Hecke operators for $GL(n)$, say $T_{1,v}, \ldots, T_{n,v}$, whose eigenvalues, suitably normalized, define the coefficients of the local factor $L(s, \Pi_v)$ at v. For the purposes of these notes we hardly need to know more about them than that. Compatibility with the local Langlands correspondence implies that $\rho_{\Pi,\ell}$ is unramified at any prime v, $v \not| \ell$, such that Π_v is unramified. The identity $L(s, \Pi_v) = L_v(s, \rho_\Pi)$ (a special case of (i) of Theorem 1.2) then implies that the eigenvalues of the $T_{i,v}$ on Π_v determine the traces of the Frobenius at v under $\rho_{\Pi,\ell}$. As for $GL(2)$, the compatible system of (semisimple) ℓ-adic representations attached to Π is thus determined up to isomorphism by the systems of eigenvalues of these Hecke operators, and indeed by the eigenvalues of the $T_{i,v}$ for a subset of places v of F of Dirichlet density 1. For example, the $T_{i,v}$ for v split over F^+ suffice.

The $G(V)$ and G' avatars are no longer needed for the construction of the arrow from right to left in (2.1). The Taylor-Wiles method depends on certain cohomology groups being free over finite subgroups of $GL(n, \mathcal{O}_v)$, and this is easiest to arrange when the cohomology is in degree 0 (or in degree 1, as in the Taylor-Wiles paper).

Thus we work with a *totally definite* unitary group over F^+, which we call G_0. Thus

$$G_0(F_\infty) = G_0(F \otimes \mathbb{R}) \xrightarrow{\sim} U(n)^d,$$

where $U(n)$ is the compact unitary group. In order for this to be of any use we need Π to descend to a G_0-avatar, i.e. a packet π_0 of automorphic representations (in practice a singleton) of G_0.

How to think about about automorphic representations of unitary groups (avatars).

From the theoretical point of view, an automorphic representation π of a unitary group G over F^+ is supposed to be parametrized possibly up to ambiguity in its L-packet, by a global Langlands parameter, which in the case that concerns us is a homomorphism $\phi = \phi_\pi$ from $Gal(\overline{\mathbb{Q}}/F^+)$ to the L-group of G,

$${}^LG = GL(n, \mathcal{K}) \rtimes Gal(F/F^+)$$

where $c \in Gal(F/F^+)$ acts on $GL(n, \mathcal{K})$ by an appropriately normalized outer automorphism. Here $\mathcal{K}$ is an algebraically closed field, often taken to be $\mathbb{C}$; in this Galois formulation we take $\mathcal{K} = \bar{\mathbb{Q}}_\ell$. The homomorphism should commute with projection of both sides to $Gal(F/F^+)$, and so

$$\phi \mid_F : Gal(\overline{\mathbb{Q}}/F)) \to GL(n, \bar{\mathbb{Q}}_\ell)$$

is just our ℓ-adic representation ρ. Recall that ρ was attached to an automorphic representation Π of $GL(n, F)$, and plays the role of the Langlands parameter of Π. Conditions (a) and (b) on ρ imply precisely that ρ extends to a Langlands parameter ϕ with values in ${}^LG$.

In practical terms. cuspidal automorphic representations of $GL(n)$ are the atoms of the theory of automorphic forms, from which all automorphic representations of other groups ultimately derive, as in Langlands' hypothetical Tannakian formalism. They are classified by their L-functions, whose basic properties were established by Jacquet, Shalika, and Piatetski-Shapiro on the one hand, and Shahidi on the other, in the 1970s and 1980s. The arithmetic properties of automorphic representations of $GL(n)$ with $n > 2$ are generally accessible only indirectly, by means of operations involving their avatars on other classical groups, usually unitary groups.

In order to understand automorphic representations of unitary groups it is best not to think of them as matrix groups but rather as abstract groups related in a certain way to $GL(n)$. This relation is exploited by the trace formula but is irrelevant for the present exposition. An automorphic representation π_0 of G_0 should be understood in terms of its local factors $\pi_{0,v}$ for primes v of F^+. This is how the relation with Π is defined. For example, suppose v splits as $w \cdot w^c$ in F. Then for any hermitian space V/F, $G(V)_v \xrightarrow{\sim} GL(n, F_w) \xrightarrow{\sim} GL(n, F_v^+)$. Condition (2) (polarization) implies that

$$\Pi_w^\vee \xrightarrow{\sim} \Pi_{w^c}.$$

Then the avatar π_0 has the property

$$\pi_{0,v} \xrightarrow{\sim} \Pi_w \tag{2.2}$$

where the isomorphism with Π_w, rather than with Π_{w^c}, depends on some implicit choices. At non-split places it is not so easy to write $\pi_{0,v}$ in terms of Π_w; there is a formula when Π_w is unramified, or at real places, but at other places this is still an open question. We solve this question by reducing to the situation where there are no such places; see below.

Any two unitary groups over F^+, relative to the quadratic extension F/F^+, are inner forms of one another; in particular, their Langlands L-groups are isomorphic. Assuming we have already descended Π to $\{\pi\}$, one can view π_0 as a functorial transfer of $\{\pi\}$ corresponding to the isomorphism ${}^LG(V) \xrightarrow{\sim} {}^LG_0$. Here we invoke the following consequence of Langlands functoriality:

Vague general principle. *(i) The only obstructions to transfer of L-packets between inner forms are local.*

(ii) Let K be a local field, G_1 a reductive group over K, G_2 an inner form of G_1 over K. Assume G_1 is quasi-split. Then there are no local obstructions to transfer of L-packets from G_2 to G_1.

(iii) Let G_1 be a reductive group over $\mathbb{R}$, G_2 an inner form, and suppose G_1 has a discrete series (in which case so does G_2). Then there is no local obstruction to transfer of discrete series L-packets from G_2 to G_1.

In other words, (i) asserts that, if one can transfer π_v to some irreducible admissible representation $\pi_{0,v}$ of $G(V_0)_v$ for every place v, then there is a global π_0. This is what happens with the Jacquet-Langlands correspondence between automorphic representations of $GL(2)$ and automorphic representations of division algebras. Of course, this is a principle, not a theorem, and until there is a completely general and explicit stable trace formula it has to be proved anew in each individual case. For the twisted unitary groups considered in [HT], this was proved in [HT] and in my earlier paper on p-adic uniformization; for untwisted unitary groups there are partial results in [HL] and in Labesse's book [L].

As for (ii), we will not encounter the quasi-split inner form of $U(n)$ over $\mathbb{R}$. Suppose v is a place of F^+ that splits in F; then we have already seen that $G(V)_v$ is a general linear group, which is certainly quasi-split. If v does not split in F, then there are two non-isomorphic unitary groups over F_v^+, the quasi-split one G_v^+, for which the hermitian form is anti-diagonal, and the non-quasi-split one G_v^- (when n is odd every unitary group over a p-adic field is quasi-split). We set $\epsilon(G_v^{\pm}) = \pm 1$; this is the *Hasse invariant* of $G_v^{\pm}$.

Finally, unitary groups over $\mathbb{R}$ all have discrete series, and the $G(*)$-avatars of our cohomological representation Π of $GL(n, F)$ are always of discrete series type at real places, so there is never a local obstruction at ∞. The discrete series is a local L-packet, but for definite groups it contains a single element. If we know $\Pi_\infty = \prod_{w \div \infty} \Pi_w$, where w runs over the (conjugate pairs of) complex prime(s) w of F, then we can determine the corresponding $\pi_{0,\infty}$, as follows. First of all, since Π is cuspidal, Shalika's theorem implies that Π_w is *generic* (has a Whittaker model) for every prime w of F. It is known that, for each irreducible finite-dimensional representation L of

$$Lie(GL(n, F_\infty))_{\mathbb{C}} = \prod_{w \div \infty} \mathfrak{gl}(n, F_w) \times \mathfrak{gl}(n, F_{w^c})$$

satisfying $L^c \xrightarrow{\sim} L^\vee$, there is a unique Π_∞ satisfying (1.1.1) for L

$$H^\bullet(\mathfrak{gl}(n, F_\infty), K_\infty; \Pi_\infty \otimes L^\vee) \neq 0, \tag{2.2.2}$$

such that all factors Π_w are generic. (See Clozel's Ann Arbor article for a discussion of this.) Say

$$L = \prod_v L_w \otimes L_w^\vee$$

where if v is the restriction of w to F^+ then $\mathfrak{gl}(n, F_w)$ acts on the first factor and $\mathfrak{gl}(n, F_{w^c})$ on the second. Then the representation π_0 of $G_0(\mathbb{R}) = \prod_v U(n)$, v running over real places of F^+, is just $\otimes_w L_w$, where there is again an implicit choice of an extension of each real v to a place w of F.

In fact, I have been concealing from you the existence of an important global obstruction, namely the obstruction to the existence of a totally definite G_0 that is quasi-split at all finite primes, hence creates no local obstructions to transfer at finite primes. Actually, we are not yet in a position to work with such a G_0, because the stable trace formula does not yet apply in this case. Instead, we recall the two non-c-conjugate places v_0 and v_1 of Theorem 1.5, and consider only Π (resp. ρ) that satisfy condition (3) (resp. (c)) at *both* v_0 and v_1 (hence at four places in all, counting the complex conjugates). We will soon show that doubling condition (3) entails no loss of generality.

By invoking Proposition 1.9, we see that, in order to prove modularity of an n-dimensional representation ρ of $Gal(\overline{\mathbb{Q}}/F)$ satisfying hypotheses (a), (b), (c) we can always reduce to the case in which

(2.4.1) d is even;
(2.4.2) ρ satisfies local hypothesis (c) at two places not conjugate under c;
(2.4.3) F/F^+ is unramified at all finite places;
(2.4.4) Every prime v at which ρ ramifies (including primes of residue characteristic ℓ) splits in F/F^+.

Indeed, we can replace F^+ by a totally real quadratic extension F_1^+ and F by $F_1 = F{\cdot}F_1^+$); we can assume v_0 splits in F_1; we can let F_1^+ absorb all the ramification of F/F^+ and split F/F^+ at every prime at which ρ ramifies. By using more general solvable extensions, we can eliminate most of the remaining ramification; only unipotent ramification cannot be absorbed by an appropriately chosen finite solvable extension.

Let Π be a cuspidal automorphic representation of $GL(n, F)$ satisfying conditions (1), (2), and (3) of Theorem 1.2. By replacing F^+ by F_1^+ as above one can arrange that Π satisfies a set of hypotheses strictly analogous to (2.4.1)-(2.4.4), and we do so henceforward, even though only (2.4.2) is really necessary, in light of Fact 2.4 (on the other hand, (2.4.3) and (2.4.4) do appear to be very helpful if one removes condition (3)).

Theorem 2.5. *Under these hypotheses,* Π *has a* G_0*-avatar, i.e. descends to an automorphic representation* π_0 *of* G_0*.*

This is due to Clozel and Labesse [CL] for the group G_0 introduced above. One way of comparing π_0 to Π is by means of their L-functions. An automorphic representation π of a unitary group G has so-called standard L-function $L(s, \pi)$, associated to the standard $2n$-dimensional representation of the L-group of G, introduced above. In Langlands' formalism it is then tautological that $L(s, \pi) = L(s, \Pi)$, at least at unramified places (and under the analogue of (2.4.4) this can be arranged at all places). Both the Langlands-Shahidi method and the doubling method of

Piatetski-Shapiro and Rallis (studied in more detail by Shimura, and more generally by Lapid and Rallis) can be used to prove that $L(s, \pi)$ admits an analytic continuation and a functional equation of the expected type, without reference to its relation to the standard L-function of an automorphic representation of $GL(n)$.

Our simplifying hypotheses determine π_0 up to isomorphism as a representation of $G_0(\mathbf{A})$. This is because compact groups over $\mathbb{R}$ have no non-trivial L-packets, and (2.4.3) and (2.4.4) have removed the potential for L-indistinguishability at finite primes. It's almost certain that Labesse's methods show that π_0 is also unique as an automorphic representation, that is, that the abstract representation occurs with multiplicity one in the automorphic spectrum of G_0. A multiplicity one theorem would improve certain results but is unnecessary for our main applications.

Up to now we have encountered automorphic forms only in the plural, as elements of automorphic representations. In the following section we work with modules of actual automorphic forms on over Hecke algebras of mixed characteristic. This theory is available for $GL(n)$ as well as for the various unitary groups we have introduced, but it works best over the totally definite unitary group G_0.