L-FUNCTIONS AND PERIODS OF ADJOINT MOTIVES

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Abstract

The article studies the compatibility of the refined Gross-Prasad (or Ichino-Ikeda) conjecture for unitary groups, due to Neal Harris, with Deligne's conjecture on critical values of L-functions. When the automorphic representations are of motivic type, it is shown that the L-values that arise in the formula are critical in Deligne's sense, and their Deligne periods can be written explicitly as products of Petersson norms of arithmetically normalized coherent cohomology classes. In some cases this can be used to verify Deligne's conjecture for critical values of adjoint type (Asai) L-functions.

Introduction

The refined Gross-Prasad conjecture, or Ichino-Ikeda conjecture, is an explicit and exact expression for certain products of special values of automorphic L-functions in terms of automorphic periods. In the situation of the present article, π and π' are automorphic representations of unitary groups $U(W)$ and $U(W')$, respectively, where W is a hermitian space of dimension n over a CM field $\mathcal{K}$ and $W' \subset W$ is a non-degenerate hermitian subspace of codimension 1. We assume π and π' admit base change to automorphic representations $BC(\pi)$ and $BC(\pi')$ of $GL(n,\mathcal{K})$ and $GL(n-1,\mathcal{K})$, respectively. The original Ichino-Ikeda conjecture is stated for inclusions of special orthogonal groups; the version for unitary groups, due to Neal Harris [NH], gives a formula for the quotient

$$\frac{L(\frac{1}{2}, BC(\pi)\times BC(\pi'))}{L(1,\pi,Ad)L(1,\pi',Ad)} \tag{0.1}$$

in terms of global periods, local integrals, and some elementary terms (see §2.1 for details). Here the numerator is a Rankin-Selberg tensor product L-function for $GL(n)\times GL(n-1)$, and the L-functions attached to the adjoint representations of the L-groups of unitary groups can be identified with the *Asai L-functions*

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$L(s, BC(\pi), As^{\pm})$, $L(1, BC(\pi'), As^{\mp})$ of the conjugate self-dual representations $BC(\pi)$, $BC(\pi')$ as follows (cf. [NH], Remark 1.4 and [GGP1], Prop. 7.4):

$$(0.2)\quad L(s,\pi,Ad) = L(s, BC(\pi), As^{(-1)^n}); L(s,\pi',Ad) = L(s, BC(\pi), As^{(-1)^{n-1}}).$$

In its formulation for special orthogonal groups, the Ichino-Ikeda conjecture is inspired by formulas for the central values of L-functions of $GL(2)$, due to Waldspurger [W] and others, and represent the culmination of several decades of work in connection with the Birch-Swinnerton-Dyer conjecture, including various attempts to generalize the Gross-Zagier formula. It is natural to focus on the central value in the numerator in the Ichino-Ikeda conjecture, and to view the L-values in the denominator as error terms. The present paper is instead primarily concerned with the denominator.

In what follows, When π is attached to a motive M of rank n over a number field, the value $L(1,\pi,Ad) = L(s, BC(\pi), As^{(-1)^n})$ is critical in Deligne's sense [D2], and is expected to be closely connected to the classification of p-adic deformations of the mod p Galois representations attached to M. For $n = 2$ this principle is well-understood and there are very precise results due to Hida, Diamond-Flach-Guo, and Dimitrov [Hi,DFG, Di]. This is the first of a series of papers whose goal is to indicate a way to prove similar results for $n > 2$. The approach suggested here is heuristic and speculative, inasmuch as the Ichino-Ikeda conjecture has only been proved in special cases, and a number of the steps rely on non-vanishing results for special values of L-functions, and ergodicity results for automorphic periods, that have yet to be studied seriously. Nevertheless, the Ichino-Ikeda conjecture, in conjunction with Deligne's conjecture on critical values of L-functions, indicate the existence of structural links between congruences among automorphic forms and the divisibility of the value $L(1,\pi,Ad)$, and these links seem worth exploring.

The function $L(s,\pi,Ad)$ is interpreted as the L-function of the Asai motive $As^{(-1)^n}(M)$ attached to M. The present paper introduces the family of cohomological realizations that should be attached to the conjectural object $As^{(-1)^n}(M)$ and explains how to relate them to automorphic forms. The main results interpret the Deligne period of $As^{(-1)^n}(M)$ in terms of coherent cohomological automorphic forms, and show how the Ichino-Ikeda conjecture can be used to prove a version of Deligne's conjecture for the critical value $L(1,\pi,Ad) = L(1, As^{(-1)^n}(M))$, assuming certain non-vanishing conjectures for twists of standard L-functions of unitary groups by finite order characters. Heuristic evidence for the non-vanishing conjectures is provided by the existence of p-adic L-functions: when π varies in a Hida family of ordinary automorphic representations with global root number $+1$, the p-adic L-function of the family is generically non-zero at the central critical point. Although the foundations are largely available for general CM fields, the main applications of the present article are limited to the case where $\mathcal{K}$ is a quadratic imaginary field and n is even; this provides for some simplification of the main formulas, while presenting the general picture. The author and L. Guerberoff hope to treat the general case in a subsequent article. Applications to congruence modules, in Hida's sense, will be treated in forthcoming joint work with C. Skinner.

The present paper can also be read as a confirmation of the compatibility between the Ichino-Ikeda conjecture and Deligne's conjecture for pairs of automorphic motives satisfying the inequalities (2.3.4), which correspond to period integrals on totally definite hermitian spaces W and W'. It appears that compatibility in general cannot be established by purely automorphic methods.

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Notation and conventions

Throughout the article, we let $\mathcal{K}$ be a CM quadratic extension of a totally real field F, $c \in Gal(\mathcal{K}/F)$ complex conjugation. Let Σ_F denote the set of real places of F, and let Σ denote a CM type of $\mathcal{K}$, a set of extensions of Σ_F to $\mathcal{K}$, so that $\Sigma \coprod c \cdot \Sigma$ is the set of archimedean embeddings of $\mathcal{K}$. If $\sigma \in \Sigma_F$, we let $\sigma_{\mathcal{K}}$ denote its extension in Σ. We let $\eta_{\mathcal{K}/F} : Gal(\bar{F}/F) \to \{\pm 1\}$ denote the Galois character attached to the quadratic extension $\mathcal{K}/F$.

Unless otherwise indicated, a discrete series representation an algebraic group G over $\mathbb{R}$ will always be assumed to be *algebraic*, in the sense that its infinitesimal character is the same as that of a finite-dimensional representation. This is of course a condition on the central character.

Let E be a number field, and let $\alpha, \beta \in E \otimes_{\mathbb{Q}} \mathbb{C}$. Following Deligne, we write

$$\alpha \sim_E \beta$$

if either $\beta \notin (E \otimes_{\mathbb{Q}} \mathbb{C})^\times$ or $\beta^{-1}\alpha \in E = E \otimes_{\mathbb{Q}} \mathbb{Q}$. In the situations that arise, if $\beta \notin (E \otimes_{\mathbb{Q}} \mathbb{C})^\times$ then we will assume $\beta = 0$.

Suppose $\mathcal{K}$ is a number field with a given embedding in $\mathbb{C}$. Then we write

$$\alpha \sim_{E,\mathcal{K}} \beta$$

if either $\beta \notin (E \otimes_{\mathbb{Q}} \mathbb{C})^\times$ or $\beta^{-1}\alpha \in E \otimes_{\mathbb{Q}} \mathcal{K} \subset E \otimes_{\mathbb{Q}} \mathbb{C}$.

1. Deligne periods of polarized regular motives

1.1. Polarized regular motives over CM fields.

Let Π be a cuspidal cohomological automorphic representation Π of $GL(n, \mathcal{K})$ that satisfies the polarization condition

$$\Pi^\vee \xrightarrow{\sim} \Pi^c. \tag{1.1.1}$$

Let $E = E(\Pi)$ denote a field of definition of Π_f.[1] This is a CM field [BHR] and in what follows we will consider c-linear automorphisms of E-vector spaces. By the

[1]To be completely accurate, although it is known that Π_f has a model over its field of rationality, it is not known that the motive we construct below has coefficients in the same field; for example, it has not been checked that the associated Galois representations can be realized over the λ-adic completions of $E(\Pi)$, because of the possibility of a non-trivial Brauer obstruction. So we will take $E(\Pi)$ to be a finite extension of the field of rationality of Π_f over which all the subsequent constructions are valid.

results of a number of people, collected in [CH], Π gives rise to a compatible system of λ-adic representations $\rho_{\Pi,\lambda} : Gal(\overline{\mathbb{Q}}/\mathcal{K}) \to GL(n, E_\lambda)$, where λ runs over places of E, with a non-degenerate pairing

$$\rho_{\Pi,\lambda} \otimes \rho^c_{\Pi,\lambda} \to E_\lambda(1-n). \tag{1.1.2}$$

To keep these Galois representations company, we postulate the existence of a pure motive $M = M_\Pi$ over $\mathcal{K}$ of rank n and weight $w = n-1$, with coefficients in E whose λ-adic realization is $\rho_{\Pi,\lambda}$ and whose other realizations can be constructed using automorphic forms. For the present purposes, all we know of M is its family of realizations, together with compatibility isomorphisms. The relation between M and Π is encapsulated in the formula

$$L(s, M) = L(s + \frac{1-n}{2}, \Pi) = L(s, \Pi \otimes (| \bullet | \circ \det)^{\frac{1-n}{2}}) \tag{1.1.3}$$

Consider the motives $RM = R_{\mathcal{K}/F}M$, $\mathcal{R}M = R_{\mathcal{K}/\mathbb{Q}}M$ over F and $\mathbb{Q}$, respectively. The base change $RM_\mathcal{K}$ of RM breaks up as $M \oplus M^c$, where the distinction between M and M^c depends on the choice of CM type Σ. Indeed, for each real embedding σ of F we can consider $RM_{B,\sigma}$ which can be interpreted as the topological cohomology $H^*(RM \times_{\sigma,\mathbb{C}} (\mathbb{C}), E)$; then

$$RM_{\mathcal{K},B,\sigma} = H^*(RM \times_{\sigma_\mathcal{K},\mathbb{C}} (\mathbb{C}), E) \oplus H^*(RM \times_{c\sigma_\mathcal{K},\mathbb{C}} (\mathbb{C}), E).$$

The polarization is a non-degenerate pairing

$$< \bullet, \bullet >_B : M \otimes M^c \to E(1-n) \tag{1.1.4}$$

whereas F_∞ is just an isomorphism of Betti realizations

$$F_\infty : M_B \xrightarrow{\sim} M^c_B \tag{1.1.5}$$

that is linear with respect to the E-module structure. We choose an E-basis $(e_1, \ldots, e_n)$ of M_B and let $e^c_i = F_\infty(e_i)$ for $i = 1, \ldots, n$.

I refer to my paper [H97] for generalities about Deligne's conjectures [D2] on special values of L-functions, as specialized to polarized regular motives. In that paper it is assumed $M \xrightarrow{\sim} M^c$, or equivalently that Π is a base change from F to $\mathcal{K}$, so that the superscripts ^{c} can be removed in (1.1.1) and (1.1.2). The arguments in general are simple modifications of this self-dual case; however, there are roughly twice as many invariants in the general case. I follow [HLSu], where these invariants are discussed in connection with automorphic forms on unitary groups.

The restriction of scalars $R_{\mathcal{K}/\mathbb{Q}}M_\Pi$ is naturally a motive of rank n over $\mathbb{Q}$ with coefficients in $E(\Pi) \otimes \mathcal{K}$. The de Rham realization of $R_{\mathcal{K}/\mathbb{Q}}M_\Pi$, denoted $M_{\mathcal{K}/\mathbb{Q},DR}(\Pi)$, is a free rank n module over $E(\Pi) \otimes \mathcal{K}$. The Hodge decomposition

$$M_{\mathcal{K}/\mathbb{Q},DR}(\Pi) \otimes \mathbb{C} \xrightarrow{\sim} \oplus_{p+q=n-1} M^{p,q}_{\mathcal{K}/\mathbb{Q}}(\Pi) \tag{1.1.6}$$

and the natural decomposition of $E(\Pi) \otimes \mathcal{K} \otimes \mathbb{C}$-modules

$$M_{\mathcal{K}/\mathbb{Q},DR}(\Pi) \otimes \mathbb{C} \xrightarrow{\sim} \oplus_{\sigma:E(\Pi)\otimes\mathcal{K}\to\mathbb{C}} M_{\mathcal{K}/\mathbb{Q},\sigma}(\Pi) \tag{1.1.7}$$

are compatible with the $E(\Pi)\otimes\mathcal{K}$-action in the sense that complex conjugation c defines anti-linear isomorphisms

$$c : M^{p,q}_{\mathcal{K}/\mathbb{Q},\sigma}(\Pi) \xrightarrow{\sim} M^{q,p}_{\mathcal{K}/\mathbb{Q},c\sigma}(\Pi) \tag{1.1.8}$$

such that

$$c(am) = c(a)c(m),\ \ a \in E(\Pi)\otimes\mathcal{K}, m \in M^{p,q}_{\mathcal{K}/\mathbb{Q},\sigma}(\Pi). \tag{1.1.9}$$

Here

$$M^{p,q}_{\mathcal{K}/\mathbb{Q},\sigma}(\Pi) = M^{p,q}_{\mathcal{K}/\mathbb{Q}}(\Pi) \cap M_{\mathcal{K}/\mathbb{Q},\sigma}(\Pi).$$

1.1.10. Formal properties of polarized regular motives. One expects the following properties to hold:

(a) For all p, q, σ, $\dim M^{p,q}_{\mathcal{K}/\mathbb{Q},\sigma}(\Pi) \leq 1$.

(b) For all p, q, $\dim M^{p,q}_{\mathcal{K}/\mathbb{Q},\sigma}(\Pi)$ is independent of the restriction of σ to $E(\Pi)\otimes 1$.

(c) Let σ be as above and denote by $w \in \Sigma_{\mathcal{K}}$ its restriction to $1\otimes\mathcal{K}$, $w^+ \in \Sigma_F$ its restriction to F. Let $\mu(w)$ be the infinitesimal character of the finite-dimensional representation W_w defined in [HLSu, §2.3] and let

$$p(w) = \mu(w) + \frac{n-1}{2}(1, 1, \ldots, 1) := (p_1(w), p_2(w), \ldots, p_n(w))$$

so that for all i, [HLSu, (2.3.2)] implies that

$$p_i(w) + p_{n+1-i}(cw) = n - 1.$$

Then $\dim M^{p,q}_{\mathcal{K}/\mathbb{Q},\sigma}(\Pi) = 1$ if and only if $(p, q) = (p_i(w), p_{n+1-i}(cw))$ for some $i \in \mathbf{n} := \{1, \ldots, n\}$.

(d) The motive $R_{\mathcal{K}/\mathbb{Q}}M_\Pi$ has a non-degenerate polarization

$$< \bullet, \bullet >: R_{\mathcal{K}/\mathbb{Q}}M_\Pi \otimes R_{\mathcal{K}/\mathbb{Q}}M_\Pi \to \mathbb{Q}(1-n)$$

that is alternating if n is even and symmetric if n is odd. The involution † on the coefficients $E(\Pi)\otimes\mathcal{K}$ induced by this polarization:

$$< ax, y >=< x, a^\dagger y >,\ \ a \in E(\Pi)\otimes\mathcal{K},\ \ x, y \in R_{\mathcal{K}/\mathbb{Q}}M_\Pi$$

coincides with complex conjugation. In particular, the polarization induces a non-degenerate hermitian pairing

$$< \bullet, \bullet >_{i,w}: M^{p_i(w),p_{n+1-i}(cw)}_{\mathcal{K}/\mathbb{Q},\sigma}(\Pi) \otimes M^{p_i(cw),n-1-p_i(w)}_{\mathcal{K}/\mathbb{Q},\sigma}(\Pi) \to \mathbb{C}$$

for each pair (i, w).

Let $q_i(w) = n - 1 - p_i(w) = p_{n+1-i}(cw)$. For each pair $(i, w) \in \mathbf{n}\times\Sigma_{\mathcal{K}}$, we let $\omega_{i,w}(\Pi) \in M^{p_i(w),q_i(w)}_{\mathcal{K}/\mathbb{Q},\tau}(\Pi)$ be the non-zero image of some F-rational class in the appropriate stage of the Hodge filtration on $M_{\mathcal{K}/F,DR}(\Pi)$ (cf. [H97,§1.4]. Via the comparison isomorphism

$$RM_B \otimes \mathbb{C} \xrightarrow{\sim} RM_{DR} \otimes \mathbb{C}$$

there is an action of F_∞ on RM_{DR}, linear with respect to the coefficients E, that exchanges M_{DR} with M^c_{DR}. Define the de Rham polarization $< \bullet, \bullet >_{DR}$ by analogy with (1.1.4). It restricts to perfect pairings

$$M^{p_i(w), n-1-p_i(w)} \otimes M^{p_{n+1-i}(cw), n-1-p_{n+1-i}(cw)} \to E(1-n).$$

Let

$$Q_{i,w}(\Pi) =< \omega_{i,w}(\Pi), F_\infty(\omega_{i,w}(\Pi)) >_{DR} \in \mathbb{R}^\times. \tag{1.1.11}$$

Here F_∞ is the action of complex conjugation on the Betti realization of $M_{\mathcal{K}/\mathbb{Q},DR}(\Pi)$, cf. [H97, (1.0.4)]. Then we may assume

$$F_\infty(\omega_{i,w}(\Pi)) = Q_{i,w}(\Pi) \cdot \omega_{n+1-i,cw}(\Pi). \tag{P}$$

For the remainder of §1 we will assume $F = \mathbb{Q}$, since the main applications will be in this setting. We can thus choose an embedding $w : \mathcal{K} \hookrightarrow \mathbb{C}$ once and for all and drop the subscripts w in what follows, writing for example ω_i for $\omega_{i,w}$.

1.2. The determinant motive.

The determinant $\det(M)$ is a rank one motive over $\mathcal{K}$ of weight $nw = n(n-1)$ with coefficients in E. Since its λ-adic realization is the Galois character $\xi_{\Pi,\lambda} = \det \rho_{\Pi,\lambda}$ we can write $\det(M) = M(\xi_\Pi)$ where

$$\xi_\Pi = \chi_\Pi \cdot || \bullet ||^{-\frac{n(n-1)}{2}} \tag{1.2.1}$$

is the indicated shift of the central character χ_Π of Π, calculated using (1.1.3).

The polarization of M defines a polarization

$$M(\xi_\Pi) \otimes M(\xi_\Pi^c) \to E(n(1-n)) \tag{1.2.2}$$

which is obviously consistent with (1.2.1). Taking $\Omega_M = \wedge_{i=1}^n \omega_i$ as E-rational basis of $\det(M)_{DR}$, and defining Ω_M^c analogously, relation (P) yields

$$F_\infty(\Omega_M) = Q_{\det(M)}\Omega_M^c;\ Q_{\det(M)} = \prod_{i=1}^n Q_i. \tag{1.2.3}$$

On the other hand, letting e_M and e_M^c denote E-rational bases of $\det(M)_B$ and $\det(M^c)_B$ respectively, we can write

$$e_M = \delta(M)\Omega_M \tag{1.2.4}$$

where following Deligne we let $\delta(M)$ denote the determinant of the comparison isomorphism

$$I_\infty : M_B \otimes \mathbb{C} \xrightarrow{\sim} M_{DR} \otimes \mathbb{C}$$

calculated in E-rational bases; $\delta(M)$ is well-defined as an element of $(E \otimes \mathbb{C})^\times / E^\times$ (cf. [H97, (1.2.2)]).[2] The determinant of the dual map

$$(I_\infty^\vee)^{-1} : M_B^\vee \otimes \mathbb{C} \xrightarrow{\sim} M_{DR}^\vee \otimes \mathbb{C}$$

[2]Deligne's δ is the determinant of the period matrix of a motive over $\mathbb{Q}$; here the motive is over $\mathcal{K}$.

equals $\delta(M)^{-1}$, up to a multiple in $E^\times$; but by the polarization we find that this is the determinant of

$$I^c(1-n)_\infty : M^c(n-1)_B \otimes \mathbb{C} \xrightarrow{\sim} M^c(n-1)_{DR} \otimes \mathbb{C}.$$

This in turn is $(2\pi i)^{n(n-1)}$ times the determinant of

$$I^c_\infty : M^c_B \otimes \mathbb{C} \xrightarrow{\sim} M^c_{DR} \otimes \mathbb{C};$$

in other words

$$\delta(M)^{-1} = (2\pi i)^{n(n-1)}\delta(M^c). \tag{1.2.5}$$

or with respect to the comparison isomorphism

$$e^c_M = (2\pi i)^{n(1-n)}\delta(M)^{-1}\Omega^c_M \tag{1.2.6}$$

Now by (1.2.3) and (1.2.4) we have

$$\Omega^c_M = Q^{-1}_{\det(M)} F_\infty(\Omega_M) = Q^{-1}_{\det(M)}\delta(M)^{-1}e^c_M$$

which combined with (1.2.6) yields

Lemma 1.2.7. *Under the hypotheses of (1.1), we have the relation*

$$\prod_{i=1}^n Q_i = Q_{\det(M)} = (2\pi i)^{n(1-n)}\delta(M)^{-2}$$

as elements of $(E\otimes\mathbb{C})^\times/E^\times$. In other words, there is an element $d(M) \in E^\times$ such that

$$\delta(M)^{-1} = d(M)^{\frac{1}{2}} \cdot (2\pi i)^{\frac{n(n-1)}{2}} \cdot Q^{\frac{1}{2}}_{\det(M)}$$

where the choice of square root $d(M)^{\frac{1}{2}}$ depends on the choice of square root of $Q_{\det(M)}$ in $(E\otimes\mathbb{C})^\times/E^\times$.

This is to be compared to [H97, Lemma 1.4.12]. There the independent definition of $\delta(M)$ suffices to determine a choice of square root of $d(M) = [d_{DR}(M)/d_B(M)]$. Presumably $d(M)$ is again a ratio of discriminants of forms attached to the polarization, and its square root can therefore be given an independent definition in an appropriate quadratic extension of E.

1.3. Asai motives.

We postulate that the adjoint motive $Ad(M) = M \otimes M^\vee$ descends to a motive over F, denoted $As(M)$ (for Asai). This is true for the ℓ-adic realizations, as explained in [GGP1], and we introduce the corresponding ad hoc descents of the de Rham and Betti realizations in order to define the Deligne periods.

More precisely, in the article [GGP1] of Gan-Gross-Prasad, there are two descents, denoted $As(M)^+$ and $As(M)^-$, that differ from one another by twist by the quadratic character $\eta_{\mathcal{K}/F}$, and are distinguished by the signature of F_∞, which is $\frac{n(n\pm1)}{2}$ on $As(M)^\pm$. Ours is the one denoted $As(M)^{(-1)^n}$, as one sees by the definition of the F_∞ action below. Because the signs interfere with the notation for Deligne's periods we write $As(M)$ instead of $As(M)^{(-1)^n}$ and $(As(M)_B)^{\pm}$ with parentheses to designate the ±1-eigenspaces of F_∞.

We denote by $\mathbb{Q}(\eta_{\mathcal{K}/F})$ the Artin motive of rank 1 over F attached to the character $\eta_{\mathcal{K}/F}$. Let e_η denote a basis vector for $\mathbb{Q}(\eta_{\mathcal{K}/F})_B$. The archimedean Frobenius F_∞ acts as -1 on $\mathbb{Q}(\eta_{\mathcal{K}/F})_B$. Let t be a rational basis of $\mathbb{Q}(1)_{DR} = \mathbb{Q}$ (see [H97, 1.1]), $t_B = 2\pi i t$ a rational basis of $\mathbb{Q}(1)_B = (2\pi i)\mathbb{Q}$; $F_\infty(t_B) = -t_B$.

We identify $Ad(M)^c \xrightarrow{\sim} Ad(M)$ by composing

$$Ad(M)^c = M^c \otimes M^{\vee,c} \xrightarrow{\sim} M^\vee(1-n)\otimes(M^c(n-1))^c = M^\vee \otimes M \xrightarrow{\sim} M \otimes M^\vee = Ad(M)$$

where the last isomorphism is just exchanging the factors and the first is defined by the polarization. As a model for $As(M)_B$ over F we take

$$As(M)_B = M_B \otimes M_B^c(1-n) \otimes \mathbb{Q}(\eta_{\mathcal{K}/F})^{\otimes n}.$$

with the action

$$\begin{aligned} F_\infty(e_i \otimes e_j^c \otimes t_B^{1-n} \otimes e_\eta^{\otimes n}) &= e_j \otimes e_i^c \otimes (-1)^{1-n} t_B^{1-n} \otimes (-1)^n e_\eta^{\otimes n} \\ &= -e_j \otimes e_i^c \otimes \otimes t_B^{1-n} e_\eta^{\otimes n}. \end{aligned}$$

Here we have exchanged the first two factors after applying complex conjugation. Thus the vectors

$$\{e_{ij}^+ = [e_i \otimes e_j^c - e_j \otimes e_i^c] \otimes t_B^{1-n} \otimes e_\eta^{\otimes n}, i < j\},$$

resp.

$$\{e_{ij}^- = [e_i \otimes e_j^c + e_j \otimes e_i^c] \otimes t_B^{1-n} \otimes e_\eta^{\otimes n}, i \leq j\},$$

form a basis for $(As(M)_B)^+$, resp. $(As(M)_B)^-$, in Deligne's notation (where we have added parentheses as explained above). In particular,

$$\dim(As(M)_B)^+ = \frac{n(n-1)}{2};\ \dim(As(M)_B)^- = \frac{n(n+1)}{2}. \tag{1.3.1}$$

But in the applications we will be interested in the special value $L(1, As(M)) = L(0, As(M)(1))$. The action of F_∞ on the Tate twist

$$As(M)(1)_B = M_B \otimes M_B^c(2-n) \otimes \mathbb{Q}(\eta_{\mathcal{K}/F})^{\otimes n}$$

is as above, with the $(1-n)$'s replaced by n's. The motive $As(M)(1)$ is pure of weight -2, and the dimension calculation shows that F_∞ acts as the scalar $+1$ on the space of $(-1,-1)$ classes; thus $As(M)(1)$ is *critical* in Deligne's sense.[3] This implies in particular that the Hodge filtration of $As(M)(1)_{DR}$ has two distingushed steps $F^{\pm}As(M)(1)_{DR}$ (cf. [H97,§1.2]) uniquely determined by the equalities

$$\dim F^{\pm}As(M)(1)_{DR} = \dim(As(M)(1)_B)^{\pm} = \frac{n(n \pm 1)}{2}$$

[3]Dick Gross has pointed out that this can be seen purely in terms of representation theory. The local L-factor at infinity $L_\infty(s, As(M))$ has no pole at $s = 1$ because discrete series parameters are generic, and no pole at $s = 0$ because the corresponding representations are in the discrete series.

where the dimension calculation follows from (1.3.1), bearing in mind that F_∞ acts as -1 on $\mathbb{Q}(1)_B$. We can similarly define steps in the filtration of $As(M)_{DR}$:

(1.3.2) $$n^\pm := \dim F^\pm As(M)_{DR} = \dim(As(M)_B)^\pm = \frac{n(n \mp 1)}{2}.$$

Thus

$$F^+As(M)_{DR} \subsetneq F^-As(M)_{DR}; F^-As(M)(1)_{DR} \subsetneq F^+As(M)(1)_{DR}.$$

With respect to the isomorphism

$$M^\vee \xrightarrow{\sim} M^c(n-1)$$

we can take the differentials $\omega_j^c(n-1) = \omega_j^c \otimes t^{\otimes n-1}$ as a basis of $M^\vee_{DR}$. It follows from the dimension calculation above that the relevant step $F^+As(M)_{DR}$ in the Hodge filtration is spanned by the classes $\omega_{ij} = \omega_i \otimes \omega_j^c(n-1)$, of Hodge type

$$H_{ij}(As(M)) := (p_i + p_j^c + 1 - n, n - 1 - p_i - p_j^c)$$

satisfying the condition

(C(+)) $$p_i + p_j^c > n - 1$$

This is equivalent to $p_i - p_{n+1-j} > 0$ and since the p_i are strictly decreasing, $C(+)$ is true if and only if $i + j \leq n + 1$. Similarly $F^-As(M)_{DR}$ is spanned by ω_{ij} satisfying

(C(−)) $$p_i + p_j^c \geq n - 1 \ (n \text{ even})$$

which holds if and only if $i + j \leq n + 1$.

We define the motives $\wedge^2 M$ and $Sym^2 M$ over $\mathcal{K}$ in the obvious way. Because we will need a uniform notation we write $S^+(M) = Sym^2 M$, $S^-(M) = \wedge^2 M$.

Write

$$\omega_j = \sum a_{ij} e_i; \ \omega_j^c = \sum a_{ij}^c e_i^c.$$

Then we have the relation

$$a_{i,n+1-j}^c = Q_j^{-1} a_{ij}.$$

Now let $\{e_{ik}^{\pm,*}\}$ denote the dual basis to the basis $\{e_{ik}^\pm\}$ of $(As(M)_B)^\pm$ introduced above. It follows from the identity (P) that we have

$$\begin{aligned} e_{ik}^{\pm,*}(\omega_{j,n+1-\ell}) &\sim [a_{ij} a_{k,n+1-\ell}^c \pm a_{kj} a_{i,n+1-\ell}^c](2\pi i)^{1-n} \\ &\sim (2\pi i)^{1-n} Q_\ell^{-1} (a_{ij} a_{k,\ell} \pm a_{kj} a_{i,\ell}) \\ &\sim (2\pi i)^{1-n} Q_\ell^{-1} e_{ik}^{\pm,*}(\omega_j \otimes \omega_\ell) \end{aligned}$$

where the $\sim$'s mean that the calculations are up to factors in the coefficient field. Now if $H_{j,n+1-\ell}(As(M))$ satisfies $(C(+))$, then $j < \ell$. The arguments of [H97, §1.5] allow us to calculate the the matrix for the Deligne period $c^+(As(M)^\vee)$ of the *dual* of $As(M)$. However, the self-duality of $Ad(M)$ easily implies that $As(M)$ is self-dual, so the calculation that follows gives an expression for $c^+(As(M))$. The entries in the matrix are given by $e_{ik}^{+,*}(\omega_{j\ell})$ as (i,k) varies over pairs with $i \leq k$ and $j \leq \ell$ if n is odd, with strict inequalities if n is even.

With $n^{\pm}$ as in (1.3.2), the determinant of the period matrix calculating $c^{\pm}(As(M))$ is equal to a certain product $Q^{\pm}(As(M))$ of factors of the form Q_ℓ^{-1}, to be determined below, multiplied by the determinant Δ of the matrix

$$(e_i \otimes e_k - e_k \otimes e_i)^*(\omega_j \otimes \omega_\ell)$$

as (i,k) ranges over pairs with $i \leq k$ and (j,ℓ) ranges over pairs with $j < \ell$, the whole multiplied by $(2\pi i)^{(1-n)n^{\pm}}$. The determinant Δ is precisely the inverse of the determinant of the full period matrix of the motive $S^{\mp}(M)$ in the implicit bases, which Deligne denotes $\delta(S^{\mp}(M))$.

The factor $Q^{\pm}(As(M))$ is determined as follows. For $1 \leq \ell \leq n$, let $m^+(\ell)$ (resp. $m^-(\ell)$) denote the number of j such that $j \leq \ell$ (resp. $j < \ell$). Then $m^+(\ell) = \ell$, $m^-(\ell) = \ell - 1$. Let

$$Q^+(M) = \prod_\ell Q_\ell^{-m^+(\ell)} = \prod_\ell Q_\ell^{-\ell};\ Q^-(M) = \prod_\ell Q_\ell^{-m^-(\ell)} = \prod_\ell Q_\ell^{1-\ell}$$

It follows that

Formula 1.3.3.

$$Q^{\pm}(As(M)) = Q^{\mp}(M)$$

This proves the first statement of the following proposition; the second statement is proved analogously.

Proposition 1.3.4. *Let M be a polarized motive satisfying the conditions of 1.1.10, and with the property that $Ad(M)$ descends to $F = \mathbb{Q}$. Then*

$$c^+(As(M)) = (2\pi i)^{(1-n)n^+} Q^-(M)\delta(S^-(M))^{-1};$$

$$c^-(As(M)) = (2\pi i)^{(1-n)n^-} Q^+(M)\delta(S^+(M))^{-1}$$

Applying formula (5.1.8) of [D2], with n^- as in (1.3.2), we have

$$c^+(As(M)(1)) = c^-(As(M))(2\pi i)^{n^-}.$$

One calculates easily that

$$\delta(S^{\pm}(M)) = \delta(\det(M)^{n\pm 1}) = \delta(M)^{n\pm 1},$$

where the last equality follows from the considerations of §1.2.

Combining the formulas of this section with Lemma 1.2.7, we can therefore write the Deligne period for the motive of interest explicitly in terms of the Q_j's and δ.

Corollary 1.3.5. *Under the above hypotheses, we have the following expression for $c^+(As(M)(1))$:*

$$\begin{aligned}c^+(As(M)(1)) &= (2\pi i)^{n^-}(2\pi i)^{(1-n)n^-}Q^+(M)\delta(S^+(M))^{-1}\\ &= d(M)^{\frac{1}{2}}(2\pi i)^{\frac{n(n+1)}{2}}[Q_{\det(M)}]^{\frac{n-1}{2}}\cdot\prod_{\ell}Q_\ell^{1-\ell}\\ &= d(M)^{\frac{1}{2}}(2\pi i)^{\frac{n(n+1)}{2}}\prod_{\ell}Q_\ell^{\frac{n+1}{2}-\ell}\end{aligned}$$

We see that $\delta(S^-(M))^{-1}$ is an odd power of $\delta(M)^{-1}$; therefore we need to include the factor $d(M)^{\frac{1}{2}}$ introduced in Lemma 1.2.7 along with the half-integral power of $Q_{\det(M)}$. The half-integral powers of the Q_ℓ that occur in the expression for even n are not meaningful individually, and have only been included for their suggestive similarity with the standard expression for the half-sum of positive roots.

Remark 1.3.6. If one defines Q^c_ℓ by analogy with the definition of Q_ℓ above, one sees easily that

$$Q^c_\ell = Q^{-1}_{n+1-\ell}.$$

It is obvious that the expression in Corollary 1.3.5 is invariant when M and M^c are exchanged, as it should be.

1.4. Tensor products.

In subsequent sections we will explore the relations between the calculations of the previous section and the Ichino-Ikeda conjecture. Here we briefly explain how a similar calculation determines the Deligne period of the tensor product of two motives of the type considered in §1.

Suppose M and M' are two motives of dimension n and n', respectively, both of the type considered above. We let ω_a, ω^c_t, e_i, e^c_i, where $1 \leq a,t,i \leq n$, be the basis vectors defined for M above. For M' we use the notation η_b, η^c_u, f_j, f^c_j, with $1 \leq b,u,j \leq n'$. The Hodge types for M are $(p_i, n-1-p_i)$; $(p^c_i, n-1-p^c_i)$ as before; for M' we write $(r_j, n'-1-r_j)$; $(r^c_j, n'-1-r^c_j)$. The tensor product motive we consider is not $RM \otimes RM'$ but rather $R(M\otimes M') = R_{\mathcal{K}/F}(M\otimes M')$, whose Betti realization is $M_B \otimes M'_B \oplus M^c_B \otimes (M')^c_B$ and whose de Rham realization breaks up analogously. In particular, the differentials $\omega_a \otimes \eta_b$ and $\omega^c_t \otimes \eta^c_u$ form a basis for $R(M\otimes M')_{DR}$.

The motive $R(M\otimes M')$ is of dimension $2nn'$ over its coefficient field and of weight $w = n+n'-2$. We will only need to consider the case when n and n' are of opposite parity; for example, when $n' = n-1$, as in the original Gross-Prasad conjecture. Then w is odd and $R(M\otimes M')$ has no $(0,0)$ classes; it follows that the value $\frac{w+1}{2} = \frac{n+n'-1}{2}$ is a critical value of the L-function $L(s, R(M\otimes M'))$.

The basis for $R(M\otimes M')^{\pm}_B$ is then $e_i \otimes f_j \pm e^c_i \otimes f^c_j$, $1 \leq i \leq n, i \leq j \leq n'$. To determine the basis for $F^+R(M\otimes M')_{DR} = F^-R(M\otimes M')_{DR}$ we need to determine the sets $A(M,M')$ (resp. $T(M,M')$) of pairs a,b (resp. t,u) such that $p_a + r_b \geq \frac{w+1}{2}$ (resp $p^c_t + r^c_u \geq \frac{w+1}{2}$). Bearing in mind Hodge duality, the cardinality

$$|A(M,M')| + |T(M,M')| = nn' = \dim F^+R(M\otimes M')_{DR}.$$

The set $\{\omega_a \otimes \eta_b \mid (a,b) \in A(M,M')\} \cup \{\omega^c_t \otimes \eta^c_u \mid (t,u) \in T(M,M')\}$ forms a basis for $F^+R(M\otimes M')_{DR}$. A calculation using the relation (P), as in §1.3, shows that

Lemma 1.4.1.

$$c^+(R(M\otimes M')^\vee) = \pm c^-(R(M\otimes M')^\vee) = \prod_{(t,u)\in T(M,M')} Q_{n+1-t}(M)^{-1}Q_{n'+1-u}(M')^{-1}\cdot\delta(M\otimes M')^{-1},$$

where δ is the determinant of the full period matrix for $M\otimes M'$, viewed as a motive over $\mathcal{K}$.

More precisely letting (i,j) run over pairs of integers, with $1\leq i\leq n, i\leq j\leq n'$, the Deligne period $c^+(R(M\otimes M'))$ is the determinant of the matrix whose first $|A(M,M')|$ columns, indexed by pairs $(a,b)\in A(M,M')$ are the vectors $(a_{ia}b_{jb})$, and whose last $|T(M,M')|$ columns, indexed by pairs $(t,u)\in T(M,M')$, are the vectors $(a^c_{it}b^c_{ju})$. Here as above, we have written

$$\omega_a=\sum a_{ia}e_i,\ \eta_b=\sum b_{jb}f_j,\ \omega^c_t=\sum a^c_{it}e^c_i,\ \eta^c_u=\sum b^c_{ju}f^c_j.$$

By identity (P) we have

$$\omega^c_t = Q_{n+1-t}(M)^{-1}\sum a_{it}e^c_i,\ \eta^c_u=Q_{n'+1-u}(M')^{-1}\sum b_{ju}f^c_j.$$

The formula for $c^+(R(M\otimes M')^\vee)$ then follows as in §1.3.

Because the Hodge types satisfy $p^c_t > p^c_{t+1}, r^c_u > r^c_{u+1}$, we have

Lemma 1.4.2. *The set $T(M,M')$ is a* **tableau***: if $(t,u)\in T(M,M')$, then for any $t'<t, u'<u$, the pairs (t',u) and (t,u') are also in $T(M,M')$.*

We can represent $T(M,M')$ geometrically as a tableau in the rectangular grid of height n and width n', whose boxes are indexed by pairs $1\leq t\leq n, 1\leq u\leq n'$. The box at position (t,u) is filled in if $(t,u)\in T(M,M')$. Then the Lemma asserts that if a given box (t,u) is filled in, all boxes above it or to the left of it are also filled in.

In the notation of the introduction, the set $T(M,M')$ determines the pair of hermitian spaces $W'\subset W$ whose automorphic periods are expressed by the Ichino-Ikeda conjecture as the quotient of the central critical value of $L(s,R(M\otimes M'))$ by a product of critical values at $s=1$ of Asai L-functions. The automorphic periods can be normalized as in [H11], where they are called *Gross-Prasad periods.* The relation between Gross-Prasad periods and motivic periods is in general not transparent, and it is therefore not clear how to establish compatibility between the Ichino-Ikeda and Deligne conjectures in general. We will return to this topic in a subsequent article. The remainder of the present article is devoted to studying a special case where compatibility of the two conjectures can be studied.

2. The Ichino-Ikeda conjecture for unitary groups

In the present section W denotes an n-dimensional hermitian space over $\mathcal{K}$, relative to conjugation over F; until the end of §2.4 we allow F to be an arbitrary totally real field. If W_1 and W_2 are two such spaces, then for almost all finite primes v of F we have

$$U(W_1\otimes F_v)\xrightarrow{\sim} U(W_2\otimes F_v) \tag{2.0.1}$$

This allows us to consider automorphic representations of all unitary groups $U(W)$ simultaneously, and to organize them into *near equivalence classes*: the automorphic representations π_1 of $U(W_1)$ and π_2 of $U(W_2)$ are nearly equivalent if, for all but finitely many v for which (2.0.1) holds, the local components $\pi_{1,v}$ and $\pi_{2,v}$ are equivalent.

The Gross-Prasad and Ichino-Ikeda conjectures concern special values of L-functions and local ε-factors for near equivalence classes of local and automorphic representations respectively. A given near equivalence class gives rise to a family of motives (or at least realizations) in the cohomology of the corresponding Shimura varieties; the details are recalled in §2.4.

All the automorphic representations in a near equivalence class are supposed to be have a common base change, say Π, an automorphic representation of $GL(n)_{\mathcal{K}}$ that satisfies the polarization condition (1.1.1). This has been proved in a great many cases (see [L] and [Wh2], for example) and will be taken as an axiom in what follows. The near equivalence class will sometimes be denoted $\Phi(\Pi)$ – convention actually dictates it should be $\Pi(\Phi)$, or even $\Pi(\Phi(\Pi))$, where Φ is supposed to suggest the Langlands parameter of Π, but since the letter Π is otherwise engaged this looks problematic.

2.1. Statement of the conjecture.

Let $W' \subset W$ a codimension one subspace on which the restriction of the hermitian form is non-degenerate, so that $W = W' \oplus W_0$ with $W_0 = W^{\perp}$. The unitary groups of W, W', and W_0 are reductive algebraic groups over F; we write $G' = U(W')$, $G_0 = U(W_0)$ and $G = U(W)$.

Let π, π', and π_0 be tempered cuspidal automorphic representations of G, G', and G_0, respectively. Let

$$\chi_\pi : Z_G(\mathbf{A})/Z_G(F) \to \mathbb{C}^\times, \chi'_\pi : Z_{G'}(\mathbf{A})/Z_{G'}(F) \to \mathbb{C}^\times$$

denote their central characters – π_0 is itself a character and assume that

$$\chi_\pi \cdot \chi'_\pi \otimes \pi_0 \mid_{Z_G(\mathbf{A})} = 1. \tag{2.1.1}$$

Fix factorizations

$$\pi \xrightarrow{\sim} \otimes'_v \pi_v,\ \pi' \xrightarrow{\sim} \otimes'_v \pi'_v,\ \pi^\vee \xrightarrow{\sim} \otimes'_v \pi_v^\vee,\ \pi^{',\vee} \xrightarrow{\sim} \otimes'_v \pi_v^{',\vee} \tag{2.1.2}$$

and likewise for the contragredients $\pi^\vee$, $\pi^{',\vee}$. We assume the factorizations (2.1.2) are compatible with factorizations of pairings

$$< \bullet, \bullet >_\pi = \prod_v < \bullet, \bullet >_{\pi_v},\ < \bullet, \bullet >_{\pi'} = \prod_v < \bullet, \bullet >_{\pi'_v},$$

where in each case the left hand side is the L_2 pairing on cusp forms and the right hand side is the product of canonical pairings between a representation and its contragredient. We define

(2.1.3)

$$P(f, f') = \int_{G'(F)\backslash G'(\mathbf{A})} f(g')f'(g')dg',\ P(f^\vee, f^{',\vee}) = \int_{G'(F)\backslash G'(\mathbf{A})} f^\vee(g')f^{',\vee}(g')dg';$$

(2.1.4)

$$Q(f,f^\vee) = \int_{G(F)\backslash G(\mathbf{A})} f(g)f^\vee(g)dg;\ Q(f',f'^{,\vee}) = \int_{G'(F)\backslash G'(\mathbf{A})} f'(g')f'^{,\vee}(g')dg'.$$

For any place v of F, write $G_v = G(F_v)$, $G'_v = G'(F_v)$. Let dg and dg' denote Tamagawa measures on $G(\mathbf{A})$ and $G'(\mathbf{A})$, respectively. We choose factorizations $dg = \prod_v dg_v$, $dg' = \prod_v dg'_v$ over the places of v with the property that

(1) For every finite v, the measures dg_v and dg'_v take rational values on open subsets of G_v and G'_v, respectively.

(2) For all v outside a finite set S, including all archimedean places and all places at which either π or π' is ramified,

$$\int_{K_v} dg_v = \int_{K'_v} dg'_v = 1$$

where K_v and K'_v are hyperspecial maximal compact subgroups of G_v and G'_v respectively.

Assume $f \in \pi$, $f' \in \pi'$, $f^\vee \in \pi^\vee$, $f'^{,\vee} \in \pi'^{,\vee}$ are factorizable vectors,

$$f = \otimes_v f_v, f_v \in \pi_v, f' = \otimes_v f'_v \text{ etc.}$$

with respect to the isomorphisms (2.1.2). In what follows

(a) $|S(\pi,\pi')|$ is an integer measuring the size of the global L-packets of π and π';

(b) Δ_G is the value at $s=0$ of the L-function of the *Gross motive* of the group G; explicitly

$$\Delta_G = \prod_{i=1}^{n} L(i, \eta^i_{K/F});$$

(c) For each finite v,

$$\begin{aligned} Z_v &= Z_v(f_v, f_v^\vee, f'_v, f_v'^{,\vee}) \\ &= \int_{G'_v} c_{f_v,f_v^\vee}(g'_v)c_{f'_v,f_v'^{,\vee}}(g'_v)dg'_v \cdot \frac{L(1,\pi_v,Ad)L(1,\pi'_v,Ad)}{L(\frac{1}{2}, BC(\pi_v)\times BC(\pi'_v))}. \end{aligned}$$

(d) For each archimedean v,

$$Z_v = Z_v(f_v, f_v^\vee, f'_v, f_v'^{,\vee}) = \int_{G'_v} c_{f_v,f_v^\vee}(g'_v)c_{f'_v,f_v'^{,\vee}}(g'_v)dg'_v.$$

(e) In (c) and (d), the notation $c_{f_v,f_v^\vee}(g'_v)$ designates the local matrix coefficient

$$c_{f_v,f_v^\vee}(g_v) = (\pi(g_v)f_v, f_v^\vee)$$

with respect to the canonical local pairing of representations (likewise for $c_{f'_v,f_v'^{,\vee}}$).

The Ichino-Ikeda conjecture is the assertion that

$$\frac{P(f,f')P(f^\vee,f'^{,\vee})}{Q(f,f^\vee)Q(f',f'^{,\vee})} = 2^{-|S(\pi,\pi')|}\Delta_G \prod_v Z_v \cdot \frac{L(\frac{1}{2}, BC(\pi)\times BC(\pi'))}{L(1,\pi,Ad)L(1,\pi',Ad)} \tag{2.1.5}$$

Here the L-functions are defined in [NH] by Euler products over finite primes only. One of the main results of [II,NH] is that for $Z_v = 1$ for all v outside a finite set S, including all archimedean places; thus convergence of the product $\prod_v Z_v$ is not an issue. We can rewrite the right-hand side

$$2^{|S(\pi,\pi')|}\Delta_G Z_{loc} \cdot \frac{L(\frac{1}{2}, BC(\pi)\times BC(\pi'))}{L(1,\pi,Ad)L(1,\pi',Ad)}$$

with $Z_{loc} = \prod_{v\in S} Z_v$.

2.2. Local vanishing and the Gross-Prasad conjecture.

The map $P : \pi\otimes\pi' \to \mathbb{C}$ of (2.1.3) is invariant under $G'(\mathbf{A})$. Its non-triviality therefore implies that, for every v, there is a bilinear map

$$P_v : \pi_v \otimes \pi'_v \to \mathbb{C} \tag{2.2.1}$$

invariant under the diagonal action of G'_v. (The integral Z_v defines a multilinear form on $(\pi_v\otimes\pi'_v)\otimes(\pi_v^\vee\otimes\pi_v'^{,\vee})$.)

The existence of G'_v-invariant maps like (2.2.1) is the subject of the Gross-Prasad conjecture [GGP1]. For the purposes of the present exposition, it will suffice to assume $\pi_v\otimes\pi'_v$ to be tempered. Assume that L-packets can be attached consistently to tempered Langlands parameters for the group $G_v\times G'_v$ and all its inner twists (cf. [M]). Let $L(\pi_v,\pi'_v)$ denote the space of G'_v-invariant maps (2.2.1).

Local Gross-Prasad Conjecture 2.2.2. *Let WD_{F_v} denote the Weil-Deligne group of F_v, and let*

$$\Phi_v\times\Phi'_v : WD_{F_v} \to {}^L(G_v\times G'_v)$$

denote a tempered Langlands parameter for the group $G_v\times G'_v$ and all its inner twists. Then

$$\sum_{W_v = W'_v\oplus W_{0,v}} \;\sum_{\pi_v\otimes\pi'_v\in\Pi(\Phi_v\times\Phi'_v;U(W_v)\times U(W'_v))} \dim\ L(\pi_v,\pi'_v) = 1.$$

Here the outer sum runs over isometry classes of pairs of hermitian spaces over F_v, as in §2.1, and the inner sum runs over the L-packet of the given inner form of $G_v\times G'_v$ attached to $\Phi_v\times\Phi'_v$.

The full Gross-Prasad conjecture treats more general inclusions of groups and gives a formula in terms of the Langlands parameter determining the unique pair $\pi_v\otimes\pi'_v$ in the L-packet for which $L(\pi_v,\pi'_v)\neq 0$. This has been proved for special orthogonal groups by Waldspurger in the tempered case and by Moeglin and Waldspurger in general; see [MW]. Conjecture 2.2.2 for unitary groups is the subject of work in progress by R. Beuzart.

Now let (π,π') be a pair of tempered cuspidal automorphic representations of G, G', as in §2.1. For each place v, Conjecture 2.2.2 asserts the existence of unique (strong) inner forms $G_{1,v}$ and $G'_{1,v}$ of G_v and G'_v, respectively, and unique representations $\pi_{1,v}$ and $\pi'_{1,v}$ of $G_{1,v}$ and $G'_{1,v}$ in the L-packets given by the local Langlands parameters of π_v, π'_v, such that $L(\pi_{1,v},\pi'_{1,v})\neq 0$. The following is a restatement of Conjecture 26.1 of [GGP1] in the present situation.

Global Gross-Prasad Conjecture 2.2.3. *With π, π' as above, the following are equivalent*

(1) *There are unitary groups $G_1 \supset G'_1$ over F with local forms the given $G_{1,v}$ and $G'_{1,v}$, automorphic representations π_1, π'_1 with the given local components, and forms $f_1 \in \pi_1$, $f'_1 \in \pi'_1$, such that the period integral $P(f_1, f'_1) \neq 0$;*
(2) *The central value $L(\frac{1}{2}, BC(\pi_1) \otimes BC(\pi'_1)) = L(\frac{1}{2}, BC(\pi) \otimes BC(\pi')) \neq 0$.*

The Ichino-Ikeda conjecture of the previous section is a refinement of Conjecture 2.2.3. As a part of their refinement of the global Gross-Prasad conjecture for special orthogonal groups, Ichino and Ikeda have proposed a refinement of the local conjecture as well. I state it here in the unitary case. (It seems not to have been stated in [NH], though it is certainly compatible with the global conjecture stated there.)

Ichino-Ikeda conjecture 2.2.4, cf. [II, Conjecture 1.3]. *Under the hypotheses of Conjecture 2.2.2 – in particular, assuming π_v and π'_v belong to tempered L-packets – we have $L(\pi_v, \pi'_v) \neq 0$ if and only if the local integral Z_v defines a non-zero multilinear form on $(\pi_v \otimes \pi'_v) \otimes (\pi_v^\vee \otimes \pi_v'^{,\vee})$. In other words, the local zeta integral defines a basis vector in the one-dimensional vector space $L(\pi_v, \pi'_v) \otimes L(\pi_v^\vee, \pi_v'^{,\vee}))$.*

If one admits these conjectures, the non-vanishing of the numerator of the quotient of L-functions on the right-hand side of (2.1.5), together with the local non-vanishing conjecture 2.2.3, picks out a unique global pair of hermitian spaces $W \supset W'$ and a unique pair of automorphic representations π, π' of the chosen inner forms $U(W)$ and $U(W')$, for which the left-hand side and the product Z_v do not vanish. The arithmetic meaning of the local conditions at finite primes is not yet understood, but the local conditions at archimedean primes can be translated into simple conditions on the relative positions of the Hodge structures attached to the motives $M(\pi)$ and $M(\pi')$. The next two sections explain these conditions when W and W' are totally definite, and interprets the expressions on the left-hand side of (2.1.5).

2.3. Hodge structures in the definite case.

When v is a real place of F and π_v and π'_v are discrete series representations of G_v and G'_v, the dimension of $L(\pi_v, \pi'_v)$ is determined in §2 of [GGP2] in terms of the local Langlands parameters. The relation with Hodge types is reduced there to a calculation of signs, which in general is rather elaborate.

The definite case is simpler. Let H denote the compact Lie group $U(n)$, the symmetry group of the hermitian form $\sum_{i=1}^n z_i \bar{z}_i$. Let $H' = U(n-1) \times U(1)$, diagonally embedded in H, and fix an irreducible representation τ of H, with highest weight $a_1 \geq a_2 \geq \cdots \geq a_n$, $a_i \in \mathbb{Z}$, in the standard normalization . The classic branching formula [FH] determines the highest weights of the representations τ' that occur in the restriction of τ to H'.

Branching formula 2.3.1. *Let τ' be the irreducible representation of H' with highest weight $(b_1, \ldots, b_{n-1}; b_n) \in \mathbb{Z}^n$, where $b_1 \geq \cdots \geq b_{n-1}$ is a highest weight for $U(n-1)$ and b_n is the weight of a character of $U(1)$: Then $L(\tau, \tau') \neq 0$ if and only if*

(1) $\sum_{i=1}^n a_i = -\sum_{i=1}^n b_i$;
(2) $a_1 \geq -b_{n-1} \geq a_2 \geq -b_{n-2} \cdots \geq a_{n-1} \geq -b_1 \geq a_n$.

Assume W is a totally definite hermitian space over $\mathcal{K}$, and let π and π' be automorphic representations of G and G', whose base changes to $GL(n,\mathcal{K})$ and $GL(n-1,\mathcal{K})$ are denoted Π, Π'. Choose a pair (w, cw) of conjugate complex embeddings of $\mathcal{K}$ over the real embedding w^+ of F, with $w \in \Sigma$, and extend w to a map $\sigma : E(\Pi)\otimes\mathcal{K} \to \mathbb{C}$ as in §1.1. Suppose $\pi_{w^+} = \tau$, $\pi'_{w^+} = \tau'$, with parameters as in (2.3.1). The formula (1.1.10)(c) determines the Hodge numbers of $R_{\mathcal{K}/\mathbb{Q}}M_\Pi$. Bearing in mind that Π is an automorphic representation whose local component Π_w has cohomology with coefficients in the *dual* representation $\tau^\vee$ of $GL(n,\mathbb{C})$, we have

(2.3.2)
$$\dim M^{p,q}_{\mathcal{K}/\mathbb{Q},\sigma}(\Pi) = 1 \Leftrightarrow (p,q) = (p_i(w), q_i(w)) = (n-i-a_{n+1-i}, i-1+a_{n+1-i}) \text{ for some } i\ .$$

Similarly,

(2.3.3)
$$\dim M^{p,q}_{\mathcal{K}/\mathbb{Q},\sigma}(\Pi') = 1 \Leftrightarrow (p,q) = (p'_i(w), q'_i(w)) = (n-1-i-b_{n-i}, i-1+b_{n-i}) \text{ for some } i\ .$$

Comparing this to 2.3.1(2), we find that

$$(2.3.4)\quad p_1(w) > p'_1(cw) \geq p_2(w) > p'_2(cw) \geq \cdots > p_{n-1}(w) > p'_{n-1}(cw) \geq p_n(w)$$

2.4. Realizations of motives in unitary group Shimura varieties.

The hermitian spaces W and W' are assumed definite at infinity, as in the previous section. Let Π be a cuspidal cohomological automorphic representation of $GL(n)_{\mathcal{K}}$ satisfying (1.1.1). We consider the near equivalence class $\Phi(\Pi)$ of automorphic representations of varying $U(W)$. The hermitian pairing $< \bullet, \bullet >_W$ on W defines an involution $\tilde{c}$ on the algebra $End_F(W)$ via

$$< a(v), v' >_W = < v, a^{\tilde{c}}(v') >_W$$

For each such W, there is a Shimura variety $Sh(W)$ attached to the rational similitude group $GU(W)$, defined as the functor on the category of $\mathbb{Q}$-algebras R by

$$GU(W)(R) = \{g \in GL(V\otimes_{\mathbb{Q}} R) \mid g\cdot\tilde{c}(g) = \nu(g) \text{ for some } \nu(g) \in R^\times\}.$$

For each automorphic representation $\pi \in \Phi(\Pi)$ of $U(W)$, we choose an extension π^+ to an automorphic representation of $GU(W)$; we can arrange that the central character χ_{π^+} of π^+ is independent of $\pi \in \Phi(\Pi)$. We summarize the discussions in §2 of [H97] (for $F = \mathbb{Q}$) and §3.2 of [HLSu], and provide a few additional details.

For each W, we fix an irreducible admissible representation $\pi_f = \pi_{f,W}$ of $U(W)(\mathbf{A}^f)$ such that $\pi_\infty \otimes \pi_f \in \Phi(\Pi)$ for some discrete series representation π_∞ of $U(W_{\mathbb{R}}) := U(W\otimes_{\mathbb{Q}}\mathbb{R})$. For each place w of $\mathcal{K}$, let (r_w, s_w) denote the signature of the hermitian space W_w, and let $d_W = \sum_{v:F\hookrightarrow\mathbb{R}} r_w\cdot s_w$, where w is one of the two extensions of v to $\mathcal{K}$ and $r_w\cdot s_w$ does not depend on the choice. Define the Shimura variety $Sh(W)$ and the local system $\tilde{W}^+(\Pi)$ over $Sh(W)$ as in [HLSu, §3.2]; here $\tilde{W}^+(\Pi)$ is attached to a finite dimensional algebraic representation $W^+(\Pi)$ of $GU(W)$. Then the motivic realization of Π on $Sh(W)$ is the motive

$$(2.4.1)\quad \begin{aligned} M(\pi_f^+) &= Hom_{GU(\mathbf{A}^f)}(\pi_f^+, H^{d_W}(Sh(W), \tilde{W}^+(\Pi))) \\ &= Hom_{GU(\mathbf{A}^f)}(\pi_f^+, H^{d_W}(Sh(W)^*, j_{!*}\tilde{W}^+(\Pi))) \end{aligned}$$

where $j : Sh(W) \hookrightarrow Sh(W)^*$ is the embedding of $Sh(W)$ in its Baily-Borel compactification.

Let M_Π be the rank n motive over $\mathcal{K}$ introduced in §1.1 and $M_{\mathcal{K}/\mathbb{Q}}(\Pi)$ for its restriction of scalars to $\mathbb{Q}$. As in [HLSu, (3.2.4)], we have

$$(2.4.1) \qquad M(\pi_f^+) \xrightarrow{\sim} \otimes_{w\in\Sigma} \wedge^{s_w}(St)M_{F/\mathbb{Q}}(\Pi) \otimes (M(\chi_{\pi^+,W})(t_W))$$

where $t_W = \frac{1}{2}\sum_{w\in\Sigma} s_w(s_w - 1)$.

All the motives $M(\pi_f^+)$ are assumed to have coefficients in a common field $E(\pi_f)$. Let $E(W)$ be the reflex field of $Sh(W)$; it is contained in the Galois closure of $\mathcal{K}$ over $\mathbb{Q}$, and of course it depends on the signatures of W at places of Σ. The de Rham realization $M_{\mathcal{K}/\mathbb{Q},DR}(\pi_f^+)$ is free over $E(W)\otimes E(\pi_f^+)$ of rank $\prod_w \binom{n}{s_w}$; the lowest non-trivial stage of its Hodge filtration $F^{max}_{\mathcal{K}/\mathbb{Q},DR}(\pi_f^+)$ is a free rank one $E(W)\otimes E(\pi_f^+)$-submodule. Let $\Omega_W(\Pi)$ be any $E(W)\otimes E(\pi_f^+)$-basis of $F^{max}_{\mathcal{K}/\mathbb{Q},DR}(\pi_f^+)$. By analogy with (1.1.11), we define

$$(2.4.2) \qquad Q_W(\Pi) =< \omega_W(\Pi), F_\infty(\omega_W(\Pi)) >_{DR} \in \ (E(W)\otimes E(\pi_f^+)\otimes\mathbb{R})^\times.$$

We now simplify formulas by assuming $F = \mathbb{Q}$. The index W is in fact superfluous in the character $\chi_{\pi^+,W}$, given the presence of the twist t_W, but we will leave it in place. In [H97] there is a parameter denoted c in the highest weight of the representation $W^+(\Pi)$, corresponding to the restriction of the central character to the diagonal subgroup $\mathbb{G}_{m,\mathbb{Q}} \subset GU(W)$. Dually, the central character χ_{π^+} of π^+ has the property that

$$(2.4.3) \qquad \chi_{\pi^+}(t) = t^{-c}, \ t\in\mathbb{R}^\times \subset Z_{GU(W)}(\mathbb{R}).$$

Let $W(\Pi)$ denote the restriction of $W^+(\Pi)$ to $U(W)$, and identify $W(\Pi)$ with the representation $\tau^\vee$ of section 2.3, with parameters as in (2.3.1). Then $c \equiv \sum_i a_i$ (mod 2). To simplify the formulas, we assume $\sum_i a_i$ to be *even* and take $c = 0$. Then $M(\chi_{\pi^+,W})$ is a motive of weight 0.

2.5 Automorphic forms on definite unitary groups.

Let $G = U(W)$, $G' = U(W')$, as in §2.1, and assume W and W' are totally definite. We can define Shimura data $(G, x) \supset (G', x')$ where $x = x'$ is the point consisting of the trivial homomorphism from $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$ to the group G'. This satisfies all the axioms of [D1, (2.1.1)] with the exception of (2.1.1.3), which is in fact unnecessary except for considerations having to do with strong approximation. All points of the corresponding Shimura varieties are defined over (the reflex field) $\mathbb{Q}$, but automorphic forms are rational over the fields of definition of their coefficients.

We can determine these fields of definition easily. Let (ρ, V) be an irreducible algebraic representation of G. An automorphic form on G of type ρ is a function $f : G(F)\backslash G(\mathbf{A}) \to V(\mathbb{C})$, locally constant with respect to $G(\mathbf{A}^f)$, and satisfying

$$(2.5.1) \qquad f(gg_\infty) = \rho^{-1}(g_\infty)f(g), \ g\in G(\mathbf{A}), g_\infty \in G_\infty = G(F\otimes_{\mathbb{Q}}\mathbb{R}).$$

Let $\mathcal{A}(G,\rho)$ denote the space of automorphic forms of type ρ. It follows from (2.5.1) that the restriction map

$$R_f : \mathcal{A}(G,\rho) \to C^\infty(G(F)\backslash G(\mathbf{A}^f), V(\mathbb{C}))$$

is an isomorphism. If V is defined over the number field E_V, then

$$M_{DR}(S(G,x),V) := C^{\infty}(G(F)\backslash G(\mathbf{A}^f), V(E_V))$$

is an E_V-rational model for $\mathcal{A}(G,\rho)$, and for any $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, there is a canonical isomorphism

$$\sigma(M_{DR}(G,V)) \xrightarrow{\sim} M_{DR}(G,\sigma(V)). \tag{2.5.2}$$

The same naturally holds for G'.

Let V_{triv} denote the trivial one-dimensional representation of G.

Lemma 2.5.3. *There is a perfect pairing*

$$M_{DR}(S(G,x),V) \otimes M_{DR}(S(G,x),V^{\vee}) \to M_{DR}(S(G,x),V_{triv})(E_V) \to E_V$$

where the first map is defined by the natural pairing on coefficients and the second map is integration with respect to Tamagawa measure. The pairings transform under $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ by the action (2.5.2) on the coefficients V.

Proof. The first map is obviously rational over E_V, and the second map is rational because the Tamagawa measure of $G(F)\backslash G(\mathbf{A})$ is a rational number. The pairing is perfect because it is essentially given by the L_2-pairing on automorphic forms (cf. [H97, Proposition 2.6.12]).

Now suppose $V \to V'$ is a projection to an irreducible G'-invariant quotient, and let $(V')^{\vee} \to V^{\vee}$ denote the dual inclusion map. The following lemma is proved in the same way as Lemma 2.5.3.

Lemma 2.5.4. *Under these hypotheses, there is a natural $E_{V,V'} = E_V \cdot E_{V'}$-rational pairing*

$$M_{DR}(S(G,x),V) \otimes M_{DR}(S(G',x'),(V')^{\vee}) \to M_{DR}(S(G',x'),V_{triv})(E_{V,V'}) \to E_{V,V'}$$

where the first map is defined by the natural pairing on coefficients and the second map is integration with respect to Tamagawa measure. The pairings transform under $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ by the action (2.5.2) on the coefficients V, V'.

Corollary 2.5.5. *Let E be a number field containing $E_{V,V'}$, and suppose $f \in M_{DR}(S(G,x),V)(E)$, $f^{\vee} \in M_{DR}(S(G,x),V^{\vee})(E)$, $f' \in M_{DR}(S(G',x'),(V')^{\vee})(E)$, and $f'^{,\vee} \in M_{DR}(S(G',x'),V')(E)$. Define $P(f,f')$, $Q(f,f^{\vee})$, $P(f^{\vee},f'^{,\vee})$, and $Q(f',f'^{,\vee})$ as in §2.1. Then the left hand side of (2.1.5)*

$$\frac{P(f,f')P(f^{\vee},f'^{,\vee})}{Q(f,f^{\vee})Q(f',f'^{,\vee})}$$

belongs to E and for any $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$,

$$\sigma(\frac{P(f,f')P(f^{\vee},f'^{,\vee})}{Q(f,f^{\vee})Q(f',f'^{,\vee})}) = \frac{P(\sigma(f),\sigma(f'))P(\sigma(f^{\vee}),\sigma(f'^{,\vee}))}{Q(\sigma(f),\sigma(f^{\vee}))Q(\sigma(f'),\sigma(f'^{,\vee}))},$$

where $\sigma(f) \in M_{DR}(S(G,x),\sigma(V))(\sigma(E))$, etc.

In [H97, (2.6.11)] it is explained how to use the highest weight Λ of V, relative to a fixed maximal torus H, to identify $\mathcal{A}(G,\rho)$, and therefore $M_{DR}(S(G,x),V)$, with a subspace of the space $\mathcal{A}(G)$ of $\mathbb{C}$-valued automorphic forms on $G(F)\backslash G(\mathbf{A})$:

$$M_{DR}(S(G,x),V) \xrightarrow{\sim} Hom_H(\mathbb{C}_{-\Lambda}, \mathcal{A}(G)_{V^\vee}) \tag{2.5.6}$$

where $\mathbb{C}_{-\Lambda}$ is the Λ^{-1}-eigenspace for H in $V^\vee$ and $\mathcal{A}(G)_{V^\vee}$ is the $V^\vee$-isotypic subspace for the action of G_∞ by right translation. The image under this identification naturally has a rational structure over the extension $E(V,\Lambda) \supset E(V)$ over which the Λ-eigenspace in V is rational, and as V and H vary the maps (2.5.6) are rational over $E(V,\Lambda)$ and transform naturally under the action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$.

Lemma 2.5.7. *The map (2.5.6) takes the pairing of Lemma 2.5.3 to a rational multiple of the L_2-pairing on $\mathcal{A}(G)$.*

Proof. This is Proposition 2.6.12 of [H97].

2.6. Fields of rationality of automorphic representations of unitary groups.

In this section F is a general totally real field. Let Π be a cohomological cuspidal automorphic representation of $GL(n,\mathcal{K})$, and let $E(\Pi)$ be the field fixed by the subgroup of $Aut(\mathbb{C})$ consisting of σ such that $\Pi_f^\sigma \xrightarrow{\sim} \Pi_f$. It is known [Cl90] that $E(\Pi)$ is a number field and that Π_f has a rational model over $E(\Pi)$. Moreover, for any σ in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ there is a (unique) cuspidal cohomological representation $\sigma(\Pi)$ with $\sigma(\Pi)_f \xrightarrow{\sim} \sigma(\Pi_f)$ (one obtains $\sigma(\Pi)_\infty$ from Π_∞ by letting σ permute the archimedean places of $\mathcal{K}$).

Suppose Π satisfies the polarization condition (1.1.1) and G is quasi-split at all finite places of v. Then [L, Thm. 5.4; Wh2, Thm. 11.1] Π descends to an L-packet $\{\pi_\alpha, \alpha \in A\}$ of G. [L, Thm 5.4]. We mean this in the following sense: let w be a finite place of $\mathcal{K}$ at which $\mathcal{K}/F$ and Π are unramified, and let v denote the restriction of w to F. If v splits in $\mathcal{K}$, we write $\Pi_v = \Pi_w \otimes \Pi_{cw}$; if v is inert, then $\Pi_v = \Pi_w$. Then for all α, $\pi_{\alpha,v}$ is spherical and the Satake parameters of Π_v are obtained from those of $\pi_{\alpha,v}$ by the stable base change map [Mi,Theorem 4.1]. It then follows that π_∞ is the unique irreducible representation of the (compact) group G_∞ with the same infinitesimal character as Π_∞ [L, Thm. 5.5].

Proposition. *If Π is a cohomological cuspidal polarized representation of $GL(n)$ that descends to an L-packet $\{\pi_\alpha\}$ of G, then the collection $\{\pi_{\alpha,f}\}$ is rational over $E(\Pi)$. Moreover, for any σ in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, the conjugate $\sigma(\Pi)$ descends to $\{\sigma(\pi)\}$.*

Proof. Let S be the set of finite primes v at which $\mathcal{K}/F$ and Π are unramified. We first note that for all $v \notin S$, the spherical representation $\pi_{\alpha,v}$ is defined over the field of definition of Π_v. Indeed, this is clear from the relation [Mi, Theorem 4.1] of Satake parameters. Now let $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. If σ fixes $E(\Pi)$, then $\sigma(\pi_{\alpha,v}) \xrightarrow{\sim} \pi_{\alpha,v}$ for all $v \notin S$. Thus by definition, the stable base change of $\sigma(\pi_f)$ is Π, so $\sigma(\pi_\alpha)$ is a $\pi_{\alpha'}$. The same argument implies the last assertion.

3. Abelian representations of $U(m)$

3.1. Existence of abelian representations.

In this section, the Weil group of a local or global field L is denoted W_L.

Let W' be an m-dimensional hermitian space over $\mathcal{K}$, $U(W')$ the unitary group. Let μ be a Hecke character of $\mathcal{K}$ extending $\eta_{\mathcal{K}/F}$:

$$\mu \mid_{\mathbf{A}_F^\times} = \eta_{\mathcal{K}/F}.$$

Let $H = U(1)^m$ and let $\xi_\mu : {}^LH \to {}^LU(W')$ be the L-homomorphism (in the Weil group form over F) considered by White in §3 of [Wh1]. On the dual group $\hat{H} = GL(1,\mathbb{C})^m$, ξ_μ is just the diagonal embedding

$$(g_1,\dots,g_m) \mapsto \ \mathrm{diag}(g_1,\dots,g_m) \in \hat{U}(W') = GL(m,\mathbb{C}).$$

The Hecke character μ defines a character

$$W_{\mathcal{K}} \to W_{\mathcal{K}}^{ab} \xrightarrow{\sim} \mathbf{A}_{\mathcal{K}}^\times/\mathcal{K}^\times \xrightarrow{\mu} \mathbb{C},$$

also denoted μ. Set $\mu_m = \mu$ if m is even, $\mu_m = 1$ (the trivial character) if m is odd.

If $w \in W_{\mathcal{K}}$, we have

(3.1.1)
$$\xi_\mu(1,1,\dots,1)\times w = \mu_m(w)\cdot I_m \times w \in GL(n,\mathbb{C})\times W_{\mathcal{K}} \subset GL(m,\mathbb{C})\ltimes W_F = {}^LU(W').$$

The map ξ_μ is characterized by these formulas and by its value on a single element of $(1\times W_F)\ \setminus\ (1\times W_{\mathcal{K}})$, as in [Wh1]; we omit the formula.

Let $\chi = (\chi_1,\dots,\chi_m)$ be an m-tuple of Hecke characters of $U(1)(\mathbf{A}_F)/U(1)(F)$; χ is an automorphic representation of H, and we can consider its functorial transfer to $U(W')$ via the L-homomorphism ξ_μ. Concretely, an automorphic representation $\pi(\chi)$ of $U(W')$ is a functorial transfer of χ if its formal base change $\Pi(\chi) = BC(\pi(\chi))$ to $GL(m)_{\mathcal{K}}$ is a (non-cuspidal) automorphic representation with the property

$$L(s,\Pi(\chi)) = \prod_{i=1}^{m} L(s+\frac{m-1}{2}, BC(\chi_i)\cdot\mu_m). \tag{3.1.2}$$

Here

$$BC(\chi)(z) = \chi(z/c(z)), z \in \mathbf{A}_{\mathcal{K}}^\times \tag{3.1.3}$$

where c denotes Galois conjugation; this was denoted $\tilde{\chi}$ in [H97]. By definition, the functorial transfers of χ to $U(W')$ form a single L-packet $\pi(\chi)$ such that, for each place v of F, π_v is a local functorial transfer of χ_v for any $\pi \in \subset(\chi)$.

An L-packet of the form $\pi(\chi)$ will be called an *abelian L-packet* of $U(W')$, and a member of $\pi(\chi)$ that occurs with non-zero multiplicity in the automorphic spectrum of $U(W')$ is called an *abelian representation*. The existence of abelian representations in this sense is considered in §11 of [Wh2], along with other cases of endoscopic transfer. More precisely, one can say that the local functorial transfers are the L-packets defined by Moeglin in [M] – we denote them $\pi(\chi_v)$ – and that if we choose one $\pi_v \in \pi(\chi_v)$ for each v, then we can ask for the multiplicity of $\otimes_v'\pi_v$ in the automorphic spectrum of $U(W')$. Under certain simplifying hypotheses, this multiplicity is either 0 or 1, and the cases of non-zero multiplicity are determined by White in [Wh2]. We return to this point in §4.3.

Let v be a real prime of F and suppose $\chi_{j,v}(e^{i\theta}) = e^{ik_j\theta}$, with $k_j \in \mathbb{Z}$. We say that k_j is the *weight* of χ_j at v (or of $\chi_{j,v}$). The Langlands parameter of $\chi_{j,v}$ is given by the homomorphism $\phi(\chi_{j,v}) : W_{\mathbb{R}} \to {}^LU(1) = GL(1,\mathbb{C}) \rtimes Gal(\mathbb{C}/\mathbb{R})$ whose restriction to $\mathbb{C}^\times = W_{\mathbb{C}}$ is

$$W_{\mathbb{C}} \ni z \mapsto (z/\bar{z})^{k_j}$$

Then $BC_{\mathbb{C}/\mathbb{R}}(\Pi(\chi_v))$ is the representation of $GL(n,\mathbb{C})$ with Langlands parameter

$$\phi(\chi_v) : W_{\mathbb{C}} \ni z \mapsto diag((z/\bar{z})^{k_1} \cdot \mu_m(z), \ldots, (z/\bar{z})^{k_m}\mu_m(z)) \in GL(m,\mathbb{C}). \tag{3.1.4}$$

This descends to a discrete series L-packet of $U(W')_v$, for any W', if and only if the k_j are all distinct (cf. [Wh1, Def. 5.3]); then the infinitesimal character of the discrete series L-packet coincides with the Langlands parameter, and we say χ_v is *regular*.

On $U(1) \subset \mathbb{C}^\times$ we write

$$\mu_m(e^{i\theta}) = e^{it_m\theta}$$

for some $t_m \in \mathbb{Z}$. We order the k_i so that

$$k_i > k_{i+1} \tag{3.1.5}$$

with k_i defined by

$$(z/\bar{z})^{k_i} \cdot \mu_m(z) = (z/\bar{z})^{k_i + \frac{t_m}{2}}; \ k_i + \frac{t_m}{2} \in \mathbb{Z} + \frac{m-1}{2}$$

The half-integrality of $k_i + \frac{t_m}{2}$ follows from the parity of μ_m and is as it should be, cf. [Cl90, §3.5].

The following Lemma follows immediately.

Lemma 3.1.6. *Suppose χ_v is regular for all real primes v. Then the local Langlands parameter $\phi(\chi_v)$ is relevant for all $U(W')_v$ and for any W' the L-packet $\pi(\chi)$ of $U(W')$ is of discrete series type at all real places.*

The definite case. Suppose now $U(W'_v)$ is the compact form of $U(m)$. Then the L-packet $\pi(\chi)$ is a singleton, which we denote τ', with highest weight $(b_1 \geq b_2 \geq \cdots \geq b_m)$, in the notation of §2.3. The relation between b_i and k_i is given by

$$b_i = k_i - \frac{-t_m + m + 1 - 2i}{2} \tag{3.1.7}$$

so that $b_i \geq b_{i+1}$, as required.

In what follows, we assume we are given a non-trivial abelian L-packet $\pi(\chi)$ and apply it in the Ichino-Ikeda conjecture. Henceforward we specialize to the case $F = \mathbb{Q}$, $m = n-1$, with n even, so $\mu_m = 1$ and

$$k_i = b_i + \frac{n}{2} - i.$$

This will suffice to illustrate the general principles guiding this work. We hope to treat the general case in a subsequent paper.

3.2. Review of CM periods.

We review the properties of the CM period invariants, as discussed in (1.10) and (3.6) of [H97]. Since the final results will only be stated when $F = \mathbb{Q}$, we only consider the CM periods attached to imaginary quadratic fields. Details of the more general CM periods have only been written up in the present language up to algebraic factors; most of the results of the present paper can be extended to general CM fields without going beyond the available literature, provided one is will to settle for rationality up to $\bar{\mathbb{Q}}^{\times}$.

Thus, $\mathcal{K}$ is an imaginary quadratic field, with chosen embedding $\mathcal{K} \to \mathbb{C}$, denoted $\mathbf{1}$. Let $\eta : \mathbf{A}_{\mathcal{K}}^{\times}/\mathcal{K}^{\times} \to \mathbb{C}$ be a Hecke character whose archimedean part is algebraic: $\eta_{\infty}(z) = z^{-a_{\mathbf{1}}} \cdot (cz)^{-a_c}, z \in \mathbb{C}^{\times}$, with the exponents in $\mathbb{Z}$. Let $E(\eta) \supset \mathcal{K}$ be the field generated by $\eta \mid_{\mathbf{A}^f{}_{\mathcal{K}}}$, and let ${}^c\eta = \eta \circ c$. There are then two period invariants

$$p(\eta, \mathbf{1}), p(\eta, c) = p({}^c\eta, \mathbf{1}) \in (E(\eta) \otimes \mathbb{C})^{\times}/E(\eta)^{\times}.$$

These invariants satisfy the multiplicative relations

$$p(\eta_1, \bullet)p(\eta_2, \bullet) \sim_{\tilde{E}(\eta_1, \eta_2)} p(\eta_1\eta_2, \bullet), \;\; \bullet = \mathbf{1}; c \tag{3.2.1}$$

and the normalization conditions (here $|| \bullet ||$ is the norm):

$$p(|| \bullet ||^a, \mathbf{1}) = p(|| \bullet ||^a, c) = (2\pi i)^{-a}; \tag{3.2.2}$$

If η is the Hecke character attached to a Dirichlet character of conductor N (with archimedean component a power of the sign character) and $\psi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^{\times}$ is an additive character, then

$$p(\eta, \mathbf{1}) = g(\eta, \psi)^{-1}, \tag{3.2.3}$$

where $g(\eta, \psi) = \sum_{b \in (\mathbb{Z}/N\mathbb{Z})^{\times}} \eta(b)\psi(b)$ is the standard Gauss sum. If $(a_{\mathbf{1}}, a_c) = (k, 0)$, with $k > 0$, then for all critical values m of the Hecke L-function $L(s, \eta)$, we have

$$L(m, \eta) = L(0, \eta \cdot ||| \bullet ||^{-m}) \sim_{E(\eta), \mathcal{K}} \;(2\pi i)^m p(\check{\eta}, \mathbf{1}) \tag{3.2.4}$$

where $\check{\eta}(z) = \eta^{-1}(cz)$. In particular, if χ is a character of the group $U(1)$ as above, then $BC(\chi) = BC(\chi)\check{}$, so for critical values

(3.2.5)

$$L(m, BC(\chi)) \sim_{E(\chi), \mathcal{K}} \;(2\pi i)^m p(BC(\chi), \mathbf{1}) \sim_{E(\chi), \mathcal{K}} \;(2\pi i)^m p(\chi^+, \mathbf{1}) p({}^c\chi^+, \mathbf{1})^{-1}$$

for any extension χ^+ of χ to an algebraic Hecke character of $\mathcal{K}$.

3.3. Asai L-functions of abelian representations.

Fix χ as in the previous section, and let $\Pi = \Pi(\chi)$. Formula (3.1.2) gives an explicit expression for the motive $M_{\Pi(\chi)}$ over $\mathcal{K}$:

$$M_{\Pi(\chi)} = \oplus_{i=1}^{n-1} M_{BC(\chi_i)}(\frac{2-n}{2}). \tag{3.3.1}$$

It then follows from the definitions that $L(s, As(M_{\Pi(\chi)}))$, which is an L-function over F $(=\mathbb{Q})$, decomposes as

$$\begin{aligned} L(s, As(M_{\Pi(\chi)})) &= \prod_{1\leq i<j\leq n-1} L(s, AI_{\mathcal{K}/F}BC(\chi_j\cdot\chi_i^{-1}))L(s,\eta_{\mathcal{K}/F})^{n-1} \\ &= \prod_{1\leq i<j\leq n-1} L(s, AI_{\mathcal{K}/F}BC(\chi_{ij}))L(s,\eta_{\mathcal{K}/F})^{n-1} \end{aligned} \tag{3.3.2}$$

where $\chi_{ij} = \chi_j/\chi_i$. Indeed,

$$L(s, Ad(M_{\Pi(\chi)})) = \prod_{1\leq i\neq j\leq n-1} L(s, BC(\chi_j\cdot\chi_i^{-1}))\zeta_{\mathcal{K}}(s)^{n-1},$$

where $\zeta_{\mathcal{K}}$ is the Dedekind zeta function. The two descents $As^{\pm}$ are distinguished by their L-functions over F; in addition to the one indicated in (3.3.2), there is the one obtained by twisting by $\eta_{\mathcal{K}/F}$, namely

$$\prod_{1\leq i<j\leq n-1} L(s, AI_{\mathcal{K}/F}BC(\chi_j\cdot\chi_i^{-1}))\zeta_F(s)^{n-1}.$$

The condition on the signature of F_∞ guarantees that (3.3.2) is the right choice for $As(M_{\Pi(\chi)})$.

We evaluate the values at $s=1$ of the factors of (3.3.2) using Blasius' result on special values of Hecke L-series (Damarell's formula in this case). As in §3.1, we assume χ_i is of weight k_i at the archimedean prime, so that χ_{ij} is of weight $-k_{ij}$, with $k_{ij} = k_i - k_j$. We assume the χ_i are ordered so that $k_{ij} > 0$ for $i<j$, as in the branching formula 2.3.1. This is the normalization used in [H97]. As in [H97, §2.9], we define

$$\chi_{ij}^{(2)} = \chi_{ij}^2\cdot(\chi_{ij,0}\circ N_{\mathcal{K}/\mathbb{Q}})^{-1},\ \chi_{ij,0} = \chi_{ij}|_{\mathbf{A}_{\mathbb{Q}}^\times}\cdot||\bullet||_{\mathbf{A}}^{-k_{ij}}. \tag{3.3.3}$$

Then (cf. [H97, (3.6.1). (3.6.3)])

$$L(1, BC(\chi_{ij})) = L(1+k_i-k_j, \chi_{ij}^{(2)}) \sim (2\pi i)^{1+k_i-k_j}p((\chi_{ij}^{(2)})^\vee, 1)$$

By using the formula $\chi_{ij}^{(2)} = \chi_j^{(2)}/\chi_i^{(2)}$ and the relations in §3.2, we find that the value at 1 of (3.3.2) is

$$\begin{aligned} &[(2\pi i)g(\eta_{\mathcal{K}/F})]^{n-1}\cdot\prod_{i<j}(2\pi i)^{1+k_i-k_j}p((\chi_{ij}^{(2)})^\vee,1) \\ &\sim [(2\pi i)g(\eta_{\mathcal{K}/F})]^{n-1}\cdot(2\pi i)^{\frac{(n-2)(n-1)}{2}}\cdot\prod_{i=1}^{n-1}[(2\pi i)^{k_i}p((\chi_i^{(2)})^\vee,1)]^{2i-n} \\ &\sim g(\eta_{\mathcal{K}/F})^{n-1}\cdot(2\pi i)^{\frac{n(n-1)}{2}}\cdot\prod_{i=1}^{n-1}[(2\pi i)^{k_i}p((\chi_i^{(2)})^\vee,1)]^{2i-n} \end{aligned} \tag{3.3.4}$$

Comparing this formula with Corollary 1.3.5 (i), it is reasonable to suppose that

$$Q_\ell = [(2\pi i)^{k_\ell}p((\chi_\ell^{(2)})^\vee,1)]^{-2},\ \ell = 1,\ldots,n-1; \tag{3.3.5}$$

so that $[(2\pi i)^{k_\ell}p((\chi_\ell^{(2)})^\vee,1)]^{2\ell-n} = Q_\ell^{\frac{(n-1)+1}{2}-\ell}$, as predicted. However, it will not be necessary to verify this formula, since the same expression reappears in the numerator of the Ichino-Ikeda formula in the applications.

4. The critical value of the Asai L-function

We continue to assume $F = \mathbb{Q}$ and n is even. Henceforward the groups G and G' are assumed to be definite. We let $f, f^\vee, f', f'^{,\vee}$ be automorphic forms as in the statement of the Ichino-Ikeda conjecture, and we assume they are all E-rational, as in the statement of Corollary 2.5.5.

We begin by studying the L-functions that occur on the right-hand side of the Ichino-Ikeda conjecture for the pair π, π'. Starting in §4.2, we will assume $\pi' \in \pi(\chi)$ for an appropriate $n-1$-tuple χ of Hecke characters. The weights of χ will be chosen so that the unitary groups that occur on the left-hand side of (2.1.5), and in the zeta integrals on the right-hand side, are necessarily definite, as in §2.3. The left-hand side is then an algebraic number, as we have seen in Corollary 2.5.5.We conclude with an expression for the value $L(1, \pi, Ad)$, which we compare to the conjectured expression from §1.3.

4.1. Elementary and local terms in the Ichino-Ikeda formula for definite groups.

The left-hand side of the Ichino-Ikeda conjecture (2.1.5) has been studied in §2.5. Corollary 2.5.5 demonstrates that it is an algebraic number that transforms as expected under Galois conjugation. Thus the Ichino-Ikeda conjecture implies that the right-hand side is also algebraic, and determines how it transforms under Galois conjugation. In this section we study the algebraicity of the elementary and local terms.

4.1.1. The power of 2 that appears as the first term is, of course, rational.

4.1.2. The normalizing factor. The abelian normalizing factor Δ_G is a product of n abelian L-functions of $\mathbb{Q}$ – either $\zeta(s)$ or $L(s, \eta_{\mathcal{K}/\mathbb{Q}})$ depending on the parity – evaluated at integer points. Each of the integer points is well known to be critical, and the formulas for the special values can be written as follows.

$$\Delta_G \sim_{\mathcal{K}} \prod_{i=1}^{n} g(\eta^i_{\mathcal{K}/\mathbb{Q}}) \cdot (2\pi i)^i = (2\pi i)^{\frac{n(n+1)}{2}} g(\eta_{\mathcal{K}/\mathbb{Q}})^{\frac{n}{2}}.$$

Here $\sim_{\mathcal{K}}$ means that the left-hand side is a $\mathcal{K}^\times$-multiple of the right hand side. By the Iwasawa main conjecture the integral properties of the quotient $\Delta_G/(2\pi i)^{\frac{n(n+1)}{2}}$ are closely related to orders of class groups of cyclotomic fields.

4.1.3. Factorization. For the next section, we need to write $f, f^\vee, f', f'^{,\vee}$ as tensor products of vectors $f = \otimes_v f_v$, $f_v \in \pi_v$, and so on. Let $E(\pi) \supset E(V)$, $E(\pi') \supset E(V')$ denote fields of definition of π and π', respectively. In particular, each factor π_v is defined over $E(\pi)$, and we can assume the isomorphisms $\pi \xrightarrow{\sim} \otimes_v \pi_v$, $\pi' \xrightarrow{\sim} \otimes_v \pi'_v$ (and the corresponding dual maps) are defined over $E(\pi)$ and $E(\pi')$ respectively. Our hypothesis is that the test vectors on the left hand side of (2.1.5) are all E-rational; thus for all v f_v, f'_v, $f^\vee_v$, $f'^{,\vee}_v$ are also E-rational.

Moreover, the canonical local pairings $< \bullet, \bullet >_{\pi_v}$, $< \bullet, \bullet >_{\pi'_v}$ are tautologically $E(\pi)$ and $E(\pi')$-rational respectively. It follows that the matrix coefficients $c_{f_v, f^\vee_v}(g_v)$ and $c_{f'_v, f'^{,\vee}_v}(g'_v)$ are E-rational. For finite v, this means that they are functions that take values in the indicated number fields. For $v = \infty$, an E-rational matrix coefficient of the algebraic representation π_∞ is an element of the affine algebra $E(G)$ of the algebraic group G; likewise for π'_∞.

4.1.4. Measures and archimedean local terms. We want to prove that the product Z_{loc} of local terms on the right-hand side of (2.1.5) is an algebraic number that transforms appropriately under Galois conjugation. We begin by reconsidering the factorization $dg' = \prod_v dg'_v$ of Tamagawa measure. For the moment F is an arbitrary totally real field, and $G_\infty = \prod_{v|\infty} G_v$ is the product of definite unitary groups. For $v \notin S$, let $K'_v \subset G'_v$ be a hyperspecial maximal compact subgroup; we recall from §2.1 that $\int_{K'_v} dg'_v = 1$ for $v \notin S$.

Lemma 4.1.4.1. *For any sufficiently small open subgroup $\prod_{v \in S} K'_v \subset \prod_{v \in S} G'_v$, the open subgroup $G'_\infty \times \prod_{v \nmid \infty} K'_v \subset G(\mathbf{A})$ acts freely (on the right) on $G'(F)\backslash G'(\mathbf{A})$ with finitely many orbits. In particular, $\int_{G'_\infty \times \prod_{v \nmid \infty} K'_v} dg$ is a rational number.*

Proof. Let $U = G'_\infty \times \prod_v K'_v$, and let $g \in G(\mathbf{A})$ be a fixed point of some $u \in U$. Thus $gu = \gamma g$ for some $\gamma \in G'(F)$, or $gug^{-1} \in gUg^{-1} \cap G'(F)$. It's well known that this intersection is trivial if U is sufficiently small (cf. the proof of Lemma 2.3.1 of [CHT08]. Finiteness of the number of orbits is clear because U is open in $G'(\mathbf{A})$ and $G'(F)\backslash G'(\mathbf{A})$ is compact. The final assertion follows from the first because the Tamagawa number of G' is rational (in fact it equals 2).

Corollary 4.1.4.2. *The volume of G'_∞ with respect to $dg_\infty = \prod_{v|\infty} dg_v$ is rational.*

Proof. Indeed,

$$\int_{G'_\infty} dg_\infty = \frac{\int_{G'_\infty \times \prod_v K'_v} dg}{\int \prod_{v \nmid \infty} K'_v}.$$

The numerator is rational by the Lemma, and the denominator is rational by conditions (1) and (2) of §2.1.

Now for simplicity we assume $F = \mathbb{Q}$, so that there is only one archimedean prime.

Corollary 4.1.4.3. *The archimedean local factor Z_∞ of Z_{loc} is an algebraic number.*

Proof. It follows from Lemma 2.5.7 that Z_∞ is a rational multiple of the integral of a product of E-rational matrix coefficients of two algebraic representations of G'_v with respect to the measure of total volume 1. By the orthogonality relations this is an element of E.

4.1.5. Non-archimedean local factors.. Let $p \in S$ be a finite prime and let E be a number field over which both π_p and π'_p are defined. Then it makes sense to speak of E-rational matrix coefficients $c_{f_p,f_p^\vee}$ and $c_{f'_p,f_p'^{,\vee}}$ of π_p and π'_p, respectively. Recall that in §2.1 we have assumed that local measures at finite primes take rational values on compact open subsets.

Lemma 4.1.5.1. *Suppose π_p and π'_p are tempered. For any E-rational matrix coefficients $c_{f_p,f_p^\vee}$ and $c_{f'_p,f_p'^{,\vee}}$ as above, the local zeta integral has the property that*

$$Z_p(f_p, f_p^\vee, f'_p, f_p'^{,\vee}) \in E.$$

In [II] and [NH] it is proved that the integral defining $Z_p(f_p, f_p^\vee, f'_p, f_p'^{,\vee})$ converges absolutely when the two representations are tempered, but no information is given about the rationality of the integral. Using Casselman's results on asymptotics of matrix coefficients, Moeglin and Waldspurger [MW, Lemma 1.7] decompose the analogous integral for pairs of special orthogonal groups (even in the non-tempered case) into a finite sum of terms that can easily be seen to be rational over E.

More precisely, we write G and G' for the local groups at p. Assume π and π' are constituents of representations induced from supercuspidal representations of the Levi components M and M' of of parabolic subgroups $P \subset G$, $P' \subset G'$, respectively, with M and M' respectively of (split) rank t, t'. Thus π and π' belong to complex families (components of the respective Bernstein centers) $C(\pi)$ and $C(\pi')$ of dimension t and t', parametrized by characters $X(M)$ of M and M', modulo the actions of the normalizers $W_M = N_G(M)/M$ and $W_{M'} = N_{G'}(M')/M'$:

$$C(\pi) = Spec(\mathbb{C}[X(M)]^{W_M}), C(\pi') = Spec(\mathbb{C}[X(M')]^{W_{M'}}). \tag{4.1.5.2}$$

These complex families have rational structures over $\mathbb{Q}$ whose E-rational points are the E-rational orbits of W_M and $W_{M'}$ on the character groups. The functions $f_p, f_p^\vee$ and $f'_p, f_p^{\prime,\vee}$ can be extended to E-rational algebraic functions on $C(\pi)$ and $C(\pi')$. The lemma proved by Moeglin and Waldspurger (in the orthogonal case, but the argument works as well for unitary groups) is then

Lemma 4.1.5.3 (Moeglin, Waldspurger). *There are polynomials*

$$D, L \in \mathbb{C}[X(M), X(M')],$$

depending on $f_p, f_p^\vee, f'_p, f_p^{\prime,\vee}$, *such that*

$$D \cdot Z_p(f_p, f_p^\vee, f'_p, f_p^{\prime,\vee}) = L.$$

For the proof of the lemma, it is not assumed that π and π' are tempered. In the tempered case, the convergence proved in [II,NH] implies that D has no pole at the point corresponding to $\pi, \pi' \in C(\pi) \times C(\pi')$.

For our purposes, the important point is that every step in the proof in [MW] is rational over E. The main reduction step is the expression of the integral as a finite sum of terms indexed by rational parabolic subgroups of G or G', in which the matrix coefficients are replaced by corresponding expressions involving the non-normalized Jacquet modules. Since the non-normalized Jacquet functor preserves rationality over $\mathbb{Q}$, the proof of Lemma 4.1.5.3 actually yields Lemma 4.1.5.3.

4.1.6. Conclusion. Combining the results obtained above with Corollary 2.5.5, we find that

$$\frac{(2\pi i)^{\frac{n(n+1)}{2}} L(\frac{1}{2}, BC(\pi) \times BC(\pi'))}{L(1, \pi, Ad)L(1, \pi', Ad)} \in \overline{\mathbb{Q}}; \tag{4.1.6.1}$$

For all $\sigma \in Gal(\overline{\mathbb{Q}}/\mathcal{K})$,
(4.1.6.2)

$$\sigma[\frac{(2\pi i)^{\frac{n(n+1)}{2}} L(\frac{1}{2}, BC(\pi) \times BC(\pi'))}{L(1, \pi, Ad)L(1, \pi', Ad)}] = \frac{(2\pi i)^{\frac{n(n+1)}{2}} L(\frac{1}{2}, BC(\sigma(\pi)) \times BC(\sigma(\pi')))}{L(1, \sigma(\pi), Ad)L(1, \sigma(\pi'), Ad)}.$$

Including the Gauss sums that appear in (4.1.2) in the expression (4.1.4.1) would allow us to assert the modified version of (4.1.6.2) for all $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. However, the subsequent calculations are taken from [H97] and have are only been proved for conjugation by $Gal(\overline{\mathbb{Q}}/\mathcal{K})$.

4.2. Tensor products involving abelian representations.

Let π and π' be automorphic representations of the definite unitary groups G and G', as in §2.3, with base changes Π and Π' to $GL(n)_{\mathcal{K}}$ and $GL(n-1)_{\mathcal{K}}$ respectively, and with central characters χ_π and $\chi_{\pi'}$. We assume $L(\tau,\tau') \neq 0$, with $\tau = \pi_\infty$, $\tau' = \pi'_\infty$; thus the highest weights of τ and τ' satisfy the branching law 2.3.1. Our goal is to understand the special value $L(1,\pi,Ad)$. This is unchanged when π is twisted by a Hecke character, so we lose no generality if we assume the highest weight of $\tau = \pi_\infty$, with parameters as in §2.3, has the form $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. It then follows from 2.3.1 that the k_j are all *negative.*

We assume $\pi' \in \Pi(\chi)$. Then (since $\mu_{n-1} = 1$)

$$L(s,\Pi\times\Pi') = \prod_{i=1}^{n-1} L(s,\Pi\otimes BC(\chi_j)) = \prod_{i=1}^{n-1} L(s,\pi\otimes\chi_i\circ\det,St). \tag{4.2.1}$$

Here St is the standard L-function of the L-group of G in the unitary normalization, as in [H97]. In the motivic normalization (cf. [H97], we then have

$$L(s,\Pi\times\Pi') = \prod_{i=1}^{n-1} L^{mot}(s+\frac{n-1}{2},\pi\otimes\chi_i\circ\det,St). \tag{4.2.2}$$

Lemma 4.2.3. *The value $s_0 = \frac{n}{2}$ is critical in Deligne's sense for each of the factors $L^{mot}(s,\pi\otimes\chi_i\circ\det,St)$.*

(If n were odd, there would be a shift of $\frac{1}{2}$ to compensate the character μ.)

Proof. The line $Re(s) = s_0$ is the axis of symmetry for the functional equation, and the integral point on the axis of symmetry of the L-function of a motive is critical whenever the motive is of odd weight. The motive in question is $M(\Pi)\otimes M(BC(\chi_i))$. Since $M(\Pi)$ is of weight $n-1$ and $M(BC(\chi))$ is of weight 0 for any algebraic Hecke character χ, the lemma follows.

We can thus use the formulas in [H97,H07] to express $L(s_0,\Pi\times\Pi')$ in terms of automorphic periods.

Lemma 4.2.4. *In the terminology of* [H97,§1.7]*, the character $BC(\chi_i)$ belongs to the ith critical interval for $M(\Pi)$, for $i = 1,\ldots,n-1$.*

Proof. Recall from [H97] that the ith critical interval is the interval

$$\begin{aligned}[n-2p_i, n-2p_{i+1}-2] &= [n-2(n-i+a_{n+1-i}), n-2(n-i+a_{n-i})] \\ &= [-2a_{n+1-i}-n+2i, -2a_{n-i}-n+2i],\end{aligned}$$

where the first equality is (2.3.2). On the other hand, up to a twist by a power of the norm character $z\bar{z}$, $BC(\chi_i)_\infty$ is of weight $2k_i = 2b_i - n + 2i$, so the lemma follows from the inequalities 2.3.1(2).

Now suppose the following hypothesis is satisfied:

Hypothesis NE. *For every inner form J of G_∞, there exists an inner form G_J of G with $G_{J,\infty} = J$ and a holomorphic automorphic representation π_J of G_J that is nearly equivalent to π; in other words, such that $\pi_{J,v} \xrightarrow{\sim} \pi_v$ for all but finitely many places v.*

Then we can apply Theorem 4.3 of [H07] and find that

$$L^{mot}(\frac{n}{2}, \pi \otimes \chi_i, St) \sim_{E(\pi,\chi_i),\mathcal{K}} (2\pi i)^{\frac{n}{2}+k_i(2i-n)} g(\eta_{\mathcal{K}/F})^{\frac{n}{2}} P^{(n-i)}(\Pi) p((\chi_i^{(2)})^\vee, 1)^{2i-n}.$$
$$\sim_{E(\pi,\chi_i),\mathcal{K}} (2\pi i)^{\frac{n}{2}} G(i,\chi) P^{(n-i)}(\Pi)$$

where we have introduced the abbreviation

$$G(i,\chi) = [(2\pi i)^{k_i} \cdot p((\chi_i^{(2)})^\vee, 1)]^{2i-n}$$

and we have chosen to ignore powers of $g(\eta_{\mathcal{K}/F})$.

The periods $P^{(s)}(\Pi)$ were defined in [H97], (2.8.2), where they were denoted $P^{(s)}(\pi, V; \beta)$. Roughly speaking, $P^{(s)}(\pi, V; \beta)$ is the normalized Petersson square norm of a holomorphic automorphic form β on the Shimura variety attached to a unitary group $GU(V)$ of a hermitian space V of signature (r, s); β is assumed rational over an appropriate coefficient field, and the period $P^{(s)}(\pi, V; \beta)$ is well-defined up to multiplication by a scalar in this coefficient field. It is proved under somewhat restrictive hypotheses in Corollary 3.5.12 of [H97] that $P^{(s)}(\pi, V; \beta)$ depends only on the near equivalence class of π (and on the signature (r, s)), and therefore only on Π. The argument used to prove Corollary 3.5.12 in [H97] can be applied to the result of Theorem 4.3 of [H07] to obtain the same statement under a much weaker hypothesis, namely when the L-functions $L^{mot}(s, \pi \otimes \chi_i, St)$ have non-vanishing critical values for some χ_i in the corresponding critical interval for Π. Since this is a consequence of Hypothesis (3) of Theorem 4.2.6, we will just assume this to be the case; thus it is legitimate to write $P^{(s)}(\Pi)$ as a function of the near-equivalence class.[4]

The statement of Theorem 4.3 of [H07] is conditional on the possibility of representing the special value in question as an integral of a holomorphic automorphic form – hence the need for Hypothesis NE – against an Eisenstein series realized by means of the Siegel-Weil formula. That this is possible for the central value is proved in [HLSu], §4.2.

In other words,

$$L^{mot}(\frac{n}{2}, \pi \otimes \pi') = \prod_{i=1}^{n-1} L^{mot}(\frac{n}{2}, \pi \otimes \chi_i, St)$$
$$\sim_{E(\pi,\{\chi_i\}),\mathcal{K}} (2\pi i)^{\frac{n(n-1)}{2}} \prod_{i=1}^{n-1} G(i,\chi) \cdot P^{(n-i)}(\Pi).$$

[4]Under Hypotheses 4.1.4, 4.1.10, and 4.1.14 of [H07b], Theorem 4.2.1 of [H07b] implies immediately that $P^{(s)}(\pi, V; \beta)$ depends only on the near equivalence class of π. The most important of these hypotheses is 4.1.10: Π is cohomological with non-trivial cohomology with coefficients in a representation of $GL(n)$ of regular highest weight).

Combining this with (3.3.4), and bearing in mind that $L(s, As(\pi')) = L(s, As(M_{\Pi(\chi)}))$, we find

$$\begin{aligned}\frac{L^{mot}(\frac{n}{2}, \pi \otimes \pi')}{L(1, As(\pi'))} &\sim_{E(\pi,\{\chi_i\}),\mathcal{K}} (2\pi i)^{\frac{n(n-1)}{2}} \frac{\prod_{i=1}^{n-1} G(i,\chi) \cdot P^{(n-i)}(\Pi)}{(2\pi i)^{\frac{n(n-1)}{2}} \cdot \prod_{i=1}^{n-1} G(i,\chi)} \\ &\sim_{E(\pi,\{\chi_i\}),\mathcal{K}} \prod_{i=1}^{n-1} P^{(n-i)}(\Pi)\end{aligned} \tag{4.2.5}$$

The following theorem then follows immediately from (4.2.5) and (4.1.6).

Theorem 4.2.6. *We admit the Ichino-Ikeda conjecture (2.1.5). Fix a representation τ of G_∞, and an automorphic representation π of G of infinity type τ. Suppose π satisfies Hypothesis NE, and suppose there exists an $n-1$ tuple χ satisfying*

1. *The L-packet $\Pi(\chi)$ on G' is non-trivial;*
2. *Let τ' denote the common archimedean component of all elements of $\Pi(\chi)$. Then τ' satisfies the inequalities of 2.3.1 (2) relative to τ, i.e. $L(\tau, \tau') \neq 0$;*
3. *For each χ_i, the central value $L^{mot}(\frac{n}{2}, \pi \otimes \chi_i, St) = L(\frac{1}{2}, \pi \otimes \chi_i, St) \neq 0$.*

Then

$$L(1, \pi, Ad) \sim_{E(\pi),\mathcal{K}} (2\pi i)^{\frac{n(n+1)}{2}} \prod_{i=1}^{n-1} P^{(n-i)}(\Pi).$$

Remarks 4.2.7.

(a) It is legitimate to replace $E(\pi, \{\chi_i\})$ by $E(\pi)$ because we can let the χ_i vary over their Galois conjugates; only π remains on the two sides.

(b) Hypotheses (1) and (3) imply that the numerator $L(\frac{1}{2}, \Pi \times BC(\Pi(\chi)))$ of (4.1.6.1) does not vanish. The Ichino-Ikeda conjecture, together with the Gross-Prasad Conjecture, then picks out a pair (G_1, G_1'), of inner forms of G, G', respectively, and automorphic representations π_1 and π_1' on G_1, G_1', with $BC(\pi_1) = \Pi$, $BC(\pi_1') = BC(\Pi(\chi))$, such that the left hand side of the identity (2.1.5) does not vanish for some choice of data $f, f', f^\vee, f'^{,\vee}$. In particular, for all places v, $L(\pi_{1,v}, \pi'_{1,v}) \otimes L(\pi^\vee_{1,v}, \pi'^{,\vee}_{1,v}) \neq 0$. Moreover, the quadruple $(G_1, G_1', \pi_1, \pi_1')$ is unique. It follows from Hypothesis (2) that $G_{1,\infty} = G_\infty$ and $G'_{1,\infty} = G'_\infty$ are compact. Since $n-1$ is odd, this implies that G_1' and G' are isomorphic. On the other hand, G_1 may well be different from G at finite places, but since $L(1, \pi_1, Ad) = L(1, \pi, Ad)$, we need not refer to π_1 in the statement of Theorem 4.2.6.

4.3. Verification of the hypotheses of Theorem 4.2.6.

Paul-James White has shown the existence of L-packets $\Pi(\chi)$ satisfying hypotheses (1) and (2) of Theorem 4.2.6 in considerable generality. The result is contained in Corollary 11.2 of [Wh2].

Proposition 4.3.1. *Suppose τ has the property that there exists τ' satisfying the inequalities of 2.3.1 (2) relative to τ such that τ' is* strictly regular, *in the sense that its highest weight is given by*

$$b_1 > \cdots > b_{n-1}$$

with strict inequalities. Then for any $(n-1)$-tuple $\chi = (\chi_1, \ldots, \chi_{n-1})$, with χ_i of weight k_i defined by (3.1.7), there exists a non-trivial L-packet of the form $\Pi(\chi)$ on G', as in the statement of Theorem 4.2.6, with archimedean component τ'.

Proof. Let τ' be as in the statement of the proposition, with highest weight $b_1 > \cdots > b_{n-1}$. Let $\chi = (\chi_1, \ldots, \chi_{n-1})$ be any $(n-1)$-tuple of Hecke characters of $\mathcal{K}$ with weights k_i defined by (3.1.7). We assume that each χ_j is a stable base change from $U(1)$; that there is a rational prime q that remains inert in $\mathcal{K}$ such that the inertial characters of the χ_i at the prime $q\mathcal{O}_{\mathcal{K}}$ are all distinct; and that at every prime q' of $\mathbb{Q}$ that ramifies the inertial characters of the χ_i at the prime of $\mathcal{K}$ dividing q' are also distinct. Then χ is tempered θ-discrete stable, in the sense of [Wh2], at the given q and each ramified q'. Let $\Pi = \chi_1 \boxplus \cdots \boxplus \chi_{n-1}$ be the corresponding automorphic representation of $GL(n-1)_{\mathcal{K}}$. Then Π satisfies the hypotheses of Corollary 11.2 of [Wh2], hence it descends to an L-packet $\Pi(\chi)$ of G' containing an automorphic representation σ with multiplicity 1.

Note that White's results are much stronger than the statement of the proposition; it is very easy to find non-trivial $\Pi(\chi)$, provided τ satisfies the superregularity condition of the proposition. This condition is moreover clearly superfluous, and can be removed once the non-tempered spectrum of the unitary group G' has been determined.

4.3.2 The non-vanishing hypothesis (3) of Theorem 4.2.6. This hypothesis is not accessible at present. One can conjecture that it is always true, given the freedom one has in choosing χ in the proof of Proposition 4.3.1. For each i one needs to find χ_i of the appropriate weight such that $L(\frac{1}{2}, \pi \otimes \chi_i, St) \neq 0$; equivalently, with χ_i fixed, one needs to find χ'_i of finite order, with trivial restriction to the idèles of $\mathbb{Q}$, such that $L(\frac{1}{2}, \pi \otimes \chi_i \cdot \chi'_i, St) \neq 0$.

The first condition is to find χ'_i such that the sign of the functional equation of $L(\frac{1}{2}, \pi \otimes \chi_i \cdot \chi'_i, St)$ is $+1$. This is a local problem and can always be solved. As explained in [HLSu], the local signs $\varepsilon(1/2, \pi_v \otimes \chi_{i,v} \cdot \cdot \chi'_i) \in \{\pm 1\}$ determine a certain Siegel-Weil Eisenstein series on a quasi-split unitary group $U(n,n)$, and the vanishing of the central value $L(\frac{1}{2}, \pi \otimes \chi_i \cdot \chi'_i, St)$ corresponds to the triviality of the pairing of this Eisenstein series with vectors in $\pi \otimes \chi_i \cdot \chi'_i \bigotimes (\pi \otimes \chi_i \cdot \chi'_i)^\vee$ in the doubling method. However, the Eisenstein series itself is non-trivial; so there are certainly representations π for which $L(\frac{1}{2}, \pi \otimes \chi_i \cdot \chi'_i, St) \neq 0$!

One would like to say that the L-function does not vanish for most π in a family of representations. For the families typically considered by analytic number theorists this also seems to be an inaccessible problem. On the other hand, one can prove such a generic non-vanishing result for Hida families of automorphic representations, provided one has well-behaved p-adic L-functions for these families. This will be explained in more detail in forthcoming work of the author with Eischen, Li, and Skinner.

4.4. Comparison of Theorem 4.2.6 with Deligne's conjecture.

It remains to compare the expression

$$(2\pi i)^{\frac{n(n+1)}{2}} \prod_{i=1}^{n-1} P^{(n-i)}(\Pi)$$

of Theorem 4.2.6 with the expression

$$d(M)^{\frac{1}{2}}(2\pi i)^{\frac{n(n+1)}{2}}[Q_{\det(M)}]^{\frac{n-1}{2}} \cdot \prod_{\ell} Q_{\ell}^{1-\ell}$$

predicted by Deligne's conjecture as expressed in Corollary 1.3.5; in other words, to justify a comparison

$$\prod_{i=1}^{n-1} P^{(n-i)}(\Pi) \sim_{\mathcal{K}} d(M)^{\frac{1}{2}}[Q_{\det(M)}]^{\frac{n-1}{2}} \cdot \prod_{\ell} Q_{\ell}^{1-\ell} \tag{4.4.1}$$

The comparison can only be heuristic, because the invariants Q_ℓ are defined in terms of a hypothetical polarized regular motive whereas the $P^{(n-i)}(\Pi)$ are normalized Petersson square norms of arithmetic holomorphic automorphic forms on Shimura varieties. We reason as §3.7 of [H97], deriving a version of (4.4.1) from the Tate Conjecture. Briefly, we stipulate that the Q_ℓ are defined for a motive $M(\Pi)$ with λ-adic realizations $\rho_{\Pi,\lambda}$, as in §1.1, while the $P^{(s)}(\Pi)$ are periods of a motive, say $M^{(s)}(\Pi)$, whose λ-adic realization is isomorphic to an explicit abelian twist of $\bigwedge^{n-s} M(\Pi)^\vee$; cf. [H97, 2.7.6.1, 2.7.7, 3.7.9] and the subsequent discussion. More precisely, in view of the Tate Conjecture, the relation of L-functions asserted as Conjecture 2.7.7 of [H97] motivates the following version of Hypothesis 3.7.9 of [H97][5]

$$\begin{aligned} M^{(s)}(\Pi) &\xrightarrow{\sim} \bigwedge^{r} M(\Pi)^\vee \otimes M(\chi_{\pi^+})(\frac{1}{2}r(r-1)),\ r = n-s \\ &\xrightarrow{\sim} (\bigwedge^{s} M(\Pi)) \otimes M(\chi_{\pi^+})^{-1}(\frac{1}{2}r(r-1)), \end{aligned} \tag{4.4.2}$$

where χ_{π^+} is the central character of any of the representations π^+ of one of the groups $GU(W) \supset U(W) = G$, the base change of whose restriction to G is Π. With χ_Π as in §1.2, we thus have

$$\chi_\Pi = \chi_{\pi^+}/\chi_{\pi^+}^c. \tag{4.4.3}$$

To be completely accurate, the restriction of π^+ to G may have several irreducible components π, but they all have the same base change to $GL(n)$. Note that the relation (4.4.3) is insensitive to the choice of extension of the central character of one such π to the center of $GU(W)$, which is isomorphic to $GL(1)_{\mathcal{K}}$. We have made the simplifying hypothesis that the parameter c of (2.4.3) equals 0, so we may assume the restriction of χ_{π^+} to the idèles of $\mathbb{Q}$ is a Hecke character of finite order, in other words a Dirichlet character χ_0.

As in [H97], (4.4.2) motivates the following relations:

$$P^{(n-i)}(\Pi) \sim_{\mathcal{K}} \prod_{\ell=1}^{n-i} Q_\ell \cdot Q(\chi_{\pi^+})^{-1}.$$

[5]Thanks to progress on the stable trace formula, especially the proof of the Fundamental Lemma, Langlands' Conjecture 2.7.7 on the cohomology of Shimura varieties attached to unitary groups is much closer to being established now than when [H97] was published. The conjecture has been proved in a number of cases, under simplifying hypotheses, the corresponding relations of automorphic representations are the subject of [CHL] and, more generally, of [Wh2].

Here $Q(\chi_{\pi^+})$ is defined by analogy with $Q_{\det M}$.

The Tate twist is invisible at this stage because the periods $P^{(s)}$ and Q_ℓ are defined with respect to the de Rham pairing, and $\mathbb{Q}(1)_{DR} = \mathbb{Q}$. Then the left hand side of (4.4.1) is

$$\sim_{\mathcal{K}} [\prod_{i=1}^{n-1}\prod_{\ell=1}^{n-i} Q_\ell] \cdot Q(\chi_{\pi^+})^{1-n} \ \sim_{\mathcal{K}} [\prod_{\ell=1}^{n-1} Q_\ell^{n-\ell}] \cdot Q(\chi_{\pi^+})^{1-n},$$

Thus the relation (4.4.1) follows from

$$(4.4.4) \qquad Q(\chi_{\pi^+}) \ \sim_{\mathcal{K}} d(M)^{\frac{1}{2}} Q^{\frac{1}{2}}_{\det M(\Pi)} \ \sim_{\mathcal{K}} d(M)^{\frac{1}{2}} Q(\xi_\Pi)^{\frac{1}{2}} = d(M)^{\frac{1}{2}} Q(\chi_\Pi)^{\frac{1}{2}}.$$

where the last relation is (1.2.1), bearing in mind that the Tate twist does not contribute to this calculation, so that $Q(\xi_\Pi) = Q(\chi_\Pi)$. By (4.4.3), (4.4.4) is equivalent to

$$(4.4.5) \qquad Q(\chi_{\pi^+}) \ \sim_{\mathcal{K}} d(M)^{\frac{1}{2}} Q(\chi_{\pi^+}/\chi^c_{\pi^+})^{\frac{1}{2}}$$

But $\chi^c_\pi = \chi_\pi^{-1}$ (since it is a character of $U(1)$), so $\chi_{\pi^+} \cdot \chi^c_{\pi^+}$ factors through the norm from $\mathcal{K}$ to $\mathbb{Q}$.

We hope to provide a hypothetical interpretation of $d(M)$ in a subsequent paper with Guerberoff. In the meantime, we may as well square the two sides of (4.4.5), which reduces the question to

$$(4.4.6) \qquad Q(\chi_{\pi^+} \cdot \chi^c_{\pi^+}) \ \sim_{\mathcal{K}} Q(\chi_0 \circ N_{\mathcal{K}/\mathbb{Q}}) \ \sim_{\mathcal{K}} 1,$$

with χ_0 as above. Finally, if we are willing to accept the analogue of the relation (3.3.5) (with $k_\ell = 0$):

$$Q(\chi_0 \circ N_{\mathcal{K}/\mathbb{Q}}) = p(([\chi_0 \circ N_{\mathcal{K}/\mathbb{Q}}]^{(2)})^\vee, 1)^{-2}$$

then we are done, because the definition implies that $\chi^{(2)}$ is trivial for any Dirichlet character χ composed with the norm.

References

[BHR] D, Blasius, M. Harris, and D.Ramakrishnan, Coherent cohomology, limits of discrete series, and Galois conjugation. *Duke Math. J.* **73** (1994), 647–685.

[CH] G. Chenevier and M. Harris, Construction of automorphic Galois representations, II, to appear.

[Cl90] L. Clozel, Motifs et formes automorphes: applications du principe de fonctorialité, In L. Clozel and J. S. Milne, eds., *Automorphic forms, Shimura varieties, and L-functions*, Vol. I , *Perspect. Math.* **10** (1990) 77–159.

[CHL] L. Clozel, M. Harris, and J.-P. Labesse, Construction of automorphic Galois representations, I, in L. Clozel, M. Harris, J.-P. Labesse, and B. C. Ngô, eds., *Stabilization of the trace formula, Shimura varieties, and arithmetic applications. Volume 1: On the stabilization of the trace formula*, Boston: International Press (2011).

[D1] P. Deligne, Variétés de Shimura : Interprétation modulaire, et techniques de construction de modèles canoniques, *Proc. Symp. Pure Math.*, **33.2** (1979) 247-290.

[D2] P. Deligne, Valeurs de fonctions L et périodes d'intégrales, *Proc. Symp. Pure Math.*, **33.2** (1979) 313-342.

[DFG] F. Diamond, M. Flach, L. Guo, The Tamagawa number conjecture of adjoint motives of modular forms. *Ann. Sci. cole Norm. Sup.*, **37** (2004) 663–727.

[Di] M. Dimitrov, On Ihara's lemma for Hilbert modular varieties. *Compos. Math.*, **145** (2009) 1114–1146.

[FH] W. Fulton and J. Harris, *Representation theory. A first course*, *Graduate Texts in Mathematics*, **129** Springer-Verlag, New York, (1991).

[GGP1] W. T. Gan, B. Gross, and D. Prasad, Symplectic local root numbers, central critical L-values, and restriction problems in the representation theory of classical groups, manuscript (2009).

[GGP2] W. T. Gan, B. Gross, and D. Prasad, Restrictions of representations of classical groups: examples, manuscript (2009).

[H97] M. Harris, L-functions and periods of polarized regular motives. *J. Reine Angew. Math.* **483** (1997), 75–161.

[H07] M. Harris, A simple proof of rationality of Siegel-Weil Eisenstein series, in W.T. Gan, S. S. Kudla, and Y. Tschinkel, eds., *Eisenstein Series and Applications*, Birkhuser: Progress in Mathematics **258** (2007) 149–186.

[H07b] M. Harris, Cohomological automorphic forms on unitary groups, II: period relations and values of L- functions, in *Harmonic Analysis, Group Representations, Automorphic Forms and Invariant Theory*, Vol. 12, Lecture Notes Series, Institute of Mathematical Sciences, National University of Singapore (special issue in honor of R. Howes 60th birthday), 89-150 (2007).

[H11] M. Harris, Beilinson-Bernstein localization over $\mathbb{Q}$ and periods of automorphic forms, (manuscript, 2011).

[HLSu] M. Harris, J.-S. Li, B.-Y. Sun, Theta correspondences for close unitary groups, in *Arithmetic Geometry and Automorphic Forms*, *Advanced Lectures in Mathematics*, **19**,(2011) 265–307.

[NH] N. Harris, A Refined Gross-Prasad Conjecture for Unitary Groups, thesis, UCSD (2011).

[Hi] H. Hida, Congruence of cusp forms and special values of their zeta functions. *Invent. Math.*, **63** (1981) 225–261.

[II] A. Ichino and T. Ikeda, On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture, *Geom. Funct. Anal.*, **19** (2010) 1378–1425.

[L] J.-P. Labesse, Changement de base CM et séries discrètes, in L. Clozel, M. Harris, J.-P. Labesse, and B. C. Ngô, eds., *Stabilization of the trace formula, Shimura varieties, and arithmetic applications. Volume 1: On the stabilization of the trace formula*, Boston: International Press (2011).

[Mi] A. Minguez, Unramified representations of unitary groups, in L. Clozel, M. Harris, J.-P. Labesse, and B. C. Ngô, eds., *Stabilization of the trace formula, Shimura varieties, and arithmetic applications. Volume 1: On the stabilization of the trace formula*, Boston: International Press (2011) 389–410.

[M] C. Moeglin, Classification et changement de base pour les séries discrètes des groupes unitaires p-adiques. *Pacific J. Math.*, **233** (2007) 159–204.

[MW] C. Moeglin and J.-L. Waldspurger, La conjecture locale de Gross-Prasad pour les groupes spéciaux orthogonaux : le cas général, preprint 2010.

[NH] N. Harris, A Refined Gross-Prasad Conjecture for Unitary Groups, thesis, UCSD (2011).

[W] J.-L. Waldspurger, Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie, *Compositio Math.*, **54** (1985) 173–242.

[Wh1] P.-J. White, Le produit tensoriel automorphe et l'endoscopie sur le groupe unitaire, thesis (2010).

[Wh2] P.-J. White, Tempered automorphic representations of the unitary group, preprint (2011).