THE TAYLOR-WILES METHOD FOR COHERENT COHOMOLOGY (PRELIMINARY DRAFT)

Michael Harris

UFR de Mathématiques
Université Paris 7

Introduction

The Taylor-Wiles method was introduced in [TW] in order to complete Wiles' proof of Fermat's Last Theorem [W]; it shows that a map from a ring of deformations of a mod p Galois representation $\bar{\rho}$ to a Hecke algebra, acting on the integral p-adic topological cohomology of modular curves, is an isomorphism when the necessary hypotheses apply, specifically that $\bar{\rho}$ admits at least one lifting that arises from modular forms. The isomorphism of rings, called an " $R = T$ theorem", then implies that any deformation of $\bar{\rho}$ that satisfies the appropriate hypotheses – in the simplest situation it has to be crystalline at p and minimally ramified at other primes – also arises from modular forms.

The technique has now been applied in a variety of higher-dimensional settings, either with minimal ramification [CHT, GT] or not (for example [SW,T,K, BGHT, BGGT]), always with the goal of proving that Galois representations arise from automorphic forms. All of these references depend on the improved version of the Taylor-Wiles method developed independently by Diamond [D] and Fujiwara [Fu]. Where the original Taylor-Wiles method was based on direct comparison of p-adic integral Hecke algebras of varying levels, the Diamond-Fujiwara version is based on a comparison of the p-adic integral modules of automorphic cohomology, whose stability properties are much easier to check. As a bonus, at least in the case of minimal ramification – the situation of [CHT], which is the only case considered in the present paper – the Diamond-Fujiwara approach implies that these modules are free over the corresponding Hecke algebras.

The purpose of this paper is to show that the Diamond-Fujiwara version of the Taylor-Wiles method can sometimes be applied, in situations of when the topological cohomology is replaced by coherent cohomology with coefficients in appropriate automorphic vector bundles. In practice, all the higher-dimensional results, with the exception of [GT] and [Pi], have been based on topological cohomology of zero-dimensional Shimura varieties. This is because the Taylor-Wiles method does not work well in the presence of torsion, and there are no general methods for comparing torsion in the cohomology of locally symmetric spaces to automorphic forms. We will be working with a Shimura variety that admits a smooth model $\mathbb{S}_K$ over a

Institut des Mathématiques de Jussieu, U.M.R. 7586 du CNRS. Membre, Institut Universitaire de France.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

p-adic integer ring $\mathcal{O}$; here the index K designates a level subgroup, an open compact subgroup in a certain adelic group. If $\mathcal{F}$ is an automorphic vector bundle over $\mathbb{S}_K$ then $H^0(\mathbb{S}_K, \mathcal{F})$ is a free $\mathcal{O}$-module; thus for coherent cohomology in degree 0 the problem of torsion does not arise. On the other hand, the Diamond-Fujiwara argument requires that $H^0(\mathbb{S}_{K_Q}, \mathcal{F})$ be free over certain group algebras (the *diamond operators*) as Q varies. This is easy to verify in the zero-dimensional case but is not at all obvious in general.

To prove that coherent H^0 is free over the diamond operators, we use a theorem of S. Nakajima [N] to which I was led by reading an article of Chinburg [C] that treats a more general situation. Nakajima's theorem implies that, under natural hypotheses, $H^0(\mathbb{S}_{K_Q}, \mathcal{F})$ is free over the diamond operators, provided the higher cohomology of $\mathcal{F}$ vanishes. An earlier draft of this paper, sent to a few colleagues in July 2010, used the vanishing theorem of Deligne-Illusie-Raynaud [DI] in conjunction with Nakajima's theorem to apply the Taylor-Wiles method to H^0 of certain classes of automorphic line bundles on PEL-type Shimura varieties attached to anisotropic unitary groups, and speculated that a version of the Deligne-Illusie theorem for Faltings' dual Bernstein-Gelfand-Gelfand should allow the treatment of more general automorphic vector bundles. A tip from Ahmed Abbes, who read this draft, led me to the striking work of Lan and Suh [LS1, LS2], which proved an optimal vanishing theorem for automorphic vector bundles on PEL-type Shimura varieties in mixed characteristic. The present draft uses the Lan-Suh vanishing theorems to apply the Taylor-Wiles method to coherent cohomology of a substantial class of automorphic vector bundles on unitary group Shimura varieties. These are the vector bundles whose group-theoretic parameters are small relative to p, in a well-defined sense, and sufficiently far from the walls of the small p-alcove (see §6.10 for a precise characterization).

In fact, thanks to the results of [LS1], we can apply the Taylor-Wiles method to coherent cohomology in positive degree as well. In characteristic zero, by combining Hodge theory with the classification of unitary representations of Lie groups with non-zero cohomology, one can assign a preferential degree $q(\mathcal{F})$ to most automorphic vector bundles, so that $H^i(\mathbb{S}_{K_Q} \otimes \mathbb{Q}, \mathcal{F}) = 0$ for $i \neq q(\mathcal{F})$, at least if $\mathbb{S}_{K_Q}$ is projective. The Lan-Suh papers shows that this remains true in mixed characteristic, again for parameters inside and away from the walls of the small p-alcove.

In degree zero, an analogous result was obtained in Vincent Pilloni's thesis for Siegel modular forms that are ordinary in Hida's sense. Pilloni's theorem, which appears as Théorème 7.2 in the article [Pi], is based on Hida theory; I do not know whether or not it can be adapted to non-ordinary forms.

We obtain no new results about Galois representations, and in fact I believe that practically everything one wants to say about automorphy of Galois representations can be obtained from the zero-dimensional case, as in [CHT], using Langlands functoriality for classical groups (see [A] and, in special cases, [CHLN]). Our purpose is rather to prove that $H^{q(\mathcal{F})}(\mathbb{S}_K, \mathcal{F})$, or rather its localizations at non-Eisenstein ideals of the Hecke algebra, is free over the (localized) Hecke algebra (always in the case of minimal ramification). In the original Taylor-Wiles argument, this property was a hypothesis, justified by arguments going back to Mazur's work on the Eisenstein ideal. One of the benefits of the Diamond-Fujiwara version of the Taylor-Wiles argument is that this property is obtained as a bonus in the course of proving the $R = T$ theorem. Using the degeneration of the BGG spectral sequence and p-

adic comparison theorems, we can also show that the (localized) middle-dimensional Betti cohomology of $\mathbb{S}_{K_Q} \otimes \mathbb{Q}$ with coefficients in (p-integral) local systems is free over the (localized) Hecke algebra, again when the parameters satisfy the Lan-Suh regularity hypotheses. Such results, both in the coherent and the topological settings, have applications to the study of p-adic families of cohomological modular forms and to the construction of p-adic L-functions. Although the structure of coherent cohomology as a Hecke module apparently has nothing to do with Galois representations, I know of no other way to prove such results.

I note that vanishing theorems for toroidal compactifications of Siegel modular varieties have been used by Stroh [St] to prove that certain Siegel modular forms lift to characteristic zero. The second Lan-Shu paper proves vanishing theorems for *interior cohomology* of coherent sheaves on non-compact PEL type Shimura varieties. It is likely that the localized coherent interior cohomology groups are also free over Hecke algebras in the non-compact case, but Nakajima's theorem does not apply directly. I plan to return to this question in the future (possibly in a later draft).

Experts are aware that in general one can only prove $R = T$ theorems in minimal level, and starting with [SW], [K] and [T] one has been content to prove weaker versions that suffice to prove automorphy of Galois representations. The problem is that Ribet's techniques of level raising and level lowering are not available in higher dimensions. The generalization of Ihara's lemma conjectured in [CHT] would suffice to prove more general $R = T$ theorems, but only if Ribet's level-lowering techniques can be similarly generalized.

The present study is an outgrowth of my ongoing work on p-adic L-functions with Li and Skinner, which is projected to continue, at least in part, with Eischen and Emerton. I thank them for many discussions about p-adic families of modular forms. I am also grateful for exchanges with Nick Shepherd-Barron and Marc Levine. I thank Ahmed Abbes for alerting me to the work of Lan and Suh and Kai-Wen Lan for sending me copies of their papers, and for correcting some of the misrepresentations of their contents that found their way into an earlier version of this manuscript. Thanks are also due to Jacques Tilouine for drawing my attention to the results of Pilloni on Taylor-Wiles systems for Siegel modular forms. Finally, I thank the Petersburg branch of the Steklov Institute of Mathematics for their hospitality during the preparation of this paper.

1. Shimura varieties and automorphic vector bundles

We will ultimately restrict our attention to Shimura varieties attached to unitary groups, but we begin by considering a the most general situation. Let (G, X) be a Shimura datum, with G a reductive algebraic group over $\mathbb{Q}$ and X a (disconnected) symmetric space for $G(\mathbb{R})$, as in Deligne's formalism. Let $E = E(G, X) \subset \mathbb{C}$ be the reflex field and let p be an *odd* prime at which G is unramified. We let $K \subset G(\mathbf{A}^f)$ be an open compact subgroup, and we assume $K = K_p \times K^p$ where $K_p \subset G(\mathbb{Q}_p)$ is a hyperspecial maximal compact subgroup. The Shimura variety $S_K(G, X)$, whose $\mathbb{C}$ points are given by

$$G(\mathbb{Q})\backslash X \times G(\mathbf{A}^f)/K,$$

has a canonical model over $E(G, X)$, and as K varies subject to the restriction at

p, the inverse limit

$$S_{K_p}(G,X) := \varprojlim_{K \supset K_p} S_K(G,X)$$

admits a continuous action of $G(\mathbf{A}_f^p)$ that coincides with the obvious one over $\mathbb{C}$ and respects the canonical models.

Choose a prime v of E of characteristic p. We assume the family of E-schemes $\{S_K(G,X), K \subset K_p\}$ admit extensions to flat $\mathcal{O}_v$ schemes $\mathbb{S}_K = \mathbb{S}_K(G,X)$ and that the action of $G(\mathbf{A}_f^p)$ extends compatibly. In the applications the extensions will be defined uniquely. If K is sufficiently small, for example if K is *neat* in the sense of [P], then the action of $G(\mathbb{Q})$ on $X \times G(\mathbf{A}^f)/K$ has no fixed points and $S_K(G,X)$ is a smooth variety over $Spec(E)$; we assume that $\mathbb{S}_K$ is then smooth over $Spec(\mathcal{O}_v)$. Unless specified otherwise, K will always be assumed neat.

The points $x \in X$ are identified with homomorphisms

$$h_x : \mathbb{S} = R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)_{\mathbb{C}} \to G_{\mathbb{R}}$$

satisfying Deligne's list of axioms [De, 2.1.1]. The centralizer in $G(\mathbb{R})$ of h_x is a reductive group K_x whose intersection with the derived subgroup $G^{der}(\mathbb{R})$ of $G(\mathbb{R})$ is a maximal compact connected subgroup. Deligne's axiom (2.1.1.1) concerns the adjoint action

$$Ad \circ h_x : \mathbb{S} \to GL(Lie(G)_{\mathbb{C}})$$

that yields an eigenspace decomposition (the *Harish-Chandra decomposition*

$$Lie(G)_{\mathbb{C}} = \mathfrak{p}_x^- \oplus Lie(K_x)_{\mathbb{C}} \oplus \mathfrak{p}_x^+. \tag{1.1}$$

Here $z \in \mathbb{S}(\mathbb{R}) \xrightarrow{\sim} \mathbb{C}^\times$ acts trivially on $Lie(K_x)$ and as $(z/\bar{z})$ (resp. $\bar{z}/z$) on $\mathfrak{p}_x^+$ (resp. $\mathfrak{p}_x^-$). The Lie subalgebras $\mathfrak{p}_x^-$ and $\mathfrak{p}_x^+$ are naturally identified, respectively, with the anti-holomorphic and holomorphic tangent spaces of X at x.

Let $\check{X}$ denote the compact dual of X, and $X \hookrightarrow \check{X}$ the Borel embedding. Concretely, $\check{X}$ is a flag variety of maximal parabolic subgroups of $G_{\mathbb{C}}$ and the image in $\check{X}$ of $x \in X$ is a maximal parabolic P_x with Levi subgroup K_x. In [H85] it is explained how to define a canonical E-rational structure on the flag variety $\check{X}$, following Deligne, and how to define a functor $\mathcal{V} \mapsto [\mathcal{V}]$ from G-equivariant vector bundles on X to $G(\mathbf{A}_f^p)$-equivariant vector bundles on $S_{K_p}(G,X)$. The latter are called *automorphic vector bundles*. The functor is compatible with the E-structure in the sense that, if $\mathcal{V}$ is defined as G-equivariant vector bundle over a field $E(\mathcal{V})$ (which can always be taken to be a number field), then for any $\sigma \in Gal(\bar{E}/E)$, we have

$$\sigma[\mathcal{V}] = [\sigma(\mathcal{V})] \tag{1.2}$$

The bracket notation in the previous paragraph was introduced in order to make reference to the functor. Automorphic vector bundles will in general be denoted $\mathcal{F}$, and we let $E(\mathcal{F})$ denote a field of definition for the corresponding equivariant vector bundle over $\check{X}$.

Fix a point $x \in X$ with stabilizer $P_x \subset G_{\mathbb{C}}$; thus

$$Lie(P_x) = Lie(K_x)_{\mathbb{C}} \oplus \mathfrak{p}_x^+ \tag{1.3}$$

in the Harish-Chandra decomposition. There is a natural equivalence of categories between G-equivariant vector bundles $\mathcal{V}$ on X and finite-dimensional representations (τ, W_τ) of P_x; W_τ is the fiber of $\mathcal{V}$ at x, and τ is the isotropy representation. In particular, the natural representation of P_x

$$ad^+ : P_x \to K_x \to Aut(\mathfrak{p}_x^+),$$

where the second arrow is the adjoint representation, defines an automorphic vector bundle canonically isomorphic to the tangent bundle $\mathcal{T}_{S(G,X)}$. Likewise $\wedge^{top}(ad^+)^\vee$, the dual of the top exterior power of the adjoint action on $\mathfrak{p}_x^+$, defines the canonical bundle $\Omega^{top}_{S(G,X)}$ as an automorphic vector bundle.

A representation (τ, W_τ) of K_x extends trivially to a representation of P_x and thus defines an automorphic vector bundle $\mathcal{F} = \mathcal{F}_\tau$ on $S_{K_p}(G,X)$ whose fiber at a point beneath $x \times g$ for any $g \in G(\mathbf{A}^f)$ can be identified with W_τ. The automorphic vector bundle $\mathcal{F}$ can also be identified with the family of bundles $\mathcal{F}_K$ on $S_K(G,X)$. If $\mathcal{F}_K$ extends to a $G(\mathbf{A}_f^p)$-equivariant vector bundle on $\mathbb{S}_K$, we denote the extension $\mathcal{F}_{K,\mathbb{S}}$; in the applications a specific extension will be defined.

In the following discussion, we assume G is anisotropic modulo center, so that $S_K(G,X)$ is a smooth projective variety for each (neat) K. Let $\gamma \in G(\mathbf{A}_f^p)$. Then $\gamma^{-1}K\gamma$ is again a neat level subgroup, and right translation by γ defines an isomorphism

$$r_\gamma : S_K(G,X) \xrightarrow{\sim} S_{\gamma^{-1}K\gamma}(G,X).$$

Let $K(\gamma) = K \cap \gamma^{-1}K\gamma$. There are natural finite morphisms

$$p_1 : S_{K(\gamma)}(G,X) \to S_K(G,X); p_2 : S_{K(\gamma)}(G,X) \to S_{\gamma^{-1}K\gamma}(G,X). \tag{1.4}$$

Let $\mathcal{F}$ be an automorphic vector bundle on $S_K(G,X)$. For any integer i, there are canonical maps

$$\begin{aligned} &p_1^* : H^i(S_K(G,X), \mathcal{F}_K) \to H^i(S_{K(\gamma)}(G,X), \mathcal{F}_{K(\gamma)}); \\ &p_{2,*} : H^i(S_{K(\gamma)}(G,X), \mathcal{F}_{K(\gamma)}) \to H^i(S_{\gamma^{-1}K\gamma}(G,X), \mathcal{F}_{\gamma^{-1}K\gamma}) \end{aligned} \tag{1.5}$$

and we define

$$T(\gamma) = (r_\gamma^{-1})_* \circ p_{2,*} \circ p_1^* : H^i(S_K(G,X), \mathcal{F}_K) \to H^i(S_K(G,X), \mathcal{F}_K). \tag{1.6}$$

Here $(r_\gamma^{-1})_*$ is the functorial map on coherent cohomology associated to the inverse of the map r_γ of schemes and the corresponding map on equviariant vector bundles.

Assume the integral models $\mathbb{S}_K$ and $\mathcal{F}_{K,\mathbb{S}}$ exist for all neat $K \supset K_p$. We let $\mathcal{O} = \mathcal{O}(\mathcal{F})$ denote the integral closure of $\mathcal{O}_v$ in a fixed p-adic completion of $E(\mathcal{F})$, and in what follows we work over $\mathcal{O}$; in particular, we replace $\mathbb{S}_K$ by $\mathbb{S}_K \times_{Spec(\mathcal{O}_v)} Spec(\mathcal{O})$. Under the structural morphism $p : \mathbb{S}_K \to Spec(\mathcal{O})$, we can identify $R^ip_*(\mathcal{F}_{K,\mathbb{S}})$ with an $\mathcal{O}$-module of finite type, denoted simply $H^i(\mathbb{S}_K, \mathcal{F}_K)$. Then for any $\gamma \in G(\mathbf{A}_f^p)$, we can define the map

$$T(\gamma) : H^i(\mathbb{S}_K, \mathcal{F}_K) \to H^i(\mathbb{S}_K, \mathcal{F}_K)$$

by the formula (1.6). The *Hecke algebra* $\mathcal{H}^i(\mathcal{F}, K)$ is the $\mathcal{O}$-subalgebra of $End_{\mathcal{O}}(H^i(\mathbb{S}_K, \mathcal{F}_K))$ generated by $T(\gamma)$ for $\gamma \in G(\mathbf{A}_f^p)$.

Let S be the set of rational primes q such that $K_q = K \cap G(\mathbb{Q}_q)$ is not a hyperspecial maximal compact subgroup. We let $G^S \subset G(\mathbf{A}_f^p)$ be the subgroup of elements $\gamma = (\gamma_q)$ whose local components γ_q are trivial at all $q \in S$. The *unramified Hecke algebra* $\mathbb{T}^i(\mathcal{F}, K)$ is the $\mathcal{O}$-subalgebra generated by $T(\gamma)$ for $\gamma \in G^S$. This is the image of the restricted tensor product of the local Hecke algebras of $G(\mathbb{Q}_q)$ relative to the hyperspecial maximal compact subgroup K_q, and hence is a commutative subalgebra of $End(H^i(\mathbb{S}_K, \mathcal{F}_K))$ The purpose of this paper is to find conditions that guarantee that $H^i(\mathbb{S}_K, \mathcal{F}_K)$ is a free $\mathbb{T}^i(\mathcal{F}, K)$-module. In practice the definition of $\mathbb{T}^i(\mathcal{F}, K)$ may include fewer Hecke operators, but there should be no difference with the present definition.

Compactifications. If G is not anisotropic modulo center, then $S_K(G, X)$ and $\mathbb{S}_K(G, X)$ are not projective. Given the additional data of a family of fans Σ, one can define a toroidal compactification $S_{K,\Sigma}(G, X)$ as in [H86, P]. Assume K is neat in the sense of [P]. Then under appropriate hypotheses, indicated in the above references, $S_{K,\Sigma}(G, X)$ is smooth, projective, and has a canonical model over $E(G, X)$ extending that on $S_K(G, X)$.

Provided K is neat, an automorphic vector bundle $\mathcal{F}_K$ on $S_K(G, X)$ admits a *canonical extension* $\mathcal{F}_K^{can}$ over $S_{K,\Sigma}(G, X)$, constructed in [H86] following Mumford. Except in a few low-dimensional cases, the action of $G(\mathbf{A}_f^p)$ only extends to the family $S_{K,\Sigma}(G, X)$ unless the Σ are also allowed to vary along with K, and in particular the algebra $\mathcal{H}_K$ of Hecke operators of level K do not act geometrically on $\mathcal{F}_K^{can}$. However, it is explained in [H90] that $\mathcal{H}_K$ does act canonically on the finite-dimensional vector space $H^i(S_{K,\Sigma}, \mathcal{F}_K^{can})$, for each i.

In certain cases it is known that $S_{K,\Sigma}(G, X)$ extends to a toroidal compactification of $\mathbb{S}_K(G, X)$ that is smooth and projective over $\mathcal{O}_v$. This has been constructed notably in the book [FC] of Faltings-Chai when G is the symplectic similitude group, and there are various additional cases in the literature, as well as some unpublished work of Vasiu. At present we cannot answer any questions regarding the cohomology of these compactifications, but it is natural to ask them in the setting of our axiomatic discussion in §3.

2. Parameters for unitary groups

Let F be a CM field, quadratic over a totally real field F^+. Let V be an n-dimensional space over F with nondegenerate hermitian form $(\bullet, \bullet)$, let $U = U(V)$ be its unitary group, and define the reductive group G over $\mathbb{Q}$ by its values on $\mathbb{Q}$-algebras R:

$$G(R) = \{g \in GL(V \otimes_{\mathbb{Q}} R) \mid (g(v), g(w)) = \nu(g)(v, w) \text{ for some } \nu(g) \in R^\times\}.$$

There is a standard way to extend G to a Shimura datum (G, X) so that $S(G, X)$ is a Shimura variety of PEL type. In subsequent sections we assume G is globally anisotropic modulo its center, which implies if $n > 2$ that the completion of V at at least one archimedean place of F is totally definite. This implies that $S_K(G, X)$ is projective for all K. However, all the results of the present section are independent of this hypothesis. We let $d_V = \dim S_K(G, X)$.

Because $S(G, X)$ is of PEL type and G is of type A, it follows from [Ko] that one of the (unlabelled) hypotheses of §1 is satisfied. Let $g = n \cdot [F : \mathbb{Q}]$.

Proposition 2.1. *Fix a prime p at which G is unramified, and fix a hyperspecial maximal compact $K_p \subset G(\mathbb{Q}_p)$. The integral models $\mathbb{S}_K$, as in §1 exist for all neat $K \supset K_p$. More precisely, if K is neat and contains K_p, then up to replacing K by a normal subgroup K' of finite index, for any prime v of $E(G,X)$ dividing p there is a smooth projective moduli scheme $\mathbb{S}_{K'}$ over $Spec(\mathcal{O}_v)$ of abelian varieties of dimension g, with a PEL structure defined in terms of the hermitian space V, whose generic fiber is isomorphic to $S_{K'}(G,X)$, and the quotient of $\mathbb{S}_{K'}$ by K/K' gives the smooth integral model $\mathbb{S}_K$.*

Before discussing how automorphic vector bundles extend over the integral model, we briefly recall their construction in characteristic zero. Fix a point $x \in X$, viewed as a subset of $\check{X}$, and let stabilizer $P_x \subset G$ denote its stabilizer. Let S_∞ denote the set of real places of F^+, and let $\tilde{S}_\infty$ denote a CM type for F, i.e. a choice for each $v \in S_\infty$ of a complex place $\tilde{v}$ of F. The CM type is in fact part of the data defining X. For each $v \in S_\infty$, there is a partition $r_v + s_v = n$ such that the hermitian space $V \otimes_{F,\tilde{v}} \mathbb{C}$ has signature (r_v, s_v); then $U(F_v^+) \xrightarrow{\sim} U(r_v, s_v)$, with the ambiguity between $U(r_v, s_v)$ and $U(s_v, r_v)$ resolved by the choice of $\tilde{v}$.

Let $K_x \subset P_x$ as in §1, and let $T \subset K_x$ be a maximal torus; then T is also a maximal torus in G. Let $\Phi = \Phi(G,T)$ denote the set of roots of G relative to T, and let Φ^+ be a system of positive roots compatible with P_x, i.e. containing the roots of $\mathfrak{p}^-$ (with apologies for the change of sign). For any m let $\mathfrak{S}_m$ be the symmetric group on m letters. Let W^x be a set of Kostant representatives for the Weyl group of G relative to the maximal parabolic P_x; this is the section $\kappa : W(G,T)/W(K_x,T) \to W(G,T)$ of the map

$$W(G,T) \to W(G,T)/W(K_x,T) \simeq \prod_{v \in S_\infty} \mathfrak{S}_n/(\mathfrak{S}_{r_v} \times \mathfrak{S}_{s_v}),$$

with the property that the length $\ell(\kappa(\bar{w}))$ is minimal among elements of $W(G,T)$ mapping to $\bar{w} \in W(G,T)/W(K_x,T)$.

The group $G_\mathbb{C}$ is isomorphic to

$$\prod_{v \in S_\infty} GL(n)_{F_v} \times GL(1)_\mathbb{C} = GL(n)_\mathbb{C}^{[F^+:\mathbb{Q}]} \times GL(1)_\mathbb{C}. \tag{2.2}$$

We may identify $T_\mathbb{C}$ with the product of subgroups of diagonal matrices in the right-hand side of (2.2):

$$T(\mathbb{C}) = (GL(1,\mathbb{C})^n)^{[F^+:\mathbb{Q}]} \times GL(1)_\mathbb{C}.$$

The character group $X(T)_\mathbb{C} = X(T)_{\overline{\mathbb{Q}}}$ of T can correspondingly be identified with

$$X(T) = (\mathbb{Z}^n)^{[F^+:\mathbb{Q}]} \times \mathbb{Z}.$$

The subset $X^+(T)$ of dominant weights for G is then identified with the set of integer parameters

$$((a_{1,\tau} \geq a_{2,\tau} \geq \cdots \geq a_{n,\tau})_{\tau \in S_\infty}; c) \tag{2.3}$$

with c the character of the final $GL(1)_{\mathbb{C}}$. This is contained in the set $X_x^+(T)$ of dominant weights for K_x, which reflects the signature:

$$((a_{1,\tau} \geq \cdots \geq a_{r,\tau}; a_{r+1,\tau} \geq \cdots \geq a_{n,\tau})_{\tau \in S_\infty}; c)$$

Let $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+}$ denote the half-sum of roots in Φ^+. Let $\mu \in X^+(T)$. For any $w \in W^x$ it is well known that

$$w * (\mu) = w(\mu + \rho) - \rho) \in X_x^+(T). \tag{2.4}$$

Let W be the finite-dimensional irreducible representation of G with highest weight μ; it is defined over the finite extension $E(W)$ of $E(G,X)$ such that $Gal(\overline{\mathbb{Q}}/E(W))$ is the stabilizer of μ in $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Let $\tilde{W}$ denote the corresponding local system in complex vector spaces over $S(G,X)$ or $S_K(G,X)$ (any $K \supset K_p$):

$$G(\mathbb{Q})\backslash W(\mathbb{C}) \times X \times G(\mathbf{A}^f)/K.$$

The cohomology of $S(G,X)$ with coefficients in $\tilde{W}$ obtains a Hodge structure (pure, because G is anisotropic modulo center) first considered in this generality by Zucker. We use Faltings' dual Bernstein-Gelfand-Gelfand resolution, written as in [H94],

$$0 \longrightarrow \tilde{W} \to \mathcal{K}^0(\tilde{W}) \to \mathcal{K}^1(\tilde{W}) \to \ldots \to \mathcal{K}^{d_V}(\tilde{W}) \to 0, \tag{2.5}$$

where for each i,

$$\mathcal{K}^i(\tilde{W}) \xrightarrow{\sim} \bigoplus_{\ell(w)=i, w \in W^x} \mathcal{E}_w(W))$$

to write the Hodge decomposition for cohomology in middle degree:

$$H^{d_V}(Sh(V), \tilde{W}) \xrightarrow{\sim} \bigoplus_{w \in W^x} H^{d_V - \ell(w)}(Sh(V), \mathcal{E}_w(W)) \otimes \mathbb{C}. \tag{2.6}$$

Here $\ell(w)$ is the length of $w \in W^x$, cf. [H94, p. 571] and $\mathcal{E}_w(W)$ is the automorphic vector bundle attached to the isotropy representation $\mathcal{E}_{w,x}(W)$ of P_x, as in §1. It is known that $\mathcal{E}_{w,x}$ is the irreducible representation of K_x with highest weight $w * (\mu)$.

In what follows, if $\alpha \in \Phi$ let $\alpha^\vee \in X_*(T) = Hom(X(T), \mathbb{Z})$ denote its coroot.

Definition 2.7 (cf. [LS1,2.32]). *Say a weight $\mu \in X^+(T)$ is p-small, resp. p-small relative to x if*

$$< \mu + \rho, \alpha^\vee > \leq \ p, \forall\ \alpha \in X^+(T) \ (resp.\ \forall\ \alpha \in X_x^+(T)).$$

For comparison with the results of §7 below, I note that §3 of [LS1] is actually their second construction of the $\mathcal{E}_w(W)_{K,\mathbb{S}}$. Their first construction, in §1 of [LS1], is based on the construction for PEL type Shimura varieties of what they call the **principal** P_x **bundle** over $\mathbb{S}_K$, which defines an exact functor from $\mathcal{O}$-representations of P_1 to $G(\mathbf{A}_f^p)$-equivariant vector bundles on $\mathbb{S}_K$. This is the integral version of the functor defined for general Shimura varieties in characteristic zero in [H85] and recalled in §1.

In what follows, we fix a prime v of $E(G,X)$ dividing p, and let $\mathcal{O}$ denote a sufficiently large finite extension of $\mathcal{O}_v$. Concretely, in the statement of Proposition 2.8, we can take $\mathcal{O}$ to be the integers in the completion of the extension $E(W_\mu) \supset E(G,X)$ at a prime dividing v.

Proposition 2.8. *Suppose $\mu \in X^+(T)$ is p-small and let $W = W_\mu$ be the corresponding irreducible representation of G. Then for all $w \in W^x$, the automorphic vector bundle $\mathcal{E}_w(W)$ extends to a vector bundle $\mathcal{E}_w(W)_{K,\mathbb{S}}$ on the integral model $\mathbb{S}_K \times_{Spec(\mathcal{O}_v)} Spec(\mathcal{O})$, whenever $K \supset K_p$ is a neat compact open subgroup of $G(\mathbf{A}^f)$. As K varies, these extensions are compatible with the action of $G(\mathbf{A}_f^p)$. In particular, for any integer i, the Hecke algebra $\mathcal{H}^i(\mathcal{E}_w(W), K)$ can be defined as in §1.*

Proof. The first assertion summarizes the discussion of §3 of [LS1]; the result is also proved in [Ro] though it is not stated as such. The second assertion is clear from the construction of [LS1] in terms of the principal P_x bundle, the point being that this bundle is itself $G(\mathbf{A}_f^p)$-equivariant, (cf. the corresponding construction in [H85]).

3. Vanishing and cyclic covers

In the Taylor-Wiles method one chooses collections $Q = \{q_1, \ldots, q_s\}$ of primes of the ground field F^+, of fixed cardinality s, such that

$$Nq_i = |\mathcal{O}_{q_i}/q_i| \equiv 1 \pmod p, \ i = 1, \ldots, s \tag{3.1}$$

and satisfying a short list of additional conditions determined by the residual Galois representation under consideration. Let Δ_{q_i} denote the maximal p-power quotient of $(\mathcal{O}_{q_i}/q_i)^\times$ and let $\Delta_Q = \prod_{i=1}^s \Delta_{q_i}$

We start with a level subgroup K as in the previous section. We write $K^U = K \cap U(\mathbf{A}^f)$, and for a prime v of $\mathbb{Q}$ write $K_v^U = K_v \cap U(\mathbb{Q}_v)$. If v is a place of $\mathbb{Q}$ such that every prime q of F^+ dividing v splits in F, then we assume we can write

$$K_v^U = \prod_{q \mid v} K_q.$$

It is always assumed that Q contains no primes dividing a prime in the set S of ramified primes considered above, and that K admits a factorization $K^Q \times K_Q$ where $K_Q = \prod_{q \in Q} K_q$. Thus for each $q \in Q$, K_q is a hyperspecial maximal compact subgroup of $U(F_q^+)$. For each $q \in Q$, we choose two open subgroups $K_1(q) \subset K_0(q) \subset K_q$, with $K_1(q)$ normal in $K_0(q)$, and with a given isomorphism

$$K_0(q)/K_1(q) \xrightarrow{\sim} (\mathcal{O}_q/q)^\times. \tag{3.2}$$

For example, when $K_q = GL(n, \mathcal{O}_q)$, we take $K_0(q)$ (resp. $K_1(q)$) as in [CHT,2.1] to be the subgroup consisting of matrices with the last row congruent to $(0, \ldots, 0, *)$ (resp. $(0, \ldots, 0, 1)$) modulo q. Let $K_0(Q) = K^Q \otimes \prod_{q \in Q} K_0(q)$ and define $K_1(Q)$ analogously.

There is a canonical map $p_\Delta : K_0(Q) \to \Delta_Q$ and we let $K_\Delta(Q) = \ker p_\Delta$; then $K_\Delta(Q) \supset K_1(Q)$. Let

$$S_0(Q) = S_{K_0(Q)}(G, X), S_\Delta(Q) = S_{K_\Delta(Q)}(G, X)$$

and define the integral models $\mathbb{S}_0(Q)$, $\mathbb{S}_\Delta(Q)$ analogously. Assuming our reference group K is neat, the natural map $S_\Delta(Q) \to S_0(Q)$ is finite étale with Galois group Δ_Q. We will always assume

Hypothesis 3.3. *The natural finite map*

$$f_Q : \mathbb{S}_\Delta(Q) \to \mathbb{S}_0(Q)$$

is étale with Galois group Δ_Q.

In the applications, f_Q is a map of moduli spaces and the hypothesis is easy to verify.

The starting point for the application of the Taylor-Wiles method to coherent cohomology is the following result of Nakajima.

Theorem 3.4 (Nakajima, [N, Theorem 2]). *Let k be a field and $f : X \to Y$ a finite étale Galois covering of projective varieties with Galois group Γ. Let $\mathcal{F}$ be a coherent sheaf on Y and assume $H^i(X, f^*(\mathcal{F})) = 0$ for all indices except $i = i_0$. Then $H^{i_0}(X, f^*(\mathcal{F}))$ is a free $k[\Gamma]$-module.*

Corollary 3.5. *Let $\mathcal{O}$ be a p-adic integer ring with residue field k, and let $f : X \to Y$ be a finite étale Galois covering of projective $\mathcal{O}$-schemes with Galois group Γ. Let $\mathcal{F}$ be a locally free sheaf on Y and assume $H^i(X, f^*(\mathcal{F})) = 0$ for all indices except $i = i_0$. Assume moreover that $H^{i_0}(X, f^*(\mathcal{F}))$ is $\mathcal{O}$-torsion-free. Then $H^{i_0}(X, f^*(\mathcal{F}))$ is a free $\mathcal{O}[\Gamma]$-module.*

Proof. The argument is standard. Let $\bar{Y}$ and $\bar{X}$ denote the special fibers over $Spec(k)$ of Y and X, respectively, $\bar{f} : \bar{X} \to \bar{Y}$ the pullback, and let $i_Y : \bar{Y} \to Y$, $i_X : \bar{X} \to X$ be the corresponding closed immersions. Let $\varpi \in \mathcal{O}$ be a uniformizer and consider the short exact sequence of sheaves

$$0 \to f^*(\mathcal{F}) \xrightarrow{\times\varpi} f^*\mathcal{F} \to i_X^* f^*\mathcal{F} = \bar{f}^* i_Y^*(\mathcal{F}) \to 0 \tag{3.5.1}$$

(the sequence is exact at the left because $\mathcal{F}$ is locally free). The vanishing hypothesis implies that the long exact cohomology sequence reduces to

$$0 \to H^{i_0}(X, f^*(\mathcal{F})) \xrightarrow{\times\varpi} H^{i_0}(X, f^*(\mathcal{F})) \to H^{i_0}(\bar{X}, i_X^* f^*\mathcal{F}) \to 0 \tag{3.5.2}$$

Applying the theorem to $\bar{f}$ and the sheaf $i_Y^*(\mathcal{F})$, we see that $H^{i_0}(\bar{X}, i_X^* f^*\mathcal{F})$ is a free $k[\Gamma]$-module, necessarily of finite type. Let $\bar{e}_i, i = 1, \dots N$ denote a $k[\Gamma]$ basis of $H^{i_0}(\bar{X}, i_X^* f^*\mathcal{F})$, and lift each $\bar{e}_i$ to an element $e_i \in H^{i_0}(X, f^*(\mathcal{F}))$. Let $M = k[\Gamma]^N$ and map N to $H^{i_0}(X, f^*(\mathcal{F}))$ by sending the generators of M to the e_i. By Nakayama's lemma this map is surjective. Let K be the kernel, so we have a short exact sequence

$$0 \to K \to M \to H^{i_0}(X, f^*(\mathcal{F})) \to 0.$$

Since the map $M \to H^{i_0}(X, f^*(\mathcal{F}))$ becomes an isomorphism mod ϖ, and since $H^{i_0}(X, f^*(\mathcal{F}))$ is ϖ-torsion free, another application of Nakayama's lemma completes the proof.

In the applications, $X = \mathbb{S}_\Delta(Q)$, $Y = \mathbb{S}_0(Q)$, $f = f_Q$ is the natural map, $\Gamma = \Delta_Q$, and $\mathcal{F} = \mathcal{F}_{K_0(Q)}$ is one of the automorphic vector bundles introduced in §1. The subscripts for $\mathcal{F}$ are understood in what follows.

We return to the notation of §1 and §2. In particular, $S(G, X)$ is a PEL-type Shimura variety attached to a unitary similitude group G. This guarantees that

Hypothesis 3.3 is satisfied.[1] We assume henceforward that G is anisotropic modulo center. We let $\mathcal{O}$ denote a sufficiently large finite extension of $\mathcal{O}_v$, as in the paragraph preceding Proposition 2.8.

Theorem 3.6 (Lan-Suh [LS1, Theorem 8.2]). *Let v be a prime of $E(G,X)$ dividing p. Let $K \supset K_p$ be a neat open compact subgroup of $G(\mathbf{A}^f)$, and let $\mathbb{S}_K$ be the smooth projective model of $S_K(G,X)$ over Spec$(\mathcal{O}_v)$. discussed in previous sections. Let $\mu \in X^+(T)$. Suppose μ is p-small, sufficiently regular in the sense of [LS1, 7.18], and satisfies the inequality [LS1, (7.22)]. Then for every $w \in W^x$,*

$$H^i(\mathbb{S}_{K,\mathcal{O}}, \mathcal{E}_w(W)_{K,\mathbb{S}}) = 0 \text{ if } i \neq d_V - \ell(w).$$

Moreover,

$$H^{d_V-\ell(w)}(\mathbb{S}_{K,\mathcal{O}}, \mathcal{E}_w(W)_{K,\mathbb{S}})$$

is a free $\mathcal{O}$ module.

The regularity hypotheses in the statement of Theorem 3.6 will be made explicit in §6.10 when G is a unitary group.

When $K = K_\Delta(Q)$, resp. $K = K_0(Q)$, we write $\mathcal{E}_w(W)_{\Delta,\mathbb{S}}$ resp. $\mathcal{E}_w(W)_{0,\mathbb{S}}$, instead of $\mathcal{E}_w(W)_{K,\mathbb{S}}$.

Proposition 3.7. *Let $K = K_0(Q)$, with Q as above. Assume μ satisfies the hypotheses of Theorem 3.6. Then for all $w \in W^x$, $H^{d_V-\ell(w)}(\mathbb{S}_\Delta(Q), \mathcal{E}_w(W)_{\Delta,\mathbb{S}})$ is a free $\mathcal{O}[\Delta_Q]$-module, and*

$$H^i(\mathbb{S}_\Delta(Q), \mathcal{E}_w(W)_{\Delta,\mathbb{S}}) = 0 \text{ if } i \neq d_V - \ell(w).$$

Proof. By Hypothesis 3.3, the map f_Q is a finite étale morphism of smooth projective schemes over $\mathcal{O}$. By Theorem 3.6, the map f_Q satisfies the vanishing hypothesis of Corollary 3.5, with $X = \mathbb{S}_\Delta$, $Y = \mathbb{S}_0$, and $i_0 = \ell - d_V$. The Proposition thus follows immediately from Corollary 3.5.

4. Taylor-Wiles systems for coherent cohomology (axiomatic treatment)

Proposition 3.7 is the main cohomological input in the Taylor-Wiles method. The remaining inputs are mostly Galois-theoretic in nature. We review the axiomatic formulation of the Diamond-Fujiwara version of the Taylor-Wiles method, roughly following [Fu]. Because we are working with coherent cohomology rather than ℓ-adic cohomology, the deformation characteristic is denoted p rather than ℓ, in contrast to [CHT]. The discussion of Galois representations attached to automorphic representations is implicitly influenced by [BG].

We fix an automorphic vector bundle $\mathcal{F}$, a neat level subgroup K, and an index i_0 as in Corollary 3.5. Write $H_\emptyset = H^{i_0}(\mathbb{S}_K, \mathcal{F})$. For any Q as above, we set

$$H_{0,Q} = H^{i_0}(\mathbb{S}_0, \mathcal{F});\ H_{\Delta,Q} = H^{i_0}(\mathbb{S}_\Delta, \mathcal{F}).$$

We assume, consistently with the hypotheses of Corollary 2.7, that

[1] Indeed, the map $\mathbb{S}_\Delta(Q) \to \mathbb{S}_0(Q)$ is the morphism of moduli spaces of abelian varieties with PEL structure corresponding to forgetting a part of the level structure at primes in Q, and the map on moduli problems is obviously étale.

Hypothesis 4.1. *$H_\emptyset$, $H_{0,Q}$, and $H_{\Delta,Q}$ are torsion-free $\mathcal{O}$-modules, and $H^i(\mathbb{S}_K,\mathcal{F}) = 0$ for $i \neq i_0$, for $K = K_{0,Q}, K_{\Delta,Q}$.*

The goal is to study the structure of $H_\emptyset$ over the Hecke algebra

$$\mathbb{T}_\emptyset = \mathbb{T}^{i_0}(\mathcal{F}, K)$$

defined in §1. We define $\mathbb{T}_{0,Q}$ and $\mathbb{T}_{\Delta,Q}$ in the obvious way. In any case, for $\mathbb{T} = \mathbb{T}_\emptyset, \mathbb{T}_{0,Q}, \mathbb{T}_{\Delta,Q}$, $\mathbb{T}$ is $\mathcal{O}$-torsion free, by hypothesis, and is thus a subalgebra of $\mathbb{T}\otimes_{\mathbb{Z}}\mathbb{Q}$. We assume

Hypothesis 4.2. *$\mathbb{T}\otimes_{\mathbb{Z}}\mathbb{Q}$ is a reduced $\mathcal{O}\otimes_{\mathbb{Z}}\mathbb{Q}$-algebra.*

In particular, the intersection of the kernels of all homomorphisms $\lambda : \mathbb{T} \to \overline{\mathbb{Q}}_p$ is trivial.

We let E^+ be a finite extension of $\mathbb{Q}$. For the applications in the present paper, E^+ will always be F^+, but one can imagine more general situations. However, we will assume the sets Q, which are sets of finite places of F^+ split in F, can be identified with sets of finite places of E^+. In the following Hypothesis we refer to a fixed reductive algebraic group $\mathcal{G}$, not necessarily connected, over $Spec(\mathcal{O})$. In practice this should be closely related to the L-group of G but in the cases treated in [CHT,T] the two groups do not coincide; cf. [BG] for an attempt at an explanation. Let S_p, resp. $S(E^+)$, denote the set of primes of E^+ dividing p, resp. dividing primes in S.

Galois Hypotheses 4.3. *Let $\mathbb{T} = \mathbb{T}_\emptyset, \mathbb{T}_{0,Q}, \mathbb{T}_{\Delta,Q}$. For every homomorphism $\lambda : \mathbb{T} \to \overline{\mathbb{Q}}_p$), there exists a continuous homomorphism*

$$\rho_\lambda : Gal(\overline{\mathbb{Q}}/E^+) \to \mathcal{G}(\mathcal{O}_{\overline{\mathbb{Q}}_p}).$$

This homomorphism satisfies the following hypotheses. In what follows, r denotes a (variable) irreducible algebraic representation of $\mathcal{G}$.

1. *For all primes v of E^+ not ramified in E and not in $S(E^+)\cup S_p \cup Q$, ρ_λ is unramified at v and, for any geometric Frobenius element $Frob_v$ at v in $Gal(\overline{\mathbb{Q}}/E^+)$,*
$$Tr\ r\circ\rho_\lambda(Frob_v) \in \lambda(\mathbb{T}).$$
2. *For any prime $v \in S_p$ with decomposition group $\Gamma_v \subset Gal(\overline{\mathbb{Q}}/E^+)$, $r\circ\rho_\lambda\mid_{\Gamma_v}$ is crystalline with Hodge-Tate weights*
$$\mu_{r,\mathcal{F}}(v) = \{\mu_{1,r,\mathcal{F}}(v),\ldots,\mu_{\dim r,r,\mathcal{F}}(v)\}$$
counted with multiplicities. The weights depend only on r and $\mathcal{F}$ and not on the specific λ.
3. *Here we assume $\mathbb{T} = \mathbb{T}_{\Delta,Q}$. Let $q \in Q$ and let Γ_q denote a decomposition group of q in $Gal(\overline{\mathbb{Q}}/E^+)$, $\rho_{\lambda,q}$ the restriction of ρ_λ to Γ_q. Then $\rho_{\lambda,q}$ is diagonalizable; that is, there is a torus $\mathcal{T}\subset\mathcal{G}$ such that $Im(\rho_{\lambda,q})\subset\mathcal{T}(\mathcal{O}_{\overline{\mathbb{Q}}_p})$. Moreover, $\rho_{\lambda,q}$ is tamely ramified, and under the canonical identification $\iota_q : (\mathcal{O}_q/q)^\times \xrightarrow{\sim} I_q^{tame}$, where I^{tame} is the tame inertia group at q, the restriction of $\rho_{\lambda,q}$ to $(\mathcal{O}_q/q)^\times$ factors through its maximal p-power quotient Δ_q.*
4. *For any prime $v\in S(E^+)$ the representation $\rho_{\lambda,q}$ is unramified at v.*

Ramification at $S(E^+)$.. Condition (4) is unnecessarily restrictive, but it is the only way to define a minimal ramification condition without introducing a great deal of extraneous notation. See §6.9 for additional comments on this condition.

4.4. Deformations. We now fix $\mathcal{F}$ and a homomorphism λ as in (4.3), and assume ρ_λ takes values in $\mathcal{G}(\mathcal{O})$

$$\rho_\lambda : Gal(\overline{\mathbb{Q}}/E^+) \to \mathcal{G}(\mathcal{O})$$

where $\mathcal{O}$ is replaced if necessary by a finite extension; the maximal ideal of $\mathcal{O}$ is denoted $\mathfrak{m}_{\mathcal{O}}$, and the residue field is still denoted k. Let

$$\bar{\rho}_\lambda : Gal(\overline{\mathbb{Q}}/E^+) \to \mathcal{G}(k)$$

denote the corresponding residual representation. Let $\mathcal{G}^0 \subset \mathcal{G}$ denote the identity component, and let E^0/E^+ be the finite Galois extension such that

$$Gal(\overline{\mathbb{Q}}/E^0) = \rho_\lambda^{-1}(\mathcal{G}^0) = \bar{\rho}_\lambda^{-1}(\mathcal{G}^0) \tag{4.4.1}$$

where the second equality is a hypothesis. We assume all primes in $Q \cup S_p$ are split in E^0/E^+. We fix algebraic characters

$$\delta : \mathcal{G}^0 \to \mathbb{G}_m, \ \nu : \mathcal{G} \to \mathbb{G}_m.$$

Let A be an Artinian local $\mathcal{O}$-algebra with maximal ideal $\mathfrak{m}_A$ and residue field k. A *lifting* of $\bar{\rho}_\lambda$ to A is a homomorphism $\rho' : Gal(\overline{\mathbb{Q}}/E^+) \to \mathcal{G}(A)$, unramified outside $S \cup S_p \cup Q$, with a given isomorphism

$$\rho' \xrightarrow{\sim} \bar{\rho}_\lambda \pmod{\mathfrak{m}_{\mathcal{O}}}$$

such that

$$\nu \circ \rho' = \nu \circ \rho. \tag{4.4.2}$$

A *deformation* of $\bar{\rho}_\lambda$ is an equivalence class of liftings up to conjugation by $\mathcal{G}(1 + \mathfrak{m}_A) := \ker(\mathcal{G}(A) \to \mathcal{G}(k))$. One defines analogously liftings and deformations to complete noetherian local $\mathcal{O}$-algebras.

Let $\mathcal{S}_S = \{\mathcal{D}_v, v \in S\}$ be a collection of liftable local deformation problems for $v \in S$, as in [CHT, Def. 2.2.6, 2.2.7]. At primes v dividing p we consider deformations ρ' satisfying the

4.4.3. Crystalline condition. For any irreducible representation r of $\mathcal{G}$, $r \circ \rho' \mid_{\Gamma_v}$ is crystalline of Hodge-Tate weights $\mu_{r,\mathcal{F}}(v)$.

If $Q \neq \emptyset$, we assume

4.4.4. Local condition at Q. For any $q \in Q$, $\rho' \mid_{\Gamma_q}$ satisfies (3) of Hypothesis 4.3.

Corresponding to condition (4) of 4.3, we assume the deformation condition at $v \in S(E^+)$ to be *unrestricted*, in the sense of [CHT, 2.4.3]. We use the notation $\mathcal{S}$ to designate the deformation conditions $\mathcal{S}_S$ together with conditions 4.4.3 and 4.4.4. Finally, we make the following two assumptions familiar from proofs of automorphy lifting theorems.

4.4.5. Representability hypothesis. *For each Q as above the functor on the category of Artinian local $\mathcal{O}$-algebras classifying deformations of $\bar{\rho}$ of type $\mathcal{S}$ is prorepresentable by a noetherian local $\mathcal{O}$-algebra $R_{\bar{\rho},Q}$. We let*

$$\rho_Q^{univ} : Gal(\overline{\mathbb{Q}}/E^+) \to \mathcal{G}(R_{\bar{\rho},Q})$$

denote (a model for) the universal deformation of $\bar{\rho}$ of type $\mathcal{S}$.

Let $\mathfrak{P}_\lambda = \ker \lambda : \mathbb{T} \to \overline{\mathbb{Q}}_p$ and let $\mathfrak{m}_\rho$ be the (unique) maximal ideal of $\mathbb{T}$ containing $\mathfrak{P}_\lambda$. Suppose for the moment $\mathbb{T} = \mathbb{T}_{\Delta,Q}$, and let $\mathbb{T}_{\bar{\rho},\Delta,Q}$ denote the localization of $\mathbb{T}$ at $\mathfrak{m}_\rho$. Then

$$\mathbb{T}_{\bar{\rho},\Delta,Q} \otimes \mathbb{Q} = \oplus_{\lambda' \equiv \lambda} \mathbb{T}_{\lambda'}$$

where $\mathbb{T}_{\lambda'}$ is the completion of $\mathbb{T}$ at the kernel of λ' and the notation $\lambda' \equiv \lambda$ just means that $\lambda' \subset \mathfrak{m}_\rho$. The sum of the $\rho_{\lambda'}$ for $\lambda' \equiv \lambda$ defines a representation

$$\rho_{\equiv} : Gal(\overline{\mathbb{Q}}/E^+) \to \mathcal{G}(\oplus_{\lambda' \equiv \lambda} \mathbb{T}_{\lambda'}),$$

hence by (4.4.5) a classifying homomorphism

$$\phi_{\bar{\rho},Q} : R_{\bar{\rho},Q} \to \mathbb{T}_{\bar{\rho},\Delta,Q} \otimes \mathbb{Q}. \tag{4.4.6.Q}$$

One similarly defines $\mathbb{T}_{\bar{\rho},\Delta,Q}$ and $\mathbb{T}_{\bar{\rho},\emptyset}$ and there is a classifying homomorphism

$$\phi_{\bar{\rho},\emptyset} : R_{\bar{\rho},\emptyset} \to \mathbb{T}_{\bar{\rho},\emptyset} \otimes \mathbb{Q}. \tag{4.4.6.$\emptyset$}$$

4.4.7. Hecke algebra hypothesis. *The classifying morphisms (4.4.6) maps $R_{\bar{\rho},Q}$ (resp. $R_{\bar{\rho},\emptyset}$) surjectively onto $\mathbb{T}_{\bar{\rho},\Delta,Q}$ (resp. $\mathbb{T}_{\bar{\rho},\emptyset}$).*

We let $H_{\bar{\rho},\emptyset}$, $H_{\bar{\rho},0,Q}$, and $H_{\bar{\rho},\Delta,Q}$ denote the completions of the respective modules of coherent cohomology at the corresponding maximal ideals $\mathfrak{m}_\rho$. For $q \in Q$, recall the canonical identification ι_q of (3) of Hypothesis 4.3. The composite

$$\phi_{\bar{\rho}} = \delta \circ (\rho_Q^{univ} \mid_{\Gamma_q}) \circ \iota_q : \Delta_q \to \mathbb{T}_{\bar{\rho},\Delta,Q}$$

defines an action of Δ_q on the module $H_{\bar{\rho},\Delta,Q}$. Letting q vary, we obtain an action of Δ_Q on $H_{\bar{\rho},\Delta,Q}$, which we call the *arithmetic* action.

4.4.8. Taylor-Wiles hypotheses, Part I.
(a) The arithmetic action coincides with the action on coherent cohomology induced by the natural geometric action of the group $K_0(Q)/K_\Delta(Q)$ on the scheme $\mathbb{S}_\Delta(Q)$.
(b) The coherent cohomology module $H_{\bar{\rho},\Delta,Q}$ is free over $\mathcal{O}[\Delta_Q]$ for the arithmetic action.

In the applications, (b) will follow from (a) and Proposition 2.6.

4.4.9. Taylor-Wiles hypotheses, Part II.
(a) There are non-negative integers $s \geq s'$ and, for each $M \geq 1$, a set of primes $Q = Q_M = \{q_{1,M}, q_{2,M}, \ldots, q_{s,M}\}$ as above, with

$$q_{i,M} \equiv 1 \pmod{p^M},\ i = 1, \ldots, s, \tag{4.4.9.1}$$

such that, letting $\mathfrak{m}_{Q_M}$ denote the maximal ideal of $R^{univ}_{\bar{\rho},Q_M}$,

$$\dim \mathfrak{m}_{Q_M}/(\mathfrak{m}_{Q_M})^2 = s'. \tag{4.4.9.2}$$

Moreover, if $d_{i,M}$ is a generator of $\Delta_{q_{i,M}}$, $i = 1, \ldots, s$, then
(4.4.9.3)
the elements $t_{i,M} = \delta \circ (\rho^{univ}_{Q_M} \mid_{\Gamma_{q_{i,M}}}) \circ \iota_{q_{i,M}}(d_{i,M}) - 1$ generate $\mathfrak{m}_{Q_M}/(\mathfrak{m}_{Q_M})^2$.

(b) For each M as above, there is a natural isomorphism of $\mathbb{T}_{\bar{\rho},0,Q_M} = \mathbb{T}_{\bar{\rho},\emptyset}$-modules

$$H_{\bar{\rho},\emptyset} \xrightarrow{\sim} H_{\bar{\rho},0,Q_M}.$$

In the cases where these hypotheses apply, 4.4.9 (b) conceals the assumption that the deformation conditions $\mathcal{S}$ are *minimal*, as in [CHT], rather than general as in [T] or [W]. It then follows from the Diamond-Fujiwara version of the Taylor-Wiles method, as in [CHT, Theorem 3.5.1], that

Theorem 4.4.10. *Under the above hypotheses, the classifying map*

$$\phi_{\bar{\rho},\emptyset} : R_{\bar{\rho},\emptyset} \to \mathbb{T}_{\bar{\rho},\emptyset}$$

is an isomorphism of complete intersections, and $H_{\bar{\rho},\emptyset}$ is a free module over $\mathbb{T}_{\bar{\rho},\emptyset}$.

Remark 4.4.11. We note that 4.4.8 (b), together with 4.4.9 (b), actually implies the existence of an isomorphism of $\mathbb{T}$-modules

$$H_{\bar{\rho},\emptyset} \xrightarrow{\sim} H_{\bar{\rho},\Delta,Q_M} \otimes_{\mathcal{O}[\Delta_{Q_M}]} \mathcal{O},$$

where Δ_{Q_M} acts trivially on $\mathcal{O}$, and it is this that is used in the proof of Theorem 4.4.10, cf. [CHT, p. 115]. The proof is slightly different in this case than in [CHT], because we are working with coherent cohomology of a space of positive dimension. Here is the argument. Because of the vanishing hypothesis in 4.1, it follows from the Hochschild-Serre spectral sequence for the action of Δ_{Q_M} on coherent cohomology that there is a canonical isomorphism

$$H^{i_0}(\mathbb{S}_0, \mathcal{F})) \xrightarrow{\sim} H^0(\Delta_{Q_M}, H^{i_0}(\mathbb{S}_\Delta, \mathcal{F})).$$

Since it is canonical and the Δ_Q action commutes with the Hecke operators away from Q_M, the isomorphism respects the $\mathbb{T}$-module structure and passes to an isomorphism after localization:

$$H_{\bar{\rho},0,Q_M} \xrightarrow{\sim} H^0(\Delta_{Q_M}, H_{\bar{\rho},\Delta,Q_M}).$$

But for any free $\mathcal{O}[\Delta_Q]$-module A there is a canonical isomorphism

$$H^0(\Delta_{Q_M}, A) \xrightarrow{\sim} A \otimes_{\mathcal{O}[\Delta_{Q_M}]} \mathcal{O}$$

given by averaging over Δ_{Q_M} inside $A \otimes \mathbb{Q}$ and projecting to $A \otimes_{\mathcal{O}[\Delta_{Q_M}]} \mathcal{O}$.

5. Automorphic representations of unitary groups

We now specialize to the case in which the Galois hypotheses 4.3 are already known. The group G is again a unitary similitude group, as in §2, with $U \subset G$ the unitary group.

We consider

$$\Pi(W) = \{\pi \mid H^\bullet(Lie(G)_\mathbb{C}, K_x; \pi \otimes W) \neq 0\}$$

where π runs over the set of irreducible unitary representations of $G(\mathbb{R})$. Let $\Pi^d(W) \subset \Pi(W)$ denote the subset of discrete series representations. The set W^x is in bijection with the *discrete series L-packet* $\Pi^d(W)$ attached to W, and the bijection can be normalized so that

$$w \leftrightarrow \pi_w(W) \;\Leftrightarrow\; H^\bullet(Lie(P_x), K_x; \pi_w(W) \otimes \mathcal{E}_{w,x}(W)) \neq 0) \tag{5.1}$$

(cf. [H90, §4]).

The following Theorem is essentially due to Labesse.

Theorem 5.2 [L, Cor. 5.3]. *Let π be an automorphic representation of G with $\pi_\infty = \pi_w(W)$, and let π^U denote any irreducible component of the restriction of π to $U(\mathbf{A})$. Suppose the infinitesimal character of W is regular (i.e., the highest weight of W does not lie on any walls of the corresponding Weyl chamber). Then there is a partition $n = n_1 + \cdots + n_r$ and a collection Π_i of cuspidal automorphic representations of $GL(n_i)_F$ satisfying*

$$\theta_{n_i}(\Pi_i) = \Pi_i, i = 1, \dots, r$$

where θ_{n_i} is the permutation of automorphic representations of $GL(n_i)_F$ induced by the outer automorphism $g \mapsto c({}^t g^{-1})$ where c is complex conjugation, such that

$$\Pi_1 \boxplus \cdots \boxplus \Pi_r$$

defines a weak base change for π^U. Moreover, each Π_i is cohomological.

Proof. It is well known and easy to check that any two choices of π^U are isomorphic at almost all places of F^+. We thus just have to remark that the condition on the infinitesimal character guarantees that π^U satisfies [L]'s condition (*) – indeed, the condition implies that $\Pi(W) = \Pi^d(W)$ – and that, for the same reason, the only discrete automorphic representations of $GL(n_i)_F$ with the corresponding cohomology are generic at all archimedean places, hence cuspidal.

The following corollary is a summary of (a part of) the results of [CHL], [Shin], [CH], [Gu].

Corollary 5.3. *Let $w \in W^x$ and let $i_0 = d_V - \ell(w)$. Let $\mathbb{T}_{\emptyset,w}(W) = \mathbb{T}^{i_0}(\mathcal{E}_w(W), K)$, in the notation of §4. Define $\mathbb{T}_{0,Q,w}(W)$ and $\mathbb{T}_{\Delta,Q,w}(W)$ analogously, and let $\mathbb{T}_w(W)$ denote one of these Hecke algebras. Suppose the infinitesimal character of W is regular. Then the Galois hypotheses 4.3 are valid for $\mathbb{T}_w(W)$, with $\mathcal{G} = \mathcal{G}_n$, the disconnected group defined in [CHT], and with $E = F$, $E^+ = F^+$. The representation*

$$\rho_\lambda \mid_{Gal(\overline{\mathbb{Q}}/E)} \to \mathcal{G}_n(\overline{\mathbb{Q}}_p)$$

takes values in the identity component $\mathcal{G}_n^0 = GL(n) \times GL(1)$ of $\mathcal{G}_n$ and projection onto the first factordefines an n-dimensional representation

$$\rho_\lambda^0 : Gal(\overline{\mathbb{Q}}/E) \to GL(n, \overline{\mathbb{Q}}_p).$$

The Hodge-Tate weights of ρ_λ^0 are determined by formula (1.6) of [CH] (cf. [HT, Theorem VII.1.9]), and in turn determine $\mu_{r,\mathcal{E}_w(W)}(v)$ for any irreducible representation r of $\mathcal{G}$. If λ corresponds to a representation π as in Theorem 5.2 with partition $n = n_1 + \cdots + n_r$, then ρ_λ^0 can be assumed semisimple and to have at least r irreducible components.

Moreover, ρ_λ satisfies conditions 4.4.3 and 4.4.4. If the (semisimplified) residual representation $\bar\rho_\lambda^0$ is irreducible – in particular, $r = 1$ in the above partition – then $\bar\rho_\lambda$ satisfies the representability hypothesis 4.4.5 and the Hecke algebra hypothesis 4.4.6.

Proof. The first part is contained in [CHL], [Shin], [CH] and the second part is essentially due to [Gu], who worked out the details in the course of his study of the Taylor-Wiles method in the case where V is a totally definite unitary group.

Corollary 5.4. *Let $K \supset K_p$ be a neat compact open subgroup of $G(\mathbf{A}_f^p)$; let*

$$\mathbb{T}_{\emptyset,w}(W) = \mathbb{T}^{d_V - \ell(w)}(\mathcal{E}_w(W), K)$$

and define $\mathbb{T}_{0,Q,w}(W)$ and $\mathbb{T}_{\Delta,Q,w}(W)$ analogously. Let $\mathbb{T}_w(W)$ denote one of these Hecke algebras. Then the Galois hypotheses 4.3 are valid for $\mathbb{T}_w(W)$, with $\mathcal{G} = \mathcal{G}_n$, the disconnected group defined in [CHT], and with $E = F$, $E^+ = F^+$. Moreover, the Galois representations $\rho_\lambda \mid_{Gal(\overline{\mathbb{Q}}/F)} \to \mathcal{G}_n(\overline{\mathbb{Q}}_p)$ satisfy the conditions of the second part of Corollary 4.6.

Proof. This follows immediately from 5.3.

As usual, we can replace $\overline{\mathbb{Q}}_p$ by a sufficiently large p-adic integer ring $\mathcal{O}$ (cf. [CHT, Lemma 2.1.5]), and we do so without comment below. It remains to verify the Taylor-Wiles hypotheses 4.4.8 and 4.4.9. This is the subject of the next section.

6. Taylor-Wiles systems for coherent cohomology (main theorems)

We illustrate our methods by applying the Taylor-Wiles construction to the cohomology groups $H^{d_V - \ell(w)}(\mathbb{S}_K(G,X), \mathcal{E}_w(W))$ when the highest weight μ of W satisfies the hypotheses of Theorem 3.6. Let $\mathbb{T} = \mathbb{T}_w(W)$ as in Corollary 5.4, $\lambda : \mathbb{T} \to \overline{\mathbb{Q}}_p$ a homomorphism as in 4.3, and for $\mathcal{O}$ a sufficiently large p-adic integer ring let $\rho = \rho_\lambda : Gal(\overline{\mathbb{Q}}/F^+) \to \mathcal{G}_n(\mathcal{O})$ be the corresponding Galois representation, whose existence is guaranteed by Corollary 5.4. Define $\mathbb{T}_{\emptyset,\bar{\rho}}$, $\mathbb{T}_{0,Q,\bar{\rho}}$, $\mathbb{T}_{\Delta,Q,\bar{\rho}}$, and the corresponding modules $H_{\emptyset,\bar{\rho}}$, $H_{0,Q,\bar{\rho}}$, and $H_{\Delta,Q,\bar{\rho}}$, as in §4.4. Our residual modularity hypothesis is that

$$H_{\emptyset,\bar{\rho}} \neq (0).$$

We assume $K = \prod_q K_q$ is a neat compact open subgroup of $G(\mathbf{A}_f^p)$, with K_p a hyperspecial maximal compact subgroup. We assume the set of places q at which K_q is not hyperspecial to consist of primes q such that, for all places v of F^+ dividing q, v splits in F, and as in §3 we make the further hypothesis that at such q $K_q^U = \prod_{v \mid q} K_v$ with $K_v \subset GL(n, \mathcal{O}_v)$. We modify our notation: the set S of bad places is now the set of primes v of F^+ split in F, such that $K_v \neq GL(n, \mathcal{O}_v)$. For each such v we choose a lifting $\tilde{v}$ to F.

The hypotheses in 4.4.8 and 4.4.9 are of different natures. The claims in 4.4.8 are valid for any set of primes Q satisfying (3.1), and can be established directly. Those in 4.4.9, on the other hand, depend on specific choices of Q_M adapted to the arithmetic of $\bar{\rho}$, and the proof recapitulates most of the Taylor-Wiles method. We begin by taking care of 4.4.8.

Lemma 6.1. *Let Q be any set of primes as in (3.1). Assume $K \supset K_p$ is a neat subgroup and let $\mathcal{F} = \mathcal{E}_w(W)$, where W is the irreducible representation of G of highest weight μ satisfying the conditions of Theorem 3.6. Then hypothesis 4.4.8 is valid.*

Proof. Part (a) of 4.4.8 is verified as on p. 115 of [CHT], specifically the discussion culminating in lines -7 and -6; it only needs to be remarked that the Galois representation ρ is attached not to the unitary group G but rather to $GL(n)$. Now Part (b) follows from part (a) and Proposition 3.7.

If L is a non-archimedean local field with integer ring A, let $Iw(n, A) \subset GL(n, A)$ denote the upper-triangular Iwahori subgroup, the group of elements congruent to upper-triangular matrices modulo the maximal ideal $m_A \subset A$, and let $Iw_1(n,A) \subset Iw(n,A)$ denote the subgroup whose diagonal elements are all congruent to 1 modulo m_A. For the remainder of this section we make a series of restrictive assumptions on λ and its associated Galois representation ρ. Specifically, we assume that $\bar{\rho}$ satisfies the hypotheses on p. 112 of [CHT], namely

(6.2) $\bar{\rho}(Gal(\bar{F}/F^+(\zeta_p)))$ is *big* in the sense of [CHT, section 2.5];

(6.3) For all $v \in S(F^+)$, $\bar{\rho}$ is unramified at v and

$$H^0(Gal(\bar{F}_{\tilde{v}}/F_{\tilde{v}}), (ad\ \bar{\rho})(1)) = (0).$$

(This corresponds to condition (4) of 4.3 and is the assumption that for all $v \in S(F^+)$, the unrestricted deformation condition is in fact minimal, cf. §6.9 below.)

(6.4) If $\tau \in S_\infty$ gives rise to a place in S_p then, in the notation of (2.3), either

$$p - 1 - n \geq a_{\tau,1} \geq \cdots \geq a_{\tau,n} \geq 0$$

or

$$p - 1 - n \geq a_{\tau\cdot c,1} \geq \cdots \geq a_{\tau\cdot c,n} \geq 0$$

I refer to [CHT] for details, specifically for the association of real and p-adic places. Next, we make an important hypothesis on the deformation problem:

Hypothesis 6.5. *We assume that the deformation conditions $\mathcal{S}$ are minimally ramified at all primes $v \in S \setminus S_p$, in the sense of [CHT,2.4.4].*

We now choose sets Q_M as on the top of p. 114 of [CHT], where the index is N rather than M. For the integers s and s' of 4.4.9 we take the integers q and q' of [CHT, p.114, line 2]. (It is proved in [CHT], and more generally in [Gu], that the integer $q' = q$ in this situation, and the reader loses nothing by assuming this to be the case, the more so inasmuch as it follows from the argument we are about to sketch.) The existence of Q_M is guaranteed by the minimality hypothesis 6.5 as well as conditions (6.2) and (6.3). The sets Q_M are assumed to have the properties indicated in Proposition 2.5.9 of [CHT], and in particular this implies conditions (4.4.9.1) and (4.4.9.2) above, as well as the local conditions 4.4.4. As for condition (4.4.9.3), it is built into the construction of the sets Q_M on p. 62 of [CHT]. This proves

Lemma 6.6. *Assume $\bar{\rho}$ satisfies (6.2) and (6.3), and the deformation conditions $\mathcal{S}$ satisfy Hypothesis 6.5. Let Q_M be as in the preceding paragraph. Then part (a) of 4.4.9 is satisfied for these Q_M.*

Lemma 6.7. *Under the above hypotheses, 4.4.9 (b) is also satisfied for Q_M.*

Proof. This is the last line of [CHT,p. 115].

Combining all our results, we obtain our main theorem.

Theorem 6.8. *Let $K = \prod_q K_q$ be a neat compact open subgroup of $G(\mathbf{A}_f^p)$, with K_p a hyperspecial maximal compact subgroup, as at the beginning of the section. For $v \in S_a$, we assume $K_v \subset GL(n, \mathcal{O}_v)$ to be the subgroup of elements k whose reduction modulo the maximal ideal $\mathfrak{m}_v$ of $\mathcal{O}_v$ are upper-triangular unipotent.*

Let W be the irreducible representation of G of highest weight and assume μ satisfies the conditions of Theorem 3.6. Define

$$\mathbb{T}_{\emptyset,w}(W) = \mathbb{T}^{d_V-\ell(W)}(\mathcal{E}_w(W), K)$$

as in §4. Let $\lambda : \mathbb{T}_{\emptyset,w}(W) \to \overline{\mathbb{Q}}_p$ be a non-trivial homomorphism and assume the corresponding Galois representation ρ_λ satisfies conditions (6.2), (6.3), and (6.4). In particular, for $v \in S$, we assume the deformation condition in $\mathcal{S}$ at v is unrestricted and minimal (cf. 6.9, below).

Then the classifying map

$$\phi_{\bar{\rho},\emptyset} : R_{\bar{\rho},\emptyset} \to \mathbb{T}_{\bar{\rho},\emptyset}$$

is an isomorphism of complete intersections, and $H_{\bar{\rho},\emptyset}$ is a free module over $\mathbb{T}_{\bar{\rho},\emptyset}$.

Proof. By Theorem 4.4.10, it remains to observe that we have verified all the hypotheses assumed in the statement of that theorem.

6.9. The minimality condition. By imposing condition (6.3), we have eliminated all complications due to ramification at places of F^+ prime to p. This is not strictly necessary, and it is possible to prove the analogue of Theorem 6.8 with more complicated ramification conditions, provided they are all assumed to be minimal. Recall that a deformation condition at a prime v of F^+, assumed split in F/F^+ is associated to a subspace $L_v \subset H^1(\Gamma_{\tilde{v}}, ad\ \bar{\rho})$. The deformation condition is *minimal* at v if

$$\dim L_v = \dim H^0(\Gamma_{\tilde{v}}, ad\ \bar{\rho}).$$

However, to define this condition on automorphic forms, we need to introduce Hecke algebras acting on coherent cohomology with coefficients in non-trivial representations of $\prod_{v\in S} K_v^+$, where K_v^+ is a hyperspecial maximal compact subgroup (in practice, $K_v \subset K_v^+ = \prod_{w\ |\ v} GL(n, \mathcal{O}_w)$). In [CHT], the group G is an inner form of a unitary group with local component at v isomorphic to the multiplicative group of a division algebra, so that $K^+v = G_v$; moreover G is anisotropic modulo center at the archimedean prime. The corresponding modules of modular forms are defined in [CHT, §3.3], especially on p. 98. Similar constructions can be done for more general unitary groups, using the theory of types. Since the purpose of this article is to illustrate the method, we have preferred to impose the artificially restrictive condition (6.3).

6.10. The regularity conditions. We conclude this section by making the regularity conditions of Theorem 3.6 explicit. Let the weight μ be given by (2.3).

6.10.1. p-small. This is the condition that guarantees that a Weyl module in characteristic zero remains irreducible mod p. In terms of the weight μ, this means that

$$0 \leq a_{i,\tau} - a_{j,\tau} < p \text{ if } i < j.$$

6.10.2. [LS1, 7.18]. The condition of *sufficient regularity* is introduced to incorporate the twisting by the smallest possible ample automorphic line bundle. Assume the hermitian space V is not totally definite. Then sufficient regularity comes down to the simple condition

$$a_{i,\tau} - a_{i+1,\tau} \geq 1\ \forall\ i, \tau.$$

In other words, the left-hand inequality in 6.10.1 becomes strict.

6.10.3. [LS1, (7.22)]. The formula, translated into our notation, is

$$|\mu|_{re} + \sum_{\tau \in S_\infty} min(1, d_\tau) max(r_\tau, s_\tau) < p.$$

Each of these terms can be located (with some difficulty) in [LS1]. The crucial term is

$$|\mu|_{re} = d_V + \sum_{\tau \in S_\infty} \sum_j (a_{i,\tau} - 2[a_{n,\tau}/2])$$

where $[\bullet]$ denotes the greatest integer in $\bullet$. The pair (r_τ, s_τ) is the signature of the hermitian space at the complex place τ, as in §2. Finally, the factor $min(1, d_\tau)$ is 1 if V_τ is indefinite, 0 otherwise.

7. Topological cohomology

7.1. Preliminaries. The regularity conditions of Theorem 3.6, made explicit in §6.10, are hypotheses on the highest weight of an irreducible representation W of G defining a local system (topological or p-adic) on the Shimura variety. The vanishing theorems of Lan and Suh for coherent cohomology imply the vanishing de Rham and topological cohomology of this local system outside the middle degree, under the same regularity conditions. Combining these results with the methods of this paper, we show that the de Rham and topological cohomology, localized at a non-Eisenstein maximal ideal and in the situation of minimal ramification, is again free over the Hecke algebra. The proof for de Rham cohomology is straightforward, whereas that for topological cohomology depends on the p-adic comparison theorem.

As in §2, let W be the finite-dimensional irreducible representation of G with highest weight μ. In this section we always assume μ to be p-small (cf. (6.10.*)). Fix a place v of $E(W)$ dividing p and let $\mathcal{O}$ denote a sufficiently large p-adic integer ring containing $\mathcal{O}_v$, with fraction field $Frac(\mathcal{O})$. The following observation is well known and is explained in §2.6 of [LS1], cf. also [LP]. We let $G_{\mathbb{Z}_p}$ denote the smooth reductive group scheme over $\mathbb{Z}_p$ corresponding to our chosen hyperspecial maximal compact subgroup $K_p \subset G(\mathbb{Q}_p)$, and by abuse of notation we write $G(\mathbb{Z}_p) = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$.

Lemma 7.1.1. *Up to homotheties, there is a unique $K_p = G(\mathbb{Z}_p)$-stable lattice $W_\mu(\mathcal{O}) \subset W(Frac(\mathcal{O}))$.*

This corresponds to the lattice denoted $V_{\mu,R}$ in [LS1]. The generic fiber $S_K(G,X)$ of the Shimura variety has a tower $S_{K_p(m)\times K^p}(G,X)$ of étale coverings for $m \in \mathbb{N}$, where $K_p(m)$ are the principal congruence subgroups

$$K_p(m) = \ker K_p \to Aut(W_\mu(\mathcal{O})/p^m \cdot W_\mu(\mathcal{O}).$$

Thus there is a map from the étale fundamental group $\pi_1(S_K(G,X),t)$ (where t is the generic point) to K_p and the representation of K_p on $W_\mu(\mathcal{O})$ gives rise to an étale local system $\tilde{W}_{ét}(\mathcal{O}) = \tilde{W}_{\mu,ét}(\mathcal{O})$ on $S_K(G,X)$, as well as a topological local system $\tilde{W}_B(\mathcal{O}) = \tilde{W}_{\mu,B}(\mathcal{O})$; the μ's are understood and will be omitted. These local systems are constructed geometrically in §4.3 of [LS1] (with $V_{[\mu]}$'s instead of $\tilde{W}$'s).

On the other hand, the identification of the integral model $\mathbb{S}_K$ with a moduli scheme of abelian varieties with PEL structure defines an exact functor ([LS1], Lemma 1.20) $\mathcal{E}_{G,\mathcal{O}}$ from the category $Rep_{\mathcal{O}}(G_{\mathbb{Z}_p})$ of representations of the group scheme $G_{\mathbb{Z}_p}$ over $\mathcal{O}$ to the category of locally free coherent sheaves on $\mathbb{S}_K$, with integrable connection and compatible $G(\mathbf{A}_f^p)$-action covering the natural action on the family $\mathbb{S}_{K\times K^p}$. Suppose μ is a p-small highest weight for G and $W_{\mathcal{O}} = W_{\mu,\mathcal{O}}$ is the corresponding object of $Rep_{\mathcal{O}}(G_{\mathbb{Z}_p})$, so that

$$W_\mu(\mathcal{O}) = W_{\mu,\mathcal{O}}(\mathcal{O}).$$

In terms of the construction (1.2) in §1 above, applied to $\mathcal{V} = W_{\mu,Frac(\mathcal{O})} \times \check{X}$ on $\check{X}$ with diagonal action by G, then

$$[W_{\mu,Frac(\mathcal{O})} \times \check{X}] = \mathcal{E}_{G,\mathcal{O}}(W_{\mu,\mathcal{O}}) \times_{\mathcal{O}} Frac(\mathcal{O}). \tag{7.1.2}$$

The $G(\mathbf{A}_f^p)$-action is not mentioned in [LS1] but the assertions regarding compatibility with this action here and below are obvious.

7.2. de Rham cohomology.

Let μ, $W = W_\mu$, and the place v dividing p be as above and fix a p-adic integer ring $\mathcal{O}$ containing $E(W)_v$. We define the integral de Rham cohomology

$$H^\bullet_{dR}(\mathbb{S}_K, W_{\mathcal{O}}) := \mathbb{H}^\bullet(\Omega^\bullet_{\mathbb{S}_K} \otimes \mathcal{E}_{G,\mathcal{O}}(W_{\mu,\mathcal{O}})). \tag{7.2.1}$$

We combine two of the main theorems of [LS1]:

Theorem 7.2.2, [LS1, Theorems 8.1, 8.2]. *Assume μ is p-small and satisfies the regularity conditions of 3.6. Then*

$$H^i_{dR}(\mathbb{S}_K, W_{\mathcal{O}}) = 0, i \neq d_V,$$

$H^{d_V}_{dR}(\mathbb{S}_K, W_{\mathcal{O}})$ is a free $\mathcal{O}$ module of finite rank, and there is a natural decreasing (Hodge) filtration $F^\bullet(H^{d_V}_{dR}(\mathbb{S}_K, W_{\mathcal{O}}))$ by $\mathcal{O}$-direct summands such that

$$Gr_F(H^{d_V}_{dR}(\mathbb{S}_K, W_{\mathcal{O}})) = \oplus_{w\in W^x} H^{d_V-\ell(w)}(\mathbb{S}_K, \mathcal{E}_w(W)).$$

The isomorphisms are functorial as K^p varies and are equivariant with respect to the action of $G(\mathbf{A}_f^p)$.

Using the $G(\mathbf{A}_f^p)$-action, we define the unramified Hecke algebra

$$\mathbb{T}^{d_V}_{dR}(W, K) \subset End(H^{d_V}_{dR}(\mathbb{S}_K, W_{\mathcal{O}}))$$

as in §1.

Theorem 7.2.3. *Assume μ is p-small and satisfies the regularity conditions of 3.6. Hypotheses are as in Theorem 6.8. Let λ and ρ be as in the statement of that theorem, and define $\mathbb{T}_{dR,\bar{\rho},\emptyset}$ as the localization at $\mathfrak{m}_\rho$ of $\mathbb{T}^{d_V}_{dR}(W, K)$. Then there is an isomorphism of complete intersections,*

$$R_{\bar{\rho},\emptyset} \to \mathbb{T}_{dR,\bar{\rho},\emptyset}$$

and the localization $H^{d_V}_{dR}(\mathbb{S}_K, W_{\mathcal{O}})_{\mathfrak{m}_\rho}$ of $H^{d_V}_{dR}(\mathbb{S}_K, W_{\mathcal{O}})$ is a free $\mathbb{T}_{B,\bar{\rho},\emptyset}$-module.

Proof. The obvious strategy is to use Theorem 7.2.2, observing that $H^{d_V}_{dR}(\mathbb{S}_K, W_{\mathcal{O}})_{\mathfrak{m}_\rho}$ has a $\mathbb{T}^{d_V}_{dR}(W, K)$-stable filtration, each of whose graded pieces is a free $\mathbb{T}$-module by Theorem 6.8. The problem with this strategy is that $\mathbb{T}$ is a moving target. There is no reason *a priori* to assert that the localized Hecke algebras attached to each of the graded pieces $H^{d_V-\ell(w)}(\mathbb{S}_K, \mathcal{E}_w(W))$ are isomorphic to one another, nor to $\mathbb{T}_{dR,\bar{\rho},\emptyset}$, although it turns out that they are all isomorphic (compatibly) to the same deformation ring $R_{\bar{\rho},\emptyset}$.

It can be shown directly, using the results of [L], that all the localized Hecke algebras are in fact isomorphic. But it is easier to recapitulate the Taylor-Wiles argument in the present situation, since we will need to do so anyway in the next section. We let Q_M be sets of Taylor-Wiles primes, as in §6, and define $\mathbb{T}^{d_V}_{dR,0,Q_M}(W, K)$, $\mathbb{T}_{dR,\rho,0,Q_M}(W, K)$, $\mathbb{T}^{d_V}_{dR,\Delta,Q_M}(W, K)$, and $\mathbb{T}_{dR,\rho,\Delta,Q_M}(W, K)$ by replacing K with $K_0(Q_M)$ and $K_\Delta(Q_M)$, respectively. Similarly, we define $H_{dR,\rho,\emptyset}$, $H_{dR,\rho,0,Q_M}$, and H_{dR,ρ,Δ,Q_M} as the localizations at $\mathfrak{m}_\rho$ of the corresponding $H^{d_V}_{dR}(\mathbb{S}_{K(*)}, W_{\mathcal{O}})$. Applying the arguments of the previous section, we see that it suffices to show that

$$H_{dR,\rho,\Delta,Q_M} \text{ is a free } \mathcal{O}[\Delta_{Q_M}]\text{-module} \tag{7.2.4}$$

for all Q_M. But under the hypotheses, Theorem 7.2.2, together with Proposition 3.7, implies that H_{dR,ρ,Δ,Q_M} has a $\mathcal{O}[\Delta_{Q_M}]$-stable filtration by $\mathcal{O}$-direct summands, whose associated graded module is free over $\mathcal{O}[\Delta_{Q_M}]$.

Although we do not know that the Hecke algebra $\mathbb{T}^{d_V}_{dR}(W, K)$ is isomorphic to one of the algebras considered in the earlier sections, we know at least, by comparison with the action on complex cohomology that it satisfies the Galois Hypotheses 4.3; the proof is identical with that of Corollary 5.3. Thus the Taylor-Wiles argument applies in this situation and allows us to conclude.

7.3. Topological cohomology.

Let $\mathcal{O}$ be a coefficient ring as in §7.1 with residue field k. Using comparison theorems of integral p-adic Hodge theory [BM], Lan and Suh prove the following analogue of Theorem 7.2.2. Lan and Suh work with étale cohomology, but I state their results in terms of topological cohomology.

Theorem 7.3.1 [LS1, Theorem 8.12]. *Assume μ is p-small and satisfies the regularity conditions of 3.6 as well as the inequality $|\mu_{comp}| \le p-2$ (see [LS1, 5.14] and 7.3.4, below). Let $K = K_p \times K^p$ with K_p hyperspecial maximal as above. Then*

1. $H^i(S_K(G,X)(\mathbb{C}), \tilde{W}_{\mu,B}(\mathcal{O})) = 0$ *for* $i \neq d_V$*;*
2. $H^i(S_K(G,X)(\mathbb{C}), \tilde{W}_{\mu,B}(k)) = 0$ *for* $i \neq d_V$*;*
3. $H^{d_V}(S_K(G,X)(\mathbb{C}), \tilde{W}_{\mu,B}(\mathcal{O}))$ *is free of finite rank over* $\mathcal{O}$*;*
4. $H^{d_V}(S_K(G,X)(\mathbb{C}), \tilde{W}_{\mu,B}(k)) \xrightarrow{\sim} H^{d_V}(S_K(G,X)(\mathbb{C}), \tilde{W}_{\mu,B}(\mathcal{O})) \otimes_{\mathcal{O}} k$.

Moreover,

$$\begin{aligned} rank_{\mathcal{O}} H^{d_V}(S_K(G,X)(\mathbb{C}), \tilde{W}_{\mu,B}(\mathcal{O})) &= \dim_k H^{d_V}(S_K(G,X)(\mathbb{C}), \tilde{W}_{\mu,B}(k)) \\ &= \dim_k H^{d_V}(S_K(G,X)(\mathbb{C}), \tilde{W}_{\mu,B}(\mathcal{O})) \otimes_{\mathcal{O}} k \\ &= rank_{\mathcal{O}} H^{d_V}_{dR}(\mathbb{S}_K, W_{\mathcal{O}}). \end{aligned}$$

The last statement is not explicit in [LS1, Theorem 8.12], but it is an immediate consequence of the comparison theorems they use.

Using the $G(\mathbf{A}_f^p)$-action, we define the unramified Hecke algebra

$$\mathbb{T}_B^{d_V}(W,K) \subset End(H^{d_V}(S_K(G,X)(\mathbb{C}), \tilde{W}_{\mu,B}(\mathcal{O})))$$

as in §1. Since the p-adic comparison theorems are compatible with correspondences, we know that

$$\mathbb{T}_B^{d_V}(W,K) \xrightarrow{\sim} \mathbb{T}_{dR}^{d_V}(W,K)$$

as $\mathcal{O}$-algebras. As in §7.2, we can construct a Taylor-Wiles system using the $H^{d_V}(S_K(G,X)(\mathbb{C}))$. The crucial condition is again 4.4.8 (b). Notation is defined by analogy to that in §7.2, with $H_{B,\rho,\emptyset}$, $H_{B,\rho,0,Q_M}$, and H_{B,ρ,Δ,Q_M} replacing the corresponding H_{dR}'s. For $* = 0, \Delta$, we also abbreviate

$$H_{B,\mu,*,Q_M} = H^{d_V}(S_{K_*(Q_M)}(G,X)(\mathbb{C}), \tilde{W}_{\mu,B}(\mathcal{O})).$$

Proposition 7.3.2. *Under the hypotheses of Theorem 7.3.1, H_{B,μ,Δ,Q_M} is a free $\mathcal{O}[\Delta_{Q_M}]$-module.*

Proof. As in the proof of Corollary 3.5, it suffices to prove that the reduction

$$\bar{H}_{B,\mu,\Delta,Q_M} := H_{B,\mu,\Delta,Q_M} \otimes_{\mathcal{O}} k$$

is a free $k[\Delta_{Q_M}]$-module. Write $Q = Q_M$. Let $W_\mu^\vee$ denote the contragredient of W_μ, and let

$$\bar{H}_{B,\mu^\vee,*,Q} = H^{d_V}(S_{K_*(Q)}(G,X)(\mathbb{C}), \widetilde{W^\vee}_{\mu,B}(k));$$

the notation is consistent by Theorem 7.3.1 (4). By Theorem 7.3.1, applied to $W_\mu^\vee$, the Hochschild-Serre spectral sequence for the action of Δ_Q on $\bar{H}_{B,\mu^\vee,\Delta,Q}$ degenerates at E_2 and yields an isomorphism

$$H^0(\Delta_Q, \bar{H}_{B,\mu^\vee,\Delta,Q}) \xrightarrow{\sim} \bar{H}_{B,\mu^\vee,0,Q}.$$

By Poincaré duality we thus have

$$\bar{H}_{B,\mu,0,Q} \xrightarrow{\sim} \bar{H}_{B,\mu,\Delta,Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O}. \tag{7.3.2.1}$$

Let $N = \dim_k \bar{H}_{B,\mu,0,Q}$. By the comparison of dimensions in Theorem 7.3.1, we find

$$\begin{aligned} \dim_k \bar{H}_{B,\mu,\Delta,Q} &= rank_{\mathcal{O}} H_{dR}^{d_V}(\mathbb{S}_{K_\Delta(Q)}, W_{\mathcal{O}}) \\ &= |\Delta_Q| \cdot rank_{\mathcal{O}} H_{dR}^{d_V}(\mathbb{S}_{K_0(Q)}, W_{\mathcal{O}}) \text{ (by (7.2.4))} \\ &= |\Delta_Q| \cdot N \text{ (by 7.3.1 again).} \end{aligned} \tag{7.3.2.2}$$

Now let $e_1, \ldots, e_N$ be a basis for $\bar{H}_{B,\mu,0,Q}$. Lift the e_i to elements $\tilde{e}_i \in \bar{H}_{B,\mu,\Delta,Q}$ using (7.3.2.1). Define a map

$$k[\Delta_Q]^N \to \bar{H}_{B,\mu,\Delta,Q} \tag{7.3.2.3}$$

by sending the generators of the left-hand side to the $\tilde{e}_i$. It follows from (7.3.2.1) and Nakayama's lemma, applied to the noetherian local ring $k[\Delta_Q]$, that (7.3.2.3) is surjective. But (7.3.2.2) implies that the two sides have the same dimension, hence (7.3.2.3) is an isomorphism.

The proof of Theorem 7.2.3 then carries over and we obtain

Theorem 7.3.3. *Assume μ is p-small and satisfies the regularity conditions of 3.6 as well as the inequality $|\mu_{comp}| \leq p-2$. Hypotheses are as in Theorem 6.8. Let λ and ρ be as in the statement of that theorem, and define $\mathbb{T}_{B,\bar{\rho},\emptyset}$ as the localization at $\mathfrak{m}_\rho$ of $\mathbb{T}_B^{d_V}(W,K)$. Then there is an isomorphism of complete intersections,*

$$R_{\bar{\rho},\emptyset} \to \mathbb{T}_{B,\bar{\rho},\emptyset}$$

and the localization $H_{B,\rho,\emptyset}$ of $H^{d_V}(S_{K_*(Q_M)}(G,X)(\mathbb{C}), \tilde{W}_{\mu,B}(\mathcal{O}))$ is a free $\mathbb{T}_{B,\bar{\rho},\emptyset}$-module.

7.3.4. The additional inequality $|\mu_{comp}| \leq p-2$ of Theorem 7.3.1 is required to apply the integral comparison theorems stated in [BM]. Whereas the other regularity conditions are related either to generalizations of the Kodaira vanishing theorem or to modular representation theory, this condition is purely geometric. Lan and Suh use integral p-adic Hodge theory for cohomology with trivial coefficients of (log)-smooth projective varieties. The local system $\tilde{W}_{\mu,B}$ over $S_K(G,X)$ can be realized as a direct factor of the direct image of the trivial sheaf with respect to the canonical map of an abelian scheme to the base $S_K(G,X)$. The constant $|\mu_{comp}|$ is roughly equal to d_V plus the relative cohomological degree in the fiber direction. If μ is written as in (2.3) then one can take $2d_V + (\sum_{i,\tau}(a_{i,\tau} - 2[\frac{a_{n,\tau}}{2}])]$ for $|\mu_{comp}|$ (cf. 6.10.3).. One can optimize $|\mu_{comp}|$ for a given μ but the bound is still probably unnecessarily strong. The correct bound should probably be roughly the Fontaine-Laffaille condition (6.4).

8. Variants

I mention two possible directions in which the results of the present paper might be generalized.

8.1. Non compact Shimura varieties. The paper [LS2] proves vanishing theorems, analogous to Theorem 3.6, for coherent cohomology of non-compact PEL-type Shimura varieties. Following [H90] they use the two natural extensions of automorphic vector bundles to toroidal compactifications, denoted $\mathcal{E}^{can}$ and $\mathcal{E}^{sub}$, and study the interior cohomology

$$H_!^\bullet(\mathbb{S}_{K,\mathcal{O}}, \mathcal{E}) := Im[H^\bullet(\mathbb{S}_{K,\mathcal{O},\Sigma}, \mathcal{E}^{sub}) \to H^\bullet(\mathbb{S}_{K,\mathcal{O},\Sigma}, \mathcal{E}^{can})$$

where $\mathbb{S}_{K,\mathcal{O},\Sigma}$ is any smooth projective toroidal compactification. This is the analogue for coherent cohomology of the image of cohomology with compact support in cohomology of a topological local system over a Shimura variety.

For the interior cohomology, Lan and Suh prove vanishing in the same setting as Theorem 3.6. There is no doubt that the results of the present paper should be valid for interior cohomology of non-compact Shimura varieties, but this would require an extension of Nakajima's theorem to interior cohomology.

8.2. Hida families.

In the ordinary setting treated by Pilloni in [Pi], the $R = T$ isomorphism extends to an isomorphism of a Hida deformation ring with Hida's Hecke algebra, and by the same token Pilloni shows that the Hida family of ordinary Siegel modular forms is a free module over the big Hecke algebra ([Pi], §7.3). In principle this requires the analogue of Theorem 6.8 when μ is no longer p-small. The question remains interesting for coherent cohomology in positive degree.

References

[A] J. Arthur, The Endoscopic Classification of Representations: Orthogonal and Symplectic Groups (book to appear).

[BGGT] T. Barnet-Lamb, D. Geraghty, T. Gee, R. Taylor, Potential automorphy and change of weight, preprint 2010, at http://www.math.harvard.edu/ rtaylor/

[BGHT] T. Barnet-Lamb, D. Geraghty, M. Harris, R. Taylor, A family of Calabi-Yau varieties and potential automorphy II, Publ. RIMS, in press.

[BM] C. Breuil and W. Messing, Torsion étale and crystalline cohomologies, *Astérisque*, **279** (2002) 81-124.

[BG] K. Buzzard, T. Gee, The conjectural connections between automorphic representations and Galois representations (preliminary version fall 2009).

[CH] G. Chenevier, M. Harris, Construction of automorphic Galois representations, II, manuscript at http://fa.institut.math.jussieu.fr/node/45.

[C] T. Chinburg, Galois structure of de Rham cohomology of tame covers of schemes, *Annals of Math.*, **139** (1994) 443-490.

[CHL] L. Clozel, M. Harris, J.-P. Labesse, Construction of automorphic Galois representations, I, chapter in [CHLN]

[CHLN] L. Clozel, M. Harris, J.-P. Labesse, B.C. Ngô, *The stable trace formula, Shimura varieties, and arithmetic applications, Book 1* (in preparation; preliminary versions available at http://fa.institut.math.jussieu.fr/node/44).

[CHT] L. Clozel, M. Harris, R. Taylor, Automorphy for some ℓ-adic lifts of automorphic mod ℓ Galois representations, *Publ. Math. IHES*, **108** (2008) 1-181.

[De] P. Deligne, Variétés de Shimura: interprétation modulaire et techniques de construction de modèles canoniques, *Proc. Symp. Pure Math.*, **33, part 2** (1979) 247-290. 379–391.

[DI] P. Deligne, L. Illusie, Relèvements modulo p^2 et décomposition du complexe de de Rham, *Inv. Math.*, **89** (1987) 247-270.

[D] F. Diamond, The Taylor-Wiles construction and multiplicity one, *Invent. Math.*, **128**, (1997), 379–391.

[FC] G. Faltings, C.-L. Chai, *Degenerations of abelian varieties* Springer-Verlag (1990).

[Fu] K. Fujiwara, Deformation rings and Hecke algebras in the totally real case, version 2.0 (preprint 1999).

[GT] A. Genestier, J. Tilouine, Systèmes de Taylor-Wiles pour $GSp4$, in "Formes automorphes. II. Le cas du groupe $GSp(4)$. *Astérisque*, **302** (2005) 177290.

[Gu] L. Guerberoff, Modularity lifting theorems for Galois representations of unitary type, preprint (2009).

[H85] M. Harris, Arithmetic vector bundles and automorphic forms on Shimura varieties: I, *Invent. Math.*, **82** (1985)-151-189; II. *Compositio Math.*, **60** (1986), 323-378.

[H86] M. Harris, Functorial properties of toroidal compactifications of locally symmetric varieties, *Proc. London Math. Soc*, **59** (1989) 1-22.

[H90] M. Harris, Automorphic forms of $\overline{\partial}$-cohomology type as coherent cohomology classes, *J. Diff. Geom*, **32** (1990), 1-63.

[H94] M. Harris, Hodge de Rham structures and periods of automorphic forms, *Proc. Symp. Pure Math.*, **55.2** (1994) 573-624.

[K] M. Kisin, Moduli of finite flat group schemes and modularity, *Ann. of Math.*, **170** (2009) 1085–1180.

[Ko] R. Kottwitz, Points on some Shimura varieties over finite fields, *JAMS* , **5** (1992) 373-444.

[L] J.-P. Labesse, Changement de base CM et séries discrètes, chapter in [CHLN].

[LP] K. W. Lan and P. Polo, Dual BGG complexes for automorphic vector bundles, manuscript 2010.

[LS] K. W. Lan and J. Suh, Liftability of mod p cusp forms of parallel weights, *IMRN*, to appear.

[LS1] K. W. Lan and J. Suh, Vanishing theorems for torsion automorphic sheaves on compact PEL- type Shimura varieties, manuscript 2010.

[LS2] K. W. Lan and J. Suh, Vanishing theorems for torsion automorphic sheaves on general PEL- type Shimura varieties, manuscript 2010.

[M] A. Mokrane and J. Tilouine, Cohomology of Siegel varieties with p-adic integral coefficients and Applications, *Astérisque*, **280** (2002) 1-95.

[N] S. Nakajima, On the Galois module structure of the cohomology groups of an algebraic variety, *Inv. Math.*, **75** (1984) 1-8.

[Pi] V. Pilloni, Modularité, formes de Siegel et surfaces abéliennes, manuscript (2010).

[P] R. Pink, Arithmetical compactification of mixed Shimura varieties, *Bonner Math. Schriften*, **209** (1990).

[Ro] S. Rozensztajn, Comparaison entre cohomologie cristalline et cohomologie étale p-adique sur certaines variétés de Shimura *Bulletin de la SMF*, **137** (2009), 297-320.

[Shin] S.-W. Shin, Galois representations arising from some compact Shimura varieties, *Ann. of Math.*, in press.

[SW] C. Skinner, A. Wiles, Residually Reducible Representations and Modular Forms,*Publ. Math. IHES* **89** (1999), 5-126.

[St] B. Stroh, Relèvement de formes modulaires de Siegel, *Proc. AMS*, **138** (2010), 3089-3094.

[T] R. Taylor, Automorphy for some ℓ-adic lifts of automorphic mod ℓ Galois representations, II, *Publ. Math. IHES*, **108** (2008) 1-181.

[TW] R. Taylor, A. Wiles, Ring-theoretic properties of certain Hecke algebras, *Annals of Math.*, **141**, 553-572 (1995).

[W] A. Wiles, Modular elliptic curves and Fermat's last theorem, *Annals of Math.*, **142**, 443-551 (1995).