A family of Calabi-Yau varieties and potential automorphy

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Introduction

In this paper we generalise the methods of [T1] and [T2] to symplectic Galois representations of dimension greater than 2. Recall that these papers showed that some quite general two dimensional Galois representations of $\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q})$ became modular after restriction to some Galois totally real field. This has proved a surprisingly powerful result.

An example of the sort of theorem we prove in this paper is the following (see theorem 3.2 below).

Theorem A *Suppose that n is an even integer and that q is a prime. Suppose that $l \neq q$ is a prime sufficiently large compared to n, and that*

$$r : \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL_2(\mathbb{Z}_l)$$

is a continuous representation which is unramified almost everywhere and which has odd determinant (i.e. $\det r(c) = -1$). Suppose that r also enjoys the following properties.

1. *r is surjective.*
2. *r is crystalline at l with Hodge-Tate numbers 0 and 1.*
3. *$r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)}$ is unramified and the ratio of the eigenvalues of Frobenius is q.*

Then there is a Galois totally real number field over which $\mathrm{Symm}^{\,n-1} r$ becomes automorphic.

The key points are that no assumption is made on whether $\mathrm{Symm}^{\,n-1} r \bmod l$ is automorphic, but we can only conclude automorphy over some number field, not necessarily over $\mathbb{Q}$.

The papers [T1] and [T2] relied on the study of certain moduli spaces of Hilbert-Blumenthal abelian varieties. The main innovation in this paper is to replace these modular families by the family

$$Y_t : \quad X_0^{n+1} + X_1^{n+1} + ... + X_n^{n+1} = (n+1)tX_0X_1...X_n$$

of projective hypersurfaces over the affine line. More precisely

$$H' = \ker(\mu_{n+1}^{n+1} \xrightarrow{\Pi} \mu_{n+1})$$

acts on this family (by multiplication of the coordinates) and we will consider the H'-invariants in the cohomology in degree $n-1$ of a fibre in this family.

Note that in the case $n = 2$ this is just a family of elliptic curves, so our theory is in a sense a natural generalisation of the $n = 2$ case.

The proof of theorem A is then intertwined with the proof of the following theorem (see theorem 3.3 below).

Theorem B *Suppose that n is an even integer and that $q \not| n+1$ is a prime. Suppose that l is a prime sufficiently large compared to n, and that*

$$r : \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GSp_n(\mathbb{Z}_l)$$

is a continuous representation which is unramified almost everywhere and has odd multiplier character. Suppose that r also enjoys the following properties.

1. *r is surjective.*
2. *r is crystalline at l with Hodge-Tate numbers $0, 1, ..., n-1$. Moreover there is an element t of the maximal unramified extension of $\mathbb{Q}_l$ with $t^{n+1} - 1$ a unit at l, such that*

   $$\overline{r} \cong H^{n-1}(Y_t \otimes \overline{\mathbb{Q}}_l, \mathbb{F}_l)^{H'}$$

   as symplectic representations of the inertia group at l.
3. *$r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)}$ is unramified and $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_q/\mathbb{Q}_q)}(\mathrm{Frob}_q)$ has eigenvalues of the form $\alpha, \alpha q, ..., \alpha q^{n-1}$.*

Then there is a Galois totally real number field over which r becomes automorphic.

As in the $n = 2$ case we expect these results to have important applications. For instance we prove the following theorems.

Theorem C *Let $E/\mathbb{Q}$ be an elliptic curve with multiplicative reduction at a prime q.*

1. *For any odd integer m there is a finite Galois totally real field $F/\mathbb{Q}$ such that $\mathrm{Symm}^m H^1(E)$ becomes automorphic over F. (One can choose an F that will work simultaneously for any finite set of odd positive integers.)*
2. *For any positive integer m the L-function $L(\mathrm{Symm}^m H^1(E)/\mathbb{Q}, s)$ has meromorphic continuation to the whole complex plane and satisfies the expected functional equation. It does not vanish in $\mathrm{Re}\, s \geq 1 + m/2$.*

3. The numbers

$$(1 + p - \#E(\mathbb{F}_p))/2\sqrt{p}$$

are equidistributed in $[-1, 1]$ *with respect to the measure* $(2/\pi)\sqrt{1-t^2}\, dt$.

(See theorems 4.1, 4.2 and 4.3 below.)

Theorem D *Suppose that* n *is an even, positive integer, and that* $t \in \mathbb{Q} - \mathbb{Z}[1/(n+1)]$. *Then the L-function* $L(V_t, s)$ *of*

$$H^{n-1}(Y_t \times \overline{\mathbb{Q}}, \mathbb{Q}_l)^{H'}$$

is independent of l, *has meromorphic continuation to the whole complex plane and satisfies the expected functional equaltion*

$$L(V_t, s) = \epsilon(V_t, s) L(V_t, n - s).$$

(See theorem 4.4 for details.)

Other applications are surely possible. For instance in the setting of theorem B one can conclude that r is part of a compatible system of l'-adic Galois representations.

The surjectivity assumptions in theorems A and B can be relaxed, but we have not been able to formulate cleanly the generality in which our method works. It derives from similar assumptions in [CHT] and [T3]. The assumption that r is crystalline with distinct Hodge-Tate numbers also derives from [CHT] and [T3]. The assumptions that the Hodge-Tate numbers are exactly $0, 1, ..., n-1$ and that the restriction of r mod l to inertia at l comes from some Y_t both derive from the particular family Y_t we work with. The second of these assumptions might be relaxed either by using different families or if one had improvements to the lifting theorems in [CHT] and [T3]. Griffiths transversality seems to provide an obstruction to finding suitable families with other Hodge-Tate weights, but this assumption might be relaxed if one had results about the possible weights of automorphic mod l representations on unitary groups ('the weight in Serre's conjecture'). The assumptions at q derive from limits to our current knowledge about automorphic forms on unitary groups. One could expect to remove them as the trace formula technology improves.

To generalise the results of [T1] and [T2] to higher dimensional representations two things were needed: generalisations of the 'modularity of lifts' theorems of Wiles [W] and Taylor-Wiles [TW] from GL_2 to GSp_n (or some similar group); and families of 'motives' with large monodromy but with $h^{i,j} \leq 1$ for all i, j.

The first of these problems is overcome in [CHT] and [T3]. When this paper was submitted only [CHT] was available. In that paper we had succeeded in generalising the arguments of [TW] to prove modularity of 'minimal' lifts but had only been able to generalise the results [W] conditionally under the assumption of a generalisation of Ihara's lemma (lemma 3.2 of [I], see conjecture A in the introduction of [CHT] for our conjectured generalisation). Thus at that time the main results of this paper were all conditional on conjecture A of the introduction of [CHT]. However, while this paper was being refereed, one of us (R.T.) found a way to apply generalisations of the arguments of [TW] directly in the non-minimal case thus avoiding the level raising arguments of [W] and the appeal to conjecture A of [CHT]. This means that the results of this paper also became unconditional.

The second of the above problems is treated in this paper. We learnt of the family Y_t from the physics literature, but have since been told that it had been extensively studied earlier by Dwork (unpublished).

In the first section of this paper we study the family Y_t. Most of the results we state seem to be well known, but, when we can't find an easily accessible reference, we give the proof. In the second section we recall some simple algebraic number theory results that we will need. The main substance of the paper is contained in section three where we prove various potential modularity theorems. In the final section we give some example applications, including theorems C and D.

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Notation

We will write μ_m for the group scheme of m^{th} roots of 1. We will use ζ_m to denote a primitive m^{th} root of 1. We will also denote by ϵ_l the l-adic cyclotomic character.

c will denote complex conjugation.

If T is a variety and t a point of T we will write $\mathcal{O}_{T,t}$ for the local ring of T at t. We will use $k(t)$ to denote its residue field and $\mathcal{O}_{T,t}^{\wedge}$ to denote its completion.

If r is a representation we will write r^{ss} for its semisimplification.

Let K be a p-adic field and $v : K^\times \twoheadrightarrow \mathbb{Z}$ its valuation. We will write $\mathcal{O}_K$ for its ring of integers and $k(K)$ or $k(v)$ for its residue field. We will denote by $|\ |_K$ the absolute value on K defined by $|a|_K = (\#k(K))^{-v(a)}$. We will also denote by W_K the Weil group of K and by I_K the inertia subgroup of W_K. We will write Frob_K or Frob_v for the geometric Frobenius element in W_K/I_K. We will write Art_K for the Artin isomorphism $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{\mathrm{ab}}$ normalised to send uniformisers to lifts of Frob_K. If $p \not| n$ then we will write $\omega_{K,n} = \omega_n$ for the character

$$\begin{array}{ccc} I_K & \longrightarrow k(K)^\times & \\ \sigma & \longmapsto & (\sigma \sqrt[p^n-1]{\varpi_K})/\sqrt[p^n-1]{\varpi_K}, \end{array}$$

where ϖ_K is a uniformiser for K. (The definition is independent of the choice of this uniformiser. Note that $\epsilon_p = \omega_{K,1}^{[I_{\mathbb{Q}_p}:I_K]}$.) If $l \neq p$, we will let $t_{K,l}$ denote a surjective homomorphism $t_{K,l} : I_K \twoheadrightarrow \mathbb{Z}_l$ (which is unique up to $\mathbb{Z}_l^\times$-multiples). By a Weil-Deligne represenation of W_K we mean a pair (r, N) where $r : W_K \to GL(V)$ is a homomorphism with open kernel and where $N \in \mathrm{End}\,(V)$ satisfies $r(\sigma)Nr(\sigma)^{-1} = |\mathrm{Art}_K^{-1}\sigma|_K N$. We will write $(r,N)^{\text{F-ss}}$ for the Frobenius semisimplification (r^{ss}, N) of (r, N). We will denote by rec the local Langlands bijection from irreducible smooth representations of $GL_n(K)$ to n-dimensional Frobenius semi-simple Weil-Deligne representations of W_K (see [HT]). If $l \neq p$ and W is a continuous finite dimensional l-adic representation of $\mathrm{Gal}\,(\overline{F}/F)$ then we write $\mathrm{WD}(W)$ for the associated Weil-Deligne representation of W_K (see for instance [TY]). We will write $\mathrm{Sp}_n(1)$ for the Steinberg representation of $GL_n(K)$.

If K is a number field (i.e. a finite extension of $\mathbb{Q}$) we will write $\mathbb{A}_K$ for its ring of adeles.

1 A family of hypersurfaces.

Let n be an even positive integer. Consider the scheme

$$Y \subset \mathbb{P}^n \times \mathbb{P}^1$$

over $\mathbb{Z}[\frac{1}{n+1}]$ defined by

$$s(X_0^{n+1} + X_1^{n+1} + \cdots + X_n^{n+1}) = (n+1)tX_0 \cdot X_1 \cdot \cdots \cdot X_n.$$

We will consider Y as a family of schemes over $\mathbb{P}^1$ by projection π to the second factor. We will label points of $\mathbb{P}^1$ with reference to the affine piece $s = 1$. If t is a point of $\mathbb{P}^1$ we shall write Y_t for the fibre of Y above t. Let $T_0 = \mathbb{P}^1 - (\{\infty\} \cup \mu_{n+1})/\mathbb{Z}[1/(n+1)]$. The mapping $Y|_{T_0} \to T_0$ is smooth. The total space $Y - Y_\infty$ is regular. If $\zeta^{n+1} = 1$ then Y_ζ has only isolated singularities at points where all the X_i's are $(n+1)^{th}$ roots of unity with product ζ^{-1}. These singularities are ordinary quadratic singularities.

If ζ is a primitive $(n+1)^{th}$ root of unity then over $\mathbb{Z}[1/(n+1), \zeta]$ the scheme Y gets a natural action of the group $H = \mu_{n+1}^{n+1}/\mu_{n+1}$ with the sub-μ_{n+1} embedded diagonally:

$$(\zeta_0, ..., \zeta_n)(X_0 : ... : X_n) = (\zeta_0 X_0 : ... : \zeta_n X_n).$$

We will let H_0 denote the subgroup of elements $(\zeta_i) \in H$ with $\zeta_0\zeta_1...\zeta_n = 1$. Then H_0 acts on every fibre Y_t. If $t^{n+1} = 1$ then H_0 permutes transitively the singularities of Y_t. The whole group H acts on Y_0.

For N coprime to $n+1$ set

$$V_n[N] = V[N] = (R^{n-1}\pi_*\mathbb{Z}/N\mathbb{Z})^{H_0},$$

a lisse sheaf on $T_0 \times \operatorname{Spec} \mathbb{Z}[1/N(n+1)]$. (Although the action of H_0 is only defined over a cyclotomic extension, the H_0 invariants make sense over $\mathbb{Z}[1/N(n+1)]$.) If $l \not| n+1$ is prime set

$$V_{n,l} = V_l = (\lim_{\leftarrow m} V[l^m]) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Similarly define

$$V = (R^{n-1}\pi_*\mathbb{Z})^{H_0}$$

a locally constant sheaf on $T_0(\mathbb{C})$ and

$$V_{\mathrm{DR}} = \mathcal{H}^{n-1}_{\mathrm{DR}}(Y/(\mathbb{P}^1 - (\{\infty\} \cup \mu_{n+1})))^{H_0}$$

a locally free coherent sheaf with a decreasing filtration $F^i V_{\mathrm{DR}}$ (and a connection) over T_0. The locally constant sheaf on $T_0(\mathbb{C})$ corresponding to V_l is $V \otimes \mathbb{Q}_l$. Note that there are natural perfect alternating pairings:

$$V[N] \times V[N] \longrightarrow (\mathbb{Z}/N\mathbb{Z})(1-n)$$

and

$$V_l \times V_l \longrightarrow \mathbb{Q}_l(1-n)$$

and

$$V \times V \longrightarrow \mathbb{Z}$$

coming from Poincare duality.

The following facts seem to be well known (see for example [K2], [LSW]). Nick Katz has told us that many of them were known to Dwork in 1960's, but he only wrote up the case $n = 3$.

Lemma 1.1 *$V[N]$, V_l and $V \otimes \mathbb{Q}$ are all locally free of rank n.*

Proof: We need only check the fibre at 0. In the case $V \otimes \mathbb{C}$ this is shown to be locally free of rank n in proposition I.7.4 of [DMOS]. The same argument works in the other cases. □

Corollary 1.2 *If $(N, n+1) = 1$ then V/NV is the locally constant sheaf on $T_0(\mathbb{C})$ corresponding to $V[N]$.*

Lemma 1.3 *Under the action of $H/H_0 \cong \mu_{n+1}$ the fibres $(V \otimes \mathbb{C})_0$ and $(V_l \otimes_{\mathbb{Q}_l} \overline{\mathbb{Q}}_l)_0$ split up as n one dimensional eigenspaces, one for each non-trivial character of μ_{n+1}.*

Proof: This is just proposition I.7.4 of [DMOS]. □

Lemma 1.4 *The monodromy of $V \otimes \mathbb{Q}$ around a point in $\zeta \in \mu_{n+1}$ has 1-eigenspace of dimension at least $n-1$.*

Proof: Let $t \in T_0(\mathbb{C})$. Picard-Lefschetz theory (see [SGA7]) gives an H_0-orbit Δ of elements of $H^{n-1}(Y_t(\mathbb{C}), \mathbb{Z})$ and an exact sequence

$$(0) \longrightarrow H^{n-1}(Y_\zeta(\mathbb{C}), \mathbb{Z}) \longrightarrow H^{n-1}(Y_t(\mathbb{C}), \mathbb{Z}) \longrightarrow \mathbb{Z}^\Delta.$$

If $x \in H^{n-1}(Y_t(\mathbb{C}), \mathbb{Z})$ maps to $(x_\delta) \in \mathbb{Z}^\Delta$ then the monodromy operator sends x to $x \pm \sum_{\delta \in \Delta} x_\delta \delta$. Taking H_0 invariants we get an exact sequence

$$(0) \longrightarrow H^{n-1}(Y_\zeta(\mathbb{C}), \mathbb{Z})^{H_0} \longrightarrow \widetilde{V}_\zeta \stackrel{d}{\longrightarrow} \mathbb{Z}$$

and the monodromy operator sends $x \in V_\zeta$ to $x \pm d(x) \sum_{\delta \in \Delta} \delta$. □

We remark that this argument works equally well for V_l or $V[l]$ over $T_0 \times \mathbb{Z}[1/l(n+1)]$.

We also want to analyse the monodromy at infinity. For simplicity we will argue analytically as in [M] and [LSW], which in turn is based on Griffith's method [G] for calculating the cohomology of a hypersurface. (Indeed the argument below is sketched in [LSW].) One of us (N.I.S-B.) has found an H_0-equivariant blow up of Y which is semistable at ∞, and it seems possible that combining this with the Rapoport-Zink spectral sequence would give an algebraic argument, which might give more precise information.

Write

$$Q_t = (X_0^{n+1} + ... + X_n^{n+1})/(n+1) - tX_0X_1...X_n,$$

and

$$\Omega = \sum_{i=0}^{n} (-1)^i X_i dX_0 \wedge ... \wedge dX_{i-1} \wedge dX_{i+1} \wedge ... \wedge dX_n.$$

Then for $i = 1, ..., n+1$

$$\omega_i' = (i-1)!(X_0X_1...X_n)^{i-1}\Omega/Q_t^i$$

is a meromorphic differential on $\mathbb{P}^n(\mathbb{C})$ with a pole of order i along Y_t. Moreover $d\omega_i'/dt = \omega_{i+1}'$. Also set $\omega_i = t^i\omega_i'$ so that ω_i is H-invariant and

$$td\omega_i/dt = i\omega_i + \omega_{i+1}.$$

Suppose that $t \notin \{\infty\} \cup \mu_{n+1}(\mathbb{C})$. We claim that for $i = 1, ..., n$ we have

$$\omega_i' \in \mathcal{H}_i(Y_t) - \mathcal{H}_{i-1}(Y_t)$$

in the notation of section 5 of [G]. If this were not the case then proposition 4.6 of [G] would tell us that $(X_0X_1...X_n)^{i-1}$ lies in the ideal generated by the $X_j^n - tX_0...X_{j-1}X_{j+1}...X_n$. Hence $(X_0X_1...X_n)^i$ would lie in the ideal generated by the $X_j^{n+1} - tX_0X_1...X_n$. Symmetrising under the action of H_0 and using the fact that $\mathbb{C}[X_0, ..., X_n]^{H_0} = \mathbb{C}[Z, Y_0, ..., Y_n]/(Z^{n+1} - Y_0...Y_n)$ (with $Y_j = X_j^{n+1}$ and $Z = X_0...X_n$), we would have that Z^i lies in the ideal generated by the $Y_j - tZ$ and $Z^{n+1} - Y_0...Y_n$ in $\mathbb{C}[Z, Y_0, ..., Y_n]$. Taking the degree i homogeneous part and using the fact that $i < n+1$ we would have that Z^i lies in the ideal generated by the $Y_j - tZ$ in $\mathbb{C}[Z, Y_0, ..., Y_n]$. Setting $Z = 1$ and $Y_0 = Y_1 = ... = Y_n = t$ would then give a contradiction, proving the claim.

Integration against ω_i' gives a linear form $H_n(\mathbb{P}^n(\mathbb{C}) - Y_t(\mathbb{C}), \mathbb{Z}) \to \mathbb{C}$. Composing this with the map $H_{n-1}(Y_t(\mathbb{C}), \mathbb{Z}) \to H_n(\mathbb{P}^n(\mathbb{C}) - Y_t(\mathbb{C}), \mathbb{Z})$ shows that ω_i' gives a class $R(\omega_i')$ in $H^{n-1}(Y_t(\mathbb{C}), \mathbb{C})^{H_0}$. According to theorem 8.3 of [G]

$$R(\omega_i') \in (F^{n-i}V_{\mathrm{DR}})_t \otimes \mathbb{C} - (F^{n+1-i}V_{\mathrm{DR}})_t \otimes \mathbb{C}.$$

Thus the $R(\omega_i')$ for $i = 1, ..., n$ are a basis of $H^{n-1}(Y_t(\mathbb{C}), \mathbb{C})^{H_0}$. Moreover we deduce the following lemma (due to Deligne, see proposition I.7.6 of [DMOS]).

Lemma 1.5 *For $j = 0, ..., n-1$ we have*

$$\dim F^jV_{\mathrm{DR}}/F^{j+1}V_{\mathrm{DR}} = 1.$$

Moreover if ζ is a primitive $(n+1)^{th}$ root of unity then H acts on

$$F^j V_{\mathrm{DR},0}/F^{j+1}V_{\mathrm{DR},0} \otimes \mathbb{Z}[1/(n+1), \zeta]$$

by

$$(\zeta_0, ..., \zeta_n) \longmapsto (\zeta_0 ... \zeta_n)^{n-j}.$$

Now assume in addition that $t \neq 0$. Then the class $[\omega_{n+1}]$ is in the span of the classes $[\omega_1], ..., [\omega_n]$. In section 4 (particularly equation (4.5)) of [G] a method is described for calculating its coefficients. To carry it out we will need certain integers $A_{i,j}$ defined recursively for $j > i \geq 0$ by

- $A_{0,j} = 1$ for all $j > 0$ and
- $A_{i+1,j} = A_{i,i+1} + 2A_{i,i+2} + ... + (j-i-1)A_{i,j-1}$.

Note that these also satisfy $A_{i,i+1} = 1$ for all i and

$$A_{i,j} = A_{i,j-1} + (j-i)A_{i-1,j-1}$$

for $j-1 > i > 0$. We claim that for all non-negative integers i and n we have

$$(i+1)^n = \sum_{j=0}^{\min(n,i)} A_{n-j,n+1}\ i!/(i-j)!.$$

This can be proved by induction on n. The case $n=0$ is clear. For general i we see that

$$\begin{array}{rl} & \sum_{j=0}^{\min(n,i)} A_{n-j,n+1} i!/(i-j)! \\ = & \sum_{j=1}^{\min(n,i)} A_{n-j,n} i!/(i-j)! + \sum_{j=0}^{\min(n-1,i)} (j+1) A_{n-j-1,n} i!/(i-j)! \\ = & \sum_{j=0}^{\min(n-1,i-1)} A_{n-j-1,n} i!(i-j+j+1)/(i-j)! + A_{n-i-1,n}(i+1)i! \\ = & (i+1) \sum_{j=0}^{\min(i,n-1)} A_{n-1-j,n} i!/(i-j)! \\ = & (i+1)^n, \end{array}$$

where we set $A_{n-i-1,n} = 0$ if $i \geq n$. Thus we see that, as polynomials in T

$$T^n = \sum_{j=0}^{n} A_{j,n+1}(T-1)(T-2)...(T+j-n).$$

Write

$$A(z) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & \frac{A_{n,n+1}}{z-1} \\ 1 & 2 & 0 & \dots & 0 & 0 & \frac{A_{n-1,n+1}}{z-1} \\ 0 & 1 & 3 & \dots & 0 & 0 & \frac{A_{n-2,n+1}}{z-1} \\ & & & \ddots & & & \\ & & & & n-2 & 0 & \frac{A_{3,n+1}}{z-1} \\ & & & & 1 & n-1 & \frac{A_{2,n+1}}{z-1} \\ & & & & 0 & 1 & n+\frac{A_{1,n+1}}{z-1} \end{pmatrix}.$$

Then expanding along the last column we see that $A(0)$ has characteristic polynomial

$$\sum_{j=0}^{n+1} A_{j,n+1}(T-1)(T-2)...(T+j-n) = T^n.$$

It also has rank $n-1$ and so has minimal polynomial T^n. Consider the differential equation

$$zdv(z)/dz = -A(z)v(z)/(n+1).$$

In a neighbourhood of zero its solutions are of the form

$$S(z)\exp(-A(0)\log(z)/(n+1))v_0$$

where $S(z)$ is a single matrix valued function in a neighbourhood of 0 and v_0 is a constant vector. (See section 1 of [M].)

We will prove by induction on i that

$$(1-t^{n+1})[\omega_{n+1}] - t^{n+1}(A_{1,n+1}[\omega_n] + A_{2,n+1}[\omega_{n-1}] + ... + A_{i,n+1}[\omega_{n+1-i}]) = (n-1-i)!t^{n+1} \left[\left(\sum_{j=i+1}^n t^{j-i}(j-i)A_{i,j}(X_0...X_j)^{j-i-1}(X_{j+1}...X_n)^{n+j-i}\right)\Omega/Q_t^{n-i}\right].$$

To prove the case $i=0$ combine formula (4.5) of [G] with the formula

$$(1-t^{n+1})(X_0...X_n)^n = \sum_{j=0}^n (X_j^n - X_0...X_{j-1}X_{j+1}...X_n)(X_0...X_{j-1})^{j-1}X_j^j(X_{j+1}...X_n)^{n+j}.$$

To prove the case $i>0$ combine the case $i-1$ and formula (4.5) of [G] with the formula

$$\sum_{j=i}^n t^{j+1-i}(j+1-i)A_{i-1,j}(X_0...X_j)^{j-i}(X_{j+1}...X_n)^{n+1+j-i} - A_{i,n+1}t^{n+1-i}(X_0...X_n)^{n-i} = \sum_{k=i+1}^n (X_k - X_0...X_{k-1}X_{k+1}...X_n)t^{k-i}A_{i,k} (X_0...X_{k-1})^{k-i-1}X_k^{k-i}(X_{k+1}...X_n)^{n+k-i}.$$

The special case $i = n$ then tells us that

$$[\omega_{n+1}] = \frac{1}{t^{-(n+1)} - 1}(A_{1,n+1}[\omega_n] + ... + A_{n,n+1}[\omega_1]).$$

Suppose that $\gamma_t \in H_{n-1}(Y_t(\mathbb{C}), \mathbb{Z})^{H_0}$ maps to $\Gamma_t \in H_n(\mathbb{P}^n(\mathbb{C}) - Y_t(\mathbb{C}), \mathbb{Z})$. Then the coefficients of γ_t with respect to the basis of $H_{n-1}(Y_t(\mathbb{C}), \mathbb{C})^{H_0}$ dual to $[\omega_1], ..., [\omega_n]$ is given by

$$v(\gamma_t) = \begin{pmatrix} \int_{\Gamma_t} \omega_1 \\ \vdots \\ \int_{\Gamma_t} \omega_n \end{pmatrix}.$$

As explained in [M] if γ_t is a locally constant section of the local system of the $H_{n-1}(Y_t(\mathbb{C}), \mathbb{Z})^{H_0}$ then the Γ_t can be taken locally constant and so

$$tdv(\gamma_t)/dt = A(t^{-(n+1)})v(\gamma_t).$$

Let z_0 be close to zero in $\mathbb{P}^1$ and let P be a loop in a small neighbourhood of 0 based at z_0 and going m times around 0. Let $\widetilde{P}$ be a lifting of this path under the map $\mathbb{P}^1 \to \mathbb{P}^1$ under which $t \mapsto t^{-(n+1)}$ starting at t_0 and ending at ht_0 for some $h \in H$. Let $\gamma \in H_{n-1}(Y_{t_0}(\mathbb{C}), \mathbb{Z})^{H_0}$. If we carry γ along $\widetilde{P}$ in a locally constant fashion we end up with an element $\widetilde{P}\gamma \in H_{n-1}(Y_{ht_0}(\mathbb{C}), \mathbb{Z})^{H_0}$ where

$$v(\widetilde{P}\gamma) = S(t_0^{-(n+1)}) \exp(\pm 2\pi i m A(0)/(n+1)) S(t_0^{-(n+1)})^{-1} v(\gamma),$$

and so

$$h^{-1}v(\widetilde{P}\gamma) = S(t_0^{-(n+1)}) \exp(\pm 2\pi i m A(0)/(n+1)) S(t_0^{-(n+1)})^{-1} v(\gamma).$$

In particular we see that the monodromy around infinity on $H_{n-1}(Y_{t_0}(\mathbb{C}), \mathbb{Z})^{H_0}$ is generated by $\exp(2\pi i A(0))$ with respect to a suitable basis. This matrix is unipotent with minimal polynomial $(T-1)^n$.

Let ζ denote a primitive $(n+1)^{th}$ root of 1. The map $t \mapsto t^{n+1}$ gives a finite Galois etale cover

$$(\mathbb{P}^1 - \{0, \infty\}) \times \operatorname{Spec} \mathbb{C} \longrightarrow (\mathbb{P}^1 - \{0, \infty\}) \times \operatorname{Spec} \mathbb{C}$$

with Galois group H/H_0. Thus the sheaf V descends to a locally constant sheaf $\widetilde{V}$ on $\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$. Note that there is a natural perfect alternating pairing:

$$\widetilde{V} \times \widetilde{V} \longrightarrow \mathbb{Z}.$$

Lemma 1.6 *The monodromy of $\widetilde{V}$ around ∞ unipotent with minimal polynomial $(T-1)^n$. The monodromy around 1 is unipotent and the 1 eigenspace has dimension exactly $n-1$. The mondromy around 0 has eigenvalues the set of nontrivial $(n+1)^{th}$ roots of 1 (each with multiplicity one).*

Proof: By the calculation of the last but one paragraph the monodromy of $V \otimes \mathbb{C}$ around ∞ can be respresented by $\exp(\pm 2\pi i A(0)/(n+1))$ with respect to some basis. The action of the monodromy at 0 follows from lemma 1.3. Because $\mathbb{P}^1 \to \mathbb{P}^1$ over $\mathbb{Z}[1/(n+1)]$ given by $t \mapsto t^{n+1}$ is etale above 1 it follows from lemma 1.4 that the monodromy at 1 has 1 eigenspace of dimension at least $n-1$. Because it preserves a perfect alternating pairing we see that it must have determinant 1. Thus 1 is its only eigenvalue. Finally it can not be the identity as else the monodromy at ∞ would be conjugate to the monodromy at 0 or its inverse. $\Box$

Corollary 1.7 *The monodromy of V around ∞ unipotent with minimal polynomial $(T-1)^n$. The monodromy around any element of $\mu_{n+1}(\mathbb{C})$ is unipotent with 1 eigenspace of dimension exactly $n-1$.*

Corollary 1.8 *Identify $\mathbb{C}((1/T)) = \mathcal{O}^{\wedge}_{\mathbb{P}^1\times\mathbb{C},\infty}(T)$. Also identify*

$$\pi_1(\mathrm{Spec}\,\mathbb{C}((1/T)) \cong \lim_{\leftarrow N} \mathrm{Gal}\,(\mathbb{C}((1/T^{1/N}))/\mathbb{C}((1/T))) \cong \prod_p \mathbb{Z}_p.$$

Then the action of $\pi_1(\mathrm{Spec}\,\mathbb{C}((1/T))$ on $V_l|_{\mathrm{Spec}\,\mathbb{C}((1/T))}$ (resp. $V[l]|_{\mathrm{Spec}\,\mathbb{C}((1/T))}$) is via $x \mapsto u^x$ for a unipotent matrix u. In the case of V_l then u has minimal polynomial $(X-1)^n$. There exists a constant $D(n)$ depending only on n such that for $l > D(n)$, this is also true in the case of $V[l]$.

Proof: A unipotent matix $u \in GL_n(\mathbb{Z})$ with minimal polynomial $(X-1)^n$ reduces modulo l for all but finitely many primes l to an unipotent matrix in $GL_n(\mathbb{F}_l)$ with minimal polynomial $(X-1)^n$. (If not for some $0 < i < n$ we would have $(u-1)^i \equiv 0 \bmod l$ for infinitely many l.) $\Box$

It seems likely that N.I.S.-B.'s resolution of Y would allow one to make explicit the finite set of l for which the last assertion fails.

We would like to thank Nick Katz for telling us that the following lemma is true and providing a reference to [K2]. Because of the difficulty of comparing the notation of [K2] with ours we have chosen to give a direct proof. If $z \in \mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ then let $Sp(\widetilde{V}_z \otimes \mathbb{C})$ denote the group of automorphisms of $\widetilde{V}_z \otimes \mathbb{C}$ which preserve the alternating form.

Lemma 1.9 *If $z \in \mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}$ then the image of $\pi_1(\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}, z)$ in $Sp(\widetilde{V}_z \otimes \mathbb{C})$ is Zariski dense.*

Proof: This follows from the previous lemma and the results of [BH]. More precisely let $\mathcal{H}$ denote the image of $\pi_1(\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}, z)$ in $Sp(\widetilde{V}_z \otimes \mathbb{C})$ and let $\mathcal{H}_r$ denote the normal subgroup generated by monodromy at 1. It follows from proposition 3.3 of [BH] that $\mathcal{H}$ is irreducible and from theorem 5.8 of [BH] that $\mathcal{H}$ is also primitive. Theorem 5.3 of [BH] tells us that $\mathcal{H}_r$ is irreducible and then theorem 5.14 of [BH] tells us that $\mathcal{H}_r$ is primitive. (In the case $n = 2$ use the fact that $\mathcal{H}_r$ is irreducible and contains a non-trivial unipotent element.) $\mathcal{H}_r$ is infinite. Then it follows from propositions 6.3 and 6.4 of [BH] that $\mathcal{H}_r$ is Zariski dense in $Sp(\widetilde{V}_z \otimes \mathbb{C})$. $\Box$

If $t \in T_0(\mathbb{C})$ let $Sp(V_t \otimes \mathbb{C})$ (resp. $Sp(V[N]_t)$) denote the group of automorphisms of $V_t \otimes \mathbb{C}$ (resp. $V[N]_t$) which preserve the alternating form.

Corollary 1.10 *If $t \in T_0(\mathbb{C})$ then the image of $\pi_1(T_0(\mathbb{C}), t)$ in $Sp(V_t \otimes \mathbb{C})$ is Zariski dense.*

Combining this with theorem 7.5 and lemma 8.4 of [MVW] (or theorem 5.1 of [N]) we obtain the following corollary. (In the spirit of full disclosure we remark that theorem 7.5 of [MVW] relies on the classification of finite simple groups and [N] does not pretend to give a complete proof of its theorem 5.1.)

Corollary 1.11 *There is a constant $C(n)$ such that if N is an integer divisible only by primes $p > C(n)$ and if $t \in T_0(\mathbb{C})$ then the map*

$$\pi_1(T_0(\mathbb{C}), t) \longrightarrow Sp(V[N]_t)$$

is surjective.

Let F be a number field and let W be a free $\mathbb{Z}/N\mathbb{Z}$-module of rank n with a continuous action of $\mathrm{Gal}(\overline{F}/F)$ and a perfect alternating pairing

$$\langle\ ,\ \rangle_W : W \times W \longrightarrow (\mathbb{Z}/N\mathbb{Z})(1-n).$$

We may think of W as a lisse etale sheaf over $\mathrm{Spec}\, F$. Consider the functor from $T_0 \times \mathrm{Spec}\, F$-schemes to sets which sends X to the set of isomorphisms between the pull back of W and the pull back of $V[N]$ which sends $\langle\ ,\ \rangle_W$ to the pairing we have defined on $V[N]$. This functor is represented by a finite etale cover $T_W/T_0 \times \mathrm{Spec}\, F$. The previous corollary implies the next one.

Corollary 1.12 *If N is an integer divisible only by primes $p > C(n)$ and if $W, \langle\ ,\ \rangle_W$ is as above, then $T_W(\mathbb{C})$ is connected for any embedding $F \hookrightarrow \mathbb{C}$, i.e. T_W is geometrically connected.*

Lemma 1.13 *Suppose that $K/\mathbb{Q}_l$ is a finite extension and that $t \in T_0(K)$. Then $V_{l,t}$ is a de Rham representation of $\mathrm{Gal}\,(\overline{K}/K)$ with Hodge-Tate numbers $\{0, 1, ., , , , n-1\}$. If $t \in \mathcal{O}_K$ and $1/(t^{n+1} - 1) \in \mathcal{O}_K$ then $V_{l,t}$ is crystalline.*

Proof: $V_{l,t} = H^{n-1}(Y_t \times \mathrm{Spec}\,\overline{K}, \mathbb{Q}_l)^{H_0}$. The first assertion follows from the comparison theorem and the fact that $H^{n-1}_{\mathrm{DR}}(Y_t/K)^{H_0}$ has one dimensional graded pieces in each of the degrees $0, 1, ..., n-1$. The second assertion follows as $Y_t/\mathcal{O}_K$ is smooth and projective. $\Box$

Lemma 1.14 *Suppose that $l \equiv 1 \bmod n+1$. Then*

$$V[l]_0 \cong 1 \oplus \epsilon_l^{-1} \oplus ... \oplus \epsilon_l^{1-n}$$

as a module for $I_{\mathbb{Q}_l}$.

Proof: It suffices to prove that

$$V_{l,0} \cong 1 \oplus \epsilon_l \oplus ... \oplus \epsilon_l^{1-n}.$$

(As $l > n$ the characters $\epsilon^0, ..., \epsilon^{1-n}$ all have distinct reductions modulo l). However because l splits in the extension of $\mathbb{Q}$ obtained by adjoining a primitive $(n+1)^{th}$ root of 1, lemma 1.3 tells us that $V_{l,0}$ is the direct sum of n characters as a $\mathrm{Gal}\,(\overline{\mathbb{Q}}_l/\mathbb{Q}_l)$-module. These characters are crystalline and the Hodge-Tate numbers are $0, 1, ..., n-1$. The results follows. $\Box$

Lemma 1.15 *Suppose $q \neq l$ are primes not dividing $n+1$, and suppose that $K/\mathbb{Q}_q$ is a finite extension. Normalise the valuation v_K on K to have image $\mathbb{Z}$. Suppose that $a \in K$ has $v_K(a) < 0$.*

1. *The semisimplification of $V_{l,a}$ and $V[l]_a$ are unramified and Frob_K has eigenvalues of the form $\alpha, \alpha\#k(K), ..., \alpha(\#k(K))^{n-1}$ for some $\alpha \in \{\pm 1\}$, where $k(K)$ denotes the residue field of K.*

2. *The inertia group acts on $V_{l,a}$ as $\exp(Nt_K)$, where N is a nilpotent endomorphism of $V_{l,a}$ with minimal polynomial X^n.*

3. *The inertia group acts on $V[l]_a$ as $\exp(v_K(a)Nt_K)$, where N is a nilpotent endomorphism of $V[l]_a$, and if $l > D(n)$ then N has minimal polynomial T^n.*

Proof: First we prove the second and third parts. Let W denote the Witt vectors of $\overline{\mathbb{F}}_q$ and let F denote its field of fractions. We have a commutative diagram:

$$\begin{array}{ccc} \pi_1(\mathrm{Spec}\,\overline{F}((1/T))) & \xrightarrow{\sim} & \prod_p \mathbb{Z}_p \\ \downarrow & & \downarrow \\ \pi_1(\mathrm{Spec}\,W((1/T))) & \xrightarrow{\sim} & \prod_{p\neq q} \mathbb{Z}_p \\ \uparrow & & \uparrow v_K(a) \\ \pi_1(\mathrm{Spec}\,FK) & \twoheadrightarrow & \prod_{p\neq q} \mathbb{Z}_p. \end{array}$$

Here the left hand up arrow is induced by $T \mapsto a$. The right hand down arrow is the natural projection and the right hand up arrow is multiplication by $v_K(a)$. The isomorphisms $\pi_1(\mathrm{Spec}\,\overline{F}((1/T))) \xrightarrow{\sim} \prod_p \mathbb{Z}_p$ and $\pi_1(\mathrm{Spec}\,W((1/T))) \xrightarrow{\sim} \prod_{p\neq q} \mathbb{Z}_p$ result from corollary XIII.5.3 of [SGA1]. More precisely

$$\pi_1(\mathrm{Spec}\,\overline{F}((1/T))) = \lim_{\leftarrow N} \mathrm{Gal}\,(\overline{F}((1/T^{1/N}))/\overline{F}((1/T)))$$

and

$$\pi_1(\mathrm{Spec}\,W((1/T))) = \lim_{\leftarrow (N,q)=1} \mathrm{Gal}\,(W((1/T^{1/N}))/W((1/T))).$$

(Note that, as the fraction field of $W[[1/T]]/(1/T)$ has characteristic zero, the tame assumption in corollary XIII.5.3 is vacuous.) The final surjection $\pi_1(\mathrm{Spec}\,FK) \twoheadrightarrow \prod_{p\neq q} \mathbb{Z}_p$ comes from

$$\pi_1(\mathrm{Spec}\,FK) \twoheadrightarrow \lim_{\leftarrow (N,q)=1} \mathrm{Gal}\,(FK(\varpi_K^{1/N})/FK),$$

where ϖ_K is a uniformiser in K.

Considering

$$W((1/T)) = \mathcal{O}^{\wedge}_{\mathbb{P}^1,\infty}(T),$$

the sheaves $V_l|_{\mathrm{Spec}\,W((1/T))}$ and $V[l]|_{\mathrm{Spec}\,W((1/T))}$ correspond to representations of $\pi_1(\mathrm{Spec}\,W((1/T)))$. Corollary 1.8 tells us that the pull back of these representations to $\pi_1(\mathrm{Spec}\,\overline{F}((1/T))) \cong \prod_p \mathbb{Z}_p$ sends 1 to a unipotent matrix. Moreover in the case V_l or in the case $V[l]$ with $l > D(n)$, we know that this unipotent matrix has minimal polynomial $(X-1)^n$. The lemma follows.

Now we prove the first part. It is enough to consider $V_{l,t}$. From the second part we see that Frob_K has eigenvalues $\alpha, \alpha\#k(K), ..., \alpha(\#k(K))^{n-1}$ for some $\alpha \in \mathbb{Q}_l^\times$. The alternating pairing shows that

$$\{\alpha, \alpha\#k(K), ..., \alpha(\#k(K))^{n-1}\} = \{\alpha^{-1}, \alpha^{-1}\#k(K), ..., \alpha^{-1}(\#k(K))^{n-1}\}.$$

Thus $\alpha = \pm 1$. $\Box$

2 Some algebraic number theory

We briefly recall a theorem of Moret-Bailly [MB] (see also [GPR]). (Luis Dieulefait tells us that he has also explained this slight strengthening of the result of [MB] in a conference in Strasbourg in July 2005.)

Proposition 2.1 *Let F be a number field and let $S = S_1 \coprod S_2 \coprod S_3$ be a finite set of places of F such that S_2 contains no infinite place. Suppose that T/F is a smooth, geometrically connected variety. Suppose also that for $v \in S_1$, $\Omega_v \subset T(F_v)$ is a non-empty open (for the v-topology) subset; that for $v \in S_2$, $\Omega_v \subset T(F_v^{\rm nr})$ is a non-empty open $\mathrm{Gal}\,(F_v^{\rm nr}/F_v)$-invariant subset; and that for $v \in S_3$, $\Omega_v \subset T(\overline{F}_v)$ is a non-empty open $\mathrm{Gal}\,(\overline{F}_v/F_v)$-invariant subset . Suppose finally that L/F is a finite extension.*

Then there is a finite Galois extension F'/F and a point $P \in T(F')$ such that

- *F'/F is linearly disjoint from L/F;*
- *every place v of S_1 splits completely in F' and if w is a prime of F' above v then $P \in \Omega_v \subset T(F'_w)$;*
- *every place v of S_2 is unramified in F' and if w is a prime of F' above v then $P \in \Omega_v \cap T(F'_w)$;*
- *and if w is a prime of F' above $v \in S_3$ then $P \in \Omega_v \cap T(F'_w)$.*

Proof: We may suppose that L/F is Galois. Let $L_1, ..., L_r$ denote the intermediate fields $L \supset L_i \supset F$ with L_i/F Galois with simple Galois group. Combining Hensel's lemma with the Weil bounds we see that T has an F_v rational point for all but finitely many primes v of F. Thus enlarging S_1 to include one sufficiently large prime that is not split in each field L_i (the prime may depend on i), we may suppress the first condition on F'.

Replacing F by a finite Galois extension in which all the places of S_1 split completely, in which the primes of S_2 are unramified with sufficiently large inertial degree and in which all the primes in S_3 give rise to sufficiently large completions, we may suppose that $S_2 \cup S_3 = \emptyset$. (We may have to replace the field F' we obtain with its normal closure over the original field F.)

Now the theorem follows from theorem 1.3 of [MB]. □

Lemma 2.2 *Let M be an imaginary CM field with maximal totally real subfield M^+, S a finite set of finite places of M and $T \supset S$ an infinite set of finite places of M with $cT = T$. Suppose that there are continuous characters:*

- $\chi_S : \mathcal{O}_{M,S}^\times \to \overline{\mathbb{Q}}^\times$,
- $\chi_+ : (\mathbb{A}_{M^+}^\infty)^\times \to \overline{\mathbb{Q}}^\times$,
- $\psi_0 : M^\times \to \overline{\mathbb{Q}}^\times$,

such that

- *if χ_+ is ramified at v then T contains some place of M above v,*
- $\psi_0|_{(M^+)^\times} = \chi_+|_{(M^+)^\times}$*, and*
- $\chi_S|_{(\mathbb{A}_{M^+})^\infty)^\times \cap \mathcal{O}_{M,S}^\times} = \chi_+|_{(\mathbb{A}_{M^+})^\infty)^\times \cap \mathcal{O}_{M,S}^\times}$.

Then there is a continuous character

$$\psi : (\mathbb{A}_M^\infty)^\times \longrightarrow \overline{\mathbb{Q}}^\times$$

such that

- *ψ is unramified outside T,*
- $\psi|_{M^\times} = \psi_0$,
- $\psi|_{\mathcal{O}_{M,S}^\times} = \chi_S$,
- *and* $\psi|_{(\mathbb{A}_{M^+}^\infty)^\times} = \chi_+$.

Proof: Choose $U_0 = \prod_{v \notin S} U_{0,v} \subset \prod_{v \notin S} \mathcal{O}_{M,v}^\times$ be an open subgroup such that $U_0 \cap (\mathbb{A}_{M^+}^\infty)^\times \subset \ker \chi_+$ and $U_{0,v} = \mathcal{O}_{M,v}^\times$ for $v \notin T$. Let $V = \prod_{v \notin S} V_v \subset \prod_{v \notin S} \mathcal{O}_{M,v}^\times$ be an open compact subgroup such that $V \cap \mu_\infty(M) = \{1\}$ and $V_v = \mathcal{O}_{M,v}^\times$ for $v \notin T$. Let U denote the subset of U_0 consisting of elements u with $c(u)/u \in V$. Then $U = \prod_{v \notin S} U_v$ with $U_v = \mathcal{O}_{M,v}^\times$ for $v \notin T$. Moreover $M^\times \cap \mathcal{O}_{M,S}^\times U (\mathbb{A}_{M^+}^\infty)^\times = (M^+)^\times$. (For if a lies in the intersection then

$$c(a)/a \in \ker(\mathbf{N}_{M/M^+} : \mathcal{O}_M^\times \longrightarrow \mathcal{O}_{M^+}^\times) \cap \mathcal{O}_{M,S}^\times V = \mu_\infty(M) \cap \mathcal{O}_{M,S}^\times V = \{1\},$$

so that $a \in (M^+)^\times$.)

Define a continuous character

$$\psi : \mathcal{O}_{M,S}^\times U (\mathbb{A}_{M^+}^\infty)^\times \longrightarrow \overline{\mathbb{Q}}^\times$$

to be χ_S on $\mathcal{O}_{M,S}^\times$, to be 1 on U and to be χ_+ on $(\mathbb{A}_{M^+}^\infty)^\times$. This is easily seen to be well defined. Extend ψ to $M^\times \mathcal{O}_{M,S}^\times U (\mathbb{A}_{M^+}^\infty)^\times$ by setting it equal to ψ_0 on $M^\times$. This is well defined because $M^\times \cap \mathcal{O}_{M,S}^\times U (\mathbb{A}_{M^+}^\infty)^\times = (M^+)^\times$. Now extend ψ to $(\mathbb{A}_M^\infty)^\times$ in any way. (This is possible as $M^\times \mathcal{O}_{M,S}^\times U (\mathbb{A}_{M^+}^\infty)^\times$ has finite index in $(\mathbb{A}_M^\infty)^\times$.) This ψ satisfies the requirements of the theorem. $\Box$

3 Potential modularity

In this section we will use the notations T_0, $V_{n,l}$, $V_n[N]$, T_W, $C(n)$ and $D(n)$ from section 1 without comment. (See the first and third paragraphs of section 1, corollary 1.11, the paragraph proceeding this corollary and lemma 1.15.)

Let F denote a totally real field and n a positive integer. Let l be a rational prime and let $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$. Let S be a non-empty finite set of finite places of F and for $v \in S$ the ρ_v be an irreducible square-integrable representation of $GL_n(F_v)$. Recall (see section 4.5 of [CHT]) that by a *RAESDC representation* π *of* $GL_n(\mathbb{A}_F)$ *of weight* 0 *and type* $\{\rho_v\}_{v\in S}$ we mean a cuspidal automorphic representation π of $GL_n(\mathbb{A}_F)$ such that

- $\pi^\vee \cong \chi\pi$ for some character $\chi : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times$ with $\chi_v(-1)$ independent of $v|\infty$;
- π_∞ has the same infinitessimal character as the trivial representation of $GL_n(F_\infty)$;
- and for $v \in S$ the representation π_v is an unramified twist of ρ_v.

We say that π has *level prime to* l if for all places $w|l$ the representation π_w is unramified.

Recall (see [TY] and section 3.3 of [CHT]) that if π is a RAESDC representation of $GL_n(\mathbb{A}_F)$ of weight 0 and type $\{\rho_v\}_{v\in S}$ (with $S \neq \emptyset$), then there is a continuous irreducible representation

$$r_{l,\imath}(\pi) : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

with the following properties.

1. For every prime $v \not| l$ of F we have
$$\mathrm{WD}(r_{l,\imath}(\pi)|_{\mathrm{Gal}\,(\overline{F}_v/F_v)})^{\text{F-ss}} = \imath^{-1}(\mathrm{rec}(\pi_v) \otimes |\mathrm{Art}\,_K^{-1}|_K^{(1-n)/2}).$$
2. $r_{l,\imath}(\pi)^\vee = r_{l,\imath}(\pi)\epsilon^{n-1}r_{l,\imath}(\chi)$. (For the notation $r_{l,\imath}(\chi)$ see [HT] or [TY].)
3. If $v|l$ is a prime of F then $r_{l,\imath}(\pi)|_{\mathrm{Gal}\,(\overline{F}_v/F_v)}$ is potentially semistable, and if π_v is unramified then it is crystalline.
4. If $v|l$ is a prime of F and if $\tau : F \hookrightarrow \overline{\mathbb{Q}}_l$ lies above v then
$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^i(r_{l,\imath}(\pi) \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 0$$
unless $i \in \{0, 1, ..., n-1\}$ in which case
$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^i(r_{l,\imath}(\pi) \otimes_{\tau,F_v} B_{\mathrm{DR}})^{\mathrm{Gal}\,(\overline{F}_v/F_v)} = 1.$$

The representation $r_{l,\imath}(\pi)$ is conjugate to one into $GL_n(\mathcal{O}_{\overline{\mathbb{Q}}_l})$. Reducing this modulo the maximal ideal and taking the semisimplification gives a semisimple continuous representation

$$\overline{r}_{l,\imath}(\pi) : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbb{F}}_l)$$

which is independent of the choice of conjugate.

We will call a representation

$$r : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbb{Q}}_l)$$

(resp.

$$\overline{r} : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_n(\overline{\mathbb{F}}_l))$$

which arises in this way for some π (resp. some π of level prime to l) and $\imath$ *automorphic of weight* 0 *and type* $\{\rho_v\}_{v\in S}$. In the case of r, if π has level prime to l then we will say that r is automorphic of *level prime to* l.

We will call a subgroup $\Delta \subset GL(V/\overline{\mathbb{F}}_l)$ *big* if the following hold.

- Δ has no l-power order quotient.
- $H^i(\Delta, \mathrm{ad}\,^0 V) = (0)$ for $i = 0$ and 1.
- For all irreducible $\overline{\mathbb{F}}_l[\Delta]$-submodules W of $\mathrm{ad}\,V$ we can find $h \in \Delta$ and $\alpha \in \overline{\mathbb{F}}_l$ with the following properties. The α generalised eigenspace $V_{h,\alpha}$ of h on V is one dimensional. Let $\pi_{h,\alpha} : V \to V_{h,\alpha}$ (resp. $i_{h,\alpha} : V_{h,\alpha} \hookrightarrow V$) denote the h-equivariant projection of V to $V_{h,\alpha}$ (resp. h-equivariant injection of $V_{h,\alpha}$ into V). (So that $\pi_{h,\alpha} \circ i_{h,\alpha} = 1$.) Then $\pi_{h,\alpha} \circ W \circ i_{h,\alpha} \neq (0)$.

Note that this only depends on the image of Δ in $PGL(V/\overline{\mathbb{F}}_l)$.

We will now prove our first potential modularity theorem. It is somewhat technical and will be essentially subsumed in later theorems, but it is needed in the proofs of these theorems. For other applications the conditions at l and q make this theorem too weak to be very useful. The reader may like to first think about the special case $F = F_0$, $r = 1$, $\mathcal{L} = \emptyset$, which will convey the essential points of both the theorem and its proof. Following the proof the reader can find some brief comments which may help in navigating the technical complexities of the argument.

Theorem 3.1 *Suppose that F/F_0 is a Galois extension of totally real fields and that $n_1, ..., n_r$ are even positive integers. Suppose that $l > \max\{C(n_i), n_i\}$ is a prime which is unramified in F and satisfies $l \equiv 1 \bmod n_i + 1$ for $i = 1, ..., r$. Let v_q be a prime of F above a rational prime $q \neq l$ such that $q \not| (n_i + 1)$ for $i = 1, ..., r$. Let $\mathcal{L}$ be a finite, $\mathrm{Gal}\,(F/F_0)$-invariant set of primes of F not containing primes above lq.*

Suppose also that for $i = 1, ..., r$

$$r_i : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GSp_{n_i}(\mathbb{Z}_l)$$

is a continuous representation which is unramified at all but finitely many primes and enjoys the following properties.

1. *r_i has multiplier $\epsilon_l^{1-n_i}$.*

2. *Let $\overline{r}_i$ denote the semisimplification of the reduction of r_i. Then the image $\overline{r}_i \mathrm{Gal}\,(\overline{F}/F(\zeta_l))$ is big (in $GL_n(\overline{\mathbb{F}}_l)$), and $\overline{F}^{\ker \mathrm{ad}\, \overline{r}_i}$ does not contain $F(\zeta_l)$.*

3. *r_i is unramified at all primes in $\mathcal{L}$.*

4. *If $w|l$ is a prime of F then $r_i|_{\mathrm{Gal}\,(\overline{F}_w/F_w)}$ is crystalline and for $\tau : F_w \hookrightarrow \overline{\mathbb{Q}}_l$ we have*
$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^j(r_i \otimes_{\tau, F_w} B_{\mathrm{DR}}) = 1$$
for $j = 0, ..., n_i - 1$ and $= 0$ otherwise. Moreover
$$\overline{r}_i|_{I_{F_w}} \cong 1 \oplus \epsilon_l^{-1} \oplus ... \oplus \epsilon_l^{1-n_i}.$$

5. *$r_i|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_{v_q}/F_{v_q})}$ is unramified and $r_i|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_{v_q}/F_{v_q})}(\mathrm{Frob}_{v_q})$ has eigenvalues of the form $\alpha, \alpha(\#k(v_q)), ..., \alpha(\#k(v_q))^{n_i-1}$.*

Then there is a totally real field F'/F which is Galois over F_0 and linearly independent from the compositum of the $\overline{F}^{\ker \overline{r}_i}$ over F. Moreover all primes of $\mathcal{L}$ and all primes of F above l are unramified in F'. Finally there is a prime w_q of F' over v_q such that each $r_i|_{\mathrm{Gal}\,(\overline{F}/F')}$ is automorphic of weight 0 and type $\{\mathrm{Sp}\,_n(1)\}_{\{w_q\}}$.

Proof: Let $E/\mathbb{Q}$ be an imaginary quadratic field. For $i = 1, ..., r$ let $M_i/\mathbb{Q}$ be a cyclic Galois imaginary CM field of degree n_i over $\mathbb{Q}$ such that

- l and the primes below $\mathcal{L}$ are unramified in M_i;

- and the compositum of E and the normal closure of $F/\mathbb{Q}$ is linearly disjoint from the compositum of the M_j's.

Choose a generator τ_i of $\mathrm{Gal}\,(M_i/\mathbb{Q})$. Choose a prime p_i which is inert but unramified in M_i and split completely in EF_0.

For $i = 1, ..., r$ choose a continuous homomorphism

$$\psi_i : (\mathbb{A}^{\infty}_{M_i})^{\times} \longrightarrow \overline{M}_i^{\times}$$

with the following properties.

- $\psi_i|_{M_i^{\times}}(a) = \prod_{j=0}^{n_i/2-1} \tau_i^j(a^j)\tau_i^{j+n_i/2}(a^{n_i-1-j})$.
- $\psi_i|_{(\mathbb{A}^{\infty}_{M_i^+})^{\times}} = \prod_v |\ |_v^{1-n_i}$.
- ψ_i is unramified at l and the primes below $\mathcal{L}$.
- $\psi_i|_{\mathcal{O}^{\times}_{M_i,p_i}} \neq \psi_i^{\tau_i^j}|_{\mathcal{O}^{\times}_{M_i,p_i}}$ for $j = 1, ..., n-1$.
- ψ_i only ramifies above rational primes which split in E.

The existence of such a character ψ_i follows easily from lemma 2.2. Let $\widetilde{M}_i$ denote a finite extension of M_i which is Galois over $\mathbb{Q}$ and contains the image of ψ_i.

Choose a prime l' which splits in $EF\widetilde{M}_1...\widetilde{M}_r(\zeta_{n_1(n_1+1)}, ..., \zeta_{n_r(n_r+1)})$ such that

- $l' > 8((n_i + 2)/4)^{n_i/2+1}$ for all i;
- $l' > C(n_i)$ for all i;
- l' does not divide the class number of E;
- each $\overline{r}_i$ is unramified above l';
- each ψ_i is unramified above l';
- $l' \not| p_i^{n_i} - 1$ for all i;
- $l' \not| q^j - 1$ for $j = 1, ..., \max\{n_i\} - 1$;
- $l' \neq l$, $l' \neq q$ and l' does not lie below $\mathcal{L}$.

Let $\widetilde{w}_{l',i}$ denote a prime of $\widetilde{M}_i$ above l' and let $w_{l',i} = \widetilde{w}_{l',i}|_{M_i}$.

Define a continuous character

$$\psi_{i,l'} : M_i^\times \backslash (\mathbb{A}_{M_i}^\infty)^\times \longrightarrow \widetilde{M}_{i,\widetilde{w}_{l',i}}^\times$$

by

$$\psi_{i,l'}(a) = \psi_i(a) \prod_{j=0}^{n_i/2-1} a_{\tau_i^{-j} w_{l',i}}^{-j} a_{\tau_i^{-j+n_i/2} w_{l',i}}^{j+1-n_i}.$$

Composing this with the Artin reciprocity map and reducing modulo $\widetilde{w}_{l',i}$ we obtain a character

$$\overline{\theta}_i : \mathrm{Gal}\,(M_i/\mathbb{Q}) \longrightarrow \mathbb{F}_{l'}^\times$$

with the following properties.

- $\overline{\theta}_i \overline{\theta}_i^c = \epsilon_{l'}^{1-n_i}$.
- $\overline{\theta}_i|_{I_{M_{i,\tau_i^j w_{l',i}}}} = \epsilon_{l'}^{-j}$ for $j = 0, ..., n/2-1$.
- $\overline{\theta}_i$ is unramified above l and the primes below $\mathcal{L}$.
- $\overline{\theta}_i|_{I_{M_{i,p_i}}} \neq \overline{\theta}^{\tau_i^j}|_{I_{M_{i,p_i}}}$ for $j = 1, ..., n-1$.
- $\overline{\theta}_i$ only ramifies above primes above rational primes which split in E.

Define an alternating pairing on $\mathrm{Ind}\,_{\mathrm{Gal}\,(\overline{M}_i/M_i)}^{\mathrm{Gal}\,(\overline{M}_i/\mathbb{Q})} \overline{\theta}_i$ by

$$\langle \varphi, \varphi' \rangle = \sum_{\sigma \in \mathrm{Gal}\,(\overline{M}_i/M_i) \backslash \mathrm{Gal}\,(\overline{M}_i/\mathbb{Q})} \epsilon(\sigma)^{n_i-1} \varphi(\sigma)\varphi'(c\sigma)$$

where c is any complex conjugation. (It is alternating because n_i is even.) This gives rise to a homomorphism

$$I(\overline{\theta}_i) : \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GSp_{n_i}(\mathbb{F}_{l'}).$$

Let K denote the compositum of the fixed fields of the $\ker \overline{r}_i$ and the $\ker I(\overline{\theta}_i)$. Let W_i be the free $\mathbb{Z}/ll'\mathbb{Z}$-module of rank n_i corresponding to $\overline{r}_i \times I(\overline{\theta}_i)$. The module W_i comes with a perfect alternating pairing

$$W_i \times W_i \longrightarrow (\mathbb{Z}/ll'\mathbb{Z})(1-n_i).$$

The scheme T_{W_i}/F is geometrically connected. Let S_1 denote the infinite primes of F, let S_2 equal $\mathcal{L}$ union the set of primes of F above ll', and let

$S_3 = \{v_q\}$. If w is an infinite place of F let $\Omega_{i,w} = T_{W_i}(F_w)$. This is non-empty as all elements of $GSp_{n_i}(\mathbb{Z}/ll'\mathbb{Z})$ of order two and multiplier -1 are conjugate. If $w \in S_2$ let $\Omega_{i,w}$ denote the set of elements of $T_{W_i}(F_w^{\mathrm{nr}})$ above $\{t \in T_0(F_w^{\mathrm{nr}}) : w(1 - t^{n_i+1}) = 0\}$. Then $\Omega_{i,w}$ is open, $\mathrm{Gal}\,(F_w^{\mathrm{nr}}/F_w)$-invariant and non-empty (as it contains a point above $0 \in T_0(F_w^{\mathrm{nr}})$). Let Ω_{i,v_q}denote the preimage in $T_{W_i}(\overline{F}_{v_q})$ of $\{t \in T_0(F_{v_q}) : \ v_q(t) < 0\}$. This set is open, $\mathrm{Gal}\,(\overline{F}_{v_q}/F_{v_q})$-invariant and non-empty. By proposition 2.1 we can find recursively totally real fields F_i'/F and point $\widetilde{t}_i \in T_{W_i}(F_i)$ such that

- F_i/F is Galois,
- F_i/F is unramified above $\mathcal{L}$ and above ll',
- F_i is linearly disjoint from $KF_1...F_{i-1}$ over F,
- and $\widetilde{t}_i$ lies in $\Omega_{i,w}$ for all $w \in S_1 \cup S_2 \cup S_3$.

Let $\widetilde{F} = F_1...F_r$, a Galois extension of F which is totally real, in which all primes of S_1 split completely and in which all primes of S_2 are unramified. Then $\widetilde{F}$ is linearly disjoint from K over F. Let $t_i \in T_0(\widetilde{F})$ denote the image of $\widetilde{t}_i$. Then $V_{n_i}[l]_{t_i} \cong \overline{r}_i|_{\mathrm{Gal}\,(\overline{F}/\widetilde{F}')}$ and $V_{n_i}[l']_{t_i} \cong I(\overline{\theta}_i)|_{\mathrm{Gal}\,(\overline{F}/\widetilde{F}')}$. Moreover Y_{n_i,t_i} has good reduction above ll' so that V_{n_i,l,t_i} is crystalline above l and unramified above l', while V_{n_i,l',t_i} is unramified above l and crystalline above l'. If w is a prime of $\widetilde{F}$ above v_q then the semisimplification of $V_{n_i,l',t_i}|_{\mathrm{Gal}\,(\overline{\widetilde{F}}_w/\widetilde{F}_w)}$ is unramified and Frob_w has eigenvalues $\beta, \beta(\#k(w)), ..., \beta(\#k(w))^{n_i-1}$ for some $\beta \in \{\pm 1\}$, which may depend on w.

Let F' denote the normal closure of $\widetilde{F}$ over F_0. It is linearly disjoint from the compositum of the $\overline{F}^{\ker \overline{r}_i}$ over F. By theorem 5.4 of [T3] we see that each V_{n_i,l',t_i} is automorphic over F' of weight 0 and type $\{\mathrm{Sp}_{n_i}(1)\}_{\{w|v_q\}}$ and level prime to l'. It also has level prime to l, so that $V_{n_i}[l]_{t_i} \cong \overline{r}_i|_{\mathrm{Gal}\,(\overline{F}'/F')}$ is also automorphic over F' of weight 0 and type $\{\mathrm{Sp}_{n_i}(1)\}_{\{w|v_q\}}$. By theorem 5.2 of [T3] we see that r_i is automorphic over F' of weight 0 and type $\{\mathrm{Sp}_{n_i}(1)\}_{\{w|v_q\}}$ and level prime to l. $\Box$

We hope that the following informal remarks may help guide the reader through the apparent complexity of the proof of theorems 3.1. This complexity is imposed upon us by three circumstances.

(a) The modularity theorems proved in [CHT] and [T3] only apply to l-adic representations which, at some finite place v, correspond under the local Langlands correspondence to discrete series representations. It is possible that further developments of the stable trace formula will make this hypothesis unnecessary.

(b) In the second place, our knowledge of the bad reduction of the Calabi-Yau hypersurfaces Y_t considered in section 1 is only sufficient to provide inertial representations of Steinberg type (with maximally unipotent mondromy), as in Lemma 1.15; this explains our local hypotheses at the primes denoted q.

(c) The monomial representations $I(\overline{\theta}_i)$ considered in the proof of theorem 3.1 can never be locally of Steinberg type, but they can be locally of supercuspidal type, and are chosen to be so at the primes denoted p_i. The local hypothesis at p_i is used in the proof of theorem 5.4 of [T3].

In a special case we now improve upon theorem 3.1, by weakening the conditions at l and q. This theorem suffices for the applications to the Sato-Tate conjecture in the next section. Its proof depends in an essential way on theorem 3.1. The reader might like to think first about the special case $t = 1$ and $\det r = \epsilon_l^{-1}$, which will convey the main points of both the statement and proof of this theorem.

Theorem 3.2 *Suppose that F is a totally real field and that $n_1, ..., n_t$ are even positive integers. Suppose also that $l > \max\{C(n_i), 2n_i + 1\}$ is a prime which is unramified in F and that v_q is a prime of F above a rational prime $q \neq l$.*

Suppose also that

$$r : \mathrm{Gal}\,(\overline{F}/F) \longrightarrow GL_2(\mathbb{Z}_l)$$

is a continuous representation which is unramified at all but finitely many primes and totally odd (in the sense that $\det r(c) = -1$ for every complex conjugation $c \in \mathrm{Gal}\,(\overline{F}/F)$). Suppose that r also enjoys the following properties.

1. *r is surjective.*
2. *If $w|l$ is a prime of F then $r|_{\mathrm{Gal}\,(\overline{F}_w/F_w)}$ is crystalline and for $\tau : F_w \hookrightarrow \overline{\mathbb{Q}}_l$ we have*
$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^{\,j}(r \otimes_{\tau, F_w} B_{\mathrm{DR}}) = 1$$
for $j = 0, 1$ and $= 0$ otherwise.
3. *There is a prime v_q of F split above q for which $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_{v_q}/F_{v_q})}$ is unramified and $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_{v_q}/F_{v_q})}(\mathrm{Frob}_{v_q})$ has eigenvalues of the form $\alpha, \alpha\# k(v_q)$.*

Then there is a Galois totally real extension F''/F in which l is unramified, and a prime w_q of F'' over v_q such that each of the representations $\mathrm{Symm}^{\,n_i-1} r|_{\mathrm{Gal}\,(\overline{F}/F'')}$ is automorphic of weight 0 and type $\{\mathrm{Sp}_{\,n}(1)\}_{\{w_q\}}$.

Proof: Let $\overline{r}$ denote the reduction $r \bmod l$.

The character $\epsilon_l \det r$ is totally even and unramified at l. Thus $\epsilon_l \det r$ has finite order. Set $F_1 = \overline{F}^{\ker \epsilon_l \det r}$. Then F_1 is totally real and l is unramified in F_1.

Choose a rational prime q' and a prime $v_{q'}$ of F above q' such that

- r is unramified above q',
- $\overline{r}(\mathrm{Frob}_{v_{q'}})$ has eigenvalues $1, \#k(v_q)$,
- $q' \not| (n_i + 1)$ for $i = 1, ..., t$,
- $q' \neq q$ and $q' \neq l$.

Also choose a prime l' which splits in $\mathbb{Q}(\zeta_{n_1+1}, ..., \zeta_{n_t+1})$ and such that

- $l' \equiv 1 \bmod n_i + 1$ for $i = 1, ..., t$,
- $l' \neq l$, q, or q',
- $l' > \max(C(n_i), n_i)$,
- l' is unramified in F_1,
- and r is unramified at l'.

Choose an elliptic curve E_1/F such that

- E_1 has good reduction above l;
- E_1 has multiplicative reduction at v_q and $v_{q'}$;
- E_1 has good ordinary reduction above l', but $H^1(E_1 \times \overline{F}, \mathbb{Z}/l'\mathbb{Z})$ is tamely ramified at l';
- $\mathrm{Gal}\,(\overline{F}/F) \twoheadrightarrow \mathrm{Aut}\,(H^1(E_1 \times \overline{F}, \mathbb{Z}/l'\mathbb{Z}))$.

The existence of such an E_1 results from the form of Hilbert irreducibility with weak approximation (see [E]). (The existence of such an E_1 over F_{v_q} (resp. $F_{v_{q'}}$) results from taking a j-invariant with $\mathrm{val}_q(j) < 0$ (resp. $\mathrm{val}_{q'}(j) < 0$). The existence of such an E_1 over $\mathbb{Q}_{l'}$ results from taking the canonical lift of an ordinary elliptic curve over $\mathbb{F}_{l'}$.)

Let W denote the free rank two $\mathbb{Z}/ll'\mathbb{Z}$ module with $\mathrm{Gal}\,(\overline{F}/F_1)$-action corresponding to $\overline{r} \times H^1(E_1 \times \overline{F}, \mathbb{Z}/l'\mathbb{Z})$ and let

$$\langle\ ,\ \rangle : W \times W \longrightarrow (\mathbb{Z}/ll'\mathbb{Z})(-1)$$

be a perfect alternating pairing. Thus W gives a lisse etale sheaf over $\mathrm{Spec}\, F_1$. Let $X_W/\mathrm{Spec}\, F_1$ denote the moduli space for the functor which takes a locally noetherian F_1-scheme S to the set of isomorphism classes of pairs (E, i), where $\pi : E \to S$ is an elliptic curve and where

$$i : W \xrightarrow{\sim} R^1\pi_*(\mathbb{Z}/ll'\mathbb{Z})$$

takes $\langle\ ,\ \rangle$ to the duality coming from the cup product. Then X_W is a fine moduli space (as $ll' > 2$). It is a smooth, geometrically connected, affine curve.

Let S_1 denote the set of places of F_1 above ∞; let S_2 denote the set of palces of F_1 above ll'; and let S_3 denote the set of primes of F_1 above v_q and $v_{q'}$. If v is an infinite place of F_1 take $\Omega_v = X_W(F_{1,v})$. It is non-empty as $GL_2(\mathbb{Z}/ll'\mathbb{Z})$ has a unique conjugacy class of elements of order 2 and determinant -1. If v is a place of F_1 above l' let $\Omega_v \subset X_W(F_{1,v}^{\mathrm{nr}})$ consist of pairs (E, i) such that E has good reduction. This set is open and $\mathrm{Gal}\,(F_{1,v}^{\mathrm{nr}}/F_{1,v})$-invariant. It is also non-empty: for instance take $E = E_1$. If v is a place of F_1 above v_q or $v_{q'}$, let Ω_v denote the open subset of $X_W(\overline{F}_{1,v})$ corresponding to elliptic curves with multiplicative reduction. It is a non-empty, $\mathrm{Gal}\,(\overline{F}_{1,v}/F_{1,v})$-invariant, open set.

If v is a place of F_1 above l let $\Omega_v \subset X_W(F_{1,v}^{\mathrm{nr}})$ consist of pairs (E, i) such that E has good reduction. This set is open and $\mathrm{Gal}\,(F_{1,v}^{\mathrm{nr}}/F_{1,v})$-invariant. It is also non-empty: From the theory of Fontaine-Lafaille we see that either $W[l]|_{I_{F_{1,v}}} \cong \omega_2^{-1} \oplus \omega_2^{-l}$ or there is an exact sequence

$$(0) \longrightarrow \mathbb{Z}/l\mathbb{Z} \longrightarrow W[l] \longrightarrow (\mathbb{Z}/l\mathbb{Z})(-1) \longrightarrow (0)$$

over $I_{F_{1,v}}$. In the first case any lift to the ring of integers of a finite extension of $F_{1,v}$ of a supersingular elliptic curve over $\overline{k(v)}$ will give a point of Ω_v. So consider the second case. Let $k/k(v)$ be a finite extension and $\overline{E}/k$ an ordinary elliptic curve such that Frob_k acts trivially on $\overline{E}[l](\overline{k})$. Let K denote the unramified extension of $F_{1,v}$ with residue field k. Enlarging k if necessary we can assume that Frob_K also acts trivially on $W^{I_{F_{1,v}}}$. Let χ give the action of $\mathrm{Gal}\,(\overline{k}/k)$ on $E[l^\infty](\overline{k})$. By Serre-Tate theory, liftings of $\overline{E}$ to $\mathcal{O}_K$ are parametrised by extensions of $(\mathbb{Q}_l/\mathbb{Z}_l)(\chi)$ by $\mu_{l^\infty}(\chi^{-1})$ over $\mathcal{O}_K$. If the l-torsion in such an extension is isomorphic (over K) to $W^\vee$, the corresponding lifting E will satisfy $H^1(E \times \overline{K}, \mathbb{Z}/l\mathbb{Z}) \cong W$. Extensions of $(\mathbb{Q}_l/\mathbb{Z}_l)(\chi)$ by $\mu_{l^\infty}(\chi^{-1})$ over $\mathcal{O}_K$ are parametrised by $H^1(\mathrm{Gal}\,(\overline{K}/K), \mathbb{Z}_l(\epsilon_l\chi^{-2}))$ (as $\chi^2 \neq 1$). The representation $W^\vee$ corresponds to a class in $H^1(\mathrm{Gal}\,(\overline{K}/K), (\mathbb{Z}/l\mathbb{Z})(\epsilon_l))$ which is 'peu-ramifié'. We must show that this class is in the image of

$$H^1(\mathrm{Gal}\,(\overline{K}/K), \mathbb{Z}_l(\epsilon_l\chi^{-2}) \longrightarrow H^1(\mathrm{Gal}\,(\overline{K}/K), (\mathbb{Z}/l\mathbb{Z})(\epsilon_l))$$

coming from the fact that $\chi^2 \equiv 1 \bmod l$. By local duality, this image is the annihilator of the image of the map

$$H^0(\mathrm{Gal}\,(\overline{K}/K), (\mathbb{Q}_l/\mathbb{Z}_l)(\chi^2)) \longrightarrow H^1(\mathrm{Gal}\,(\overline{K}/K), \mathbb{Z}/l\mathbb{Z})$$

coming from the exact sequence

$$(0) \longrightarrow \mathbb{Z}/l\mathbb{Z} \longrightarrow (\mathbb{Q}_l/\mathbb{Z}_l)(\chi^2) \stackrel{l}{\longrightarrow} (\mathbb{Q}_l/\mathbb{Z}_l)(\chi^2) \longrightarrow (0).$$

Because χ^2 is unramified, this image consists of unramified homomorphisms, which annihilate any 'peu-ramifié' class.

By proposition 2.1 we can find a finite Galois extension F'/F containing F_1 and an elliptic curve E/F' with the following properties.

- F' is linearly disjoint from $\overline{F}^{\ker(\mathrm{Gal}\,(\overline{F}/F)\to \mathrm{Aut}\,(W))}$ over F_1.
- F' is totally real.
- All primes above ll' are unramified in F'.
- E has good reduction at all places above l.
- E has good reduction at all places above l'.
- E has split multiplicative reduction above v_q and $v_{q'}$.
- $H^1(E \times \overline{F}, \mathbb{Z}/l\mathbb{Z}) \cong \overline{r}|_{\mathrm{Gal}\,(\overline{F}/F')}$.
- $H^1(E \times \overline{F}, \mathbb{Z}/l'\mathbb{Z})$ is tamely ramified above l'.

By theorem 3.1 we see that there is a totally real field F''/F' and a prime $w_{q'}$ of F'' above $v_{q'}$ such that:

- F''/F is Galois.
- l and l' are unramified in F''.
- F'' is linearly disjoint over F' from $F'\overline{F}^{\ker(\mathrm{Gal}\,(\overline{F}/F)\to \mathrm{Aut}\,(W))}$ (and hence F'' is linearly disjoint over F_1 from $\overline{F}^{\ker \overline{r}}$).
- Each $\mathrm{Symm}^{n_i-1} H^1(E \times \overline{F}, \mathbb{Z}_{l'})$ is automorphic over F'' of weight 0, type $\{\mathrm{Sp}_{n_i}(1)\}_{\{w_{q'}\}}$ and level prime to l'.

(To check the second condition of theorem 3.1 apply corollary 1.5.4 of [CHT] and the fact that $PSL_2(\mathbb{F}_l)$ is simple for $l > 3$.) Let w_q be a prime of F'' above v_q. Each $\mathrm{Symm}^{n_i-1} H^1(E \times \overline{F}, \mathbb{Z}_{l'})$ is also automorphic over F'' of weight 0, type $\{\mathrm{Sp}_{n_i}(1)\}_{\{w_q\}}$ and level prime to l. Thus each $\mathrm{Symm}^{n_i-1} H^1(E \times \overline{F}, \mathbb{Z}/l\mathbb{Z}) \cong \mathrm{Symm}^{n_i-1} \overline{r}|_{\mathrm{Gal}(\overline{F}/F''')}$ is automorphic over F'' of weight 0 and type $\{\mathrm{Sp}_{n_i}(1)\}_{\{w_q\}}$. By theorem 4.3 of [T3] we see that each $\mathrm{Symm}^{n_i-1} r$ is automorphic over F'' of weight 0 and type $\{\mathrm{Sp}_{n_i}(1)\}_{\{w_q\}}$. (Again we use corollary 1.5.4 of [CHT] and the simplicity of $PSL_2(\mathbb{F}_l)$ for $l > 3$.) □

We remark that the auxilliary prime q' is needed because we have not assumed that $q \not| n_i + 1$ for $i = 1, ..., t$.

Finally in this section we go back and prove the following improvement on theorem 3.1. (The key point is the weakening of the conditions at l and q.) Again the reader might like to consider first the case that r has multiplier ϵ_l^{1-n}, which will convey the main points of both the statement and proof of this theorem.

Theorem 3.3 *Suppose that F is a totally real field and that n is an even positive integer. Suppose that $l > \max\{C(n), D(n), n, 3\}$ is a rational prime which is unramified in F. Let v_q be a prime of F above a rational prime $q \not| (n+1)l$.*

Suppose also that

$$r : \mathrm{Gal}(\overline{F}/F) \longrightarrow GSp_n(\mathbb{Z}_l)$$

is a continuous representation which is unramified at all but finitely many primes and which is totally odd (in the sense that $r(c)$ has multiplier -1 for all complex conjugations c). Suppose moreover it enjoys the following properties.

1. *Letting $\overline{r}$ denote the semisimplification of the reduction of r, the image $\overline{r}\mathrm{Gal}(\overline{F}/F(\zeta_l))$ is big (in $GL_n(\overline{\mathbb{F}}_l)$) and $\overline{F}^{\ker \mathrm{ad}\,\overline{r}}$ does not contain $F(\zeta_l)$. This will be satisfied if r is surjective.*

2. *If $w|l$ is a prime of F then $r|_{\mathrm{Gal}(\overline{F}_w/F_w)}$ is crystalline and for $\tau : F_w \hookrightarrow \overline{\mathbb{Q}}_l$ we have*
$$\dim_{\overline{\mathbb{Q}}_l} \mathrm{gr}^j (r \otimes_{\tau, F_w} B_{\mathrm{DR}}) = 1$$
for $j = 0, ..., n-1$ and $= 0$ otherwise. Moreover there is a point $t_w \in \mathcal{O}_{F_w^{\mathrm{nr}}}$ with $w(t_w^{n+1} - 1) = 0$ such that
$$\overline{r}|_{I_{F_w}} \cong V_n[l]_{t_w}.$$

3. $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_{v_q}/F_{v_q})}$ *is unramified and* $r|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_{v_q}/F_{v_q})}(\mathrm{Frob}_{v_q})$ *has eigenvalues of the form* $\alpha, \alpha(\#k(v_q)), ..., \alpha(\#k(v_q))^{n-1}$.

Then there is a totally real extension F''/F *and a place* w_q *of* F'' *above* v_q *such that* $r|_{\mathrm{Gal}\,(\overline{F}/F'')}$ *is automorphic of weight* 0 *and type* $\{\mathrm{Sp}_n(1)\}_{\{w_q\}}$.

Proof: Let ν denote the multiplier character of r. Then $\nu\epsilon^{n-1}$ is trivial on all complex conjugations and unramified above l. Thus $\nu\epsilon^{n-1}$ has finite order. Set $F_1 = \overline{F}^{\ker \nu\epsilon_l^{n-1}}$. Then F_1 is totally real and l is unramified in F_1.

Choose a rational prime l' which splits in $\mathbb{Q}(\zeta_{n+1})$ such that

- l' is unramified in F_1 and r is unramified above l',
- $l' > \max\{n, C(n)\}$, and
- $l' \not\equiv 1$
  $bmod n + 1$.

Choose $t_1 \in F$ with the following properties.

- If $w|ll'$ then $w(t_1^{n+1} - 1) = 0$.
- If $w|l'$ then $V_n[l']_{t_1}|_{I_{F_w}} \cong 1 \oplus \epsilon_{l'}^{-1} \oplus ... \oplus \epsilon_{l'}^{1-n}$.
- $\mathrm{Gal}\,(\overline{F}/F) \to GSp(V_n[l']_{t_1})$ is surjective.

The existence of such an t_1 results from the form of Hilbert irreducibility with weak approximation (see [E]). (One may achieve the second condition by taking t_1 to be l'-adically close to zero.)

Let W be the free rank two $\mathbb{Z}/ll'\mathbb{Z}$-module with $\mathrm{Gal}\,(\overline{F}/F_1)$-action corresponding to $\overline{r} \times V_n[l']_{t_1}$. It comes with a perfect alternating pairing

$$\langle\ ,\ \rangle : W \times W \longrightarrow (\mathbb{Z}/ll'\mathbb{Z})(1-n).$$

The scheme T_W is geometrically connected. Let S_1 denote the places of F_1 above ∞; let S_2 denote the set of places of F_1 above ll'; and let S_3 denote the set of places of F_1 above v_q. For w an infinite place of F_1 let $\Omega_w = T_W(F_w)$ which is non-empty as all elements of order two in $GSp_n(\mathbb{Z}/ll'\mathbb{Z})$ with multiplier -1 are conjugate. If $w|ll'$ let $\Omega_w \subset T_W(F_{1,w}^{\mathrm{nr}})$ denote the preimage of $\{t \in T_0(F_{1,w}^{\mathrm{nr}}) :\ w(t^{n+1} - 1) = 0\}$. It is open, $\mathrm{Gal}\,(F_{1,w}^{\mathrm{nr}}/F_{1,w})$-invariant and non-empty. If w is a place of F_1 above v_q, let $\Omega_w \subset T_W(\overline{F}_{1,w})$ denote the open subset of points lying above $\{t \in T_0(F_{1,w}) : w(t) < 0\}$. It is non-empty, $\mathrm{Gal}\,(\overline{F}_{1,w}/F_{1,w})$-invariant and open.

Thus we may find a finite Galois totally real extension F'/F containing F_1 and a point $t \in T_0(F')$ with the following properties.

- l and l' are unramified in F'.
- F' is linearly disjoint from $\overline{F}^{\ker(\mathrm{Gal}\,(\overline{F}/F)\to \mathrm{Aut}\,(W))}$ over F_1.
- $V_n[l]_t \cong \overline{r}|_{\mathrm{Gal}\,(\overline{F}/F')}$.
- $V_{n,l',t}$ is unramified above l and crystalline above l'.
- If w is a place of F' above l' then $V_n[l']_t|_{I_{F'_w}} \cong 1 \oplus \epsilon_{l'}^{-1} \oplus ... \oplus \epsilon_{l'}^{1-n}$.
- If w is a place of F' above v_q then $V_{n,l',t}|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}'_w/F'_w)}$ is unramified and Frob_w has eigenvalues of the form $\alpha, \alpha\#k(v_q), ..., \alpha(\#k(v_q))^{n-1}$ for some α.

According to theorem 3.1 we can find a totally real extension F''/F' and a place $w_q|v_q$ of F'' with the following properties.

- F''/F is Galois.
- l and l' are unramified in F''.
- $V_{n,l',t}$ is automorphic over F'' of weight 0, type $\{\mathrm{Sp}\,_n(1)\}_{\{w_q\}}$ and level prime to ll'.

(To check the second assumption of theorem 3.1 use lemma 1.5.5 of [CHT] and the simplicity of $PSp_n(\mathbb{F}_l)$ for $l > 3$.) Hence $V_n[l]_t$ and $\overline{r}$ are automorphic over F'' of weight 0 and type $\{\mathrm{Sp}\,_n(1)\}_{\{w_q\}}$. Finally theorem 4.3 of [T3] tells us that r is automorphic over F'' of weight 0 and type $\{\mathrm{Sp}\,_n(1)\}_{\{w_q\}}$. $\Box$

4 Applications

Suppose that F and $L \subset \mathbb{R}$ are totally real fields and that A/F is an abelian scheme equipped with an embedding $i : L \hookrightarrow \mathrm{End}\,^0(A/F)$. Recall (e.g from proposition 1.10, proposition 1.4 and the discussion just before proposition 1.4 of [R]) that A admits a polarisation over F whose Rosati involution acts trivially on iL. Thus if λ is a prime of L above a rational prime l then

$$\det H^1(A \times \overline{F}, \mathbb{Q}_l) \otimes_{L_l} L_\lambda = L_\lambda(\epsilon_l^{-1}).$$

Suppose also that m is a positive integer. For each finite place v of F there is a two dimensional Weil-Deligne representation $\mathrm{WD}_v(A, i)$ over $\overline{L}$ such that for each prime λ of L with residue characteristic l different from the residue characteristic of v we have

$$\mathrm{WD}(H^1(A\times\overline{F},\mathbb{Q}_l)|_{\mathrm{Gal}\,(\overline{F}_v/F_v)}\otimes_{L_l}L_\lambda)\cong \mathrm{WD}_v(A,i).$$

We define an L-series

$$L(\mathrm{Symm}^m(A,i)/F,s)=\prod_{v\not|\infty}L(\mathrm{Symm}^m\mathrm{WD}_v(A,i),s).$$

It converges absolutely, uniformly on compact sets, to a non-zero holomorphic function in $\mathrm{Re}\, s>1+m/2$. We say that $\mathrm{Symm}^m(A,i)$ is automorphic of type $\{\rho_v\}_{v\in S}$, if there is a RAESDC representation of $GL_{m+1}(\mathbb{A}_F)$ of weight 0 and type $\{\rho_v\}_{v\in S}$ such that

$$\mathrm{rec}(\pi_v)|\mathrm{Art}_K^{-1}|_K^{-m/2}=\mathrm{Symm}^m\mathrm{WD}_v(A,i)$$

for all finite places v of F.

Note that the following are equivalent.

1. $\mathrm{Symm}^m(A,i)$ is automorphic over F of type $\{\rho_v\}_{v\in S}$.
2. For all finite places λ of L, if l is the residue characteristic of λ, then $\mathrm{Symm}^m(H^1(A\times\overline{F},\mathbb{Q}_l)\otimes_{L_l}L_\lambda)$ is automorphic over F of weight 0 and type $\{\rho_v\}_{v\in S}$.
3. For some rational prime l and some place $\lambda|l$ of L the representation $\mathrm{Symm}^m(H^1(A\times\overline{F},\mathbb{Q}_l)\otimes_{L_l}L_\lambda)$ is automorphic over F of weight 0 and type $\{\rho_v\}_{v\in S}$.

(The first statement implies the third. The second statement implies the first (by the strong multiplicity one theorem). We will check that the third implies the second. Suppose that $\mathrm{Symm}^m(H^1(A\times\overline{F},\mathbb{Q}_l)\otimes_{L_l}L_\lambda)$ arises from an RAESDC representation π and an isomorphism $\imath:\overline{L}_\lambda\xrightarrow{\sim}\mathbb{C}$. Let l' be a rational prime and let $\imath':\overline{\mathbb{Q}}_{l'}\xrightarrow{\sim}\mathbb{C}$. Let λ' be the prime of L above l' corresponding to $(\imath')^{-1}\circ\imath|_L$. Then from the Cebotarev density theorem we see that

$$r_{l',\imath'}(\pi)\cong\mathrm{Symm}^m(H^1(A\times\overline{F},\mathbb{Q}_{l'})\otimes_{L_{l'}}L_{\lambda'}).$$

Thus $\mathrm{Symm}^m(H^1(A\times\overline{F},\mathbb{Q}_{l'})\otimes_{L_{l'}}L_{\lambda'})$ is also automorphic over F of weight 0 and type $\{\rho_v\}_{v\in S}$.)

Theorem 4.1 *Let F and L be totally real fields. Let A/F be an abelian variety of dimension $[L:\mathbb{Q}]$ and suppose that $i: L \hookrightarrow \mathrm{End}^0(A/F)$. Let $\mathcal{N}$ be a finite set of even positive integers. Fix an embedding $L \hookrightarrow \mathbb{R}$. Suppose that A has multiplicative reduction at some prime v_q of F.*

There is a Galois totally real field F'/F such that for any $n \in \mathcal{N}$ and any intermediate field $F' \supset F'' \supset F$ with F'/F'' soluble, $\mathrm{Symm}^{n-1} A$ *is automorphic over F''.*

Proof: Twisting by a quadratic character if necessary we may assume that A has split multiplicative reduction at v_q i.e. Frob_{v_q} has eigenvalues 1 and $\#k(v_q)$ on $H^1(A \times \overline{F}, \mathbb{Q}_l)|^{\mathrm{ss}}_{\mathrm{Gal}\,(\overline{F}_{v_q}/F_{v_q})}$ for all l different from the residue characteristic of v_q.

Choose l sufficiently large that

- l is unramified in F,
- $l > \max\{n, C(n)\}_{n \in \mathcal{N}}$,
- A has good reduction at all primes above l,
- $\mathrm{Gal}\,(\overline{F}/F) \twoheadrightarrow \mathrm{Aut}\,(H^1(A \times \overline{F}, \mathbb{Z}/l\mathbb{Z})/\mathcal{O}_L/l\mathcal{O}_L)$,
- and l splits completely in L.

(If this were not possible then for all but finitely many primes l which split completely in L there would be a prime $\lambda | l$ of L such that $\mathrm{Gal}\,(\overline{F}/F) \to \mathrm{Aut}\,(H^1(A \times \overline{F}, \mathbb{Z}/l\mathbb{Z}) \otimes_{\mathcal{O}_L} \mathcal{O}_L/\lambda\mathcal{O}_L)$ is not surjective. Note that for almost all such l the determinant of the image is $(\mathbb{Z}/l\mathbb{Z})^\times$ (look at inertia at l) and the image contains a non-trivial unipotent element (look at inertia at v_q). Thus for all but finitely many primes l which split completely in L there is a prime $\lambda | l$ of L such that the image of $\mathrm{Gal}\,(\overline{F}/F) \to \mathrm{Aut}\,(H^1(A \times \overline{F}, \mathbb{Z}/l\mathbb{Z}) \otimes_{\mathcal{O}_L} \mathcal{O}_L/\lambda\mathcal{O}_L)$ is contained in a Borel subgroup of $GL_2(\mathbb{Z}/l\mathbb{Z})$ and its semisimplification has abelian image. It follows from theorem 1 of section 3.6 of [Se2] that the image of $\mathrm{Gal}\,(\overline{F}/F) \to \mathrm{Aut}\,(H^1(A \times \overline{F}, \mathbb{Q}_l) \otimes_L L_\lambda)$ is abelian for all l and λ. This contradicts the multiplicative reduction at v_q.) Choose a prime $\lambda | l$ of L.

Theorem 3.2 tells us that there is a Galois totally real field F'/F in which l is unramified and a prime w_q of F' above v_q such that for any $n \in \mathcal{N}$, $\mathrm{Symm}^{n-1}(H^1(A \times \overline{F}, \mathbb{Q}_l) \otimes_{L_l} L_\lambda)$ is automorphic over F' of weight 0, type $\{\mathrm{Sp}_n(1)\}_{\{w_q\}}$ and level prime to l. By lemma 3.3.2 of [CHT] we see that $\mathrm{Symm}^{n-1}(H^1(A \times \overline{F}, \mathbb{Q}_l) \otimes_{L_l} L_\lambda)$ is also automorphic over any F'' as in the theorem of weight 0, type $\{\mathrm{Sp}_n(1)\}_{\{w_q\}}$ and level prime to l. Hence $\mathrm{Symm}^{n-1} A$ is automorphic over F''. $\Box$

Theorem 4.2 *Let F and L be totally real fields. Let A/F be an abelian variety of dimension $[L:\mathbb{Q}]$ and suppose that $i: L \hookrightarrow \mathrm{End}^0(A/F)$. Fix an embedding $L \hookrightarrow \mathbb{R}$. Suppose that A has multiplicative reduction at some prime v_q of F.*

Then for all $m \in \mathbb{Z}_{\geq 1}$ the function $L(\mathrm{Symm}^m(A,i),s)$ has meromorphic continuation to the whole complex plane, satisfies the expected functional equation and is holomorphic and non-zero in $\mathrm{Re}\, s \geq 1 + m/2$.

Proof: We argue by induction on m. The assertion is vacuous if $m < 1$. Suppose that $m \in \mathbb{Z}_{\geq 1}$ is odd and that the theorem is proved for $1 \leq m' < m$. We will prove the theorem for m and $m+1$. Apply theorem 4.1 with $\mathcal{N} = \{2, m+1\}$. Let F'/F be as in the conclusion of that theorem. Write

$$1 = \sum_j a_j \mathrm{Ind}_{\mathrm{Gal}(F'/F_j)}^{\mathrm{Gal}(F'/F)} \chi_j$$

where $a_j \in \mathbb{Z}$, $F' \supset F_j \supset F$ with F'/F_j soluble, and χ_j is a homomorphism $\mathrm{Gal}(F'/F_j) \to \mathbb{C}^\times$. Then $(A,i) \times \overline{F}_j$ is automorphic arising from an RAESDC representation σ_j of $GL_2(\mathbb{A}_{F_j})$, and $\mathrm{Symm}^m(A,i) \times \overline{F}_j$ is automorphic arising from an RAESDC representation π_j of $GL_2(\mathbb{A}_{F_j})$. Then we see that

$$L(\mathrm{Symm}^m(A,i),s) = \prod_j L(\pi_j \otimes (\chi_j \circ \mathrm{Art}_{F_j}), s)^{a_j}$$

and

$$L(\mathrm{Symm}^{m+1}(A,i),s) L(\mathrm{Symm}^{m-1}(A,i), s-1) = \prod_j L((\pi_j \otimes (\chi_j \circ \mathrm{Art}_{F_j})) \times \sigma_j, s)^{a_j}$$

and

$$L(\mathrm{Symm}^2(A,i),s) = \prod_j L((\mathrm{Symm}^2 \pi_j) \otimes (\chi_j \circ \mathrm{Art}_{F_j}), s)^{a_j}$$

(See [T2] for similar calculations.) Our theorem for m and $m+1$ follows (for instance) from [CP] and theorem 5.1 of [Sh] (and in the case $m+1=2$ also from [GJ]). $\Box$

Theorem 4.3 *Let F be a totally real field. Let E/F be an elliptic curve with multiplicative reduction at some prime v_q of F. The numbers*

$$(1 + \mathbf{N}v - \#E(k(v)))/2\sqrt{\mathbf{N}v}$$

as v ranged over the primes of F are equidistributed in $[-1,1]$ with respect to the measure $(2/\pi)\sqrt{1-t^2}\, dt$.

Proof: This follows from theorem 4.2 and the corollary to theorem 2 of [Se1], as explained on page I-26 of [Se1]. $\Box$

Now fix an even positive integer n. Finally let us consider the L-functions of the motives V_t for $t \in \mathbb{Q}$. More precisely for each pair of rational primes l and p there is a Weil-Deligne representation $\mathrm{WD}(V_{l,t}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)})$ of $W_{\mathbb{Q}_p}$ associated to the $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-module $V_{l,t}$ (see for instance [TY]). Moreover for all but finitely many p there is a Weil-Deligne representation $\mathrm{WD}_p(V_t)$ of $W_{\mathbb{Q}_p}$ over $\overline{\mathbb{Q}}$ such that for each prime $l \neq p$ and each embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_l$ the Weil-Deligne representation $\mathrm{WD}_p(V_t)$ is equivalent to the Frobenius semi-simplification $\mathrm{WD}(V_{l,t}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)})^{\text{F-ss}}$. Let $S(V_t)$ denote the finite set of primes p for which no such representation $\mathrm{WD}_p(V_t)$ exists. It is expected that $S(V_t) = \emptyset$. If indeed $S(V_t) = \emptyset$, then we set $L(V_t, s)$ equal to

$$2^{n/2}(2\pi)^{n(n-2)/8}(2\pi)^{-ns/2}\Gamma(s)\Gamma(s-1)...\Gamma(s+1-n/2)\prod_p L(\mathrm{WD}_p(V_t), s)$$

and

$$\epsilon(V_t, s) = i^{-n/2}\prod_p \epsilon(\mathrm{WD}_p(V_t), \psi_p, \mu_p, s),$$

where μ_p is the additive Haar measure on $\mathbb{Q}_p$ defined by $\mu_p(\mathbb{Z}_p) = 1$, and $\psi_p : \mathbb{Q}_p \to \mathbb{C}$ is the continuous homomorphism defined by

$$\psi_p(x+y) = e^{-2\pi i x}$$

for $x \in \mathbb{Z}[1/p]$ and $y \in \mathbb{Z}_p$. The function $\epsilon(V_t, s)$ is entire. The product defining $L(V_t, s)$ converges absolutely uniformly in compact subsets of $\mathrm{Re}\, s > 1 + m/2$ and hence gives a holomorphic function in $\mathrm{Re}\, s > 1 + m/2$.

Theorem 4.4 *Suppose that $t \in \mathbb{Q} - \mathbb{Z}[1/(n+1)]$. Then $S(V_t) = \emptyset$ and the function $L(V_t, s)$ has meromorphic continuation to the whole complex plane and satisfies the functional equation*

$$L(V, s) = \epsilon(V, s)L(V, n-s).$$

Proof: Choose a prime q dividing the denominator of t. By lemma 1.15 and, for instance, proposition 3 of [Sc] (see also [TY]), we see that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts irreducibly on $V_{l,t}$. Let G_l denote the Zariski closure of the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in $GSp(V_{l,t})$ and let G_l^0 denote the connected component of the identity in G_l. Then G_l^0 is reductive and (by lemma 1.15) contains a unipotent element with minimal polynomial $(T-1)^n$. Moreover as the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $V_{l,t}$ has multiplier ϵ^{1-n}, we see that the multiplier map from G_l^0 to $\mathbb{G}_m$ is dominating. By theorem 9.10 of [K1] (see also [Sc] for a more conceptual argument due to Grojnowski) we see that G_l^0 is either GSp_n or $(\mathbb{G}_m \times GL_2)/\mathbb{G}_m$ embedded via $(x, y) \mapsto x\mathrm{Symm}^{n-1}y$. (Here $\mathbb{G}_m \hookrightarrow \mathbb{G}_m \times GL_2$ via $z \mapsto (z^{1-n}, z)$.) In either case we also see that $G_l = G_l^0$. (In the second case use the fact that any automorphism of SL_2 is inner.) Let Γ_l denote the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in $PGSp(V[l]_t)$. The main theorem of [L] tells us there is a set S of rational primes of Dirichlet density zero, such that if $l \notin S$ then either

$$PSp(V[l]_t) \subset \Gamma_l \subset PGSp(V[l]_t)$$

or

$$\mathrm{Symm}^{n-1} PSL_2(\mathbb{F}_l) \subset \Gamma_l \subset \mathrm{Symm}^{n-1} PGL_2(\mathbb{F}_l).$$

Choose a prime $l \notin S$ such that $l \not| \mathrm{val}_q(t)$, $l > \max\{2n+1, D(n)\}$ and $l \neq q$. By lemma 1.15 we see that the image of $I_{\mathbb{Q}_q}$ in $Sp(V[l]_t)$ contains a unipotent element with minimal polynomial $(T-1)^n$. Combining the above discussion with corollary 1.5.4 and lemma 1.5.5 of [CHT], we see that the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_l))$ in $GSp(V[l]_t)$ is big. Using the simplicity of $PSL_2(\mathbb{F}_l)$ and $PSp_n(\mathbb{F}_l)$ we also see that $\zeta_l \notin \overline{\mathbb{Q}}^{\ker V[l]_t}$. Thus theorem 3.3 tells us that we can find a Galois totally real field $F/\mathbb{Q}$ such that $V_{l,t}|_{\mathrm{Gal}(\overline{F}/F)}$ is automorphic of weight 0 and type $\{\mathrm{Sp}_n(1)\}_{\{v|q\}}$.

If F' is any subfield of F with $\mathrm{Gal}(F/F')$ soluble, we see that there is a RAESDC representation $\pi_{F'}$ of $GL_n(\mathbb{A}_{F'})$ of weight 0 and type $\{\mathrm{Sp}_n(1)\}_{\{v|q\}}$ such that for any rational prime l and any isomorphism $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ we have

$$r_{l,\imath}(\pi_{F'}) \cong V_{l,t}|_{\mathrm{Gal}(\overline{F}'/F')}.$$

As a virtual representation of $\mathrm{Gal}(F/\mathbb{Q})$ write

$$1 = \sum_j a_j \mathrm{Ind}_{\mathrm{Gal}(F/F_j)}^{\mathrm{Gal}(F/\mathbb{Q})} \chi_j,$$

where $a_j \in \mathbb{Z}$, where $F \supset F_j$ with $\mathrm{Gal}(F/F_j)$ soluble, and where $\chi_j : \mathrm{Gal}(F/F_j) \to \mathbb{C}^\times$ is a homomorphism. Then, for all rational primes l and for all isomorphisms $\imath : \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, we have (as virtual representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$)

$$V_{l,t} = \sum_j a_j \mathrm{Ind}_{\mathrm{Gal}(F/F_j)}^{\mathrm{Gal}(F/\mathbb{Q})} r_{l,\imath}(\pi_{F_j} \otimes (\chi_j \circ \mathrm{Art}_{F_j})).$$

We deduce that, in the notation of [TY], $\mathrm{WD}(V_{l,t}|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)})^{\mathrm{ss}}$ is independent of $l \neq p$. Moreover by theorem 3.2 (and lemma 1.3(2)) of [TY], we see that

$\mathrm{WD}(V_{l,t}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)})^{\text{F-ss}}$ is pure. Hence by lemma 1.3(4) of [TY] we deduce that $\mathrm{WD}(V_{l,t}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)})^{\text{F-ss}}$ is independent of $l \neq p$, i.e. $S(V_t) = \emptyset$. Moreover

$$L(V_t, s) = \prod_j L(\pi_{F_j} \otimes (\chi_j \circ \mathrm{Art}_{F_j}), s)^{a_j},$$

from which the rest of the theorem follows. □

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