OCCULT PERIOD INVARIANTS AND CRITICAL VALUES OF THE DEGREE FOUR L-FUNCTION OF $GSp(4)$

Michael Harris[1]

UFR de Mathématique
Université Paris 7
2 Pl. Jussieu
75251 Paris cedex 05 FRANCE

To Joe Shalika

Introduction

Let G be the similitude group of a four-dimensional symplectic space over $\mathbb{Q}$:

$$G = \{g \in GL(4) \mid {}^t gJg = \lambda(g)J\} \tag{0.1}$$

where $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ is the standard alternating form of dimension 4 and the homomorphism $\lambda : G \to \mathbb{G}_m$ is defined by (0.1). Let π be a cuspidal automorphic representation of G, $\pi = \pi_\infty \otimes \bigotimes_p \pi_p$, where p runs over rational primes. Let S be the set of finite primes for which π_p is not a spherical representation, together with the archimedean prime. The Langlands dual group of G can be identified with G itself, hence it makes sense to speak of the Langlands Euler product

$$L^S(s,\pi) = \prod_{p \notin S} L(s, \pi_p, r), \tag{0.2}$$

where $r : G \to GL(4)$ is the tautological representation. When π_∞ is in the holomorphic discrete series and S is empty, an integral representation for $L(s,\pi)$ was discovered by Andrianov [An], with a functional equation (at least) when all π are unramified. An adelic version of Andrianov's construction, valid in principle for all π, and over any number field, was discovered by Piatetski-Shapiro a few years later, but was only published in 1997 [PS]. The article [PS] defines local Euler factors at ramified primes as well, and obtains local and global functional equations as in Tate's thesis. More generally, if μ is a Hecke character of $\mathbf{A}^\times/\mathbb{Q}^\times$, then one can define the twisted (partial) L-function $L^S(s,\pi,\mu) = \prod_{p \notin S} L(s, \pi_p, \mu_p, r)$. The Andrianov-Piatetski-Shapiro method gives integral representations and functional equations for these L-functions as well.

[1]Institut de Mathématiques de Jussieu, U.M.R. 7586 du CNRS. Membre, Institut Universitaire de France

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

The constructions of [An] and [PS] are based on the Fourier expansion of forms in π. Let $P \subset G$ be the Siegel parabolic

$$P = \{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\} \subset G,$$

$U \subset P$ the unipotent radical, isomorphic to the additive group S_2 of symmetric 2×2 matrices. Let $\psi : \mathbf{A}/\mathbb{Q} \to \mathbb{C}^\times$ be a non-trivial additive character, $\beta \in S_2(\mathbb{Q})$, $\psi_\beta : U(\mathbf{A})/U(\mathbb{Q}) \to \mathbb{C}^\times$ the character $u \to \psi(tr(\beta u))$, where $u \in U(\mathbf{A})$ is viewed as an element of $S_2(\mathbf{A})$ as above and tr is the usual trace. For $f \in \pi$, we can write

$$f(ug) = \sum_\beta f_\beta(g)\psi_\beta(u), u \in U(\mathbf{A}), \tag{0.3}$$

where f_β is a smooth function on $G(\mathbf{A})$. We say β is in the *support* of π if $f_\beta \neq 0$ for some $f \in \pi$.

Write $P = MU$, with M the Levi component, isomorphic to $GL(2) \times \mathbb{G}_m$. We assume $\det(\beta) \neq 0$, and let $D = D_\beta \subset M$ denote the identity component of the stabilizer of the linear form $u \to tr(\beta u)$ under the adjoint action. Then there is a unique quadratic semi-simple algebra $K = K_\beta$ over $\mathbb{Q}$ such that $D = R_{K/\mathbb{Q}}\mathbb{G}_m$. (For all this, as well as what follows, see [PS].) Let $N_\beta = \{n \in U \mid tr(\beta n) = 0\}$, and define $R = DU \subset P \subset G$. Let H be the subgroup of $R_{K/\mathbb{Q}}GL(2)$ defined by the following cartesian diagram:

$$\begin{array}{ccc} H & \longrightarrow & R_{K/\mathbb{Q}}GL(2) \\ \downarrow & & \downarrow{\scriptstyle \det} \\ \mathbb{G}_{m,\mathbb{Q}} & \longrightarrow & R_{K/\mathbb{Q}}\mathbb{G}_{m,K} \end{array}$$

There is an embedding $H \to G$ such that $H \cap R = DN$ [PS, Prop. 2.1]. We let λ_H denote the composition of the similitude character λ with this embedding.

One chooses a β in the support of π and a Hecke character ν of $D(\mathbf{A})/D(\mathbb{Q})$, and constructs a standard Eisenstein series $E = E^\Phi(h;\mu,\nu,s)$ with $h \in H_\beta(\mathbb{Q})\backslash H_\beta(\mathbf{A})$, meromorphic in s, and depending on additional data Φ to be specified below. Let Z_G denote the center of G. Then the family of integrals

$$Z(f,\Phi,\mu,\nu,s) = \int_{Z_G(\mathbf{A})H(\mathbb{Q})\backslash H(\mathbf{A})} f(h)E^\Phi(h;\mu,\nu,s)dh \tag{0.4}$$

has an Euler product whose local factors are almost everywhere given by $L(s,\pi_p,r)$:

$$Z(f,\Phi,\mu,\nu,s) = a(\pi,\beta,\nu)\prod_{w\in S} Z_w(f,\Phi,\mu,\nu,s)\prod_{p\notin S} L(s,\pi_p,\mu,r), \tag{0.5}$$

assuming of course f and Φ to be factorizable data. Here $a(\pi,\beta,\nu)$ is a coefficient to be explained below. Note that the L-function on the right-hand side of (0.5) does not depend on the choice of ν. Starting in §1, β will be assume isotropic, so D will be a split torus and ν can be written as a pair of Hecke characters of $\mathbf{A}^\times/\mathbb{Q}^\times$.

The group G is attached to a 3-dimensional Shimura variety Sh, isomorphic to the Siegel modular variety of genus 2, with a canonical model over $\mathbb{Q}$ (see §2 for the formula). Holomorphic automorphic forms on G can be regarded as sections of automorphic vector bundles on Sh, and possess a rational structure over $\mathbb{Q}$. In the setting of [An], in which π is holomorphic discrete series, β is necessarily a positive-definite (or negative-definite) symmetric matrix, K is an imaginary quadratic field, and $H(\mathbb{R})$ is basically the same as $GL(2,\mathbb{C})$, so no Shimura variety is attached to H. Thus the formula (0.5) admits no clear interpretation in terms of algebraic geometry at integral points s. This is not surprising, since Deligne's conjecture expresses the critical values of $L(s,\pi)$ in terms of the determinant of a 2×2 matrix of periods, involving forms from distinct elements of the L-packet (conjecturally) attached to π. In particular, the critical values should involve periods of functions in $\pi' = \pi_\infty^{nh} \otimes \bigotimes_p \pi'_p$ where $\pi'_p = \pi_p$ for almost all p but π_∞^{nh} belongs to the non-holomorphic discrete series associated to π_∞. Anyway, no one knows how to construct periods of general cohomological automorphic forms on G.

On the other hand, suppose β is of signature $(1,-1)$, so that π_∞ is in the non-holomorphic discrete series. Then K is either a real quadratic field or $K = \mathbb{Q}\oplus\mathbb{Q}$, H corresponds to a Shimura subvariety $Sh_H \subset Sh$, and for special values of s and appropriate choices of the auxiliary data the Eisenstein series $E^\Phi(h;\mu,\nu,s)$ are *nearly holomorphic* automorphic forms, in Shimura's sense [S2]. The modest goal of this note is to show that, for certain choices of f, the global zeta integral in (0.4), when $s = m$ is a critical value in Deligne's sense, can be interpreted as a cup product in coherent cohomology on Sh_H. This then expresses the special values of the corresponding L-function in terms of intrinsic coherent cohomological invariants of π, the occult period invariants of the title, related to the coefficients $a(\pi,\beta,\nu)$ for varying ν. These invariants are doubly occult: in the first place, because they cannot be defined merely by reference to the abstract representation π but depend on its realization in coherent cohomology; in the second place, because they are not (yet) known to be non-trivial in any specific case.

The invariants $a(\pi,\beta,\nu)$ are obtained by comparing a rational structure on the space π defined in terms of coherent cohomology with one defined in terms of the Bessel model of π attached to the pair (β,ν). More precisely, in the absence of a natural choice of archimedean local data, the invariant is the product $a(\pi,\beta,\nu)Z_\infty(f,\Phi,\mu,\nu,m)$. Coherent cohomological invariants of this type already appeared in [H2] in connection with Rankin-Selberg L-functions for Hilbert modular forms. Similar invariants, making use of the rational structure on topological cohomology of non-hermitian locally symmetric spaces, were related by Hida [Hida] to Rankin-Selberg L-functions for $GL(2)$ over totally imaginary number fields; Grenié [G] has recently obtained a partial generalization of Hida's work to $GL(n)$ for $n > 2$. The most intriguing discovery in the present paper is that odd and even critical values are related to $a(\pi,\beta,\nu)$ for different choices of ν, an observation consistent with Deligne's conjecture.

The ideas in this note date back to 1988. The appearance of [PS] has made their publication more reasonable, and in view of the conjectures of Furusawa and Shalika [FS] on the products of two central special values of $L(s,\pi)$, publication may actually be of some use. The reader should nevertheless bear in mind that nothing in this paper should be considered definitive. In particular, the heuristic arguments presented here are vacuous unless one knows, first, that the archimedean zeta integrals in (0.5) do not vanish for a cohomological choice of data; and more crucially, that the global invariant (Bessel coefficient) $a(\pi,\beta,\nu)$ does not vanish for arithmetically interesting characters ν. Local non-vanishing should not be too hard to establish, but I haven't tried to do so. On the other hand, I have no idea how to prove non-vanishing of the global invariant.

I began writing this article in the summer of 2001, during a visit to the Hong Kong University of Science and Technology; I thank Jian-Shu Li and the HKUST for their generous hospitality. I also thank David Soudry for his very pertinent answers to a number of questions about Piatetski-Shapiro's article and its relation to other integral representations of automorphic L-functions, Masaaki Furusawa for comments regarding his conjectures with Shalika, and Daniel Bertrand for the reference for the discussion in (2.7). I thank the referee for a careful reading. Finally, it is indeed a pleasure to dedicate this paper to Joe Shalika, in whose Princeton algebra course I first encountered the notion of a group representation, and who continues to inspire and surprise.

1. Motives for $GSp(4)$

Notation is as in the introduction. We make the following departure from convention. Let $K_\infty = Z_G(\mathbb{R})U(2)$ be the standard maximal compact (mod center) *connected* subgroup of $G(\mathbb{R})$, the stabilizer of the point $i \cdot I_2$ in the Siegel upper half space of genus 2. For us, it is more convenient to let a cuspidal automorphic representation of G be an irreducible $(\mathfrak{g}, K_\infty) \times G(\mathbf{A}_f)$-submodule of the cusp forms on $G(\mathbb{Q})\backslash G(\mathbf{A})$. In the cases of interest to us, it generally takes two automorphic representations of this kind to make one of the usual kind, simply because $G(\mathbb{R})$ is disconnected.

Henceforward we assume β to be isotropic over $\mathbb{Q}$, and write H for H_β. (There is also a theory for general negative-definite β, but it seems to give less complete results.) Then up to isomorphism,

$$H = \{(g_1, g_2) \in GL(2) \times GL(2) \mid det(g_1) = det(g_2)\}.$$

Thus, letting $Sh(GL(2))$ denote the standard Shimura variety attached to $GL(2)$ (the tower of modular curves of all levels), there is a natural embedding

$$Sh_H \hookrightarrow Sh(GL(2)) \times Sh(GL(2))$$

rational over $\mathbb{Q}$. Let pr_1 and pr_2 denote the two projections of Sh_H to the Shimura variety $Sh(GL(2))$ attached to $GL(2)$ (the tower of modular curves), corresponding to the composition of the $H \to GL(2) \times GL(2)$ with projection on the first and second factors respectively. Automorphic vector bundles on $Sh(GL(2))$ are denoted $F_{k,d}$ for pairs of integers $k \equiv d \pmod 2$. Given a triple $(k, \ell, -c)$ in $\mathbb{Z}$, with $k + \ell \equiv -c \pmod 2$, $F_{(k,\ell,-c)}$ on Sh_H is the pullback via (pr_1, pr_2) of the external tensor product $F_{k,d_1} \otimes F_{\ell,d_2}$ for any pair of integers (d_1, d_2) such that $d_1 \equiv k \pmod 2$, $d_2 \equiv \ell \pmod 2$, and $d_1 + d_2 = -c$.

To any irreducible finite-dimensional representation (ρ, V_ρ) of G we can associate a local system $\tilde{V}_\rho$ in $\mathbb{Q}$-vector spaces over the Shimura variety Sh. Note that ρ can be realized over $\mathbb{Q}$. We write

$${}_KSh(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbf{A})/K_\infty K,$$

$$Sh(\mathbb{C}) = \varprojlim_K {}_KSh(\mathbb{C}),$$

where $K_\infty = Z_G(\mathbb{R})U(2)$ is as above and K runs over open compact subgroups of $G(\mathbf{A}_f)$. Then

$$\tilde{V}_{\rho,\ell}(\mathbb{C}) = \varprojlim_K G(\mathbb{Q})\backslash G(\mathbf{A}) \times V_\rho(\mathbb{Q})/K_\infty K, \tag{1.1}$$

where $G(\mathbb{Q})$ acts diagonally on $G(\mathbf{A})\times V_\rho(\mathbb{Q})$. There are compatible right actions of $G(\mathbf{A}_f)$ on Sh and $\tilde{V}_\rho$, and hence on the cohomology of the former with coefficients in the latter. The middle dimensional interior cohomology

$$H^3_!(Sh, \tilde{V}_\rho) = Im[H^3_c(Sh, \tilde{V}_\rho) \otimes \overline{\mathbb{Q}} \to H^3(Sh, \tilde{V}_\rho) \otimes \overline{\mathbb{Q}}]$$

decomposes as the direct sum of irreducible $\overline{\mathbb{Q}}[G(\mathbf{A}_f)]$-modules with finite multiplicities. One expects that, for the "general" $\overline{\mathbb{Q}}[G(\mathbf{A}_f)]$-module π_f, the space

$$M(\pi_f) = H^3_!(Sh, \tilde{V}_\rho)[\pi_f] = Hom_{G(\mathbf{A}_f)}(\pi_f, H^3_!(Sh, \tilde{V}_\rho))$$

is four-dimensional, and moreover one has

$$H^3_!(Sh, \tilde{V}_\rho)[\pi_f] = Hom_{G(\mathbf{A}_f)}(\pi_f, H^3_{cusp}(Sh, \tilde{V}_\rho))$$

where H^3_{cusp} is the image in $H^3_!$ of the cuspidal cohomology. This is still not completely known, although significant partial results have been proved by Taylor, Laumon, and Weissauer. We assume $\pi = \pi_\infty \otimes \pi_f$ to have this property, which we call *stability* (at infinity); it also presupposes a multiplicity one property about which relatively little is known.

Tensoring over $\overline{\mathbb{Q}}$ with $\mathbb{C}$, the interior cohomology acquires a Hodge decomposition

$$H^3_!(Sh, \tilde{V}_\rho) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} = \bigoplus_{i=0}^{3} H^i_!(Sh, E^{3-i}_\rho) \otimes_{\mathbb{Q}} \mathbb{C},$$

hence $M(\pi_f)$ has the analogous decomposition

$$M(\pi_f) = \bigoplus_{i=0}^{3} H^i_!(Sh, E^{3-i}_\rho) \otimes_{\mathbb{Q}} \mathbb{C}[\pi_f]. \tag{1.2}$$

Here E^j_ρ, $j = 0, 1, 2, 3$, is a locally free coherent sheaf (automorphic vector bundle) over Sh, defined over $\mathbb{Q}$, which we can describe explicitly in terms of the highest weights of ρ. We choose a maximal compact (mod center) torus $T \subset K_\infty$ and a positive root system as in [HK], to which the reader is referred for details of the following construction. Suppose $(\rho, \tilde{V}_\rho)$ has highest weight (a, b, c), with $a \geq b \geq 0$ in $\mathbb{Z}$ and c an integer congruent to $a + b$ modulo 2. An automorphic vector bundle is associated to an irreducible algebraic representation of K_∞, hence to a triple of integers (a', b', c') with $a' \geq b'$ and the same parity condition. Then E^j_ρ is associated to the triple Λ^j_ρ, where

(1.3)

$$\Lambda^0_\rho = (a, b, c);\ \Lambda^1_\rho = (a, -b-2, c);\ \Lambda^2_\rho = (b-1, -a-3, c); \Lambda^3_\rho = (-b-3, -a-3, c)$$

[HK, Table 2.2.1].

For future reference, we note that the Hodge numbers corresponding to the Hodge decomposition (1.2) are given by

(1.4) $(a+b+3+\delta,\delta),\ (a+2+\delta,b+1+\delta),\ (b+1+\delta,a+2+\delta),\ (\delta,a+b+3+\delta)$

where $\delta = \frac{c-a-b}{2}$ and the weight is $w = 3+c$. It will be most convenient to fix $c = a+b$; then $\delta = 0$ and the weight is $a+b+3$.

The notation $H^i_!$ is slightly abusive. Indeed, to define $H^i_!(Sh, E^j_\rho)$, one has first to replace ${}_KSh$ at finite level K by a smooth projective toroidal compactification ${}_KSh_\Sigma$ such that ${}_KSh_\Sigma - {}_KSh$ is a divisor with normal crossings, and to replace E^j_ρ by a pair of canonically defined extensions, $E^{j,sub}_\rho \subset E^{j,can}_\rho$ to vector bundles over Sh_Σ. (Here and below, the subscript Σ designates an unspecified datum used to define a toroidal compactification, as in [H1].) Then [H1, §2]

$$H^i_!(Sh, E^j_\rho) = \varinjlim_K Im[H^i({}_KSh_\Sigma, E^{j,sub}_\rho) \to H^i({}_KSh_\Sigma, E^{j,can}_\rho)]$$

is independent of the choices of Σ (hence the direct limit makes sense) and is an admissible $G(\mathbf{A}_f)$-module, with a canonical $\mathbb{Q}$-rational structure. Thus any irreducible admissible $G(\mathbf{A}_f)$-module π_f that occurs in $H^i_!(Sh, E^j_\rho)$ can be realized over $\overline{\mathbb{Q}}$, and we can define $H^i_!(Sh, E^{3-i}_\rho) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}[\pi_f]$ as well as $H^i_!(Sh, E^{3-i}_\rho) \otimes_{\mathbb{Q}} \mathbb{C}[\pi_f]$. A strengthening of Hodge theory due to various people in various forms (Zucker, Faltings, ...) then yields the decomposition (1.2). Stability at infinity comes down to

Hypothesis (1.5). *For $i = 0,1,2,3$,*

$$H^i_!(Sh, E^{3-i}_\rho) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}[\pi_f] = H^i_{cusp}(Sh, E^{3-i}_\rho) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}[\pi_f]$$

is of dimension one.

Here H^i_{cusp} is the image of the cusp forms in $H^i_!$ (cf. [H1]). Equality of the two spaces in (1.5) can be taken as part of the hypothesis; in any case, it will be automatic in the applications (see the next paragraph). In Hypothesis (1.5) we can replace $\overline{\mathbb{Q}}$ by $\mathbb{C}$; the two versions are equivalent. Actually, for most of what we have in mind it suffices to assume the different Hodge components have the same dimension. In any case, there is a discrete series L-packet $(\pi^j_\rho, j = 0,1,2,3)$ of $(\mathfrak{g}, K_\infty)$-modules and an isomorphism

$$(1.6) \qquad H^i_{cusp}(Sh, E^{3-i}_\rho) \otimes_{\mathbb{Q}} \mathbb{C}[\pi_f] = Hom_{(\mathfrak{g},K_\infty)\times G(\mathbf{A}_f)}(\pi^i_\rho \otimes \pi_f, \mathcal{A}_{cusp}(G))$$

where $\mathcal{A}_{cusp}(G)$ is the space of cusp forms on $G(\mathbb{Q})\backslash G(\mathbf{A})$ [BHR]. For future reference, we let $H^i_{cusp}(Sh, E^{3-i}_\rho) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}(\pi_f)$ denote the image of any non-zero $G(\mathbf{A}_f)$-morphism $\pi_f \to H^i_{cusp}(Sh, E^{3-i}_\rho) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$; by Hypothesis (1.5) such a morphism, an element of $H^i_{cusp}(Sh, E^{3-i}_\rho) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}[\pi_f]$ (note the difference in notation!), is unique up to (algebraic) scalar multiples.

We now return to π as in the introduction and assume $\pi = \pi^2_\rho \otimes \pi_f$, so π contributes to coherent cohomology in degree 2, i.e. to $H^2_{cusp}(Sh, E^1_\rho) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. We assume $a > b > 0$ (strict inequality), so that $H^2_{cusp}(Sh, E^1_\rho) = H^2_!(Sh, E^1_\rho)$ (cf. [MT], 2.1, Proposition 1). Moreover, we have a natural inclusion [H1, §3]

$$(1.7) \qquad H^2_{cusp}(Sh, E^1_\rho) \subset \varinjlim_K H^2({}_KSh_\Sigma, E^{1,sub}_\rho).$$

The space on the right-hand side is denoted $\tilde{H}^2(Sh, E_\rho^{1,sub})$, as in [H1,HK].

Let Sh_H be as in the introduction. We may assume T to be a maximal torus in $H = H_\beta$; then $T(\mathbb{R})$ is a maximal connected compact (mod center) subgroup of $H(\mathbb{R})$, and automorphic vector bundles on Sh_H correspond to weights of the torus T. Specifically, we let $T = H \cap K_\infty$, the stabilizer of the point (i,i) in the product of two upper half-planes; we occasionally also write $K_{H,\infty} = T(\mathbb{R})$. Thus any triple of integers $\Lambda^\# = (r,s,c)$, with $c \equiv r+s \pmod 2$, defines an automorphic vector bundle $F_{\Lambda^\#} = F_{(r,s,c)}$ on Sh_H. As in (1.6), we can define toroidal compactifications Sh_{H,Σ_H} of Sh_H, and canonical and subcanonical extensions $F_{\Lambda^\#}^{can}$ and $F_{\Lambda^\#}^{sub}$ over Sh_{H,Σ_H}. For $i \in \mathbb{Z}$, we define $\Lambda^\#(i) = (a-i, i-b-2, c)$ (for $\rho = (a,b,c)$ as above). Let $\iota : Sh_H \to Sh$ be the embedding. For $0 \le i \le a+b+2$ there are homomorphisms [HK, (2.6.3)]

$$\iota^*((E_\rho^1)^{sub}) \to F_{\Lambda^\#(i)}^{sub}$$

giving rise to homomorphisms

$$\psi_i : \tilde{H}^2_{cusp}(Sh, (E_\rho^1)^{sub}) \to \tilde{H}^2(Sh_H, F_{\Lambda^\#(i)}^{sub}). \tag{1.8}$$

It follows from (1.7) that one can actually lift ψ_i to a homomorphism

$$H^2_{cusp}(Sh, E_\rho^1) \to \tilde{H}^2(Sh_H, F_{\Lambda^\#(i)}^{sub}) \xrightarrow{\sim} [\tilde{H}^0(Sh_H, F_{(i-a-2,b-i,-c)}^{can})]^*, \tag{1.9}$$

where the isomorphism is given by Serre duality ([H1],Corollary 2.3; see §8 of [H1] for the shift by 2). The right-hand side is isomorphic to the space of (pairs of) classical holomorphic modular forms on Sh_H, including Eisenstein series, of weight $(a+2-i, i-b)$. Unless $b+1 \le i \le a+1$ the right-hand side is therefore uninteresting for our purposes.

Let $f \in \pi$ belong to the lowest K_∞-type subspace of π_∞, which has highest weight $(a+3, -b-1, c)$ [HK, Table 2.2.1]. We assume f is a weight vector for T with character $(a+2-i, i-b, c)$, hence upon restriction to H can pair non-trivially with a section $g \in \tilde{H}^0(Sh_H, F_{(i-a-2,b-i,-c)}^{can})$, to yield a complex number $< f, g >$. If both f and g are rational[2] over $\overline{\mathbb{Q}}$, then so is $< f, g >$. We want to identify the zeta integral of (0.4) with such a pairing. Unfortunately, the Eisenstein series we need is in general not a holomorphic section of $F_{(i-a-2,b-i,-c)}^{can}$, so a priori the pairing cannot be interpreted in terms of coherent cohomology.

(1.10) Maass operators. Our main results are based on the algebraic interpretation of the Maass operators due to Katz, and developed in the present language in [H2] and elsewhere. In this section k and ℓ are positive integers. Let Ω_i^1, $i = 1,2$ denote the pullback via pr_i to Sh_H of the cotangent bundle of $Sh(GL(2))$, and let $jet^{r_1,r_2}(F_{(-k,-\ell,-c)})$ denote the pullback via (pr_1, pr_2) of $jet^{r_1}F_{-k,d_1} \otimes jet^{r_2}F_{-\ell,d_2}$. Let

$$j^{r_1,r_2} : F_{(-k,-\ell,-c)}^{can} \to jet^{r_1,r_2}(F_{(-k,-\ell,-c)}^{can})$$

[2] Cohomology classes in $H^2_{cusp}(Sh, E_\rho^1)$ are defined by vector-valued automorphic forms via a normalized Dolbeault isomorphism, as in [H1]. By f being rational we mean f is the weight component of character $(a+2-i, i-b, c)$ of a rational element of $H^2_{cusp}(Sh, E_\rho^1)$.

denote the canonical differential operator of order $r_1 + r_2$, and let
(1.10.1)

$$\underline{Split}(r_1, r_2) : jet^{r_1,r_2}(F^{can,\infty}_{(-k,-\ell,-c)}) \to [Sym^{r_1}(\Omega^1_1) \otimes Sym^{r_2}(\Omega^1_2) \otimes F_{(-k,-\ell,-c)}]^{can,\infty} \xrightarrow{\sim} F^{can,\infty}_{(-k-2r_1,-\ell-2r_2,-c)}$$

be the canonical splitting of the Hodge filtration in the category of $H(\mathbf{A}_f)$-equivariant C^∞ vector bundles (cf. [H2,2.5]). Then $\delta_{r_1,r_2} = \underline{Split}(r_1, r_2) \circ j^{r_1,r_2}$ corresponds to the classical Maass operator in two variables. Explicitly, let E_k and E_ℓ be holomorphic modular forms on $GL(2, \mathbb{Q})\backslash GL(2, \mathbf{A})$ of weights k and ℓ, respectively, corresponding (via the trivializations of [H2]) to sections $f_k \in H^0(Sh(GL(2)), F_{-k,d_1})$ and $f_\ell \in H^0(Sh(GL(2)), F_{-\ell,d_2})$, respectively. Then $\delta_{r_1,r_2}(pr_1, pr_2)^*(f_k \otimes f_\ell)$ is the C^∞ section of $F^{can,\infty}_{(-k-2r_1,-\ell-2r_2,-c)}$ corresponding to the restriction to H of

$$D^{r_1}_k E_k \otimes D^{r_2}_\ell E_\ell$$

where for any j, d_j is the first order differential operator on the upper half-plane

$$d_j = \frac{1}{2\pi i}(\frac{d}{dz} + \frac{j}{2iy})$$

and $D^{r_1}_k = d_{k+2r_1-2} \circ \dots d_{k+2} \circ d_k$, $D^{r_2}_\ell = d_{\ell+2r_2-2} \circ \dots d_{\ell+2} \circ d_\ell$.

The map (1.10.1) is a splitting in the C^∞ category of a short exact sequence of automorphic vector bundles:

$$(1.10.2) \quad 0 \to [Sym^{r_1}(\Omega^1_1) \otimes Sym^{r_2}(\Omega^1_2) \otimes F_{(-k,-\ell,-c)}]^{can} \to jet^{r_1,r_2}(F_{(-k,-\ell,-c)})^{can} \to jet^{r_1-1,r_2-1}(F_{(-k,-\ell,-c)})^{can} \to 0$$

where the final term is defined by analogy with that in the middle.

Proposition 1.10.3. *Let dh denote a Haar measure on $Z_G(\mathbf{A})H(\mathbb{Q})\backslash H(\mathbf{A})$. There is a constant $c(dh) \in \mathbb{R}^\times$ with the following property. Let $\pi \subset \mathcal{A}_0(G)$ be as in the introduction. Suppose $f \in \pi$ defines a $\overline{\mathbb{Q}}$-rational cohomology class (also denoted f) in $H^2_{cusp}(Sh, E^1_\rho)$. Let i be an integer, $b + 1 \le i \le a + 1$, and suppose*

$$a + 2 - i = k + 2r_1, \qquad i - b = \ell + 2r_2$$

with positive (resp. non-negative) integers k, ℓ (resp. r_1, r_2). Let E_k, E_ℓ be holomorphic modular forms on $GL(2, \mathbb{Q})\backslash GL(2, \mathbf{A})$, as above, and suppose the corresponding f_k and f_ℓ are $\overline{\mathbb{Q}}$-rational. Then

$$c(dh) \cdot \int_{Z_G(\mathbf{A})H(\mathbb{Q})\backslash H(\mathbf{A})} f(h) D^{r_1}_k E_k \otimes D^{r_2}_\ell E_\ell(h) dh$$

lies in $\overline{\mathbb{Q}}$.

Proof. With k, ℓ, r_j, and i as in the statement of the proposition, the first nonzero term on the left of (1.10.2) can be identified with $F^{can}_{(-k-2r_1,-\ell-2r_2,-c)} = F^{can}_{(-2-a+i,i-b)}$. Tensoring with $F^{sub}_{\Lambda^\#(i)} = F^{sub}_{(a-i,i-b-2,c)}$ we obtain an exact sequence

$$0 \to F^{sub}_{(2,2)} \to [jet^{r_1,r_2}(F_{(-k,-\ell,-c)}) \otimes F_{(a-i,i-b-2,c)}]^{sub} \to [jet^{r_1-1,r_2-1}(F_{(-k,-\ell,-c)}) \otimes F_{(a-i,i-b-2,c)}]^{sub} \to 0,$$

which we rewrite

$$0 \to F^{sub}_{(2,2)} \to (J^{r_1,r_2})^{sub} \to (J^{r_1-1,r_2-1})^{sub} \to 0.$$

Now $F^{sub}_{(2,2)} \xrightarrow{\sim} \Omega^1_{Sh_H}$ is the dualizing sheaf. Taking the long exact sequence of cohomology, we have

$$\begin{aligned}(1.10.4)\quad \ldots \to \tilde{H}^1(Sh_H,(J^{r_1-1,r_2-1})^{sub}) &\to \tilde{H}^2(Sh_H,\Omega^1_{Sh_H}) \\ &\to \tilde{H}^2(Sh_H,(J^{r_1,r_2})^{sub}) \to \tilde{H}^2(Sh_H,(J^{r_1-1,r_2-1})^{sub}) \to 0\end{aligned}$$

which by Serre duality (cf. [H1], Corollary 2.3 for the duality between ^{sub} and ^{can}) yields

$$\begin{aligned}0 \to \tilde{H}^0(Sh_H,\Omega^1_{Sh_H} \otimes (J^{r_1-1,r_2-1})^{sub,*}) &\to \tilde{H}^0(Sh_H,\Omega^1_{Sh_H} \otimes (J^{r_1,r_2})^{sub,*}) \\ \to \tilde{H}^0(Sh_H,\mathcal{O}^{can}_{Sh_H}) &\to \tilde{H}^1(Sh_H,\Omega^1_{Sh_H} \otimes (J^{r_1-1,r_2-1})^{sub,*}) \to \ldots.\end{aligned}$$

Here we use the superscript * to denote duality. Now $\Omega^1_{Sh_H} \otimes (J^{r_1-1,r_2-1})^{sub,*}$ has a finite filtration whose associated graded object is a sum of line bundles of the form

$$\Omega^1_{Sh_H} \otimes F^{sub,*}_{(i+a-2+2e_j,b-i+2f_j)} = F^{can}_{(2-a-i-2e_j,i-b-2f_j)}$$

where $0 \leq e_j \leq r_1 - 1$, $0 \leq f_j \leq r_2 - 1$. By our choice of r_1 and r_2, each term is of the form $F_{\alpha,\beta}$ with $\alpha,\beta \geq 2$. It is known (cf. [H1,§8]) that $H^1(Sh_H, F^{can}_{\alpha,\beta}) = 0$ for $\alpha,\beta \geq 2$, hence (1.10.4) becomes

$$\begin{aligned}(1.10.5)\quad 0 \to \tilde{H}^0(Sh_H,\mathcal{O}^{can}_{Sh_H})^* &\to \tilde{H}^2(Sh_H,(J^{r_1,r_2})^{sub}) \\ &\to \tilde{H}^0(Sh_H,\Omega^1_{Sh_H} \otimes (J^{r_1-1,r_2-1})^{sub,*}) \to 0.\end{aligned}$$

The term $\tilde{H}^0(Sh_H,\mathcal{O}^{can}_{Sh_H})^*$ is a sum of one-dimensional representations of $G(\mathbf{A}_f)$. On the other hand, by filtering $\Omega^1_{Sh_H} \otimes (J^{r_1-1,r_2-1})^{sub,*}$ as before, one sees that the $H(\mathbf{A}_f)$-representation on $\tilde{H}^0(Sh_H,\Omega^1_{Sh_H} \otimes (J^{r_1-1,r_2-1})^{sub,*})$ has a filtration by representations corresponding to holomorphic modular forms of positive weight, hence its Jordan-Hölder series contains no one-dimensional constituents. (One can make the filtration finite by restricting to the subrepresentation generated by vectors of fixed level $K \subset H(\mathbf{A}_f)$). It follows that the natural map from the middle term of (1.10.5) to its $H(\mathbf{A}_f)$-coinvariants factors through a non-trivial, $\overline{\mathbb{Q}}$-rational map from the middle term

$$\begin{aligned}(1.10.6)\qquad I_{r_1,r_2}:\ \tilde{H}^2(Sh_H,(J^{r_1,r_2})^{sub}) &\to [\tilde{H}^0(Sh_H,\mathcal{O}^{can}_{Sh_H})^*]_{H(\mathbf{A}_f)} \\ &= [\tilde{H}^0(Sh_H,\mathcal{O}^{can}_{Sh_H})^*]^{H(\mathbf{A}_f)},\end{aligned}$$

where the right-hand side is a one-dimensional space generated by the constant function 1. In particular, if ϕ is a rapidly decreasing Dolbeault cocycle representing a class $[\phi] \in \tilde{H}^2(Sh_H,(J^{r_1,r_2})^{sub})$ [H1], then $\phi \mapsto I_{r_1,r_2}[\phi]$ factors through projection on $K_{H,\infty} = T(\mathbb{R})$-invariants.

Putting together all these maps, we obtain a $\mathbb{Q}$-rational, $H(\mathbf{A}_f)$-invariant pairing

$$H^2_{cusp}(Sh,(E^1_\rho)^{sub})\otimes\tilde{H}^0(Sh_H,F_{(-k,-\ell,-c)})$$
$$\stackrel{\iota^*\otimes j^{r_1,r_2}}{\longrightarrow}\ \tilde{H}^2(Sh_H,\iota^*((E^1_\rho)^{sub}))\otimes\tilde{H}^0(Sh_H,jet^{r_1,r_2}(F_{(-k,-\ell,-c)})^{can})$$
$$\stackrel{\psi_i}{\longrightarrow}\tilde{H}^2(Sh_H,F^{sub}_{\Lambda^\#(i)})\otimes\tilde{H}^0(Sh_H,jet^{r_1,r_2}(F_{(-k,-\ell,-c)})^{can})\stackrel{\cup}{\to}\tilde{H}^2(Sh_H,(J^{r_1,r_2})^{sub})$$
$$\stackrel{I_{r_1,r_2}}{\longrightarrow}[\tilde{H}^0(Sh_H,\mathcal{O}^{can}_{Sh_H})^*]^{G(\mathbf{A}_f)}\stackrel{\sim}{\longrightarrow}\mathbb{C}.$$

Applying this composition to the rapidly decreasing Dolbeault cocycle represented by $f\otimes E_{k,\ell}$, we find that

$$I_{r_1,r_2}(\psi_i(f)\otimes j^{r_1,r_2}(E_{k,\ell}))\in\overline{\mathbb{Q}}$$

But we have seen that I_{r_1,r_2} factors through projection on the T-invariants. Thus I_{r_1,r_2} factors through

$$\psi_i(f)\otimes j^{r_1,r_2}(E_{k,\ell})\mapsto(1\otimes\underline{Split}(r_1,r_2))(\psi_i(f)\otimes j^{r_1,r_2}(E_{k,\ell}))=\psi_i(f)\otimes\delta^{r_1,r_2}E_{k,\ell}.$$

Finally, the Serre duality pairing is expressed in terms of integration of Dolbeault cocycles with growth conditions [H1,Proposition 3.8]. The Proposition now follows from the definitions.

2. Interlude on Deligne's conjecture

For motives, their L-functions, and their Deligne period invariants, we refer to [D]. Let $w=a+b+3=c+3$ and let μ be a Hecke character of finite order. Let π be as above, with central character $\xi_\pi=\xi_{0,\pi}\cdot|\bullet|^{-c}$, where $|\bullet|$ is the idèle norm; then $\xi_{0,\pi}$ is a Hecke character of finite order. We postulate the existence of a motive $M(\pi_f)$ with coefficients in some number field $E(\pi_f)$ of rank four, unramified outside S, of weight w, such that

$$L^S(s,M(\pi_f))=L^S(s-\frac{3}{2},\pi)\tag{2.1}$$

as Euler products away from S. The Hodge numbers are given by (1.4), with $\delta=0$. If we are satisfied to work with motives for absolute Hodge cycles, as in [D], then the existence of $M(\pi_f)$ as indicated is roughly equivalent to Hypothesis 1.5, once one has overcome scruples regarding cohomology with support, given results of Taylor, Laumon, and Weissauer on the cohomology of the genus two Siegel modular variety. The functional equation of [PS], relating (the completed L-function) $L(s,\pi,\mu)$, to $L(1-s,\hat{\pi},\mu^{-1})=L(c+1-s,\pi,\mu^{-1}\cdot\xi_{0,\pi}^{-1})$, becomes an equation relating $L(s,M(\pi_f),\mu)$ (one completes using the local factors of [PS]) to $L(w+1-s,M'(\pi_f),\mu^{-1})$, where

$$M'(\pi_f)=\hat{M}(\pi_f)(-w)=M\otimes M(\xi_{0,\pi}^{-1})\tag{2.2}$$

where $\hat{*}$ designates duality and (w) denotes Tate twist. Indeed, there is a nondegenerate bilinear pairing

$$M(\pi_f)\otimes M(\pi_f)\to M(\xi_{0,\pi})(-w).\tag{2.3}$$

in any realization, where $M(\xi_{0,\pi})$ is the rank one motive motive attached to the Dirichlet character $\xi_{0,\pi}$ (note that the values of $\xi_{0,\pi}$ are contained in the coefficient field $E(\pi_f)$). For instance, in the ℓ-adic realizations, it suffices by Chebotarev's density theorem to verify this locally for all $p \notin S$, and this follows from (2.1) and the characteristic fact that

$$\hat{\pi}_f \xrightarrow{\sim} \pi_f \otimes \lambda \circ \xi_\pi^{-1} \tag{2.4}$$

which reflects the fact that $GSp(4)$ is its own Langlands dual group.

From (2.3) we derive the isomorphism (2.2). More generally, we let $M(\pi_f, \mu)$ be the motive whose L-function is $L(s - \frac{w}{2}, \pi, \mu)$ (one can obtain the ℓ-adic realization by twisting by μ composed with the similitude character, since the Galois representation takes values in $GSp(4)$); then

$$\hat{M}(\pi_f, \mu) \xrightarrow{\sim} M(w) \otimes M((\mu \cdot \xi_{0,\pi})^{-1}).$$

Let m be any integer. By standard calculations (as in [D] (5.1.8)) one verifies the following relations for the Deligne periods:

$$c^+(M(\pi_f, \mu)(m)) = (2\pi i)^{2m} g(\mu)^2 c^{\pm}(M(\pi_f)) \tag{2.5}$$

where $g(\mu)$ is a Gauss sum and $\pm = (-1)^{m+e(\mu_\infty)}$ where $e(\mu_\infty) = 0$ if μ_∞ is trivial and $= 1$ otherwise.

By (1.4) and standard hypotheses (e.g. [D,5.2]) the archimedean Euler factors in the functional equation for $L(s, M(\pi_f), \mu)$ are given, independently of μ, by $\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s - b - 1)$, with $\Gamma_{\mathbb{C}}(s) = 2 \cdot (2\pi)^{-s}\Gamma(s)$ for the Euler Gamma function. The *critical values* of $L(s, M(\pi_f), \mu)$, in Deligne's sense, are then the integers $m \in [b+2, a+2]$. The right half of the critical set, accessible by combining the geometric considerations of §1 with the calculations in terms of Bessel models, is then the set of integers in $[\frac{a+b}{2} + 2, a + 2]$. Note that the central value $m = \frac{a+b}{2} + 2$ is critical if and only if $c = a + b$ is even.

By (2.5), Deligne's conjecture for the special values of $L(s, M(\pi_f, \mu))$ can be stated uniformly in terms of the Deligne periods $c^{\pm}(M(\pi_f))$ and elementary factors. Let $E(\pi_f, \mu)$ be the field generated by $E(\pi_f)$ and the values of μ. We consider $L(s, M(\pi_f, \mu))$ as a function with values in $\mathbb{C} \otimes E(\pi_f, \mu)$, as in [D]. Then we have

2.6 (Deligne's conjecture). *: For $m \in [\frac{a+b}{2} + 2, a + 2] \cap \mathbb{Z}$,*

$$L(m, M(\pi_f, \mu))/(2\pi i)^{2m} g(\mu)^2 c^{\pm}(M(\pi_f)) \in E(\pi_f, \mu),$$

with $\pm = (-1)^{m+e(\mu_\infty)}$.

(2.7) Remark. When μ is fixed, Deligne's conjecture thus relates the odd and even critical values to distinct, presumably transcental, invariants. When the motive does not have additional symmetries one expects the periods $c^+(M(\pi_f))$ and $c^-(M(\pi_f))$ to be algebraically independent. For example, suppose $M(\pi_f)$ is of the form $Sym^3(M)$, where M is the motive attached to an elliptic modular form of weight $k > 2$.[3] Deligne's calculations in [D, Prop. 7.7] identify $c^+(Sym^3(M))$

[3]Combining the proof by Kim and Shahidi of the existence of the symmetric cube lift from $GL(2)$ to $GL(4)$ with any of a number of methods (e.g. [GRS], or earlier unpublished results of Jacquet, Piatetski-Shapiro, and Shalika) for associating generic representations of classical groups to self-dual forms on $GL(4)$, one can construct at least part of the motive $Sym^3(M)$ on $GSp(4)$. See [KS, §9].

(resp. $c^-(Sym^3(M))$) with $(2\pi i)^{-1}c^+(M)^3c^-(M)$ (resp. $(2\pi i)^{-1}c^-(M)^3c^+(M)$), up to rational factors. On the other hand, it follows from a generalization of a theorem of Th. Schneider, due to Bertrand (and subsequently vastly generalized by Wüstholz), that when M is the motive attached to a modular form of weight 2, then $c^+(M)/c^-(M)$ is transcendental (cf. [B], Corollary 1, p. 35). In this case it follows easily that $c^+(Sym^3(M))$ and $c^-(Sym^3(M))$ are also linearly independent over $\overline{\mathbb{Q}}$.

3. Bessel models and zeta integrals

Notation is as in the Introduction. Recall that we are assuming β isotropic over $\mathbb{Q}$, so that $K = \mathbb{Q} \oplus \mathbb{Q}$. If $h = (g_1, g_2) \in H(F)$, for some field F, then $\lambda_H(h) = det(g_1) = det(g_2)$. Let μ and $\nu = (\nu_1, \nu_2)$ be Hecke characters of $\mathbf{A}^\times/\mathbb{Q}^\times$ and $(\mathbf{A}^\times/\mathbb{Q}^\times)^2$, respectively. Let V denote the free K-module K^2, $\mathcal{S}(V_{\mathbf{A}})$ the space of Schwartz-Bruhat functions on the adeles of V. To $\Phi \in \mathcal{S}(V_{\mathbf{A}})$ one can assign an Eisenstein series $E^\Phi(h; \mu, \nu, s)$ as a function of $(h, s) \in H(\mathbb{Q})\backslash H(\mathbf{A}) \times \mathbb{C}$, meromorphic in s but holomorphic for $Re(s) >> 0$; the normalizations are given in [PS,p. 270]. With this notation, the zeta integral $Z(f, \Phi, \mu, \nu, s)$ of (0.4) is defined.

We define an adelic character $\alpha_{\nu,\beta}$ of $R = DU$ by

$$\alpha_{\nu,\beta}(du) = \nu(d)\psi_\beta(u), d \in D(\mathbf{A}), u \in U(\mathbf{A}).$$

With this definition, and for $f \in \pi$, let[4]

$$W_f(g) = W_f^{\beta,\nu}(g) = \int_{Z_G(\mathbf{A})R(\mathbb{Q})\backslash R(\mathbf{A})} f(rg)\alpha_{\nu,\beta}^{-1}(r)dr. \tag{3.1}$$

The map $f \mapsto W_f$ is a $G(\mathbf{A})$-equivariant homomorphism from π to the space of functions W on $G(\mathbf{A})$ satisfying

$$W(rg) = \alpha_{\nu,\beta}(r)W(g). \tag{3.2}$$

If this map is non-zero, it is called a (β, ν)-Bessel model ([PS] refers to it as a generalized Whittaker model, but the terminology "Bessel model" appears in other articles of Piatetski-Shapiro and seems to be more widely used).

Let ξ_ν denote the restriction of ν to the ideles of $\mathbb{Q}$, embedded (diagonally) in the ideles of K. Let ξ_π denote the central character of π, also a Hecke character of $\mathbb{Q}$. If π has a global (β, ν)-Bessel model, then necessarily $\xi_\nu = \xi_\pi$. Moreover, each local component π_w has a local (β, ν_w)-Bessel model, i.e. a map ℓ_{β,ν_w} to the space of functions on $G(\mathbb{Q}_w)$ satisfying the analogue of (3.2). One knows (cf. Theorem 3.1 of [PS] and the references given there) that local Bessel models are unique. Thus if $f = \otimes f_w \in \pi = \bigotimes \pi_w$ is a factorizable function, we can factor $W_f = \otimes_w W_{f,w} = \otimes_w W_{f,w}^{\beta,\nu}$, and this gives rise to the Euler product factorization of the zeta integral (0.4). For details see [PS, §5], and the discussion below.

[4] As in [PS], we refrain from normalizing measures, only requiring that measures on groups over non-archimedean local fields take algebraic values. The formulas to follow only hold for consistent choices of measures. The interesting question is to normalize the archimedean measure in an arithmetically meaningful way, in connection with hypothesis (3.2.2) below. Since we do not calculate the archimedean zeta integral explicitly, we do not address this question.

(3.3) Hypotheses.

(3.3.1) *β is isotropic over $\mathbb{Q}$.*
(3.3.2) *μ is a character of finite order.*
(3.3.3) *π has a (β, ν)-Bessel model.*

These hypotheses are not all of the same nature. Hypotheses (3.3.1), already introduced in §1, and (3.3.2) carry no commitment, whereas (3.3.3) is an existence hypothesis.

The $\mathbb{Q}$-isotropic non-degenerate symmetric matrices β form a single conjugacy class under the adjoint action of the rational points of the Levi component of P. If π is a theta lift from maximally isotropic $O(4)$ or $O(6)$ then β is in the support of π for any β in this conjugacy class [R, (I) §3]. Such a π thus has a (β, ν)-Bessel model for some ν, but not necessarily the ones we introduce below.

(3.4) Arithmetic Eisenstein series.

By (3.3.1), we have Let $B \subset GL(2)$ be the standard Borel subgroup, and choose an Iwasawa decomposition $GL(2, \mathbf{A}_f) = B(\mathbf{A}_f)K_f$, with $K_f = \prod_p GL(2, \mathbb{Z}_p)$. We choose a pair of integers (k, γ) with $k \equiv \gamma \pmod 2$, $k > 0$, and a Dirichlet character $\bar{\mu}$, viewed as an adelic Hecke character of finite order, satisfying

$$\bar{\mu}_\infty(-1) = (-1)^k. \tag{3.4.1}$$

Define the character

$$\chi_{k,\gamma,\bar{\mu}} : B(\mathbf{A}) \to \mathbb{C}^\times;\ \chi_{k,\gamma,\bar{\mu}} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = |ad|^{\frac{k+\gamma}{2}} |d|^{-k} \bar{\mu}(d), \tag{3.4.2}$$

where $|\bullet|$ is the idèle norm. Let $I_{k,\gamma,\bar{\mu}}$ be the induced representation $Ind_{B(\mathbf{A})}^{GL(2,\mathbf{A})} \chi_{k,\gamma,\bar{\mu}}$ where here and below we work with *non-normalized* induction.

Write $K_{GL(2),\infty}$ for the stabilizer $Z_{GL(2)} \cdot SO(2)$ of the point i in the upper half-plane. Let $\phi \in I_{k,\gamma,\bar{\mu}}$ and write $\phi = \phi_\infty \otimes \phi_f$, and always assume ϕ_∞ to be $K_{GL(2),\infty}$-finite. By the Iwasawa decomposition for $GL(2, \mathbb{R})$, ϕ_∞ is determined by its restriction to $K_{GL(2),\infty}$, and can be written as a finite sum

$$\phi_\infty(t) = \sum_\kappa a_\kappa \kappa(t),$$

where $a_\kappa \in \mathbb{C}$ and κ runs through characters of $T(\mathbb{R})$ whose restriction to the center $K_{GL(2),\infty} \cap B(\mathbb{R})$ of $GL(2, \mathbb{R})$ coincides with $\chi_{k,\gamma,\bar{\mu}}$. Say ϕ is *pure* (of type κ) if ϕ_∞ is isotypic for character κ (i.e., $a_{\kappa'} = 0$ for $\kappa' \neq \kappa$). For each κ as above, the unique pure ϕ_∞ of type κ with $\phi_\infty(1) = 1$ is called a *canonical automorphy factor*, and is denoted ϕ_κ. For exactly one κ (namely $\kappa = k$, in an appropriate normalization; cf. [H2,§3]) ϕ_κ is a holomorphic automorphy factor; if ϕ is pure for this κ we call ϕ holomorphic.

Suppose $k > 2$. Then for any $K_{GL(2),\infty}$-finite function $\phi \in I_{k,\gamma,\bar{\mu}}$ we can define an Eisenstein series $E_{k,\gamma,\bar{\mu}}(\phi)$ on $GL(2, \mathbb{Q})\backslash GL(2, \mathbf{A})$ by the absolutely convergent formula

$$E_{k,\gamma,\bar{\mu}}(\phi, g) = \sum_{\alpha \in B(\mathbb{Q})\backslash GL(2,\mathbb{Q})} \phi(\alpha g).$$

When ϕ_∞ is a holomorphic automorphy factor then $E_{k,\gamma,\bar\mu}(\phi)$ is a holomorphic Eisenstein series (of classical weight k); we denote it $E_{k,\gamma,\bar\mu}(\phi_f)$ to stress that ϕ_∞ is fixed. When $k = 1$, one can define a holomorphic Eisenstein series $E_{k,\gamma,\bar\mu}(\phi)$ by analytic continuation, and when $k = 2$ one can still define $E_{k,\gamma,\bar\mu}(\phi)$ by analytic continuation, provided $\bar\mu$ is a non-trivial character, which we will henceforth assume for simplicity.

We say ϕ is *arithmetic* if ϕ_∞ is pure of type κ with $a_\kappa \in \overline{\mathbb{Q}}$, and if ϕ_f takes values in $\overline{\mathbb{Q}}$; we then say $E_{k,\gamma,\bar\mu}(\phi)$ is arithmetic (though not necessarily holomorphic). The following Lemma is a special case of the results used in §3 of [H2].

Lemma 3.4.3. *Suppose ϕ is arithmetic and holomorphic. Then, with normalizations as in [H2], the normalized holomorphic Eisenstein series $(2\pi i)^{\frac{k+\gamma}{2}} E_{k,\gamma,\bar\mu}(\phi_f)$ corresponds, as in §1, to a $\overline{\mathbb{Q}}$-rational section of the line bundle $F_{-k,-\gamma}$ on $Sh(GL(2))$.*

Write $\underline{k}$ for the triple $(k, \gamma_1, \bar\mu_1)$. For two pairs $(\underline{k}_1, \phi_1)$ and $(\underline{k}_2, \phi_2)$, with $\underline{k}_1 = (k, \gamma_1, \bar\mu_1)$, $\underline{k}_2 = (\ell, \gamma_1, \bar\mu_1)$, we let $E_{\underline{k}_1,\underline{k}_2}(\phi_{1,f}, \phi_{2,f})$ denote the restriction of $E_{\underline{k}_1}(\phi_{1,f}) \otimes E_{\underline{k}_2}(\phi_{2,f})$ to $H(\mathbf{A}) \subset GL(2,\mathbf{A}) \times GL(2,\mathbf{A})$. For any pair (r_1, r_2) of non-negative integers, we let

$$E^{(r_1,r_2)}_{\underline{k}_1,\underline{k}_2}(\phi_{1,f}, \phi_{2,f}) = D_k^{r_1} \otimes D_\ell^{r_2} E_{\underline{k}_1,\underline{k}_2}(\phi_{1,f}, \phi_{2,f}).$$

where $D_k^{r_1}$ and $D_\ell^{r_2}$ are the Maass operators on the first and second factors of $GL(2,\mathbf{A}) \times GL(2,\mathbf{A})$, respectively, normalized as in §1. Then $E^{(r_1,r_2)}_{\underline{k}_1,\underline{k}_2}(\phi_{1,f}, \phi_{2,f})$ is a (special value of a) real analytic Eisenstein series on H, which is *nearly holomorphic* in Shimura's sense – in other words, is contained in a representation generated by holomorphic representation of classical weight $(k+2r_1, \ell+2r_2)$ in the two variables. In any case, $E^{(r_1,r_2)}_{\underline{k}_1,\underline{k}_2}(\phi_{1,f}, \phi_{2,f})$ belongs to a representation of $H(\mathbf{A})$ isomorphic to the restriction of $Ind_{B(\mathbf{A})}^{GL(2,\mathbf{A})} \chi_{\underline{k}_1} \otimes Ind_{B(\mathbf{A})}^{GL(2,\mathbf{A})} \chi_{\underline{k}_2}$, independent of the choice of (r_1, r_2).

One can define arithmeticity for Eisenstein series on $H(\mathbf{A})$ corresponding to $K_{H,\infty}$-finite functions in (the restriction to $H(\mathbf{A})$ of) $I_{k,\gamma_1,\bar\mu_1} \otimes I_{\ell,\gamma_2,\bar\mu_2}$; holomorphy has already been defined.

Corollary 3.4.4. *Suppose $\phi_{1,f}$ and $\phi_{2,f}$ take values in $\overline{\mathbb{Q}}$. Then $(2\pi i)^{r_1+r_2} E^{(r_1,r_2)}_{\underline{k}_1,\underline{k}_2}(\phi_{1,f}, \phi_{2,f})$ is arithmetic.*

This follows from standard formulas for the action of Maass operators on canonical automorphy factors (cf. [S1]); the power of $2\pi i$ comes from our normalization of these operators.

Let B' denote the upper triangular subgroup of H. The integral representation (0.4) of $L(s, M(\pi_f), \mu)$, taking into account the shift (2.1), uses an Eisenstein series $E(h, \Phi, \mu, \nu, s)$ induced from the character

$$\chi_{s,\mu,\nu}\left(\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}\right) = \mu(a_1/d_2) \cdot |a_1/d_2|^{s-1} \cdot \nu_1^{-1}(d_1)\nu_2^{-1}(d_2) \tag{3.4.5}$$

of $B'(\mathbf{A})$, where we have the relation $a_1 d_1 = a_2 d_2$. Here μ, ν_1, and ν_2 are Hecke characters of $\mathbb{Q}^\times \backslash \mathbf{A}^\times$, satisfying the single relation

$$\nu_1 \cdot \nu_2 = \xi_\pi = \xi_{\pi,0} \cdot |\bullet|^{-c}. \tag{3.4.6}$$

We assume μ to be of finite order, as above, and write $\nu_i = \nu_{i,0} \cdot |\nu_i|$, $i = 1, 2$, where $\nu_{i,0}$ is of finite order and $|\nu_i(t)| = |t|^{\alpha_i}$ is a power of the norm; then (3.4.6) implies $\alpha_1 + \alpha_2 = -c$. The argument Φ belongs to the space $\mathcal{S}(V_{\mathbf{A}})$ of Schwartz-Bruhat functions on $V_{\mathbf{A}}$, where $V = (\mathbb{Q}^2)^2$ corresponding to the realization of H as a subgroup of $GL(2, \mathbb{Q}^2)$. In what follows, we let (x_i, y_i), $i = 1, 2$, denote the standard rational coordinates on V, so that the identity subgroup of the stabilizer T of the quadratic form $Q(v) = x_1^2 + y_1^2 + x_2^2 + y_2^2$, with $v = ((x_1, y_1), (x_2, y_2))$, is the stabilizer of the point (i, i) in the product of two upper half-planes. The Schwartz-Bruhat function Φ_∞ defines a $K_{H,\infty}$-finite Eisenstein series $E(h, \Phi, \mu, \nu, s)$, for our choice of $K_{H,\infty}$, provided it is of the form $P(v)e^{-\pi Q(v)}$, where $P(v)$ is a polynomial and $e^{-\pi Q(v)}$ is the standard Gaussian. More precisely, let I_K denote $\mathbf{A}^\times \times \mathbf{A}^\times$, as in [PS], and define

(3.4.7)

$$f^\Phi(h; \mu, \nu, s) = \mu(\det h)|\det h|^{s-1} \int_{I_K} \Phi((0, t)h)|t_1 t_2|^{s-1} \mu(t_1 t_2)\nu_1(t_1)\nu_2(t_2) d^\times t$$

as in [PS,§5], where $t = (t_1, t_2) \in \mathbf{A}^\times \times \mathbf{A}^\times$ and $|\bullet|$ is the idèle norm; we have incorporated the shift (2.1). It is then clear that $\Phi_\infty = e^{-\pi Q(v)}$ gives rise to a vector fixed by the maximal compact subgroup $K^c_{H,\infty}$ of $K_{H,\infty}$. More generally, $K_{H,\infty}$ acts linearly on the space of polynomials on $V(\mathbb{R})$, and if $P(v)$ is isotypic for $K_{H,\infty}$ then $\Phi_\infty = P(v)e^{-\pi Q(v)}$ yields a vector isotypic of the same type (for $K^c_{H,\infty}$). We let $\mathcal{E}(\chi_{s,\mu,\nu})$ denote the space of Eisenstein series of the form $E(h, \Phi, \mu, \nu, s)$.

We only consider Schwartz-Bruhat functions $\Phi(v_1, v_2)$ on $V(\mathbf{A}) = \mathbf{A}^2 \oplus \mathbf{A}^2$ that factor as $\Phi((v_1, v_2)) = \Phi_1(v_1)\Phi_2(v_2)$, and such that each Φ_i factors as $\prod_w \Phi_{i,w}$ over the places of $\mathbb{Q}$. It follows from (3.4.7) that

$$f^\Phi(1; \mu, \nu, s) = \int_{I_K} \Phi((0, t))|t_1 t_2|^{s-1} \mu(t_1 t_2)\nu(t) d^\times t$$

factors as a product of local Tate integrals

$$\begin{aligned} & \prod_{i=1}^{2} \prod_{w} \int_{\mathbb{Q}_w^\times} \Phi_{i,w}(0, t_{i,w})|t_{i,w}|^{s-1} \mu(t_{i,w})\nu_{i,w}(t_{i,w}) d^\times t_{i,w} \\ (3.4.8) \qquad & = \prod_{i=1}^{2} \prod_{w \in S} Z_w(\Phi_{i,w}, s, \mu_{i,w}, \nu_{i,w}) \cdot L^S(s-1, \mu \cdot \nu_i), \end{aligned}$$

where S is a finite set of bad primes, including the archimedean primes, $Z_w(\Phi_{i,w}, s, \mu_{i,w}, \nu_{i,w})$ is just the local factor on the first line of (3.4.8), and $L^S(s-1, \mu \cdot \nu_i)$ is the partial Dirichlet L-series.

(3.4.9). Set $\bar{\mu}_i = (\mu \cdot \nu_{i,0})^{-1}$, $i = 1, 2$. For any automorphic representation σ of $H(\mathbf{A})$ and any Hecke character ξ of $\mathbf{A}^\times$, we write $\sigma \otimes \xi$ for $\sigma \otimes \xi \circ \lambda$. We fix positive integers k and ℓ as above. Comparing (3.4.2) and (3.4.5), we obtain

Lemma 3.4.9.1. *Let $s = m$. For any (r_1, r_2), $E^{(r_1,r_2)}_{\underline{k}_1,\underline{k}_2}(\phi_{1,f}, \phi_{2,f})$ belongs to $\mathcal{E}(\chi_{m,\mu,\nu}) \otimes \mu^{-1}$ provided*

$$(3.4.9.2) \qquad m - 1 = k - \alpha_1 = \ell - \alpha_2 = \frac{k + \ell + \gamma_1 + \gamma_2}{2}.$$

Moreover, any holomorphic vector in $\mathcal{E}(\chi_{s,\mu,\nu})\otimes\mu^{-1}$ *is of the form* $E_{\underline{k}_1,\underline{k}_2}(\phi_{1,f},\phi_{2,f})$ *for some choice of* ϕ_1,ϕ_2.

The first part is a trivial computation, whereas the second part follows from the fact that the holomorphic subspace of the archimedean component of $Ind_{B'(\mathbf{A})}^{H(\mathbf{A})}\chi_{m,\mu,\nu}\otimes\mu^{-1}$ is of dimension one. Given s and ν,the weight (k,ℓ) of the holomorphic vector is determined by (3.4.9.2), as is the sum $\gamma_1+\gamma_2$; the individual γ_i are only visible on $GL(2)\times GL(2)$, and not on the subgroup H.

More precisely, it follows from (3.4.6) and (3.4.9.2) that

$$\gamma_1+\gamma_2=-(\alpha_1+\alpha_2)=c;\ \alpha_1=\frac{k-\ell-c}{2};\ \alpha_2=\frac{\ell-k-c}{2}. \tag{3.4.9.3}$$

This and the congruences

$$\gamma_1\equiv k \pmod 2, \gamma_1\equiv\ell \pmod 2 \tag{3.4.9.4}$$

are the only restrictions on our choices. In order to obtain nearly holomorphic Eisenstein series, we also need to suppose (3.4.1), which is equivalent to

$$\bar{\mu}_{i,\infty}(-1)=(-1)^{\gamma_i}, \tag{3.4.9.5}$$

As the reader will verify, (3.4.9.5) is compatible with (3.4.6).

Say the (factorizable) Schwartz-Bruhat function $\Phi((v_1,v_2))=\prod_{i=1}^2\prod_w\Phi_{i,w}(v_{i,w})$ is *arithmetic* if $\Phi_{i,\infty}$ is of the form $P_i(v_i)e^{-\pi Q_i(v)}$ with P_i a homogeneous polynomial of fixed degree h_i with $\overline{\mathbb{Q}}$ coefficients, and if $\Phi_{i,v}$ for finite primes v takes values in $\overline{\mathbb{Q}}$. It follows easily from (3.4.8), (3.4.9.5), and the classical formulas for special (critical) values of Dirichlet L-functions that, if Φ is arithmetic, then there is a constant c_∞^1, depending only on μ_∞, $\nu_{i,\infty}$, the h_i, and m, such that the Eisenstein series $E(h,\Phi,\mu,\nu,m)$ is arithmetic, in the sense introduced above (3.4.3). Indeed, c_∞^1 can be taken to be an integral power of $2\pi i$ (more precisely, a product of two integral powers of $2\pi i$, one coming from the factors $L^S(m-1,\mu\nu_i)$, the other coming from the archimedean zeta integrals), which can easily be determined explicitly. It's pointless to be more precise, though, since our final result will involve an archimedean zeta integral about which nothing is known.

Lemma 3.4.9.6. *Suppose the Schwartz-Bruhat function* Φ *is arithmetic, and suppose the Eisenstein series* $E(h,\Phi,\mu,\nu,m)$ *and as* $E_{\underline{k}_1,\underline{k}_2}^{(r_1,r_2)}(\phi_{1,f},\phi_{2,f})$ *for some pair of non-negative integers* r_1,r_2 *and (any) finite data* $\phi_{i,f}$ *are of the same* $K_{H,\infty}^c$*-type. Then under (3.4.9.2), there is a constant* $c_\infty^2(m)$*, depending only on* μ_∞*,* $\nu_{i,\infty}$*, the* h_i*, and* m*, such that* $c_\infty^2(m)E(h,\Phi,\mu,\nu,m)$ *corresponds to the image under* $D_k^{r_1}\otimes D_\ell^{r_2}$ *of a* $\overline{\mathbb{Q}}$*-rational section of the line bundle* $F_{(-k,-\ell,-c)}$ *on* Sh_H.

Proof. An easy consequence of Lemma 3.4.9.1 and Corollary 3.4.4; the twist by μ^{-1} in (3.4.9.1) has no effect on the rationality over $\overline{\mathbb{Q}}$.

(3.5) The main theorem.

(3.5.0) Hypotheses, recalled. As above, $a>b>0$ is a pair of positive integers, E_ρ^1 the automorphic vector bundle on Sh associated to the K_∞-type with highest weight $\Lambda_\rho^1=(a,-b-2,a+b)$, and π is a cuspidal automorphic representation of

G such that $H^2_!(Sh, E^1_\rho)[\pi_f]$ is of dimension one. Let μ be a Hecke character of $\mathbf{A}^\times/\mathbb{Q}^\times$ of finite order, and $\nu = (\nu_1, \nu_2)$ a pair of Hecke characters of $\mathbf{A}^\times/\mathbb{Q}^\times$, satisfying the relation (3.4.6). Let β be a $\mathbb{Q}$-isotropic symmetric 2×2-matrix with non-zero determinant; we assume π admits a (β, ν)-Bessel model. Various choices of ν will be made in the following discussion, depending on the special value in question and the sign of μ. Finally, the integers $k, \ell, \gamma_1, \gamma_2$, are chosen subject to the restrictions (3.4.9.3) and (3.4.9.4).

To motivate the main theorem, we first work it out in the special cases

(3.5.1(A)) $$\mu(-1) = 1,\ a - b \equiv 0 \pmod 4;$$

(3.5.1(B)) $$\mu(-1) = 1,\ a - b \equiv 2 \pmod 4.$$

In both cases $c = a+b$ is even, so that $\xi_{0,\pi} = \nu_{1,0} \cdot \nu_{2,0}$. Odd and even critical values are treated by two separate calculations: (i) $k \equiv \ell \equiv 1 \pmod 2$ and (ii) $k \equiv \ell \equiv 2 \pmod 2$. In case A(i) or B(ii), we choose $i = \frac{a+b}{2} + 1$, so that $a + 2 - i = i - b$, i.e. the middle of the range considered in Proposition 1.10.3, and choose $k = \ell$ and $r_1 = r_2$ subject to the hypotheses of that proposition.

(3.5.1 A(i)/B(ii)) $$1 \leq k = \ell \leq \frac{a-b}{2} + 1,\ 2r_1 = 2r_2 = \frac{a-b}{2} + 1 - k.$$

In case A(ii), we choose $i = \frac{a+b}{2}$, take $k = \ell$ and $r_2 = r_1 - 1$, so that

(3.5.1 A(ii)/B(i)) $$2 \leq k = \ell \leq \frac{a-b}{2},\ 2r_1 = 2r_2 - 2 = \frac{a-b}{2} - k.$$

In case A(i)/B(ii) (resp. A(ii)/B(i)) we choose γ_1 and γ_2 odd (resp. even) subject to (3.4.9.4), and we *fix* ν_1 and ν_2 satisfying (3.4.9.2) and (3.4.9.5); in any case the weights α_i are determined by (3.4.9.3). We let ν^{odd} (resp. ν^{even}) denote the fixed pair (ν_1, ν_2) in case A(i)/B(ii) (resp. A(ii)/B(i)). We define m by (3.4.9.2); then corresponding to (3.5.1) we have

(3.5.2 A(i)/B(ii)) $$m = \frac{a+b}{2} + k + 1 = \frac{a+b}{2} + 2, \frac{a+b}{2} + 4, \ldots a + 2;$$

(3.5.2 A(ii)/B(i)) $$m = \frac{a+b}{2} + k + 1 = \frac{a+b}{2} + 3, \frac{a+b}{2} + 5, \ldots a + 1.$$

The union of these two sets is precisely the right half of the critical set for $L(s, M(\pi_f, \mu))$, as determined in §2.

Assume Φ and $f \in \pi$ satisfy the hypotheses of Lemma 3.4.9.6 and Proposition 1.10.3, respectively, with the choices of $k, \ell, \ldots$ as above. In particular, f is identified with a class in $H^2_{cusp}(Sh, E^1_\rho) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}(\pi_f)$. Assume $f = \otimes_w f_w$ is factorizable, with f_w in the (abstract) representation π_w, as in the discussion preceding (3.3), and write $W_f(g) = \prod_w W^{\beta,\nu}_{f_w}(g_w)$, where W_f is the generalized "Bessel function" of type (β, ν) defined by (3.1). The function $W_{f_w}(g_w)$ can be defined as $\ell_{\beta,\nu}(\pi_w(g_w) f_w)$, where $\ell_{\beta,\nu,w}$ is a (fixed) Bessel functional on π_w, as defined above. Recall that f is of K_∞-type $(a+3,-b-1,c)$. This is the lowest K_∞-type τ_{π_∞} in π_∞, hence is of multiplicity one. Moreover, we have assumed f to be a weight vector for $T(\mathbb{R})$ with character $(a+2-i,i-b,c)$ – with the i just specified. The corresponding weight subspace $\tau_{\pi_\infty}(a+2-i,i-b,c)$ is of dimension one. We arbitrarily choose a non-zero vector $f_{\pi_\infty}(a+2-i,i-b,c) \in \tau_{\pi_\infty}(a+2-i,i-b,c)$, and let $W_{\pi_\infty}^{(a+2-i,i-b,c)}$ denote its image under the (also arbitrarily chosen) non-zero (β,ν_∞)-Bessel functional $\ell_{\beta,\nu,\infty}$.

Now it follows as in [BHR] that the representations π_p, for p finite, can all be realized over $\overline{\mathbb{Q}}$. Moreover, the local Hecke characters ν_p take algebraic values. It follows that the local Bessel functionals $\ell_{\beta,\nu,p}$ can be chosen to take $\overline{\mathbb{Q}}$-rational vectors in π_w to functions W_{f_p} in $C^\infty(G(\mathbb{Q}_p),\overline{\mathbb{Q}})$. Recall that we have defined $H^2_{cusp}(Sh,E^1_\rho)\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}(\pi_f) \subset H^2_{cusp}(Sh,E^2_\rho)\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}$ following (1.6), and that it is isomorphic to π_f. It then follows easily from the unicity of the Bessel model that

Proposition 3.5.2. *There exists a constant $a(\pi,\beta,\nu)\in\mathbb{C}^\times$, well-defined up to $\overline{\mathbb{Q}}^\times$-multiples, such that the global Bessel functional $f\mapsto W_f$ takes $H^2_{cusp}(Sh,E^1_\rho)\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}(\pi_f) \xrightarrow{\sim} \pi_f$ to the space of (β,ν)-Bessel functions on $G(\mathbf{A})$ of the form*

$$a(\pi,\beta,\nu)\cdot W_{\pi_\infty}^{(a+2-i,i-b,c)}\otimes W_{\pi_{fin}}$$

where $W_{\pi_{fin}}(g_f)\in\overline{\mathbb{Q}}$ for all $g_f\in G(\mathbf{A}_f)$.

The constant $a(\pi,\beta,\nu)$ is the occult period invariant of the title. Under Hypothesis (1.5) it depends only on π_f. Using the argument that follows, one can show that this remains true even without Hypothesis (1.5), provided there are non-vanishing special values in the critical range.

We can now explain the Euler factorization in (0.5). For $p\notin S$ we have arranged that Φ_p as well as W_{f_p} are standard unramified data and the local zeta integral is just the local Euler factor $L_p(s,M(\pi_f,\mu)) = L(s-\frac{3}{2},\pi_p,\mu,r)$. For $p\in S$ finite we have

$$Z_p(f,\Phi,\mu,\nu,s) = \int_{N_p\backslash H_p} W_{f_p}(h_p)\Phi_p((0,1)h_p)\mu(\det h_p)|\det h_p|^{s-1}dh_p \tag{3.5.3}$$

Here $H_p = H(\mathbb{Q}_p)$, $N_p = N_\beta(\mathbb{Q}_p)$, with N_β as in the introduction, and dh_p is a rational-valued Haar measure. The integral converges absolutely for $Re(s)$ sufficiently large, and extends analytically to a rational function of s, still denoted $Z_p(f,\Phi,\mu,\nu,s)$.

Lemma 3.5.4. *Suppose Φ_p is arithmetic and W_{f_p} takes algebraic values. Then for any integer m, $Z_p(f,\Phi,\mu,\nu,m)\in\overline{\mathbb{Q}}$. Moreover, for an appropriate choice of arithmetic data Φ and W_{f_p} we can arrange that $Z_p(f,\Phi,\mu,\nu,m)\in\overline{\mathbb{Q}}^\times$.*

Proof. The first assertion is proved by in [H2, Lemma 3.4.2]. The non-vanishing of the local zeta integral at m for some (not necessarily arithmetic) choice of data is implicit in Proposition 3.2 of [PS], and is proved by standard arguments. Since the arithmetic data define $\overline{\mathbb{Q}}$-structures on the Schwartz-Bruhat and Bessel spaces, and since the zeta integral is bilinear as a function of W_{f_p} and Φ_p, the second assertion then follows from the first.

Finally, we let

(3.5.5)

$$Z_\infty(f,\Phi,\mu,\nu,s) = \int_{N_\beta(\mathbb{R})\backslash H(\mathbb{R})} W_{\pi_\infty}^{(a+2-i,i-b,c)}(h_\infty)\Phi_\infty((0,1)h_\infty)\mu(\det h_\infty)|\det h_\infty|^{s-1}dh_\infty$$

With these choices, the Euler product in the form (0.5) follows from Proposition 3.5.2.

Combining (1.10.3), (3.4.9.6), (3.5.2), (3.5.4), and (0.5), we obtain our main theorem

Theorem 3.5.5. *Let m be in the right hand half of the critical set of the L-function $L(s, M(\pi_f, \mu))$. Suppose $\mu(-1) = 1$ and $c = a + b$ is even. There is a constant $c^3_\infty(m) \in \mathbb{C}^\times$, well defined up to $\overline{\mathbb{Q}}^\times$-multiples, with the following property. In case A(i)/B(ii), for m in the list (3.5.2) A(i)/B(ii), we have*

$$(c^3_\infty(m))^{-1} a(\pi, \beta, \nu^{odd}) Z_\infty(f, \Phi, \mu, \nu^{odd}, m) L(m, M(\pi_f, \mu)) \in \overline{\mathbb{Q}}.$$

In case A(ii)/B(i), for m in the list (3.5.2) A(ii)/B(i), we have

$$(c^3_\infty(m))^{-1} a(\pi, \beta, \nu^{even}) Z_\infty(f, \Phi, \mu, \nu^{odd}, m) L(m, M(\pi_f, \mu)) \in \overline{\mathbb{Q}}.$$

We have incorporated the constant $c(dh)$ of Proposition 1.10.3 into our new constant $c^3_\infty(m)$.

Remarks. Note that this theorem is roughly compatible with Deligne's conjecture, in that, up to the "elementary factor" $(c^2_\infty(m))^{-1} Z_\infty(f, \Phi, \mu, \nu^*, m)$, the special value is determined by the parity of m. Here and below $*$ denotes "odd" or "even". Of course this theorem is vacuous if (3.3.3) fails for all ν satisfying (3.4.9.4) and (3.4.9.5). Even if the appropriate Bessel model exists, the theorem is still vacuous if our normalized archimedean zeta factor $Z_\infty(f, \Phi, \mu, \nu^*, m)$ vanishes. It should not be too difficult to determine at least whether the non-holomorphic discrete series π_∞ has non-vanishing (β, ν_∞) Bessel models, and then the calculation of the archimedean zeta factor should not be too taxing. We suspect the theorem is not vacuous, because of the formal fit with Deligne's conjecture, but we have not carried out the necessary archimedean calculations, and we have nothing to say about the global hypothesis (3.3.3).

The product $a(\pi, \beta, \nu^*) Z_\infty(f, \Phi, \mu, \nu^*, m)$ does not depend on the choice of the vector $W^{(a+2-i, i-b, c)}_{\pi_\infty}$ (of given K_∞ and T-type) in the (β, ν_∞)-subspace, but one expects there is a natural choice for which $Z_\infty(f, \Phi, \mu, \nu^*, m)$ is an algebraic multiple of some power of π for the indicated values of m. Then $a(\pi, \beta, \nu^*)$ should be directly related to $c^\pm(M(\pi_f))$, where the relation of $*$ to $\pm$ depends on the parity of $\frac{a+b}{2}$.

(3.5.6) The remaining cases. We now assume $c = a + b$ odd, so $w = c + 3$ is even. As above, we distinguish two cases:

(3.5.6.1(C)) $$\mu(-1) = 1, \ a - b \equiv 1 \pmod 4;$$

(3.5.6.1(D)) $$\mu(-1) = 1, \ a - b \equiv 3 \pmod 4.$$

It is then natural to choose $i = \frac{a+b+3}{2}$, so that $a + 2 - i = i - b - 1$, one of two points closest to the middle of the range of Proposition 1.10.3. Then k and ℓ necessarily have opposite parity: k is odd in case (C) and even in case (D). Again there are two calculations, according as (i) $k = \ell - 1$ or (ii) $k = \ell + 1$. We have
(3.5.6.1 C(i))

$$(k, \ell) = (1, 2), (3, 4), \ldots, (\frac{a-b+1}{2}, \frac{a-b+3}{2}), \quad 2r_1 = 2r_2 = \frac{a-b+1}{2} - k;$$

(3.5.6.1 D(i))
$$(k,\ell)=(2,3),(4,5),\dots,(\frac{a-b+1}{2},\frac{a-b+3}{2}),\quad 2r_1=2r_2=\frac{a-b+1}{2}+1-k;$$

(3.5.6.1 C(ii))
$$(k,\ell)=(3,2),(5,4),\dots,(\frac{a-b+1}{2},\frac{a-b-1}{2}),\quad 2r_1=2r_2-2=\frac{a-b+1}{2}-k;$$

(3.5.6.1 D(ii))
$$(k,\ell)=(2,1),(4,3),\dots,(\frac{a-b+1}{2},\frac{a-b-1}{2}),\quad 2r_1=2r_2-2=\frac{a-b+1}{2}+1-k.$$

In case C(i)/D(i) (resp. C(ii)/D(ii)) we have $(\alpha_1,\alpha_2)=(\frac{-1-c}{2},\frac{1-c}{2})$ (resp. $(\alpha_1,\alpha_2)=(\frac{1-c}{2},\frac{-1-c}{2})$). We fix ν consistent with these values of α_i and satisfying (3.4.9.5), as before, and denote them $\nu^{(i)}$ and $\nu^{(ii)}$, respectively. Defining m by (3.4.9.2), we obtain

(3.5.6.2 C(i))
$$m=k+1-\alpha_1=\frac{a+b+3}{2}+1,\frac{a+b+3}{2}+4,\dots,a+2;$$

(3.5.6.2 D(i))
$$m=\frac{a+b+3}{2}+2,\frac{a+b+3}{2}+5,\dots a+2;$$

(3.5.6.2 C(ii))
$$m=\frac{a+b+3}{2}+2,\dots a+1;$$

(3.5.6.2 D(ii))
$$m=\frac{a+b+3}{2}+1,\dots a+1;$$

The analogue of Proposition 3.5.2 remains true, under the hypotheses (3.3), and we conclude

Theorem 3.5.7. *Let m be in the right hand half of the critical set of the L-function $L(s,M(\pi_f,\mu))$. Suppose $\mu(-1)=1$ and $c=a+b$ is odd. In case C(i)/D(i), for m in the corresponding lists (3.5.6.2), we have*

$$(c^3_\infty(m))^{-1}a(\pi,\beta,\nu^{(i)})Z_\infty(f,\Phi,\mu,\nu^{(i)},m)L(m,M(\pi_f,\mu))\in\overline{\mathbb{Q}}.$$

In case C(ii)/D(ii), for m in the corresponding lists (3.5.6.2), we have

$$(c^3_\infty(m))^{-1}a(\pi,\beta,\nu^{(ii)})Z_\infty(f,\Phi,\mu,\nu^{(ii)},m)L(m,M(\pi_f,\mu))\in\overline{\mathbb{Q}}.$$

(3.6) Period relations and the case $\mu(-1)=-1$. We note first that Theorems 3.5.5 and 3.5.6.3 make no reference to the finite part of the character ν. In other words, assuming there are non-vanishing special values, we can wrote $a(\pi,\beta,\nu)=a(\pi,\beta,\nu_\infty)$ for the ν_∞ in question, which in turn is determined by the signs of $\bar{\mu}_i/\mu$ and the pair (α_1,α_2). Presumably there are refinements, in which $a(\pi,\beta,\nu)$ is determined up to $\mathbb{Q}$ rather than $\overline{\mathbb{Q}}$, which would be sensitive to the full character ν.

We have chosen to assume $|k-\ell| \leq 2$, with i close to the center of the available range, in order to cover the largest possible number of special values. This choice determines ν_∞ via (3.4.9.2) and (3.4.9.5). However, we can repeat the above argument, using other values of i, or using other sequences of pairs (k, ℓ) satisfying the appropriate congruence conditions. For example, in case A(ii)/B(i) we can take $\ell = k+2$. This leads to a different value for ν_∞, hence to a relation between the corresponding $a(\pi, \beta, \nu)$ (assuming they do not vanish). We have no interpretation to propose for this phenomenon.

The characters ν_∞ used in cases A(i)/B(ii) and in A(ii)/B(i) differ only in the signs. In both cases $k = \ell$ and $\alpha_1 = \alpha_2 = -\frac{c}{2}$; however $\nu_{i,\infty}(-1) = -1$ in cases (A(i)/B(ii)), whereas $\nu_{i,\infty}(-1) = 1$ in cases A(ii)/B(i). Now suppose $\mu(-1) = -1$. Then (3.4.9.5) requires that the signs change; i.e. that $\nu_{i,\infty}(-1) = 1$ (resp. $= -1$) in cases A(i)/B(ii) (resp. in cases A(ii)/B(i)). We leave it to the reader to verify, using (2.5), that this is completely consistent with what is predicted by Deligne's conjecture, and to check the cases (C) and (D).

References

[An] A. N. Andrianov, Zeta functions and the Siegel modular forms, Proc. Summer School Bolyai-Janos Math. Soc., Budapest (1970), Halsted, N.Y. (1975), 9-20.

[B] D. Bertrand, Endomorphismes de groupes algébriques; applications arithmétiques, Birkhaüser, *Progress in Math.*, **31** (1983), 1-45.

[BHR] D. Blasius, M. Harris, D. Ramakrishnan, Coherent cohomology, limits of discrete series, and Galois conjugation, *Duke. Math. J.*, **73** (1994), 647-685.

[D] P. Deligne, Valeurs de fonctions L et périodes d'intégrales, *Proc. Symp. Pure Math.*, **XXXIII**, part 2 (1979), 313-346.

[FS] M. Furusawa, J. Shalika, The fundamental lemma for the Bessel and Novodvorsky subgroups of $GSp(4)$ I, *C.R.A.S. Paris*, **328** (1999) 105-110; II, *C.R.A.S. Paris*, **331** (2000), 593-598.

[GRS] D. Ginzburg, S. Rallis, and D. Soudry, On explicit lifts of cusp forms from GL_m to classical groups, *Ann. of Math.* **150** (1999), 807-866.

[G] L. Grenié, Valeurs spciales de fonctions L de $GL(r) \times GL(r)$, Thesis, Université Paris 7 (January, 2000).

[H1] M. Harris, Automorphic forms of $\bar{\partial}$-cohomology type as coherent cohomology classes, *J. Diff. Geom.*, **32** (1990), 1-63.

[H2] M. Harris, Period invariants of Hilbert modular forms, I, *Lect. Notes Math.*, **1447** (1990), 155-202.

[HK] M. Harris, S. S. Kudla, Arithmetic automorphic forms for the nonholomorphic discrete series of $GSp(2)$, *Duke Math. J.* (1992), 59-121.

[Hida] H. Hida, On the critical values of L-functions of $GL(2)$ and $GL(2) \times GL(2)$, *Duke Math. J.*, **74** (1994), 431-529.

[KS] H. H. Kim, F. Shahidi, Functorial products for $GL_2 \times GL_3$ and the symmetric cube for GL_2, *Ann. of Math.*, **155** (2002), 837-893.

[MT] A. Mokrane, J. Tilouine, Cohomology of Siegel varieties with p-adic integral coefficients and applications, Prépublications Mathématiques de l'Université Paris 13, Fév. 2000.

[PS] I.I. Piatetski-Shapiro, L-functions for GSp_4, *Pacific. J. Math.*, Olga Taussky-Todd memorial issue (1997), 259-275.

[R] S. Rallis, On the Howe duality conjecture, *Compositio Math.*, **51** (1984) 333-394.

[S1] G. Shimura, The special values of the zeta functions associated with cusp forms, *Comm. Pure Appl. Math.*, **29** (1976), 783-804.

[S2] G. Shimura, On a class of nearly holomorphic automorphic forms, *Ann. of Math.*, **123** (1986), 347-406.