COHOMOLOGICAL AUTOMORPHIC FORMS ON UNITARY GROUPS, II: PERIOD RELATIONS AND VALUES OF L-FUNCTIONS

Michael Harris

UFR de Mathématiques
Université Paris 7
2 Pl. Jussieu
75251 Paris cedex 05, FRANCE

To Roger Howe

Introduction

This is the fourth in a series of articles devoted to the study of special values of L-functions of automorphic forms contributing to the cohomology of Shimura varieties attached to unitary groups, for the most part those attached to hermitian vector spaces over an imaginary quadratic field $\mathcal{K}$. Since these L-functions are all supposed to be motivic, all their values at integer points are conjectured to be arithmetically meaningful. For the finite set of integer points which are *critical* in the sense of Deligne,[1] the values are predicted by Deligne's conjecture [D] to be algebraic multiples of certain determinants of periods of algebraic differential forms against rational homology classes. One of the main results of [H3] is that the Deligne period of the polarized motives associated to the standard L-functions of a self-dual cohomological automorphic representation π of a unitary group G of signature $(n-1,1)$ can be expressed up to rational multiples as products of square norms of arithmetically normalized Dolbeault cohomology classes on the Shimura variety attached to G. The other main result of [H3] is an expression of the critical values *in the range of absolute convergence* in terms of Petersson norms of (arithmetically normalized) holomorphic automorphic forms, nearly equivalent to π but realized in general on unitary groups of different signatures. The article [H5] removes the restriction to the range of absolute convergence in the most important cases, corresponding to the existence of non-trivial theta lifts to larger unitary groups.[2]

Deligne's conjecture is thus reduced to a series of conjectural relations among the Petersson norms of cohomological forms in nearly equivalent automorphic representations of unitary groups of varying signature. The article [H4], to which this article

Institut des Mathématiques de Jussieu, U.M.R. 7586 du CNRS. Membre, Institut Universitaire de France.

[1] The same set had been identified independently by Shimura, though without the motivic interpretation.

[2] Shimura has obtained similar results in [S1] and [S2], by rather different methods.

is an immediate sequel, studied the rationality properties of the theta correspondence relating these nearly equivalent automorphic representations to one another, and to similar automorphic representations on larger unitary groups. The final section of [H4] explained how to deduce Deligne's conjecture from a collection of hypotheses of supplementary results that had not yet been verified. The hypotheses had mostly to do with the existence of stable base change from G to $GL(n, \mathcal{K})$, and are reproduced below as Hypothesis 4.1.1 and 4.1.10. The supplementary results fall into three categories:

(i) Extension of the main theorems of [H3] beyond the range of absolute convergence, now accomplished in [H5], under a hypothesis on the non-triviality of theta lifts considered in the present article;
(ii) Determination up to rational factors of a variety of archimedean zeta integrals, which we have not considered; and
(iii) An extension of the Rallis inner product formula, which relates square norms of automorphic forms to square norms of their theta lifts and special values of standard L-functions.

The main contributions of the present article are the the analysis of local and global obstructions to the non-triviality of the theta correspondence, used to verify (i) and the verification of a version of (iii). This completes the proof of Deligne's conjecture for the L-functions considered in [H3], up to determination of the archimedean zeta integrals, and under certain genericity hypotheses. More precise version of our main results are provided in §4.2.

Strictly speaking, the results of [H3] and [H5] only concern a slight weakening of Deligne's conjecture, in which determination up to rational multiples is replaced by determination up to $\mathcal{K}^{\times}$-multiples. When Deligne's conjecture is mentioned in the article we will understand it to mean this weakened version.

We write $U(n)$ to designate the unitary group of a hermitian vector space of dimension n over $\mathcal{K}$, regardless of signature or other local properties. The main object of study in the present article is the theta correspondence between automorphic representations of $U(n)$ and $U(n+1)$; we prove that any tempered cohomological automorphic representation π of $U(n)$ of holomorphic type admits a non-trivial lift (for some choice of splitting character, see below) to a *positive-definite* $U(n+1)$, provided π_∞ is sufficiently general (the minimal K-type of π_∞ is not too small). The condition on π_∞ is quite mild; for example, for $n = 2$, it applies to holomorphic modular forms of weight $k > 2$. Actually, the same methods work, and more easily, to show that π lifts to a positive-definite $U(n+r)$ for any $r > 0$, but the archimedean condition excludes more π as r grows, and the theta lift of π becomes more pathological; for example, for tempered π, the theta lift is only tempered for $r = 0, 1$. Nevertheless, this technique has potentially interesting applications to the construction of non-trivial extensions of Galois representations, a topic to which I will return in a subsequent paper.

We are only concerned with π for which π_∞ belongs to the discrete series, It is then conjectured that π is globally tempered; this is verified under a local hypothesis in [HL], and in general should be a consequence of the stabilization of the trace formula for $U(n)$ and the related calculation of the zeta functions of the associated Shimura varieties. We also assume π admits a base change to a cuspidal automorphic representation of $GL(n, \mathcal{K})$. There are both local and global obstructions to non-triviality of the theta lift of tempered π to $U(n+r)$, where temporarily we allow

$r = 0$. We first note that the theta lift for unitary groups depends on the choice of an additive character ψ, as usual, and also on a pair of multiplicative characters (*splitting characters*) χ, χ', whose properties are recalled in §2.1. The local obstructions for $r = 0$ are studied in detail in [HKS]. For a given prime v there are two possible local $U(n)_v$, denoted $U^{\pm}$, where we bear in mind that what counts for the theta correspondence is not just the isomorphism class of $U(n)$ but also that of the hermitian space to which it is attached. In [HKS] we conjectured that exactly one of these $U^{\pm}$ admitted a non-zero local lift from a given π_v. This *dichotomy conjecture* was proved in [HKS] for most π_v and was completed by a conjecture relating the sign of the lucky $U^{\pm}$ to a local root number constructed from π_v as well as χ_v, χ'_v, and ψ_v. In this paper we assume (cf. (2.1.5) that the weak Conservation Relation proved in [KR] for orthogonal/symplectic dual reductive pairs is also valid for unitary dual reductive pairs, and show that this easily implies the dichotomy conjecture. By making a judicious choice of splitting characters, we can thus show that the local obstruction is trivial for $r = 0$, with no condition on π_∞ when the latter is holomorphic. Unfortunately, the global obstruction is the vanishing of the central value of the standard L-function. It is expected that, up to twisting by a character of finite order, one can guarantee that this value is non-zero, but this is not known for $n > 2$. Thus we cannot use the theta lift from $U(n)$ to (positive-definite) $U(n)$ to obtain period relations by means of the Rallis inner product formula. We note, however, that there is no global obstruction for $r > 0$, because of our hypothesis on base change and because the L-function of a cusp form on $GL(n, \mathcal{K})$ does not vanish to the right of the critical line. We also show that, up to replacing $U(n+r)$ by an inner form at some set of finite primes, there is also no local obstruction for $r > 1$ when π is tempered. Our first main technical result (Corollary 3.3) is then that for π tempered, there is always a non-trivial theta lift to some definite $U(n+r)$ when $r \geq 2$. This allows us to apply the results of [H5] to verify (i) above for all critical values strictly to the right of the critical line.

Our next main technical result (Theorem 3.4) considers the lifting from $U(n)$ to (definite) $U(n + 1)$. Here the global obstruction is still trivial, and one can eliminate the local obstruction at the price of replacing the splitting characters. This extends the application of [H5], and thus (i) above to the critical line itself, for most character twists; this is sufficient for applications to period relations. The results of [H5], reproduced below as Theorem 3.2, are all based on Ichino's extension of the Siegel-Weil formula beyond the range of absolute convergence [I1,I2]. These also suffice for application of Theorem 3.2 to critical values at the center of the critical strip. However, they do not suffice for the study of the value at the near central point (corresponding in the unitary normalization to the value at $s = 1$) for π with trivial lift to all definite $U(n+1)$; this is because, like the results of Kudla and Rallis for orthogonal/symplectic pairs, they say nothing about the special values of Eisenstein series that are *incoherent* (i.e., not obtained by the Siegel-Weil formula for purely local reasons).

The final section is concerned with applications to Deligne's conjecture. Section 4.1 reconsiders the hypotheses made in somewhat excessive generality[3] in the final section of [H4]. In light of the results of §3, most of these hypotheses can now be considered theorems, at least in slightly weaker versions that still suffice for the arguments of [H4]. Section 4.2 derives the applications to period relations and

[3]Cf. Section E.3 at the end of the introduction.

Deligne's conjecture on critical values. In the interests of complete disclosure, I mention that the list of hypotheses has been lengthened: in particular there is a multiplicity one hypothesis and an assumption that the automorphic representation π is tempered (cf. 4.1.4). Both of these hypotheses are surely unnecessary but their inclusion shortens some of the arguments considerably. More to the point, I have no doubt they are not only true but will be verified in the near future, so I see no reason not to use them.

Much of the local theory presented in §2 is an outgrowth of my article [HKS] with Kudla and Sweet, and I thank Steve Kudla for numerous discussions of this material over the years. Although he is not listed as an author of this paper, I would like to make it clear that I consider him equally responsible for the dichotomy theorem (Theorem 2.1.7). Matthieu Cossutta read an earlier version of the manuscript carefully and pointed out misprints and one substantial error. I also thank Jian-Shu Li for providing references for the erratum E.1, Guy Henniart and A. Minguez for informing me of the results of the latter's thesis in preparation [M], and Erez Lapid for bringing the results of [LR] to my attention. I am particularly grateful to A. Ichino for keeping me informed of his work on the extended Siegel-Weil formula, and for writing up some of his results in [I2] that were essential to the present project.

The influence of Roger Howe's approach to theta functions as a means for relating automorphic forms on different groups on every line of the present paper is too obvious to require explanation. This article, and its author, have a subtler debt to Roger as well: by pointing out the hidden premises to a question I asked him at Oberwolfach in 1985 he convinced me of the possibility of a viable theory of coherent cohomology of Shimura varieties combining arithmetic and analytic techniques; this observation structures the main calculations in [H4]. For these and many other reasons, it is a great pleasure to dedicate the present paper to Roger Howe.

Errors and misprints in [H4]

While preparing the present article I discovered a substantial number of misprints in [H4]; these are listed at the end of this section. There are unfortunately also some mathematical errors.

E.1. The least important is explained in the following paragraph, that was inadvertently omitted from [H4] (a "Note added in proof" is announced on p. 108 but the text is missing).

Note that should have been added in proof in [H4]. The argument in IV. 5. is incomplete. Specifically, the first paragraph of the proof of Lemma IV.5.15 implicitly assumes, without proof, that small automorphic representations have multiplicity one in the automorphic spectrum. It is possible to prove this multiplicity one assertion using results of J.-S. Li on singular forms.

The results of Li to which the previous paragraph refers are contained in [L1, L2]. More precisely, the small automorphic representations π of [H4] have the property that the local components π_v are singular, and in fact of rank 1, for all places v; indeed, π is constructed as a global theta lift of a character, say η, of $U(1)$, and each π_v is thus a local theta lift from $U(1)$. Theorem A of [L2] then asserts that the multiplicity of π in the space of automorphic forms on $U(n)$ is bounded by the automorphic multiplicity of the character η on $U(1)$, which is obviously 1.

E.2. More serious is that the proof of Theorem V.1.10 of [H4], which is the main theorem, is apparently inadequate. The problem is that the oscillator representation depends on the choice of additive character, specifically at finite primes, and therefore the final rationality result needs to take this into account. I thought I had found a trick to avoid this problem (see [H4, p. 167]) but it now seems that it returns in the induction step. Specifically, Lemma IV.2.3.2 of [H4] (rationality of non-archimedean zeta integrals for $U(1)$) is expressed in terms of the rationality of local sections defining Eisenstein series, whereas Theorem V.1.10 is expressed in terms of rationality of local Schwartz-Bruhat functions, which are related to the Eisenstein series via the Siegel-Weil formula. The map denoted $\phi \mapsto \mathcal{F}_\phi$, defined in §I.2, is rational over the maximal cyclotomic extension of $\mathbb{Q}$ but not necessarily over $\mathbb{Q}$. The two kinds of rationality can be related, but (apparently) not on both sides of a seesaw, and so I can only claim the following weaker version of Theorem V.1.10:

Theorem E.2.1 (Theorem V.1.10 of [H4], corrected). *Notation is as in* [H4, §V.1]. *Let* $\alpha \in H_!^{2(r'-1)}(Sh(V^{(2)}, [W_{\Lambda(2)}])$ *be a coherent cohomology class rational over the number field L. Let* $\mathcal{F}$ *and* $F \in \pi(2)$ *be as in the statement of* [H4, V.1.10]. *Let* $\varphi \in \mathcal{S}(\mathbb{X}(\mathbf{A}_f))$ *be an algebraic Schwartz-Bruhat function, rational over* $\mathbb{Q}^{ab}(\pi, \chi^+, \chi'^{,+})$*, with* φ_∞ *determined by* [H4, IV (4.6.ii)]. *Then*

$$\mathbf{p}(\pi, V, V', \chi_0)^{-1}\underline{Lift}^{-1}(\theta_{\varphi,\chi,\chi',\psi_f}(F))$$

is a $\mathbb{Q}^{ab}(\pi, \chi^+, \chi'^{,+})$*-rational class in* $H^0(Sh(V'^{,(2)}), [E_{\Lambda(2)}])$. *Moreover, for any* $\sigma \in Gal(\overline{\mathbb{Q}}/\mathcal{K})$*, we have*

$$[\mathbf{p}(\pi, V, V', \chi_0)^{-1}]\underline{Lift}^{-1}(\theta_{\sigma(\varphi),\sigma(\chi),\sigma(\chi'),\sigma(\psi_f)}(\sigma(F))) \\ = \sigma(\mathbf{p}(\pi, V, V', \chi_0)^{-1}\underline{Lift}^{-1}(\theta_{\varphi,\chi,\chi',\psi_f}(F))).$$

The action of σ on χ and χ' is actually defined by means of the actions on the algebraic Hecke characters χ^+ and $\chi'^{,+}$. The difference with the statement in [H4] is that the additive character has been incorporated into the notation for the theta lift, and is affected by the Galois action. This only becomes relevant in the study of non-archimedean zeta integrals, specifically in Lemma IV.2.3.2, on p. 171 (just above formula (5.5)), and again in V. (3.13); see Lemma 3.5.12, below. The map $\phi \mapsto \mathcal{F}_\phi$ is Galois equivariant in the following sense: if we incorporate ψ_f into the notation in the obvious way, then for any $\phi \in \mathcal{S}(\mathbb{X}(\mathbf{A}_f), \overline{\mathbb{Q}})$ (the Schwartz-Bruhat space, as in §I.2)

$$\sigma(\mathcal{F}_{\phi,\psi_f}) = \mathcal{F}_{\sigma(\phi),\sigma(\psi_f)}, \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q}). \tag{E.2.2}$$

With this in mind, the argument of Lemma IV.2.3.2 takes care of the non-archimedean zeta integrals, and the rest of the proof is unaffected.

The notation $\sigma(\psi_f)$ is actually abusive because it is not generally the finite part of a global additive character. Thus

$$\theta_{\sigma(\varphi),\sigma(\chi),\sigma(\chi'),\sigma(\psi_f)}(\sigma(F)) \tag{E.2.3}$$

requires explanation. In fact, the dependence on ψ_f of the theta kernel attached to φ, or to $\sigma(\varphi)$, is in fact determined by the values of ψ_f on an open compact subgroup of $\mathbf{A}_f$; this is obvious if one rewrites the theta kernel in classical language. In other words, the theta kernel only depends on the classical additive character modulo N attached to ψ_f, where $N = N(\varphi)$ depends on φ. Call this finite additive character $\psi_{N(\varphi)}$. To define the theta kernel for $\sigma(\varphi)$ attached to $\sigma(\psi_f)$, one lets $\sigma(\psi,\phi)$ be any additive character whose associated character mod $N(\varphi)$ is $\sigma(\psi_{N(\varphi)})$. We will then have $\sigma(\psi)_\infty \neq \psi_\infty$, but this change is invisible because the archimedean Schwartz function φ_∞ (defined by [H4, IV.(4.6.ii)] – this has been added to the statement of Theorem E.2.1, and should have been recalled more clearly in [H4,§V.1]) is defined in terms of its behavior under the local theta correspondence, hence in terms of the choice of ψ_∞. In particular, changing ψ_∞ has no effect on the local zeta integral attached to φ_∞. This correction requires us to relax our hypothesis on ψ_∞ (cf. [H4, p. 137, p. 162]).

E.2.4. *The additive character ψ_∞ is assumed to be of the form $\psi_\infty(x) = e^{2\pi i\alpha x}$ for some $\alpha \in \mathbb{Q}^\times$, $\alpha > 0$.*

I stress that, for applications to period relations, Theorem E.2.1 is just as good as the incorrect version Theorem V.1.10 stated in [H4]! The main induction step is sketched in [H4, §V.3], under a series of hypotheses, most of which are verified in the present paper. At the end, instead of the relation (3.28.2), one obtains the following version, which we write out at length:

$$A(r',\pi_0) = \frac{P^{(r')}(\pi_0)}{\prod_{j=1}^{r'} Z_{\infty,j}Q_j(\pi_0)} \in \mathbb{Q}^{ab}(\pi_0,\beta^{(2)},\chi_0^+); \qquad \text{(E.2.5)}$$

$$\forall\sigma \in Gal(\overline{\mathbb{Q}}/\mathcal{K}), \sigma(A(r',\pi_0)) = A(r',\sigma(\pi_0)). \qquad \text{(E.2.6)}$$

Note that ψ does not appear in (E.2.6) or the left-hand side of (E.2.5) – no more than β or χ_0^+. Thus, assuming the hypotheses of [H4, §V.3], and assuming one can eliminate the $Z_{\infty,j}$, (E.2.6) suffices to imply the Deligne conjecture (up to $\mathcal{K}^\times$), as explained in [H3,H4].

E.3. Finally, and this is perhaps the main point of the present paper, for the moment some of the hypotheses used in the sketch of the proof of period relations in [H4,§V.3] can only be verified under slightly restrictive conditions. Specifically, Hypotheses (3.10)(b) and (3.20)(a) of [*loc. cit.*] assert that the relations between special values of the standard L-function of a cuspidal automorphic representation π of holomorphic type on a unitary group and Petersson norms of arithmetic holomorphic forms in π remain valid down to the center of symmetry of the functional equation; in [H3] these relations were proved in the range of absolute convergence of the integral representation of the L-function. For applications, we are most interested in the special values at the point $s = 1$ in the unitary normalization; this is a critical value under an additional hypothesis on the infinity type (corresponding to (3.20)(c) of [*loc. cit.*]). If π is an automorphic representation of some form of $U(n)$, then the results of [H5] allow us to prove this under a set of hypotheses that imply that some theta lift of π to a definite $U(n+1)$ is non-trivial. When I wrote [H4, §V.3] I was under the impression that it was sufficient to show non-triviality of the local theta correspondences everywhere, since the Rallis inner product formula would then imply that the global theta lift is non-zero provided the L-function was non-vanishing at $s = 1$, and this was guaranteed by a standard hypothesis on base change to $GL(n)$, included here as Hypothesis 4.1.1 (in [H4] there was only a hypothesis on non-vanishing, cf. V. (3.10)(a)). However, the Rallis inner product formula only applies when the local theta correspondences patch together into a theta correspondence with a global $U(n+1)$. In general there is a parity obstruction, and the correct assertion is Theorem 3.4 of the present paper.

The relevant modifications to the hypotheses in question are explained in §4.1. It should be added that there are numerous misprints in [H4, §V.3], especially in the crucial section 3.23, almost all of which were undoubtedly in my original manuscript. Those I have found are corrected below.

Misprints and omissions in [H4].

p. 113, (1.13) The formula for the local integral in [H4] followed [HKS] in omitting a factor $\chi^{-1}(det(g'))$. This is corrected in (1.3.3) below. The missing factor plays no role in any arguments.

p. 177, (1.5): $E_{\Lambda(2)}$ should be $E_{\Lambda'(2)}$.

p. 178, Theorem 1.10: The number field L was introduced in the beginning of the statement but forgotten in the conclusion. It should be assumed that $L \supset \mathcal{K}$. On the sixth line of the statement, $\mathbb{Q}(\pi, \chi^+, \chi^{',+})$ should be $L \cdot \mathbb{Q}(\pi, \chi^+, \chi^{',+})$; on the next line, $Gal(\overline{\mathbb{Q}}/\mathcal{K})$ should be $Gal(\overline{\mathbb{Q}}/L)$.

p. 187, line 9: $P^{(r'),*}(\pi_0, \beta)$ should be just $P^{(r'),*}(\pi_0)$

p. 190, (3.23.1): The indices on the β_i (resp. γ_j) should run from 1 to s (resp. 1 to r.

p. 190, line -4: The condition $\mu - \frac{m-1}{2} > \frac{1}{2}$ should be $\mu - \frac{m-v}{2} \geq 0$, translating the condition from 3.20(a) that μ be strictly to the right of the center of symmetry of the L-function.

p. 191, (3.23.2): The second inequality should be $-b_s - \frac{1}{2}(n-2-\alpha(\chi')) \leq -\mu - v$.

p. 191, line 11: The congruence should be $v - \alpha(\chi) \equiv n - \alpha(\chi') - 3 \pmod 2$, which implies the relation $v \equiv m \pmod 2$ asserted on the following line.

p. 191, Lemma 3.23.4. $L^{mot}(\mu, \pi', \nu^\iota, St)$ should be $L^{mot}(\mu, \pi', \check{\nu}, St)$.

p. 192, (3.23.5): The last inequality on the first line should be $\geq n - 2p_{r'} - (2k - \alpha(\chi)) + 1$.

p. 193, (3.26.2). The argument should be $\frac{1}{2}(2 - v + m - n - 1) - j$.

p. 194, (3.26.7) On the left hand side, ν^ι should be $\check{\nu}^\iota$. On the right-hand side, $P((\chi^{+,'})^\vee \cdot \nu, \iota)^s$ should be $p((\chi^{+,'})^\vee \cdot \nu, \iota)^s$.

p. 194, (3.26.8) $L^{mot,S}(\frac{1}{2}(v+n), \pi, (\check{\chi}')^+, \nu^\iota, St)$ should be $L^{mot,S}(\frac{1}{2}(n - v + 1), \pi, (\check{\chi}')^+, \check{\nu}^\iota, St)$.

0. Preliminary notation

Let E be a totally real field, $\mathcal{K}$ a totally imaginary quadratic extension of E. Let V be an n-dimensional $\mathcal{K}$-vector space, endowed with a non-degenerate hermitian form $< \bullet, \bullet >_V$, relative to the extension $\mathcal{K}/E$. We let Σ_E, resp. $\Sigma_{\mathcal{K}}$, denote the set of complex embeddings of E, resp. $\mathcal{K}$, and choose a CM type $\Sigma \subset \Sigma_{\mathcal{K}}$, i.e. a subset which upon restriction to E is identified with Σ_E. Complex conjugation in $Gal(\mathcal{K}/E)$ is denoted c.

The hermitian pairing $< \bullet, \bullet >_V$ defines an involution $\tilde{c}$ on the algebra $End(V)$ via

$$< a(v), v' >_V = < v, a^{\tilde{c}}(v') >, \quad (0.1)$$

and this involution extends to $End(V \otimes_{\mathbb{Q}} R)$ for any $\mathbb{Q}$-algebra R. We define $\mathbb{Q}$-algebraic groups $U(V) = U(V, < \bullet, \bullet >_V)$ and $GU(V) = GU(V, < \bullet, \bullet >_V)$ over $\mathbb{Q}$ such that, for any $\mathbb{Q}$-algebra R,

$$U(V)(R) = \{g \in GL(V \otimes_{\mathbb{Q}} R) \mid g \cdot \tilde{c}(g) = 1\}; \tag{0.2}$$

$$GU(V)(R) = \{g \in GL(V \otimes_{\mathbb{Q}} R) \mid g \cdot \tilde{c}(g) = \nu(g) \text{ for some } \nu(g) \in R^{\times}\}. \tag{0.3}$$

Thus $GU(V)$ admits a homomorphism $\nu : GU(V) \to \mathbb{G}_m$ with kernel $U(V)$. There is an algebraic group $U_E(V)$ over E such that $U(V) \xrightarrow{\sim} R_{E/\mathbb{Q}} U_E(V)$, where $R_{E/\mathbb{Q}}$ denotes Weil's restriction of scalars functor. This isomorphism identifies automorphic representations of $U(V)$ and $U_E(V)$.

All constructions relative to hermitian vector spaces carry over without change to skew-hermitian spaces.

The quadratic Hecke character of $\mathbf{A}_E^{\times}$ corresponding to the extension $\mathcal{K}/E$ is denoted

$$\varepsilon_{\mathcal{K}/E} : \mathbf{A}_E^{\times}/E^{\times} N_{\mathcal{K}/E} \mathbf{A}_{\mathcal{K}}^{\times} \xrightarrow{\sim} \pm 1.$$

For any hermitian or skew-hermitian space, let

$$GU(V)(\mathbf{A})^+ = \ker \varepsilon_{\mathcal{K}/E} \circ \nu \subset GU(V)(\mathbf{A}). \tag{0.4}$$

For any place v of E, we let $GU(V)_v^+ = GU(V)(E_v) \cap GU(V)(\mathbf{A})^+$. If v splits in $\mathcal{K}/E$, then $GU(V)_v^+ = GU(V)(E_v)$; otherwise $[GU(V)(E_v) : GU(V)_v^+] = 2$, and $GU(V)_v^+$ is the kernel of the composition of ν with the local norm residue map. We define $GU(V)^+(\mathbf{A}) = \prod'_v GU(V)_v^+$ (restricted direct product), noting the position of the superscript; we have

$$GU(V)(E) \cdot GU(V)^+(\mathbf{A}) = GU(V)(\mathbf{A})^+. \tag{0.5}$$

1. Eisenstein series on unitary similitude groups

(1.1) Notation for Eisenstein series. The present section is largely taken from [H3, §3] and [H4, §I.1]. Let E and $\mathcal{K}$ be as in §0. Let $(W, <,>_W)$ be any hermitian space over $\mathcal{K}$ of dimension n. Define $-W$ to be the space W with hermitian form $- <,>_W$, and let $2W = W \oplus (-W)$. Set

$$W^d = \{(v, v) \mid v \in W\}, \quad W_d = \{(v, -v) \mid v \in W\}$$

These are totally isotropic subspaces of $2W$. Let P (resp. GP) be the stablizer of W^d in $U(2W)$ (resp. $GU(2W)$). As a Levi component of P we take the subgroup $M \subset U(2W)$ which is stablizer of both W^d and W_d. Then $M \simeq GL(W^d) \xrightarrow{\sim} GL(W)$, and we let $p \mapsto A(p)$ denote the corresponding homomorphism $P \to GL(W)$. Similarly, we let $GM \subset GP$ be the stabilizer of both W^d and W_d. Then $A \times \nu : GM \to GL(W) \times \mathbb{G}_m$, with A defined as above, is an isomorphism. There is an obvious embedding

$$i_W : U(W) \times U(W) = U(W) \times U(-W) \hookrightarrow U(2W).$$

We use the same notation for the inclusion $G(U(W) \times U(-W)) \subset GU(2W)$, where as in [H3] $G(U(W) \times U(-W)) = GU(2W) \cap GU(W) \times GU(-W)$, the intersection in $GL(2W)$.

In this section we let $H = U(2W)$, viewed alternatively as an algebraic group over E or, by restriction of scalars, as an algebraic group over $\mathbb{Q}$. We choose a maximal compact subgroup $K_\infty = \prod_{v \in \Sigma_E} K_v \subset H(\mathbb{R})$; specific choices will be determined later. We also let $GH = GU(2W)$.

Let v be any place of E, $|\cdot|_v$ the corresponding absolute value on $\mathbb{Q}_v$, and let

$$(1.1.1) \qquad \delta_v(p) = |N_{\mathcal{K}/E} \circ \det(A(p))|_v^{\frac{n}{2}} |\nu(p)|^{-\frac{1}{2}n^2}, \ p \in GP(E_v).$$

This is the local modulus character of $GP(E_v)$. The adelic modulus character of $GP(\mathbf{A})$, defined analogously, is denoted $\delta_{\mathbf{A}}$. Let χ be a Hecke character of $\mathcal{K}$. We view χ as a character of $M(\mathbb{A}_E) \xrightarrow{\sim} GL(W^d)$ via composition with det. For any complex number s, define

$$\delta^0_{P,\mathbf{A}}(p, \chi, s) = \chi(\det(A(p))) \cdot |N_{\mathcal{K}/E} \circ \det(A(p))|_v^s |\nu(p)|^{-ns}$$
$$\delta_{\mathbf{A}}(p, \chi, s) = \delta_{\mathbf{A}}(p)\delta^0_{P,\mathbf{A}}(p, \chi, s).$$

The local characters $\delta_{P,v}(\cdot, \chi, s)$ and $\delta^0_{P,v}(\cdot, \chi, s)$ are defined analogously.

Let σ be a real place of E. Then $H(E_\sigma) \xrightarrow{\sim} U(n, n)$, the unitary group of signature (n, n). As in [H97, 3.1], we identify $U(n, n)$, resp. $GU(n, n)$, with the unitary group (resp. the unitary similitude group) of the standard skew-hermitian matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Let $K_{n,n} = U(n) \times U(n) \subset U(n, n)$ in the standard embedding, $GK_{n,n} = Z \cdot K_{n,n}$ where Z is the diagonal subgroup of $GU(n, n)$, and let $X_{n,n} = GU(n, n)/GK_{n,n}$, $X^+_{n,n} = U(n, n)/K_{n,n}$ be the corresponding symmetric spaces. The space $X^+_{n,n}$, which can be realized as a tube domain in the space $M(n, \mathbb{C})$ of complex $n \times n$-matrices, is naturally a connected component of $X_{n,n}$; more precisely, the identity component $GU(n, n)^+$ stabilizes $X^+_{n,n}$ and identifies it with $GU(n, n)^+/GK_{n,n}$. Writing $g \in GU(n, n)$ in block matrix form

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with respect to bases of W^d_σ and $W_{d,\sigma}$, we identify GP with the set of $g \in GU(n, n)$ for which the block $C = 0$. In the tube domain realization, the canonical automorphy factor associated to GP and $GK_{n,n}$ is given as follows: if $\tau \in X_{n,n}$ and $g \in GU(n, n)^+$, then the triple

$$(1.2) \qquad J(g, \tau) = C\tau + D, \ J'(g, \tau) = \bar{C}^t\tau + \bar{D}, \nu(g)$$

defines a canonical automorphy factor with values in $GL(n, \mathbb{C}) \times GL(n, \mathbb{C}) \times GL(1, \mathbb{R})$ (note the misprint in [H97, 3.3]).

Let $\tau_0 \in X^+_{n,n}$ denote the unique fixed point of the subgroup GK_∞ and write $J(g) = J(g, \tau_0)$. Given a pair of integers (m, κ), we define a complex valued function on $GU(n, n)^+$:

$$(1.1.3) \qquad J_{m,\kappa}(g) = \det J(g)^{-m} \cdot \det(J'(g))^{-m-\kappa} \cdot \nu(g)^{n(m+\kappa)}$$

More generally, let GH^+ denote the identity component of $GH(\mathbb{R})$, and define $\mathbf{J}_{m,\kappa} \to GH^+ \to \mathbb{C}^\times$ by

$$\mathbf{J}_{m,\kappa}((g_\sigma)_{\sigma\in\Sigma_E} = \prod_{\sigma\in\Sigma_E} J_{m,\kappa}(g_\sigma) \tag{1.1.4}$$

We can also let m and κ denote integer valued functions on σ and define analogous automorphy factors. The subsequent theory remains valid provided the value $2m(\sigma)+\kappa(\sigma)$ is independent of σ. However, we will only treat the simpler case here.

(1.2) Formulas for the Eisenstein series. Consider the induced representation

$$I_n(s,\chi) = \mathrm{Ind}(\delta^0_{P,\mathbf{A}}(p,\chi,s)) \xrightarrow{\sim} \otimes_v I_{n,v}(\delta^0_{P,v}(p,\chi,s)), \tag{1.2.1}$$

the induction being normalized; the local factors I_v, as v runs over places of E, are likewise defined by normalized induction. Explicitly,
(1.2.2)

$$I_n(s,\chi) = \{f : H(\mathbf{A}) \to \mathbb{C} \mid f(pg) = \delta_{P,\mathbf{A}}(p,\chi,s)f(g), p \in P(\mathbf{A}), g \in H(\mathbf{A})\}.$$

At archimedean places we assume our sections to be K_∞-finite. For a section $\phi(h,s;\chi) \in I_n(s,\chi)$ (cf. [H99, I.1]) we form the Eisenstein series

$$E(h,s;\phi,\chi) = \sum_{\gamma\in P(E)\backslash U(2V)(E)} \phi(\gamma h,s;\chi) \tag{1.2.3}$$

If χ is unitary, this series is absolutely convergent for $\mathrm{Re}(s) > \frac{n-1}{2}$, and it can be continued to a meromorphic function on the entire plane. *Assume henceforward* that

$$\chi\mid_{\mathbf{A}} = \varepsilon^m_{\mathcal{K}} \tag{1.2.4}$$

for a fixed positive integer m. Usually, but not invariably, we will assume $m \geq n$. The main result of [Tan] states that the possible poles of $E(h,s;\phi,\chi)$ are all simple, and can only occur at the points in the set

$$\frac{n-\delta-2r}{2},\ r = 0,\ldots,[\frac{n-\delta-1}{2}], \tag{1.2.5}$$

where $\delta = 0$ if m is even and $\delta = 1$ if m is odd. We will be concerned with the residues of $E(h,s_0;\phi,\chi)$ for s_0 in the set indicated in (1.2.5), and with the values when the Eisenstein series is holomorphic at s_0.

We write $I_n(s,\chi) = I_n(s,\chi)_\infty \otimes I_n(s,\chi)_f$, the factorization over the infinite and finite primes, respectively.

We follow [H97, 3.3] and suppose the character χ has the property that

$$\chi_\sigma(z) = z^\kappa, \chi_{c\sigma}(z) = 1, \forall\sigma\in\Sigma \tag{1.2.6}$$

Then the function $\mathbf{J}_{m,\kappa}$, defined above, belongs to $I_n(m-\frac{n}{2},\chi)_\infty$ (cf. [H97,(3.3.1)]). More generally, let

$$\mathbf{J}_{m,\kappa}(g,s+m-\frac{n}{2}) = \mathbf{J}_{m,\kappa}(g)|\det(J(g)\cdot J'(g))|^{-s} \in I_n(s,\chi)_\infty. \tag{1.2.7}$$

When $E=\mathbb{Q}$, these formulas just reduce to the formulas in [H97].

(1.3) Zeta integrals. We recall the discussion of the doubling method, applied to unitary groups; details are in [H3, (3.2)] and [H4, I.1]. Let π, π' be cuspidal automorphic representations of $GU(W) = GU(-W)$, $\phi \in \pi$, $\phi' \in \pi'$. Let $G = G(U(W) \times U(-W))$, as above, and define the Piatetski-Shapiro-Rallis zeta integral
(1.3.1)

$$Z^+(s, \phi, \phi', \chi, \varphi) = \int_{Z(\mathbf{A})\cdot G(\mathbb{Q})\backslash G(\mathbf{A})^+} E(i_W(g, g'), s, \varphi, \chi)\phi(g)\phi'(g')\chi^{-1}(g')dgdg'$$

with $G(\mathbf{A})^+$ as in (0.5) and the Eisenstein series as above. Let $\delta \in GU(2W)$ be a representative of the dense orbit of G on $GP\backslash GU(2W)$, cf. [KR1, (7.2.6)] for the analogous case of symplectic groups. The integral (1.3.1) is Eulerian; if $\phi = \otimes_v \phi_v$ and $\phi' = \otimes_v \phi'_v$ are factorizable vectors then for some finite set S containing archimedean places we have

$$(1.3.2) \qquad d_n^S(\chi, s)Z^+(s, \phi, \phi', \chi, \varphi) = \begin{cases} 0, \pi' \not\simeq \pi^\vee; \\ L^S(s + \frac{1}{2}, \pi, St, \chi)Z_S(s, \phi, \phi', \chi, \varphi), \end{cases}$$

where

$$(1.3.3) \quad Z_S(s, \phi, \phi', \chi, \varphi) = \int_{\prod_{v\in S} U(W_v)} (\pi(g)\phi, \phi')\chi_v^{-1}(det(g)) \cdot \varphi_v(\delta \cdot (g, 1), s)dg.$$

and

$$(1.3.4) \qquad d_n^S(\chi, s) = \prod_{r=0}^{n-1} L^S(2s + n - r, \varepsilon_{\mathcal{K}}^{n-1+r})$$

is a product of abelian L-functions of E with the factors at S removed. The pairing inside the integral (1.3.3) is the L_2 pairing on $GU(W)(E)\backslash GU(W)(\mathbf{A})^+/Z(\mathbf{A})$, where Z is the center of $GU(W)$. As in [H3, (3.2.4), (3.2.5)], the factor (1.3.3) can be rewritten as a product of local zeta integrals, multiplied by a global inner product. Finally, the partial L-function $L^S(s + \frac{1}{2}, \pi, St, \chi)$ is defined as in [H3, (2.7)].

Lemma 1.3.5. *The zeta integral $Z^+(s, \phi, \phi', \chi, \varphi)$ has at most simple poles on the set (1.2.5). In particular, if $L^S(s + \frac{1}{2}, \pi, St, \chi)$ has a pole at $s = s_0$ then s_0 is in the set (1.2.5), $Z_S(s, \phi, \phi', \chi, \varphi)$ is holomorphic at $s = s_0$, and*
(1.3.6)

$$res_{s=s_0} Z^+(s, \phi, \phi', \chi, \varphi) = d_n^S(\chi, s_0)^{-1} Z_S(s_0, \phi, \phi', \chi, \varphi) \cdot res_{s=s_0} L^S(s + \frac{1}{2}, \pi, St, \chi).$$

Proof. The first assertion follows from the theorem of Tan quoted in (1.2). The second assertion is then an obvious consequence of the fact that all the s_0 in (1.2.5) are positive and therefore in the range of absolute convergence of the Euler product $d_n^S(s)$.

The following lemma of Lapid and Rallis will be used specifically in the discussion of archimedean zeta integrals:

Lemma 1.3.6 [LR]. *Assume π is tempered. Then the zeta integral (1.3.3) converges absolutely for Re $s \geq 0$.*

Proof. If π is square integrable this is Lemma 3 of [LR]. In the general case this follows from multiplicativity of zeta integrals [LR, Proposition 2].

2. The local theta correspondence

(2.1) Review of local theta dichotomy.

In this section F is a non-archimedean local field of characteristic zero, K a quadratic extension of F, $\varepsilon_{K/F}$ the associated quadratic character. We fix an additive character $\psi : F \to \mathbb{C}^\times$. We will usually, but not always, assume the residue characteristic of F different from 2. Let W be an n-dimensional skew-hermitian space over K, relative to the extension K/F, with isometry group $G = U(W)$. Let m be a positive integer. In this section we recall the outlines of the theory [KS, HKS] of the local theta correspondence from irreducible admissible representations of G to representations of $U(V)$, where $(V, (,)_V)$ is a variable hermitian space over K of dimension m.

As in [KS] and [HKS] we choose a splitting character χ of $K^\times$ satisfying the local version of (1.2.4):

$$\chi\mid_{F^\times} = \varepsilon_{K/F}^m.$$

We also consider the doubled skew-hermitian space $2W$, as in §1. Multiplying the skew-hermitian form on W, or on $2W$, by an element $\delta \in K$ with trace zero to E transforms it into a hermitian form, without changing the isometry group. Up to isometry, the skew-hermitian space $2W$ is independent of the choice of W. The degenerate principal series $I_n(s, \chi)$ of $U(2W)$ is defined in the local setting by analogy with the global definition (1.2.2).

Define

$$\epsilon(V) = \varepsilon_{K/F}((-1)^{\frac{m(m-1)}{2}} \det V) \in \{\pm 1\} \tag{2.1.1}$$

where $\det V = \det((x_i, x_j)_V)$, for any K-basis $\{x_1, \ldots, x_m\}$ is a well defined element of $F^\times / N_{K/F} K^\times$. Then V is determined up to isometry by the sign $\epsilon(V)$, and we let V_m^+ and V_m^- denote the corresponding pair of hermitian spaces, up to isometry.

Let $s_0 = \frac{m-n}{2}$. We refer to §4 of [HKS] for the following facts, which are proved in [KS]. Let $\mathcal{S}(V^n)$ denote the Schwartz-Bruhat space of locally constant compactly supported functions on V^n. There is a map

$$p_{V,n} : \mathcal{S}(V^n) \to I_n(s_0, \chi); \Phi \mapsto \phi_\Phi \tag{2.1.2}$$

whose definition is recalled below, with image denoted $R_n(V, \chi)$. The following results are proved in [KS], and summarized as Proposition 4.1 of [HKS]:

Proposition (2.1.3).

(i) *If $m \leq n$, then $R_n(V, \chi)$ is irreducible, and*

$$R_n(V_m^+, \chi) \oplus R_n(V_m^-, \chi)$$

is the maximal completely reducible submodule of $I_n(s_0, \chi)$.

(ii) *If $m = n$ then*

$$I_n(0, \chi) = R_n(V_n^+, \chi) \oplus R_n(V_n^-, \chi).$$

(iii) *If $n < m \leq 2n$, then*

$$I_n(s_0, \chi) = R_n(V_m^+, \chi) + R_n(V_m^-, \chi)$$

and

$$Soc_{n,m}(\chi) = R_n(V_m^+, \chi) \cap R_n(V_m^-, \chi)$$

is the unique irreducible submodule of $I_n(s_0, \chi)$. *Moreover, if* $m < 2n$ *there is a natural isomorphism*

$$R_n(V_m^{\pm}, \chi)/Soc_{n,m}(\chi) \xrightarrow{\sim} R_n(V_{2n-m}^{\pm}, \chi).$$

If $m = 2n$ *then* $I_n(s_0, \chi) = R_n(V_{2n}^+, \chi)$ *and* $I_n(s_0, \chi)/R_n(V_{2n}^-, \chi)$ *is the one-dimensional representation* $\chi \circ \det$.

(iv) *If* $m > 2n$, *then* $I_n(s_0, \chi) = R_n(V_m^{\pm}, \chi)$ *is irreducible.*

The choice of splitting character χ, together with the additive character ψ, defines a Weil representation $\omega_{V,W,\chi}$ of the dual reductive pair $U(W) \times U(V)$ on an appropriate Schwartz-Bruhat space $\mathcal{S}_{V,W}$. For details, see [HKS]. When W is replaced by $2W$ (resp. V by $2V$) one obtains actions $\omega_{V,2W,\chi}$ of $U(2W) \times U(V)$ (resp. $\omega_{2V,W,\chi}$ of $U(W) \times U(2V)$) on the space $\mathcal{S}(V^n)$ introduced in (2.1.2) above, and the map $p_{V,n}$ of (2.1.2) is defined by

$$\phi_\Phi(g) = p_{V,n}(\Phi)(g) = (\omega_{V,2W,\chi}(g)\Phi)(0), g \in U(2W).$$

These actions can be chosen in such a way that

$$Res^{G \times U(V) \times G \times U(V)}_{G \times G \times U(V)^\Delta}(\omega_{V,W,\chi} \otimes \omega^\vee_{V,W,\chi}) = Res^{U(2W) \times U(V)}_{G \times G \times U(V)} \omega_{V,2W,\chi} \tag{2.1.4}$$

(cf. [HKS,(3.5)], or [H4,I.5]). Here $U(V)^\Delta$ is the diagonal subgroup in $U(V) \times U(V)$.

2.1.5. The Conservation Relation. Let π be an irreducible admissible representation of G. We define the representation $\Theta_\chi(V, \pi)$ of $U(V)$ by

$$\Theta_\chi(V, \pi) = [\omega_{V,W,\chi} \otimes \pi]_G = Hom_G(\omega^\vee_{V,W,\chi}, \pi). \tag{2.1.5.1}$$

Note that this is the full theta correspondence, which often gives a result larger than the Howe correspondence.

Let $m^?_\chi(\pi)$ denote the minimum integer μ such that $\Theta_\chi(\pi, V^?_\mu) \neq 0$. The analogue for unitary dual reductive pairs of the Conservation Relation, stated as Conjecture 3.6 in [KR2], is

$$m^+_\chi(\pi) + m^-_\chi(\pi) = 2n + 2. \tag{2.1.5.2}$$

For orthogonal-symplectic pairs Kudla and Rallis prove the analogue of the weaker relation

$$m^+_\chi(\pi) + m^-_\chi(\pi) \geq 2n + 2 \tag{2.1.5.3}$$

as Theorem 3.8.

The proof of (2.1.5.3) follows readily from the special case in which π is the trivial representation, denoted **1**:

$$m^+_\chi(\mathbf{1}) = 0, m^-_\chi(\mathbf{1}) = 2n + 2. \tag{2.1.5.4}$$

The analogue of (2.1.5.4) is Lemma 4.2 of [KR2]. In what follows, we will admit here that the proof of this lemma, and the derivation from (2.1.5.4) of (2.1.5.3), work in the unitary case as in [KR2]. This generalization is the subject of work in progress of Z. Gong.

2.1.6. Generic behavior of the theta correspondence.

Recall from [PSR, §2] the description of the set of $G \times G$-orbits on the projective variety $F(W) = P\backslash U(2W)$. The flag variety $F(W)$ is naturally the space of totally isotropic n-spaces in $2W$ for the natural hermitian form. As in [*loc. cit*], the $G \times G$-orbit of an isotropic n-plane L is completely determined by the invariant

$$d(L) = \dim(L \cap W) = \dim(L \cap (-W)).$$

Thus there are $r_0 + 1$ distinct orbits in $F(W)$, where $r_0 = [\frac{n}{2}]$ is the Witt index of W. The open orbit consists of L with $d(L) = 0$. Let $L \in F(W)$ and suppose $d(L) = r > 0$, and let $St_r = Stab(L) \subset G \times G$. If $\delta_r \in U(2W)$ is a representative of L in $F(W)$, then $\delta_r(St_r)\delta_r^{-1} \subset P$. As in [HKS, (4.10)], define a character $\xi_{r,s} : St_r \to \mathbb{C}^\times$, for any $s \in \mathbb{C}$, by

$$\xi_{r,s}(\gamma) = \chi(A(\delta_r(\gamma)\delta_r^{-1})) \cdot |\det(A(\delta_r(\gamma)\delta_r^{-1}))|^{s+\rho_n}.$$

Let $Q^r(s,\chi) = Ind_{St_r}^{G\times G}\xi_{r,s}$.

Definition 2.1.6.1. *For any $r > 0$, the irreducible representation π of G occurs in the rth boundary stratum at the point $s = s_0$ (for χ) if $Hom_{G\times G}(Q^r(s_0,\chi), \pi \otimes \chi\pi^\vee) \neq 0$. We say π does not occur in the boundary at the point $s = s_0$ (for χ) if it does not occur in the rth boundary stratum for any $r > 0$.*

We consider the principal series of G as an infinite union $\mathcal{P}$ of complex affine spaces, as in Bernstein's parametrization, of dimension equal to the split rank of G; the irreducible components of a given principal series representation are represented by the same point in $\mathcal{P}$. More precisely, let $P_0 \subset G$ be a minimal parabolic subgroup, $P_0 = M_0 \cdot A_0 \cdot N_0$, with A_0 a product of copies of $K^\times$, M_0 anisotropic semisimple, and N_0 unipotent. Let r be the split rank of G, so that $A_0 \xrightarrow{\sim} K^{\times,r}$. Let $U(1)$ denote the kernel of the norm map from $K^\times$ to $F^\times$, viewed indifferently as a p-adic Lie group or as an anisotropic torus over F, and let $U(2)$ be the unitary group of the unique anistropic hermitian space of dimension 2 over K. We have $M_0 \xrightarrow{\sim} 1$ if $n = 2r$ is even and G is quasi-split; otherwise $M_0 \xrightarrow{\sim} U(n-2r)$, where $2r = n-1$ if n is odd, $2r = n-2$ if n is even.

To any character $\alpha : A_0 \cdot M_0 \to \mathbb{C}^\times$ we associate the principal series

$$I(\alpha) = Ind_{P_0}^G \alpha$$

(normalized induction). These representations are generally not irreducible, and we say an irreducible admissible representation of G or G' *belongs to the principal series* if it is an irreducible constituent of some principal series representation. When $n = 2r$, so G is split, we write $\alpha = (\alpha_1, \ldots, \alpha_r)$ is an r-tuple of characters of $K^\times$; otherwise, $\alpha = (\alpha_1, \ldots, \alpha_r; \gamma)$ where the α_i are characters of $K^\times$ whereas γ is a character of the anisotropic $U(n-2r)$. The connected components of $\mathcal{P}$ are indexed by the inertial class of α, i.e. the restrictions of the α_i to the units in $K^\times$, together with the ramified character γ when $n - 2r > 0$. The connected component corresponding to a given inertial class can be parametrized by the r-nonzero complex numbers which are values of the α_i on a fixed uniformizing parameter of K, say ϖ_K. Each connected component is thus identified with an r-tuple of elements of $\mathbb{C}^\times$, but the whole is only well-defined up to permutation of the α_i. By "almost all" principal series τ of G we mean a subset $\mathcal{P}^0$ of $\mathcal{P}$ containing an open dense subset of every inertial class. Some r-tuples correspond to reducible principal series, and therefore to several distinct irreducible representations; however almost all principal series, in the above sense, are irreducible.

Lemma 2.1.6.2. *(i) For every connected component $\mathcal{P}_\alpha$ of $\mathcal{P}$, the set of representations π of G in $\mathcal{P}_\alpha$ occurring in the boundary at any point $s = s_0$ for any fixed splitting character χ is a proper closed subset; i.e. almost all π in $\mathcal{P}$ do not occur in the boundary.*

(ii) For any irreducible principal series representation π and any fixed s_0, the set of splitting characters χ such that π occurs in the boundary at s_0 for χ is finite.

(iii) For $r > 1$, any π occurring in the rth boundary stratum is non-tempered.

Proof. The standard maximal parabolic subgroups P_r of G, $r = 0, \dots, r_0$, are the stabilizers of isotropic subspaces of dimension r. The Levi factor L_r of P_r is isomorphic to $GL(r, K) \times U(n-2r)$, where $U(n-2r)$ is the unitary group of a vector space of dimension $n - 2r$ in the same Witt class as W. We claim that, for an appropriate choice of δ_r, St_r contains $GL(r,K) \times GL(r,K) \times \Delta(U(n-2r)) \subset L_r \times L_r \subset G \times G$, where the first two factors are embedded in the obvious way and the factor $\Delta(U(n-2r))$ is the diagonal in $U(n-2r) \times U(n-2r)$. Indeed, this is verified as in [KR2], §1.

Let N_r denote the unipotent radical of P_r, and let the subscript N_r denote the unnormalized Jacquet module. As in [KR2], Lemma 1.5, it follows from the above description of St_r that, if π occurs in the boundary at the point s_0 then there is an integer $r \in \{1, r_0\}$ and a character ξ'_{r,s_0} of $GL(r,K)$, derived by restriction from ξ_{r,s_0}, such that

$$Hom_{GL(r,K)}(\pi_{N_r}, \xi'_{r,s_0}) \neq 0. \tag{2.1.6.3}$$

For fixed s_0 and any r, this latter condition defines a proper closed subset of any irreducible component of $\mathcal{P}$. The remaining assertions follow immediately from the characterization (2.1.6.3).

The following theorem is joint with Steve Kudla. I recall that we are assuming the validity of the weak conservation relation (2.1.5.3) in the unitary case.

Theorem (2.1.7). *(i) For any π,*

$$Hom_{G\times G}(R_n(V,\chi), \pi \otimes (\chi \cdot \pi^\vee)) \neq 0 \Rightarrow \Theta_\chi(V,\pi) \neq 0.$$

(i') Assume the residue characteristic of F is different from 2, or π is supercuspidal, or more generally that $\Theta_\chi(V,\pi)$ admits an irreducible quotient. Then the implication in (i) is an equivalence.

(ii) Assume that π does not occur in the boundary at $s_0 = \frac{m-n}{2}$ for χ, or that $\Theta_\chi(V,\pi)$ is a non-zero irreducible representation. Then

$$d_\chi(V,\pi) = \dim Hom_{G\times G}(R_n(V,\chi), \pi \otimes (\chi \cdot \pi^\vee)) = 1.$$

(iii) If π is any admissible irreducible representation of $G = U(W)$ and $m \geq n$, then there exists V with $\dim V = m$ such that

$$\Theta_\chi(\pi, V) \neq 0 \text{ and } Hom_{G\times G}(R_n(V,\chi), \pi \otimes (\chi \cdot \pi^\vee)) \neq 0.$$

(iv) (Theta dichotomy) *With notation as in (iii), if $m \leq n$, then there is at most one such V. In particular, if $m = n$, then there is* **exactly** *one V of dimension n such that $\Theta_\chi(\pi, V) \neq 0$, determined by the following root number criterion:*

$$\Theta_\chi(\pi, V) \neq 0 \Leftrightarrow \varepsilon(\frac{1}{2}, \pi, \chi, \psi) = \varepsilon_{K/F}(-2)^n \varepsilon_{K/F}(\det V), \tag{2.1.7.1}$$

where the signs ε are defined as in [HKS] §6.

Proof. Assertion (i') is [HKS, Prop. 3.1], and (iii) is contained in [HKS,Cor. 4.5 and Cor. 4.4]. Only the first part of (iii) is asserted in [loc. cit.] but the proofs are based on the second assertion. On the other hand, the proof in [loc. cit.] of the implication in (i) does not make use of the special hypotheses in (i'). If π does not occur in the boundary at s_0 for χ, then (ii) is Theorem 4.3 (ii) of [HKS]. If $\Theta_\chi(V,\pi)$ is irreducible (i) is an equivalence, by (i'). In particular, $d_\chi(V,\pi) \geq 1$ and we have to show that it is also ≤ 1 if $\Theta_\chi(V,\pi)$ is irreducible. But the dimensions of the spaces in the last three lines of (3.5) in [HKS] are all equal, in particular

$$d_\chi(V,\pi) = \dim Hom_{G\times G\times U(V)^\triangle}(\omega_{V,W,\chi}\otimes\omega^\vee_{V,W,\chi}, \pi\otimes\pi^\vee\otimes Triv)$$

where $Triv$ denotes the trivial representation of $U(V)$. The space on the right-hand side of the last formula is just

$$\begin{aligned} &Hom_{U(V)^\triangle}(\Theta_\chi(V,\pi)\otimes\Theta_\chi(V,\pi^\vee), Triv)\\ =\ &Hom_{U(V)^\triangle}(\Theta_\chi(V,\pi)\otimes\Theta_\chi(V,\pi)^\vee, Triv)\\ =\ &Hom_{U(V)}(\Theta_\chi(V,\pi),\Theta_\chi(V,\pi)) \end{aligned}$$

where the first equality follows as in [loc. cit.] from a theorem of Mœglin-Vignéras-Waldspurger. Assertion (ii) thus follows from the irreducibility hypothesis.

Finally, when $m \leq n$, $2m < 2n+2$, so the first part of (iv) is an immediate consequence of (2.1.5.3), and the second part then follows from (iii). The final assertion is Theorem 6.1 (ii) of [HKS], bearing in mind our sign conventions.

Let m be as in the preceding proposition. We say π is *unambiguous* (relative to m and χ) if there is at most one V of dimension m such that $\Theta_\chi(V,\pi)\neq 0$, and *ambiguous* otherwise. For $m \leq n$, (2.1.7)(iv) asserts that every π is unambiguous. In contrast, (2.1.7)(iii) guarantees that there is always at least one V of dimension m such that $\Theta_\chi(V,\pi)\neq 0$ when $m > n$, and when $m > 2n$ (the so-called stable range) (2.1.3) (iv) implies the well-known result that any π is ambiguous. The following proposition indicates that ambiguity is the rule rather than the exception for $n < m \leq 2n$.

Proposition (2.1.8). *Let π be an irreducible admissible representation of $G = U(W)$. Let $n < m < 2n$, and assume π is unambiguous for m (and χ). Let $V_m^?$ be such that $\Theta_\chi(\pi, V_m^?)\neq 0$. Then $\Theta_\chi(\pi, V_{2n-m}^?)\neq 0$.*

Conversely, if $\Theta_\chi(\pi, V_{2n-m}^?)\neq 0$ for some choice of $? \in \{\pm 1\}$, then π is unambiguous for m, i.e. $\Theta_\chi(\pi, V_m^{-?}) = 0$.

Proof. We first note that the first assertion follows if we can construct a non-zero $G\times G$-equivariant pairing

$$I_n(s_0,\chi)\otimes[\pi\otimes(\chi\cdot\pi^\vee)]^\vee \to \mathbb{C}. \tag{2.1.8.1}$$

Indeed, such a pairing restricts to pair of $G\times G$-equivariant homomorphisms

$$\lambda^\pm : R_n(V_m^\pm,\chi)\to\pi\otimes(\chi\cdot\pi^\vee).$$

Since π is unambiguous $\lambda^{\pm} \neq 0$ if and only if $\pm = ?$. In particular, $f^{-?} = 0$. Since $\lambda^{?}$ and $\lambda^{-?}$ have the same restriction to $Soc_{n,m}(\chi)$, it follows that $\lambda^{?}$ factors through

$$\lambda^* \in Hom_{G\times G}(R_n(V_m^?, \chi)/Soc_{n,m}(\chi), \pi \otimes (\chi \cdot \pi^\vee)) = Hom_{G\times G}(R_n(V_{2n-m}^?, \chi), \pi \otimes (\chi \cdot \pi^\vee))$$

by (2.1.3)(iii). Then (2.1.7) (i) implies that $\Theta_\chi(\pi, V_{2n-m}^?) \neq 0$.

It thus suffices to observe [HKS, (6.27)] that the normalized zeta integral defines the required non-zero $G \times G$-equivariant pairing. For future reference, we recall the definition of the zeta integral:

$$(2.1.8.2) \qquad Z(s, \varphi, \varphi', f, \chi) = \int_G f((g, 1); \chi, s)(\pi(g)\varphi, \varphi')dg$$

where $f \in I_n(s, \chi)$ is restricted as above to $G \times G$, $\varphi \in \pi$, and $\varphi' \in \chi \cdot \pi^\vee$. The normalized zeta integral is

$$(2.1.8.3) \qquad \tilde{Z}(s, \varphi, \varphi', f, \chi) = d_n(s, \chi)L(s, \pi, St, \chi)^{-1}Z(s, \varphi, \varphi', f, \chi)$$

where $L(s, \pi, St, \chi)$ is the standard local L-function of π, twisted by χ (cf. [HKS,(6.20)]) and $d_n(s, \chi)$ is the normalizing factor $\prod_{r=0}^{n-1} L(2s + n - r, \varepsilon_{\mathcal{K}}^{n-1+r})$, where $L(s, \bullet)$ is the local abelian Euler factor. The factor $d_n(s, \chi)$ can be ignored since it has no poles at the point s_0.

The converse follows easily from (2.1.5.3), as in [KR2]: if, say $m_\chi^+(\pi) \leq 2n - m$, then $m_\chi^- \geq m + 2$, and so the only non-vanishing lift is to V_m^+; likewise with $+$ and $-$ exchanged.

Corollary (2.1.9). *Let π be an irreducible admissible representation of $G = U(W)$. Suppose π is ambiguous for $m > n$ and χ. Assume either that π does not occur in the boundary for $s_0 = \frac{m-n}{2}$ and χ or that $\Theta_\chi(\pi, V_m^{\pm})$ is an irreducible representation of $U(V_m^{\pm})$, e.g. that π is supercuspidal. Then the normalized zeta integral defines a non-zero pairing between $Soc_{n,m}(\chi)$ and $\pi \otimes (\chi \cdot \pi^\vee)$.*

Proof. The normalized zeta integral restricts to a non-zero pairing in

$$Hom_{G\times G}(R_n(V_m^?, \chi), \pi \otimes (\chi \cdot \pi^\vee))$$

for at least one $? \in \{\pm\}$. By the multiplicity one statement in (2.1.7)(ii), this is the unique pairing for this sign. Hence if its restriction to the socle is trivial, the pairing defines a theta lift to the corresponding $V_{2n-m}^?$. But this implies π is unambiguous, which contradicts the hypothesis.

For future reference, we describe the Galois equivariance of the spaces $R_n(V, \chi)$. Note that (1.2.4) implies that χ is a continuous character of a compact profinite group, hence takes values in $\overline{\mathbb{Q}}$ (even $\mathbb{Q}^{ab}$). The induced representation $I_n(s_0, \chi)$ in principle has a natural model, as a space of functions on $U(2W)$, defined over $\mathbb{Q}(\chi, \sqrt{p})$, where p is the residue characteristic of F; the square root may occur because of the appearance of square roots of the norm in the modulus character. We assume s_0 is chosen so that in fact these square roots only occur to even powers; this will always be the case in the applications (cf. the proof of Theorem 4.3, below). In what follows we write $R_{n,\psi}(V_m^{\pm}, \chi)$ for $R_n(V_m^{\pm}, \chi)$, including the additive character in the index. Then we have

Lemma (2.1.10). *For any $\sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$,*

$$\sigma(R_{n,\psi}(V_m^{\pm},\chi)) = R_{n,\sigma(\psi)}(V_m^{\pm},\sigma(\chi)) \subset I_n(s_0,\sigma(\chi)).$$

Proof. The oscillator representation, split as a representation of $U(2W) \times U(V)$ is defined over $\mathbb{Q}^{ab}(\chi) = \mathbb{Q}^{ab}$. Indeed, it has a Schrödinger model over $\mathbb{Q}^{ab}$ consisting of Schwartz-Bruhat functions with values in $\mathbb{Q}^{ab}$. The assertion follows from the explicit formulas [K] for the action on the Schrödinger model.

(2.2) Theta lifts of principal series.

In this section K/F is an unramified quadratic extension and V and W are maximally isotropic, and $n = \dim W > 1$, $m = \dim V$. We write $G = U(W)$, as above, and $G' = U(V)$. Let $P_0 \subset G$ and $P_0' \subset G'$ be minimal parabolic subgroups, $P_0 = M_0 \cdot A_0 \cdot N_0$, $P_0' = A_0' \cdot M_0' \cdot N_0'$, with A_0, M_0, A_0', M_0' tori as in (2.1.6) and N_0, N_0' unipotent. Let $r = [\frac{n}{2}]$, so that r is the split rank over F of the torus A_0.

For $t = 1, 2, \ldots, r$ let $Q_t \subset G$ be the standard maximal parabolic stabilizing an isotropic t-plane in W. The Levi component M_t of Q_t can be identified with $GL(t,K) \times G_{n-2t}$ where $G_{n-2t} = U(W_{n-2t})$ is the unitary group of the space W_{n-2t} of dimension $n-2t$ in the same Witt class (i.e., maximally isotropic) as W. We define $Q_t' \subset G'$ analogously, for $1 \leq t \leq [\frac{m}{2}]$. We let r_t denote the normalized Jacquet functor from representations of G to representations of M_t. For any irreducible admissible representation π of G we let $t(\pi)$ denote the maximum t such that $r_t(\pi)$ has a non-zero quotient of the form $\sigma_t \otimes \pi_t$ with σ_t a supercuspidal representation of $GL(t,K)$ and π_t a representation of G_{n-2t} ([MVW], p. 69). We also define $u(\pi)$ to be the maximum u such that $r_u(\pi)$ has a non-zero quotient of the form $\sigma_t \otimes \pi_t$ with π_t a supercuspidal representation of G_{n-2t}.

We define the principal series $\alpha : M_0 \to \mathbb{C}^\times$ we associate the principal series $I(\alpha), I(\alpha')$ of G, G' respectively, as in (2.1.6). If π belongs to the principal series then $t(\pi) = 1$ and $u(\pi) = r$.

We choose a second splitting character χ' of $K^\times$ such that

$$\chi' \mid_{F^\times} = \varepsilon_{K/F}^n.$$

Our objective is to determine when a principal series representation of G is in the image of the theta lift from G' for some pair (χ,χ') of splitting characters. By definition when $m = 0$ G' is the trivial group and the unique theta lift to G from G' is the trivial representation. The first observation is

Lemma (2.2.1). *Suppose $b = n - m \geq 2$ and π is an irreducible admissible representation of G that belongs to the tempered principal series. Then π cannot be a theta lift from G'.*

Proof. When $n = 2$ this is true by definition, and when $n = 3$ this is due to Gelbart and Rogawski [GR]. We may thus assume $n \geq 4$, so $r \geq 2$. In general the lemma is a consequence of Kudla's induction principle [K]. This is worked out for unitary groups by Moeglin, Vignéras, and Waldspurger in [MVW, Chapter 3, IV.4], though the result there is less complete than in the cases treated in [K]. Specifically, suppose π is a theta lift of π' for some irreducible admissible representation π' of G'. Since π belongs to the principal series, the index $t(\pi)$ defined above equals 1. Then the main theorem on p. 69 of [MVW] calculates the exponents of the normalized Jacquet module $r_1(\pi)$ of π relative to the maximal parabolic P_1 of G with Levi component $K^\times \times G_{n-2}$, where G_{n-2} is an isotropic unitary group of dimension $n-2$. In the remainder of the proof, we follow the conventions of [MVW] and consider the theta correspondence on covering groups rather than on the groups themselves using the splitting characters. Since the splitting characters are unitary they are irrelevant to determination of whether or not a representation is tempered.

Say the Jordan-Hölder constituents of $r_1(\pi)$ are of the form $\sigma_1^j \otimes \pi_{n-2}^j$ where σ_1^j are characters of $K^\times$ and π_{n-2}^j are irreducible representations of G_{n-2}. (Note that in [MVW] the groups are denoted by their Witt indices rather than by the dimensions of the corresponding hermitian spaces.) We also let P_1' denote the maximal parabolic subgroup of G' with Levi component $K^\times \times G'_{m-2}$ where G'_{m-2} is a unitary group of dimension $m-2$ in the same Witt class as G'.

There are two choices. In case 2 (b)(i) every σ_1^j equals $|\bullet|^{\frac{b-1}{2}}$, and by the usual characterization of tempered representations by exponents this is impossible unless $b=1$ (in particular, case 2 (b) cannot even happen when $n=m$). Indeed, when π is contained in a tempered principal series every σ_1^j and π_{n-2}^j is again tempered. In case 2 (b)(ii) π' is a quotient of the induced representation $I_{P_1'}^{G'}\sigma_1^{j,*} \times \pi'_{m-2}$ for some j where π'_{m-2} is a representation of G'_{m-2}, such that π_{n-2} is a theta lift from G_{m-2} (of π'_{m-2}, in fact). In this case we conclude by induction on m. (For conventions when $n \leq 1$ see the discussion following (2.2.4) below.)

Corollary (2.2.2). *Suppose π is an irreducible tempered principal series constituent. Then π is ambiguous relative to every $m > n+1$ (and every χ).*

Proof. Let $m > n+1$ and fix χ. If $m > 2n$ then there is nothing to prove, by Proposition 2.1.3.(iv). If π were unambiguous for m and χ then π would admit a theta lift to some V_{2n-m}. If $m = 2n$ this means that π is the trivial representation. If $m < 2n$ then the previous lemma implies π is non-tempered.

The goal of this section is to prove the following proposition:

Proposition (2.2.3). *Let π be an irreducible principal series consituent. Fix $m > n$ and χ'. The set of splitting characters χ such that π is unambiguous relative to m, χ', and χ is finite.*

When $m > n+1$ and π is tempered we have seen that this set is even empty. Henceforth we assume $m = n+b$, $H = U(V_{n-b})$ where V_{n-b} is an $n-b$-dimensional hermitian space with $\epsilon(V_{n-b}) = \epsilon(V)$. By Proposition 2.1.7 it suffices to show that the set of χ for which π is a theta lift from H with splitting character χ (and fixed χ') is finite. We first observe:

Lemma (2.2.4). *Suppose $\tau = \Theta_\chi(\pi, V_{n-b}) \neq 0$. Then τ belongs to the principal series.*

Proof. This follows by induction from the Main Theorem of [MVW], p. 69.

Lemma (2.2.4) represents π as $\Theta_\chi(\tau, W)$, and it thus remains to determine the theta lift of a principal series representation τ of H to G. The expression depends on the parity of n. We represent the contragredient principal series $\tau^\vee$ as an induced representation $I(\alpha)$ as above.

We let $|r_r(\tau^\vee)|$ denote the set of Jordan-Hölder constituents of the normalized Jacquet module $r_r(\tau^\vee)$ as above and write $|r_1(\tau^\vee)| = \{\sigma_r^j \otimes \tau_{n-b-2r}^j\}$. The σ_r^j

belong to the set $\{\alpha_1, \dots, \alpha_r\}$. In order to make the induction work we specify that when $n = 3$ (resp. $n = 2$) τ_{n-2}^j (resp. σ_1^j) are representations of the trivial group which by convention are just considered trivial.

The main induction step is the following variant of the Main Theorem of [MVW], p. 69, which is itself an extension of Theorem 2.5 of [K]. In what follows τ is a principal series representation of $H = U(V_{n-b})$, corresponding to the representation π of $U(W) = U(W_n)$. In particular, the indices have changed from the beginning of the section.

Proposition (2.2.5). *(a) Under the above hypotheses, let $\alpha_i \otimes \tau_{n-b-2} \in |r_1(\tau^\vee)|$. Suppose $n > 3$. Then π is a quotient of one of the following induced representations:*

(i) $I_{P_1}^G(|\bullet|^{\frac{b-1}{2}} \cdot \chi^{-1} \otimes \pi_{n-2})$ *for some* π_{n-2}*;*

(ii) $I_{P_1}^G(\alpha_i \cdot (\chi'/\chi) \otimes \pi_{n-2})$ *and* $\Theta_\chi(\pi_{n-2}, V_{n-b-2}) = \tau_{n-b-2}$.

(b) If $n - b = 1$, $r_1(\tau^\vee)$ is just $\tau^\vee = \gamma$. If γ is ramified then its theta lift to G is not in the principal series, hence cannot equal π. Suppose $n = 2$. Then if γ is trivial then $\pi = I(\chi'/\chi)$, where χ'/χ is viewed as a representation of the Levi component $GL(1, K)$ of the Borel subgroup of G. If $n > 2$ then we are necessarily in situation (i) of (a).

(c) If $n - b = 0$ and $n > 1$ then we are necessarily in situation (i) of (a).

Cases (i) and (ii) of (a) correspond to the two cases (i) and (ii) of 2(b) of the Main Theorem of [MVW]. Our conventions differ from those of [MVW] in that our theta lift of τ corresponds to their theta lift of $\tau^\vee$, and we have used the splitting character to obtain lifts directly from one unitary group to another. Our use of the contragredient in the definition of the theta lift explains the absence of the contragredient (σ_t^*) in the final result. The splitting character intervenes in (a) in the calculation of coinvariants of the metaplectic representation in the mixed Schrödinger model ([MVW] 3, V). Specifically, if $n = 2r$ is even and if $P \subset G$ is the stabilizer of a maximal isotropic lattice, then we have the usual Schrödinger model on the Schwartz-Bruhat space of V_{n-1}^r, with $H = U(V_{n-1})$. The action of the subgroup $GL(r) \subset P$ is multiplied by $\chi^{-1} \circ \det$ compared to the action described on p. 75 of [MVW] (cf. [Kudla, splittings]), whereas the action of H is the linear action on the argument V_{n-1}^r, twisted by the character χ'. In contrast, if $n = 2r+1$ is odd, and $n - b$ is even, then it is simplest to consider the lift in the opposite direction, which multiplies the action of $GL(r)$, now viewed as a Levi factor in H, by the character $\chi'^{,-1} \circ \det$, whereas the action of G is the linear action twisted by the character χ. The case where both n and $n-b$ are even is a bit more complicated to describe, since it requires a mixed model, but is treated in the same way. The signs are explained by our conventions regarding contragredients.

The special cases (b) and (c) are proved in just the same way. We mention them separately because of their role in the induction, as follows:

Proof of Proposition 2.2.3. Recall that χ' is fixed. When $n \leq 3$, (2.2.5)(a) and (b) describe π explicitly as an induced representation, depending on χ. We can thus recover χ from the Jacquet module of π with respect to the minimal parabolic P_0; in particular, χ belongs to a finite set. For general n, we apply (a). In case (i) we can already recover χ; in case (ii) we conclude by induction. We note that when $m = n$ there is no obvious restriction on the set of χ, since the twist by χ can be compensated by a twist of τ; however, there are subtle restrictions related to root numbers [HKS].

By induction on 2.2.5 we can calculate π explicitly in terms of τ for almost all τ, in the sense of (2.1.6). Reducible τ can be ignored in the following discussion, where by "almost all" τ such that $\Theta_\chi(\tau, W) \neq 0$ we mean the set of all τ in the intersection of the domain of the theta correspondence with $U(W)$ and some subset $\mathcal{P}^0$ as in (2.1.6). With this convention, it is possible for this intersection to be empty, hence "almost all" τ can refer to the empty set if the set of τ with $\Theta_\chi(\tau, W) \neq 0$ does not itself contain an open subset of $\mathcal{P}$.

Example 2.2.6. In the situation of Proposition 2.2.5 (a), case (ii) holds for almost all τ. Indeed, the situation in which case (i) applies is described explicitly in [MVW] in terms of the Jacquet module of τ.

The first two parts of the following corollary are essentially due to Kudla [K], and represent a special case of the analogue for unitary groups of his Theorem 2.5. Since his article does not treat unitary groups, we derive it from the results already quoted from [MVW].

Corollary 2.2.7. *(1) Suppose $n-b$ is even. For almost all τ, with $\tau^\vee = I(\alpha)$, such that $\pi = \Theta_\chi(\tau, W) \neq 0$, π is a subquotient of the principal series representation*

$$I(\alpha_1 \cdot (\chi'/\chi), \alpha_2 \cdot (\chi'/\chi), \ldots, \alpha_r(\chi'/\chi), |\bullet|^{\frac{b-1}{2}} \cdot \chi^{-1}, |\bullet|^{\frac{b-3}{2}} \cdot \chi^{-1}, \ldots, \nu),$$

where $\nu = |\bullet|^{\frac{1}{2}} \cdot \chi^{-1}$ or $\nu = \chi^{-1}$ as b is even or odd.

(2) Suppose $n-b$ is odd and γ is trivial. For almost all τ, with $\tau^\vee = I(\alpha)$, such that $\pi = \Theta_\chi(\tau, W) \neq 0$, π is a subquotient of the principal series representation

$$I(\alpha_1 \cdot (\chi'/\chi), \alpha_2 \cdot (\chi'/\chi), \ldots, \alpha_r(\chi'/\chi), |\bullet|^{\frac{b-1}{2}} \cdot \chi^{-1}, |\bullet|^{\frac{b-3}{2}} \cdot \chi^{-1}, \ldots, \nu'),$$

where $\nu' = |\bullet|^{\frac{1}{2}} \cdot \chi^{-1}$ if b is even and where $\nu' = \chi'/\chi$ if b is odd.

Proof. We may assume $n > 3$, the remaining cases being available in the literature as mentioned above. The proof then proceeds by induction on n, where at each step, until $n - b \leq 1$, we are allowed by (2.2.6) to assume we are in case (ii) of (a). At this point the proof divides into two cases, depending on whether n is even (i.e., we end up in case (c)) or n is odd (and we end up in case (b). If either b or $n-b$ is even, we are always in situation (i) of (a), and the induction proceeds as before. If b and $n-b$ are both odd, we are in situation (i) of (a) until the last step, where we are in case (b), with $H = U(1)$ and $G = U(2)$ (here the notation designates the quasi-split $U(2)$).

Remark 2.2.8. When the residue characteristic of F is odd, m is even, τ is spherical, and K/F and W are unramified, this result is obtained by Ichino [I1,Proposition 2.1] by adapting a method of Rallis. However, it does not seem that this result necessarily determines when a given τ is in the domain of the Howe correspondence. We address this question in the appendix (2.A).

2.2.9. The split case.

When $K = F \oplus F$ then $G \cong GL(n, F)$, $G' \cong GL(m, F)$, and an unramified splitting character is trivial. For $m \geq n$ it is known that every irreducible τ has a non-trivial theta lift to G' [MVW, p. 64, Lemme 4, Lemme 5], which we denote π If $m = n$, the calculation on p. 65 of [MVW] shows that $\pi = \tau^\vee$. In general, the formulas in Corollary 2.2.7 remain true in the split case; this is a special case of a result of Minguez [M, Theorem 4.1.2].

(2.3) The archimedean theta lift.

We consider two unitary groups $G = U(M)$ and $G' = U(N)$ with signatures to be determined later. We will assume $N > M$ except in the statement of Theorem 2.3.5. The theta lifts of discrete series representations of $U(M)$ to cohomological representations of $U(N)$ were determined explicitly by Li in [Li90] in great generality. Certain cases of Li's results were applied in [H4].

We identify the unitary groups G, G' with the classical unitary groups of given signatures $G = U(s,r)$, $G' = U(a,b)$, with maximal compact subgroups $K = U(s) \times U(r)$, $K' = U(a) \times U(b)$, respectively. Li worked with the metaplectic representation, and considered liftings from the double cover $\tilde{G}$ of G to the double cover $\tilde{G}'$ of G'; the corresponding double of K and K' are denoted $\tilde{K}$ and $\tilde{K}'$, respectively. As in [KS] and [HKS], we can replace the metaplectic double covers by the more canonical $U(1)$-covering groups, which can be split over G and G' after choosing splitting characters $\chi, \chi' : \mathbb{C}^\times \to U(1)$ as in the non-archimedean case. However, Li's formulas for the theta lifts are more transparent (and easier to remember) using parameters for the double covers.

The discrete series representation $\tilde{\pi}$ of $\tilde{G}$ is determined by the highest weight $\tilde{\tau}$ of its lowest $\tilde{K}$-type, which we represent as an (s,r)-tuple of half-integers

$$\begin{aligned}
&\tilde{\tau} = \tilde{\tau}_+ \otimes \tilde{\tau}_-;\\
(2.3.1)\qquad &\tilde{\tau}_+ = (\tilde{\beta}_1 \geq \cdots \geq \tilde{\beta}_{u(+)} \geq \frac{a-b}{2} \geq \cdots \geq \frac{a-b}{2} \geq -\tilde{\gamma}_{v(+)} \geq \cdots \geq -\tilde{\gamma}_1)\\
&\tilde{\tau}_- = (\tilde{\delta}_1 \geq \cdots \geq \tilde{\delta}_{u(-)} \geq \frac{b-a}{2} \geq \cdots \geq \frac{b-a}{2} \geq -\tilde{\alpha}_{v(-)} \geq \cdots \geq -\tilde{\alpha}_1)
\end{aligned}$$

Here $u(+) + v(+) \leq s$, $u(-) + v(-) \leq r$. We set

$$C_i = \tilde{\gamma}_i + \frac{a-b}{2};\ B_i = \tilde{\beta}_i - \frac{a-b}{2};\ A_i = \tilde{\alpha}_i + \frac{b-a}{2};\ D_i = \tilde{\delta}_i - \frac{b-a}{2}.$$

The A_i, B_i, C_i, D_i are all positive integers.

We also consider the Harish-Chandra parameter of $\tilde{\pi}$:

$$\tilde{\lambda} = (a_1, \ldots, a_{k(+)}; -b_{\ell(+)}, \ldots, -b_1) \otimes (c_1, \ldots, c_{k(-)}; -d_{\ell(-)}, \ldots, -d_1)$$

Here $k(+) + \ell(+) = s$, $k(-) + \ell(-) = r$, and all the a_i, b_i, c_i, d_i are positive integers or half-integers (depending on the parity of N as well as M). Moreover, since $\tilde{\pi}$ is in the discrete series, all of the a_i, b_i, c_i, d_i are distinct and none equals zero (this is also a consequence of the half-integral shift). The relation between $\tilde{\lambda}$ and $\tilde{\tau}$ is explicit (see [Li90, (53)] for example) and will be recalled below in a specific case of interest. The notation here is parallel but not identical to that of [Li90].

Assume

$$(2.3.2)\qquad k(+) + \ell(-) \leq a;\ k(-) + \ell(+) \leq b.$$

Then the theta lift $\Theta(\tilde{G} \to \tilde{G}'; \tilde{\pi}^\vee)$ to G' of the **contragredient** $\tilde{\pi}^\vee$ of $\tilde{\pi}$ is a nontrivial cohomological representation $\tilde{\pi}'$ with lowest $\tilde{K}'$-type $\tilde{\tau}' = \tilde{\tau}'_+ \otimes \tilde{\tau}'_-$, where (2.3.2(+))

$$\tilde{\tau}'_+ = (B_1 + \frac{s-r}{2}, \ldots, B_{u(+)} + \frac{s-r}{2}, \frac{s-r}{2}, \ldots \frac{s-r}{2}, -A_{v(-)} + \frac{s-r}{2}, \ldots, -A_1 + \frac{s-r}{2});$$

(2.3.2(-))

$$\tilde{\tau}'_{-} = (D_1+\frac{r-s}{2},\dots,D_{u(-)}+\frac{r-s}{2},\frac{r-s}{2},\dots\frac{r-s}{2},-C_{v(+)}+\frac{r-s}{2},\dots,-C_1+\frac{r-s}{2}).$$

In other words, the negative parameters (shifted by the difference of signatures) migrate from one part of the signature to the other, the positive parameters stay put, and zeroes are added as necessary. The representation $\tilde{\pi'}$ is identified in [Li90] explicitly as a cohomologically induced module $A_{\mathfrak{q}}(\lambda)$, determined by $\tilde{\tau}'$ and the infinitesimal character which can also be written explicitly.

On the other hand, when (2.3.2) is not satisfied, then $\Theta(\tilde{G} \to \tilde{G}'; \tilde{\pi}^{\vee}) = 0$.

Now choose splitting characters $\chi, \chi' \to \mathbb{C}^{\times} \to U(1)$ as in the non-archimedean case:

$$\chi \mid_{\mathbb{R}^{\times}} = \varepsilon^{N}_{\mathbb{C}/\mathbb{R}};\ \chi' \mid_{\mathbb{R}^{\times}} = \varepsilon^{M}_{\mathbb{C}/\mathbb{R}}. \tag{2.3.3}$$

The characters χ and χ' are determined by their restrictions to $U(1) = S^1 \subset \mathbb{C}^{\times}$, and can thus be indexed by integers (winding numbers) $\alpha(\chi), \alpha(\chi') \in \mathbb{Z}$:

$$\chi(u) = u^{\alpha(\chi)};\ \chi'(u) = u^{\alpha(\chi')}, u \in U(1).$$

By (2.3.3) we have $\alpha(\chi) \equiv N \pmod 2$, $\alpha(\chi') \equiv M \pmod 2$. We define

$$\tau = \tilde{\tau} + \frac{1}{2}\alpha(\chi)(1,\dots,1);\ \tau' = \tilde{\tau}' + \frac{1}{2}\alpha(\chi')(1,\dots,1).$$

We define $\tau_{\pm}$ and $\tau'_{\pm}$ likewise. Then τ (resp. τ' is an $M = (r,s)$-tuple (resp. an $N = (a,b)$-tuple) of integers defining the highest weight of a finite-dimensional representation of K (resp. K'), also denoted τ (resp. τ), which is in turn the minimal K-type (resp. K'-type) of a representation π (resp. π') of G (resp. G'). As in [H4,(3.12)-(3.13)], we have

$$\Theta(G \to G'; \pi^{\vee}) = \pi' \tag{2.3.4}$$

The following theorem is due to A. Paul [Pa].

Theorem 2.3.5. *Suppose $N \leq M$. Then there is at most one signature (a,b) such that $\Theta(G \to G' = U(a,b); \tilde{\pi}^{\vee})$ is non-trivial. If $N = M$ there is exactly one such signature (a,b).*

This is an archimedean refinement of the principle of theta dichotomy studied in [HKS] for unitary groups over p-adic fields. For discrete series the signature (a,b) is determined uniquely by (2.3.2), when $N = M$. The existence and uniqueness of $U(a,b)$ when $N = M$ is stated as Theorem 0.1 of [Pa]. The uniqueness assertion when $N < M$ is not stated explicitly by Paul, but it follows immediately from her method. Indeed, suppose $\tilde{\pi}^{\vee}$ lifts non-trivially to $U(a,b)$ and to $U(a',b')$, with $a+b = a'+b' \leq M$. Then the proof of Theorem 2.9 of [Pa] shows that the trivial representation $\mathbf{1}$ has non-zero lift to $U(a+b', a'+b)$. But Lemma 2.10 of [Pa] shows that this is impossible unless either $a+b' = a'+b$, in which case $a = a', b = b'$, or $a + b' \geq M$ and $a' + b \geq M$, which is excluded.

On the other hand, when $N > M$ then, provided the parameters satisfy the inequalities (2.3.1) and (2.3.2), there are generally several possible signatures allowing non-trivial theta lifts. The same reasoning as in the proof of 2.3.5 (a) yields the following assertion:

Proposition 2.3.6. *Under the above hypotheses, suppose $N \geq M$. Then there are at most $N - M + 1$ signatures (a, b) such that $\Theta(G \to G' = U(a,b); \tilde{\pi}^\vee)$ is non-trivial.*

For example, if $N = M + 1$ and $b = 0$ and $b' \neq 0$ then the inequality $a' + b \geq M$ from Lemma 2.10 of [Pa] implies $(a', b') = (M, 1)$. We will be most interested in the case where $v(+) = u(-) = 0$. In this case $\tilde{\pi}^\vee$ is an antiholomorphic representation and the assumption that it is in fact an antiholomorphic discrete series imposes additional inequalities on the parameters.

In the present paper we are concerned with the case where $N = M + 1$, $U(M) = U(s, r)$ has signature (s, r), $U(N) = U(0, N)$ is negative-definite, and we wish to lift antiholomorphic discrete series representations of $U(M)$ – i.e., representations corresponding to antiholomorphic cohomology classes – non-trivially to (finite-dimensional irreducible) representations of $U(N)$.

Sign conventions are as in [H3] and [H4]. An antiholomorphic representation Π of $U(M)$ is determined by the highest weight τ of its minimal $K_\infty = U(s) \times U(r)$-type:

(2.3.7) $\tau = (-\beta_s, \ldots, -\beta_1; \alpha_1, \ldots, \alpha_r), \beta_1 \geq \beta_2 \geq \cdots \geq \beta_s \geq -\alpha_r \cdots \geq -\alpha_1.$

All the α_i and β_j are non-negative integers and are positive when Π is in the discrete series. One checks that this implies that the dual $\tau^\vee$ of τ is the highest weight of an irreducible finite dimensional representation of $U(M)$ with the usual choice of positive roots. In order to have a lifting to $U(0, N)$, with respect to the splitting characters χ and χ', we need

$$-\beta_s \leq -\frac{N - \alpha(\chi')}{2} = -q_1(\chi'), \alpha_r \geq \frac{N + \alpha(\chi')}{2} = q_2(\chi') \tag{2.3.8}$$

(In [Li90] one replaces $q_1(\chi')$ and $q_2(\chi')$ by $\frac{N}{2} = \frac{N-0}{2}$ where $(0, N)$ is the signature. Here the shift by $\frac{\alpha(\chi')}{2}$ has been incorporated, as in [H4, II.3], in order to obtain integral parameters.) For τ satisfying (2.3.8), the theta lift of the corresponding Π is the finite-dimensional representation

(2.3.9) $\Theta_{\chi,\chi'}(\Pi) = \Pi' =$

$(\beta_1 - q_1(\chi'), \ldots, \beta_s - q_1(\chi'), 0, -\alpha_r + q_2(\chi'), \ldots, -\alpha_1 + q_2(\chi')) + q'(\chi)(1, \ldots, 1)$

where $q'(\chi) = \frac{r-s+\alpha(\chi)}{2}$.

One easily derives the following result from examination of the parameters above, but the methods of [Pa] provide a simpler proof:

Lemma 2.3.11. *Suppose Π is an antiholomorphic discrete series representation of $U(s, r)$ that admits a non-trivial theta lift to $U(0, r + s + 1)$ for the splitting characters χ, χ'. Then Π does not lift to $U(0, r + s - 1)$.*

Proof. We write $M = r + s$. As in the proof of Theorem 2.9 of [Pa], the assertion is reduced to the verification that, in the notation of [loc. cit.], $\theta_{M-1,M+1}(\mathbf{1}) = 0$, where $\mathbf{1}$ is the trivial representation of $U(M - 1, M + 1)$. But this follows again from Lemma 2.10 of [loc. cit.]

This has the following consequence for local zeta integrals. Let $V(a, b)$ be the hermitian space of signature (a, b), $a + b = M + 1$, and let

$$R_M(V(a,b), \chi) \subset I_M(\frac{1}{2}, \chi)_\infty$$

be the space of K_∞-finite vectors in the image of $\mathcal{S}(V(a,b)^M)_\infty$ under the analogue of the map (2.1.2).

Proposition 2.3.12. *Let* Π *be an antiholomorphic discrete series representation of* $U(s,r)$ *that admits a non-trivial theta lift to* $U(0,r+s+1)$ *for the splitting characters* χ,χ'. *Then there is a function* $f \in R_M(V(M+1,0),\chi)$ *such that, if it is extended to a section* $f(h;\chi,s) \in I_M(s+\frac{1}{2},\chi)_\infty$ *(cf. (1.2.7)), then the local zeta integral* $Z_\infty(s,\phi,\phi',\chi,f)$ *does not vanish at* $s=\frac{1}{2}$ *when* $\phi\otimes\phi'$ *is the antiholomorphic vector in* $\Pi(2)$.

Proof. By Lemma 1.3.6, the local zeta integrals for discrete series representations converge absolutely for $Res \geq 0$, and in particular down to the unitary axis, for archimedean as well as non-archimedean local fields. The point is now the following. It follows from the results of [LZ] that

$$I_M(\frac{1}{2},\chi)_\infty = \sum_{a+b=M+1} R_M(V(a,b))$$

(this is mainly contained in Theorem 5.5 and Proposition 5.8 of [LZ], though one has to read the notation carefully and compare it to the earlier results of Lee to derive this claim). Now the evaluation at $s=\frac{1}{2}$ of $Z_\infty(s,\phi,\phi',\chi,f)$ is non-zero for some choice of $f \in I_M(\frac{1}{2},\chi)_\infty$. If $f \in R_M(V(a,b))$, then the non-triviality of the local zeta integral yields a non-trivial theta lift from Π to $U(a,b)$, as in the non-archimedean case (cf. the proof of Proposition 2.1.8). On the other hand, by Proposition 2.3.6, if $a+b=M+1$, the theta lift of Π to $U(a,b)$ is zero unless $(a,b)=(M+1,0)$ or $(a,b)=(M,1)$. It follows from Theorem 5.5 of [LZ] that $R_M(V(M+1,0)) \subset R_M(V(M,1))$, so we can assume $f \in R_M(V(M,1))$.

Now suppose the proposition is false. Then as in the proof of Proposition 2.1.8, the zeta integral gives a non-trivial (bilinear!) pairing

$$\Pi(2)\otimes R_M(V(M,1))/R_M(V(M+1,0)) \to \mathbb{C}.$$

Again by Theorem 5.5 of [LZ], this quotient is isomorphic to $R_M(V(M-1,0))$. Thus (by duality) Π lifts non-trivially to $U(0,M-1)$. But this contradicts Lemma 2.3.11.

This result can be refined as follows. The space $R_M(V(M+1,0),\chi)$ is a holomorphic unitarizable representation of $U(M,M)$ (because it is a theta lift from a positive-definite unitary group). Its restriction to $U(s,r)\times U(r,s)$ decomposes discretely with multiplicity one as a sum of holomorphic representations of $U(s,r)\times U(r,s)$ – the discrete decomposition follows from unitarity, and the multiplicity one is the content of Lemma 3.3.7 of [H3].

Corollary 2.3.13. *In the statement of Proposition 2.3.12, one can take the function* f *to be a non-zero holomorphic vector in the unique subspace of the restriction to* $U(s,r)\times U(r,s)$ *of* $R_M(V(M+1,0),\chi)$ *isomorphic to the dual of* $\Pi(2)$.

Proof. Obviously only the dual $\Pi(2)^\vee$ of $\Pi(2)$ can pair non-trivially with $\Pi(2)$, and since the holomorphic K_∞-type subspace of $\Pi(2)^\vee$ is of multiplicity one, the corollary is clear.

Appendix. Generic calculation of the unramified correspondence.

In this section the residue characteristic p is *odd*. A formula determining the correspondence between spherical representations of unramified unitary groups has been known for some time, but no proof seems to be available in the literature. We derive the formula in the case of odd residue characteristic from the results of Howe, Rallis, and Kudla presented in [MVW]. This is sufficient to characterize the global correspondence at almost all places.

Notation is as in §2.2; however here we do not assume $m = \dim V \leq n = \dim W$. We choose self-dual lattices $L \subset W$, $L' \subset V$, as in [MVW], Chapter 5, and let $U = Stab(L) \subset G$, $U' = Stab(L') \subset G'$. Then U and U' are hyperspecial maximal compact subgroups of the respective unitary groups. We let $\mathcal{H} = \mathcal{H}(G//U)$ denote the Hecke algebra of U-biinvariant compactly supported functions on G, $\mathcal{H}' = \mathcal{H}(G'//U')$. Both algebras are commutative, and the Satake isomorphism identifies their spectra $Z = Spec(\mathcal{H})$, $Z' = Spec(\mathcal{H}')$ with affine spaces of dimensions $[\frac{m}{2}]$, $\frac{n}{2}]$, respectively.

Let $\mathcal{O}$ denote the ring of integers in K, $A = L \otimes_{\mathcal{O}} L' \subset W \otimes_K V$. The splitting characters $\chi, \chi'$are assumed to be unramified; this determines them uniquely, so they will be omitted from the notation. The additive character $\psi : F \rightarrow \mathbb{C}^\times$, implicit in all our constructions, is also unramified in the sense that ψ is trivial on the integer ring $\mathcal{O}_F$ of F but on no larger $\mathcal{O}_F$-submodule of F. Recall that W is supposed skew-hermitian and V hermitian, though this is inessential for what follows; the lattice L remains self-dual for the hermitian form obtained by means of the trace-zero element δ introduced at the beginning of §2.1 is a unit, as long as the δ is a unit .

The natural skew-hermitian form on $W \otimes_K V$, composed with the trace $Tr_{K/F}$, defines a non-degenerate alternating form $< \bullet, \bullet >$ on $\mathcal{V} = R_{K/F} W \otimes_K V$. The metaplectic group $\widetilde{Sp}(\mathcal{V})$ in our setting is an extension of the symplectic group $Sp(\mathcal{V})$ by $\mathbb{C}^\times$, as in [HKS]. The oscillator representation $\omega_{V,W,\chi,\chi'}$ of $G \times G'$ is defined by means of an embedding

$$i_{V,W,\chi,\chi'} : G \times G' \hookrightarrow \widetilde{Sp}(\mathcal{V})$$

determined by the (unique) choice of splitting characters, composed with the oscillator representation of $\widetilde{Sp}(\mathcal{V})$ in one of its models. In the earlier section we worked with the Schrödinger model or mixed Schrödinger model, to which we will return below. For our present purposes, we need to use the lattice model, which is a representation of $\widetilde{Sp}(\mathcal{V})$ on the space

$$\mathcal{T} = \mathcal{T}_\psi(\mathcal{V}) = \{f \in C_c^\infty(\mathcal{V}) \mid \ f(a+v) = \psi(\frac{< v, a >}{2}) f(v), \ a \in A, v \in \mathcal{V}\}.$$

We let $t_A \in \mathcal{T}$ denote the characteristic function of A.

Let π be an irreducible admissible representation of G. The lattice version of the theta correspondence is defined as in (2.1.5), with $\omega_{V,W,\chi}$ replaced by the isomorphic lattice model $\mathcal{T}$:

$$\Theta^T(V, \pi) = [\mathcal{T} \otimes \pi]_G.$$

We assume π is in the domain of the Howe correspondence; $\Theta^T(V, \pi) \neq 0$. Let π' be an irreducible admissible representation of G' for which there exists a surjective $G' \times G$-invariant map $p : \mathcal{T} \rightarrow \pi' \otimes \pi$; in other words, π' is the representation of G' associated to π by the Howe correspondence.

Theorem 2.A.1. *Assume π is spherical, i.e. $\pi^U \neq 0$. Then*
(a) $(\pi')^{U'} \neq 0$;
(b) $(\pi')^{U'} \otimes \pi^U = p(t_A)$.

Proof. The first assertion is Theorem 7.1 (b) of [Howe], proved as Theorem 5.I.10 of [MVW]. Assertion (b) is in fact a special case of Lemma 5.I.8 of [MVW], also due to Howe.

The oscillator representation restricts to a representations of $\mathcal{H}$ and $\mathcal{H}'$ on the subspace $\mathcal{T}^{U\times U'}$. Let H, H' denote their respective images. These correspond to closed affine subschemes $Z_V = Spec(H) \subset Z$ and $Z'_W = Spec(H') \subset Z'$.

Theorem 2.A.2 (Howe).
(a) *The subalgebras H and H' of $End(\mathcal{T}^{U\times U'})$ are equal. Moreover,*
(b) $\mathcal{T}^{U\times U'} = \mathcal{H} \cdot t_A = \mathcal{H}' \cdot t_A$.

Proof. The first assertion is Theorem 7.1 (c) of [Howe], whose proof was first published as Proposition 5.I.11 of [MVW]. Part (b) is the main step of the proof of the latter proposition, on p. 107 of [MVW].

Corollary 2.A.3. *The Howe correspondence takes spherical representations of G (in its domain) to spherical representations of G', and is defined on Satake parameters by a morphism $Z_V \to Z'$ of affine schemes. More precisely, the domain Z_V^0 of the Howe correspondence is a subset of the closed subscheme $Z_V \subset Z$, and the Howe correspondence on Satake parameters is defined by the restriction to Z_V^0 of a morphism $Z_V \to Z'$*

Proof. Let $h : Z_V \xrightarrow{\sim} Z'_W$ be the isomorphism determined by Theorem 2.A.2. It suffices to show that $z_{\pi'} = h(z_\pi)$, whenever π and π' correspond as above, where $z_\pi \in Z_V$, $z_{\pi'} \in Z'_W \subset Z'$ are their respective Satake parameters; and that every π with $z_\pi \in Z_V$ is in the domain of the Howe correspondence. Now it follows from 2.A.1(b) and 2.A.2 (b) that the action z_π of $\mathcal{H}$ on π^U factors through the action of the quotient H on $\mathcal{T}^{U\times U'}$, and in particular that $z_\pi \in Z_V$; likewise for $z_{\pi'}$. The first claim now follows from the definition of h.

To our knowledge the exact domain Z_V^0 of the spherical Howe correspondence has not been determined in the literature. The results of [HKS] provide some qualitative information about Z_V^0. By symmetry, it suffices to assume $n \geq m$.

Proposition 2.A.5. *Suppose $n = m$. Then the domain Z_V^0 of the Howe correspondence equals Z. In particular, $Z_V = Z$.*

Proof. We apply the dichotomy criterion Theorem 6.1 (ii) of [HKS] . In our situation, both V and W are assumed to have self-dual lattices, hence $\det(V)$ and $\det(W)$ are units and so

$$\epsilon_{K/F}(\det(V)) = \epsilon_{K/F}(\det(W)) = 1.$$

Bearing in mind that K/F is unramified and p is odd, the dichotomy criterion then states

$$\Theta_\chi(\pi, V) \neq 0 \Leftrightarrow \epsilon(\frac{1}{2}, \pi, \chi, \psi) = 1.$$

The root number on the right is that defined by the Piatetski-Shapiro-Rallis doubling method, and is known to equal 1 when π, χ, and ψ are all unramified. Thus the dichotomy criterion, as strengthened by (2.1.7)(iv), implies that an unramified π belongs to Z_V^0

Corollary 2.A.6. *Suppose $n = m + 2b$, where $b \geq 0$. Then the domain Z_V^0 of the Howe correspondence equals Z.*

Proof. This follows from Kudla's persistence principle [K,MVW] and from (2.A.5).

Lemma 2.A.7. *Suppose n is even and $m < n$, or n is odd and $m < n-1$. Then the set Z_V^0 is contained in a proper Zariski closed subset of Z.*

Proof. Inspection of the parameters in Corollary 2.2.7 – where m is replaced by $n - b$ – shows that at least one parameter defining π is fixed, independently of τ, once $n > m$ or n is odd and $n > m + 1$.

Remark 2.A.8. Lemma 2.A.7 remains valid whether or not V contains a self-dual lattice. Indeed, Corollary 2.2.7 calculates the image of the Howe correspondence from a unitary group of lower rank almost everywhere, avoiding the subset where the condition (a)(i) of Proposition 2.2.5 applies. However, it is obvious that the theta lifts from that subset also define a proper closed subset when $n > m$ (or $n > m + 1$ when n is odd).

Corollary 2.A.9. *Suppose n is even and $m \geq n$. Then the set Z_V^0 is a Zariski open subset of Z.*

Proof. When m is even, this is (2.A.6). By Kudla's persistence principle, it suffices to treat the case $m = n + 1$, and it suffices to show that any π outside a proper closed subset of Z is ambiguous for $n + 1$ and χ. By Proposition 2.1.8, it suffices to show that, for π outside a proper closed subset of Z, all theta lifts of π to $n-1$-dimensional spaces vanish. But this follows from Lemma 2.A.7 and Remark 2.A.8.

Corollary 2.A.10. *Suppose n is odd and $m \geq n-1$. Then the set Z_V^0 is a Zariski open subset of Z.*

Proof. As in (2.A.9), it suffices to consider the case $m = n - 1$. Say $n = 2r + 1$. It follows from (2.A.9) that $(Z')_W^0$ is open in Z', hence is a scheme of dimension r. It then follows from Howe's theorem (2.A.2) that Z_V^0 is also of dimension r, hence is open in Z.

Corollary 2.A.11. *Let p be an odd prime. Suppose $m \geq n$ or n is odd and $m = n - 1$. Then*

(a) The domain of the Howe correspondence contains a Zariski open subset of the space of unramified representations;

(b) On this Zariski open subset. the Howe correspondence on spherical representations is given by the formulas in Corollary 2.2.7.

Proof. Corollary 2.2.7 asserts that the Howe correspondence is given by the formulas for almost all unramified representations in the domain of the correspondence. Under the present hypotheses, (a) follows from (2.A.9) and (2.A.10). Now Corollary 2.A.3 implies that the spherical Howe correspondence is given by a morphism of schemes. The formulas in (2.2.7) are thus valid on all points of the domain of the correspondence.

3 Applications to special values of L-functions

In this section we assume $E = \mathbb{Q}$, for applications to the questions considered in [H3, H4]. We begin by introducing notation, as in [H5]. Let $G = GU(W)$, the unitary similitude group of a hermitian space $W/\mathcal{K}$ with signature (r, s) at infinity. We associate to $GU(W)$ a Shimura variety $Sh(W)$ of dimension rs, following the conventions of [H3].

Let S be a finite set of finite places of $\mathbb{Q}$. We define the motivically normalized standard L-function, with factors at S (and archimedean factors) removed, to be

$$L^{mot,S}(s, \pi \otimes \chi, St, \alpha) = L^S(s - \frac{n-1}{2}, \pi, St, \alpha) \tag{3.1.}$$

The motivically normalized standard zeta integrals are defined by the corresponding shift in (2.1.7.2). A variant of the following theorem is stated as Theorem 4.3 of [H5], and generalizes Theorem 3.5.13 of [H3], to which we refer for unexplained notation.

Theorem 3.2. *Let $G = GU(W)$, a unitary group with signature (r, s) at infinity, and let π be a cuspidal automorphic representation of G. We assume $\pi \otimes \chi$ occurs in anti-holomorphic cohomology $\bar{H}^{rs}(Sh(W), E_\mu)$ where μ is the highest weight of a finite-dimensional representation of G. Let χ, α be algebraic Hecke characters of $\mathcal{K}^\times$ of type η_k and η_κ^{-1}, respectively. Let s_0 be an integer which is critical for the L-function $L^{mot,S}(s, \pi \otimes \chi, St, \alpha)$; i.e. s_0 satisfies the inequalities (3.3.8.1) of [H3]:*

$$\frac{n-\kappa}{2} \leq s_0 \leq min(q_{s+1}(\mu) + k - \kappa - \mathcal{Q}(\mu), p_s(\mu - k - \mathcal{P}(\mu)), \tag{(**)}$$

Define $m = 2s_0 - \kappa$. Let α^* denote the unitary character $\alpha/|\alpha|$ and assume

$$\alpha^* \mid_{\mathbf{A}_\mathbb{Q}^\times} = \varepsilon_\mathcal{K}^m. \tag{3.2.1}$$

Suppose there is a positive-definite hermitian space V of dimension m, a factorizable section $\phi_f(h, s, \alpha^*) \in I_n(s, \alpha^*)_f$, factorizable vectors $\varphi \in \pi \otimes \chi$, $\varphi' \in \alpha^* \cdot (\pi \otimes \chi)^\vee$, and a finite set S of finite primes such that

- (a) *For every finite v, $\phi_v \in R_n(V_v, \alpha^*)$;*
- (b) *For every finite v in S, π_v does not occur in the boundary at s_0 for α_v^*, and π_v is ambiguous for m and α^*;*
- (c) *For every finite v, $\Theta_{\alpha^*}(\pi_v \otimes \chi_v, V_v) \neq 0$;*
- (d) *For every finite v outside S, all data (π_v, χ_v, α_v, and the additive character ψ_v) are unramified.*

Then

(i) One can find ϕ_f, φ, φ' satisfying (a) and (b) such that ϕ_f takes values in $(2\pi i)^{(s_0+\kappa)n} L \cdot \mathbb{Q}^{ab}$, and such that φ, φ' are arithmetic over the field of definition $E(\pi)$ of π_f.

(ii) Suppose φ is as in (i). Then

$$L^{mot,S}(s_0, \pi \otimes \chi, St, \alpha) \sim_{E(\pi,\chi^{(2)}\cdot\alpha);\mathcal{K}} c(\pi_\infty) P(s_0, k, \kappa, \pi, \varphi, \chi, \alpha)$$

where $P(s_0, k, \kappa, \pi, \varphi, \chi, \alpha)$ is the period

$$(2\pi i)^{s_0 n - \frac{nw}{2} + k(r-s) + \kappa s} g(\varepsilon_\mathcal{K}^{[\frac{n}{2}]} \cdot \pi^c P^{(s)}(\pi, *, \varphi) g(\alpha_0)^s p((\chi^{(2)} \cdot \alpha)^\vee, 1)^{r-s}$$

appearing in Theorem 3.5.13 of [H3] and $c(\pi_\infty)$ is a non-zero complex constant depending only on π_∞.

(iii) If π_∞ belongs to the integrable discrete series, then we can take $c(\pi_\infty) = 1$.

Comments (3.2.2). (i) We have replaced the value m of [H3] by the value s_0, since m is here reserved for the dimension of an auxiliary space. The notation (r, s) for the signature of W should not interfere with the use of s as a variable.

(ii) The hypothesis (3.2.1) guarantees that α^* can be used as a splitting character for the pair $(U(W), U(V))$, hence the condition (a) makes sense. The extension of the theta correspondence to unitary similitude groups is explained in detail in [H4,I.4] and will not be repeated here. The hypothesis should of course not be necessary, and in the cases treated by Shimura in [S1,S2] – in which E_μ is a line bundle – there is apparently no such hypothesis. However, the cases treated here are sufficient for the applications to period relations.

(iii) In [H5] conditions (a)-(d) is replaced by conditions (a) and

(c') The normalized local zeta integrals $\tilde{Z}_v^{mot}(s, \varphi_v, \varphi'_v, \phi_v, \alpha_v^*)$ do not vanish at $s = s_0$.

We derive (c') from (a)-(d) as in [HKS]. Indeed, if the normalized local zeta integral at $s = s_0$ is non-zero then, under hypothesis (a), it defines a non-trivial element of the space

$$Hom_{U(W)\times U(W)}(R_n(V_v, \alpha^*), \pi \otimes (\chi \cdot \pi^\vee)) \tag{3.2.3}$$

and hence the theta lift is non-trivial by Proposition 3.1 of [HKS]. Conversely, the non-triviality of the theta lift is equivalent to non-triviality of (3.2.3) [*loc. cit*]. Now the normalized local zeta integral is non-zero for some choice of data. If v is not in S, then (d) implies the nonvanishing of the normalized local zeta integral at v. If $v \in S$, then it follows from (b) and Proposition 2.1.9 that the normalized zeta integral is non-trivial for some choice of φ_v satisfying (a). Thus Theorem 3.2 does indeed follow from Theorem 4.3 of [H5].

One should not have to assume that π_v does not occur in the boundary, but for the time being we do not know how to prove that the dimension in (3.2.3) is at most one without making such a hypothesis. The following Corollary shows it is in fact unnecessary, at least for tempered π, except when $m = n + 1$:

Corollary 3.3. *Under the hypotheses of Theorem 4.3, assume π_f tempered at all but at most finitely many places. Suppose $m - n \geq 2$. Then hypotheses (a) and (c') are automatically satisfied for some choice of positive-definite V of dimension m. In particular, the conclusion of Theorem 3.2 holds unconditionally for such π and χ whenever $s_0 - \frac{n-\kappa}{2} \geq 1$.*

Proof. To simplify the exposition, we write π instead of $\pi \otimes \chi$ in what follows. Let S be the set of finite places v of $\mathbb{Q}$ that do not split in $\mathcal{K}$ and for which π_v does not belong to the principal series, together with the prime 2 and any places at which π_v is not tempered. By Proposition 2.1.6 (iii) for every $v \in S$ there exists a hermitian space V_v (up to isomorphism) such that

$$Hom_{G\times G}(R_n(V_v, \alpha^*), \pi \otimes (\alpha^* \cdot \pi^\vee)) \neq 0 \tag{*.}$$

(When v is split there is only one choice of V_v and (*) remains true, cf. 2.2.9.) Moreover, the proof of 2.1.7 (iii) in [HKS; 4.5 and p. 973] shows that the normalized local zeta integral gives a non-zero pairing in (*); hence condition (c') of (3.2.2)(iii) is satisfied. Let V be any positive-definite hermitian space of dimension m with V_v satisfying $*$ for all $v \in S$. For $v \notin S$, v not split in $\mathcal{K}$ π_v is a tempered principal series con stituent, hence is ambiguous for m and α^* by Corollary 2.2.2; thus there is no local obstruction at such v. Finally, for v split in $\mathcal{K}$ there is only one V_v up to isomorphism and the analogue of $R_n(V_v, \alpha^*)$ coincides with the whole of $I_n(s_0, \alpha^*)$ [KS, Theorem 1.3]. The claim then follows from Corollary 2.1.9.

The case $m = n$ corresponds to the central critical value, where the L-function can vanish, and requires slightly different methods, as in [HK] and [KR1]. There the global induced representation breaks up as an infinite direct sum $\oplus_J I_J$ over finite sets J of finite primes. The sets J of even cardinality correspond to hermitian spaces V_J of dimension $m = n$; and the conditions (a) and (b) apply with $\phi_f \in I_J$ if and only if the theta lift to V_J is non-trivial, provided the L-function does not vanish at s_0. (If the L-function vanishes there is nothing more to prove.) On the other hand, when the cardinality of J is odd then the Eisenstein series corresponding to sections in I_J vanish. One can thus obtain an unconditional result in this case as well.

It remains to treat the case $m = n+1$, corresponding to the "near central point" of the L-function. This is more subtle, and even for tempered π we do not have unconditional results, except for "most" α:

Theorem 3.4. *Under the hypotheses of Theorem 3.2, assume π_f tempered at all but finitely many places and $m = n + 1$. Fix a finite place v that does not split in $\mathcal{K}$ such that π_v is tempered. There is a finite set of characters A_v of $K_v^\times$ satisfying (4.3.1) such that, if α is a splitting character such that $\alpha_v^* \notin A_v$, then there exists a hermitian space V of dimension $n+1$ for which hypotheses (a) and (b) of Theorem 3.2 are satisfied for π, χ, α, and s_0. In particular, for such α the conclusions of Theorem 3.2 hold unconditionally.*

Proof. We let A_v be the set of splitting characters ξ such that $\pi \otimes \chi$ is unambiguous relative to $m = n+1$, χ', and ξ, where χ' is a fixed splitting character for $U(W)$. By Proposition 2.2.3 A_v is a finite set. Choose α such that $\alpha_v^* \notin A_v$. Now Proposition 2.1.6 implies that, for every finite place $w \neq v$ there is a hermitian space V_w (up to isomorphism) of dimension $m + 1$ such that

$$Hom_{G \times G}(R_n(V_w, \alpha_w^*), (\pi_w \otimes \chi_w) \otimes (\alpha_w^* \cdot (\pi_w \otimes \chi_w)^\vee)) \neq 0 \tag{3.4.1}$$

and contains the normalized zeta integral at $s = s_0$ as a non-zero element. There is exactly one global positive definite hermitian space V such that V_w is in the chosen isomorphism class at every finite prime $w \neq v$, the localization V_v at v being determined by the vanishing of the product of the local Hasse invariants. But π_v is ambiguous for m and α_v^*, hence (3.4.1) holds at v as well, and again the normalized zeta integral defines a non-zero pairing. The same is true for any Galois conjugate of α. We now conclude as in the proof of Corollary 3.3. Galois equivariance is guaranteed by Lemma 2.1.10.

Under additional hypotheses, the proof gives the following refinement:

Corollary 3.4.2. *Suppose π admits a global base change to a cuspidal automorphic representation Π of $GL(n, \mathcal{K})$. Then there exists a positive-definite hermitian space V over $\mathcal{K}$ of dimension $n + 1$ and a pair of splitting characters α^*, χ' such that $\Theta_{\alpha^*,\chi'}(G' \to U(V); \pi \otimes \chi) \neq 0$.*

Proof. We argue as above. If π admits a base change then so does $\pi \otimes \chi$. To simplify notation we replace π by $\pi \otimes \chi$. Under the hypothesis, we have

$$L^S(1, \pi, St, \alpha^*) = L^S(1, \Pi \otimes \alpha^* \circ \det) \neq 0; \tag{3.4.2.1}$$

the non-vanishing of $L^S(1, \Pi \otimes \alpha^* \circ \det)$ is the result of Jacquet-Shalika.

Given a choice of local hermitian spaces V_v for all places v of E (only the non-split places matter), there exists a global hermitian space V with localizations V_v for all v if and only if the product of the Hasse invariants $\prod_v \varepsilon(V_v) = 1$. We choose χ' arbitrarily. Let S be as in the statement of Theorem 3.2, and choose α satisfying conditions (b) and (d). It follows from Lemma 2.1.6.2 and Proposition 2.2.3 that the set of such α is infinite. Let $v_0 \in S$ so that π_{v_0} is ambiguous. As above, for each non-split prime v, there exists some V_v such that $\Theta_{\chi_v, \chi'_v}(G'_v \to U(V_v); \pi_v) \neq 0$. If necessary, we switch the form V_{v_0} so that the product of the Hasse invariants is 1; then we can assume that the V_v all correspond to localizations of a global hermitian space V.

Thus the local theta lifts are non-trivial everywhere and condition (b) of [H5, Theorem 4.3] is satisfied. It remains to show that the global theta lift is non-zero. This follows from the Rallis inner product formula, which will be stated below, and (3.4.2.1). However, there is no need to quote the Rallis inner product formula *in extenso.* The nonvanishing (3.4.2.1), together with the non-vanishing of local zeta integrals, implies that the global zeta integral

$$Z^+(s, \phi, \phi', \alpha, \varphi) = \int_{Z(\mathbf{A}) \cdot G(\mathbb{Q}) \backslash G(\mathbf{A})^+} E(i_V(g, g'), \alpha, s, \varphi) \phi(g) \phi'(g') dg dg' \tag{3.4.2.2}$$

does not vanish. But (3.4.2.2) is the pairing of a Siegel-Weil Eisenstein series lifted from $U(V)$ against the vector $\varphi \otimes \phi' \in \pi \otimes (\alpha^* \cdot \pi^\vee)$. By seesaw duality, the theta lift of this vector to $U(V) \times U(V)$ is non-trivial.

3.5. Extensions of the Rallis inner product formula.

We recall the main theorems of [I1, I2], in the form in which they are presented in [H5]. Let $G = GU(W)$, $H = GU(2W)$, $G' = GU(V)$, with $\dim W = m$, $\dim V = n$ (note the switch!) and choose splitting characters χ, χ' as above. We consider theta lifts from G' to H. The extension to similitude groups as in [H4, H5]: to a pair consisting of an automorphic form F on $G'(\mathbf{A})$ and a function $\phi \in \mathcal{S}(V^m)$ we obtain a function

$$\theta_{\phi, \chi, \chi'}(F) : H(E) \backslash H(\mathbf{A})^+ \to \mathbb{C}$$

where $H(\mathbf{A})^+ = GU(2W)(\mathbf{A})^+$ is as defined in (0.4).

Theorem 3.5.1 [I1]. *Suppose $n < m$, and let $s_0 = \frac{m-n}{2}$, $r_0 = 2s_0$. Let $V' = V \oplus \mathbb{H}^{r_0}$, where $\mathbb{H}$ is a hyperbolic hermitian space of dimension 2. There is an explicit real constant c_K such that, for any K_H-finite $\Phi \in \mathcal{S}((V')^n(\mathbf{A}))$,*

$$Res_{s=s_0} E(h, s, \phi_\Phi, \chi) = c_K \theta_{\pi_{V'}^V, \pi_K(\Phi), \chi, triv, \psi}(h)$$

for all $h \in H(\mathbf{A})^+$.

The index K in c_K denotes a maximal compact subgroup of $G'(\mathbf{A})$, and c_K is a factor of proportionality relating two Haar measures, calculated explicitly in

[I1,§9] We refer to [I1, §4] and Remark 3.5.15, below, for the notation $\pi^{V}_{V'}$ (which Ichino calls $\pi^{Q}_{Q'}$) and π_K. Note in particular that $I_{\chi,triv,\psi}(*)$ denotes the *regularized* theta integral, see (3.5.4.3), below. The extension to the subgroup $H(\mathbf{A})^+$ of the similitude group is as in Corollary 3.3.2 of [H5]. For $n \geq m$ Ichino has proved the following theorem; we follow the statement of [H5, Corollary 3.3.2].

Theorem 3.5.2 [I2]. *Suppose $n \geq m$ and V is positive definite. Then the Eisenstein series $E(h, s, \phi_\Phi, \chi)$ has no pole at $s = s_0$, and*

$$I_{\chi,triv,\psi}(\Phi)(h) = c \cdot E(h, s_0, \phi_\Phi, \chi)$$

for $h \in GH(\mathbf{A})^+$, where $c = 1$ if $m = n$ and $c = \frac{1}{2}$ otherwise.

In particular, we have the Rallis inner product formula: setting

$$Z^+(s, \phi, \phi', \chi, \varphi) = \int_{Z(\mathbf{A})\cdot G(\mathbb{Q})\backslash G(\mathbf{A})^+} E(i_W(g, g'), \chi, s, \varphi)\phi(g)\phi'(g')dgdg'$$

we have

$$Z^+(s, \phi, \phi', \chi, \varphi) = c \cdot I_{\Delta,\chi,\chi'}(\theta_{\varphi,\chi,\chi'}(\phi \otimes \phi')).$$

An almost immediate consequence of Theorem 3.5.1 is the characterization of poles of standard L-functions of automorphic representations of G. The analogous case of standard L-functions of symplectic groups has been considered by Kudla and Rallis in [KR1], and the proof goes over without change. We derive the regularized Rallis inner product formula as in [KR1,§8], and compare it to the formula proposed in [H4, V.3]. Let π be a cuspidal automorphic representation of G, χ and χ' a pair of splitting characters, $\pi(2)$ any irreducible extension to $G(G \times G^-)$ of the representation $\pi \otimes [\check{\chi} \circ \det \otimes \check{\pi}]$. The normalized diagonal period integrals $I_{\Delta,\chi}$ and $I_{\Delta,\chi'}$ are defined as in [loc. cit.]. Let $F \in \pi(2)$, and let $\phi(s) = \phi(h, s; \chi) \in I_m(s, \chi)$ be a section defining an Eisenstein series as in §1. The factor of the global zeta integral corresponding to the collection S of bad primes is defined, as in (1.3.3), to be

$$I_{\Delta,\chi} Z_S(\phi)(s)(F) = \int_{G_S} I_{\Delta,\chi}\pi(2)(g, 1)F \cdot \phi_S(\delta \cdot (g, 1), s)dg. \tag{3.5.3}$$

Here $G_S = \prod_{w\in S} G(E_w)$, $\delta \in GH$ is a representative of the dense orbit of $G \times G$ on $GP\backslash GH$, and the notation $I_{\Delta,\chi}$ on the left-hand side is a reminder that we are integrating along the diagonal inside the integral over G_S.

Theorem 3.5.4. *With the above notation, let $F \in \pi(2)$ be factorizable. Assume $n < m$. Suppose $\theta_{\varphi,\chi,\chi'}(F)$ is cuspidal on $G(G' \times G'^{,-})$. Then there is a finite set S of finite places of E such that*

(3.5.4.1)

$$c_K I_{\Delta,\chi'}(\theta_{\varphi,\chi,\chi'}(F)) = Res_{s=s_0} d^S_m(s, \chi)^{-1} I_{\Delta,\chi}(Z_S(\Phi)(s)(F)) \cdot L^S(s + \frac{1}{2}, \pi_{\mathcal{K}} \cdot \chi)$$

where $\varphi = \pi^V_{V'}\pi_K(\Phi)$ and $d^S_m(s)$ is given by (1.3.4).

Proof. Notation is as in [H4, V.3]. We suppose we have a form $F \in \pi(2)$ of discrete series type on $G(G\times G^-)$, with $G = GU(V)$, with $\dim V = m$. We define $\theta_{\varphi,\chi,\chi'}(F)$

on $G(G' \times G')$ as in *loc. cit.*, with $G' = GU(V')$, $\dim V' = n$; under the hypotheses of *loc. cit.*, this theta lift is holomorphic.

We recall Ichino's regularization of the theta integral in Theorem 3.5.1. We assume $n \leq m$. Let v be a finite place of E of odd residue characteristic that is inert in $\mathcal{K}$ and unramified over $\mathbb{Q}$ We assume moreover that the splitting characters χ, χ' and the additive character ψ are unramified at v, that the unitary groups G and G' are quasi-split at v, and that the function Φ is fixed by a hyperspecial maximal compact subgroup $K_v \times K'_v \subset H(E_v) \times G'(E_v)$ at v. Ichino also assumes K'_v is well placed with respect to a chosen Siegel set in $G'(\mathbf{A})$. Then [I1,§2] there exist K'_v-biinvariant functions $\alpha \in C_c^\infty(G'(E_v), \mathbb{Q}(q_v^{\frac{1}{2}}))$ and, for each such α, a non-zero constant $c_\alpha \in \mathbb{Q}(q_v^{\frac{1}{2}})^\times$ with the property that the integral defining $I_{\chi,triv,\psi}(\omega_{V,W,\chi}(\alpha)\pi^V_{V'}\pi_K(\Phi))(h)$ is absolutely convergent and, if $I_{\chi,triv,\psi}(\pi^V_{V'}\pi_K(\Phi))(h)$ is already absolutely convergent, then

$$I_{\chi,triv,\psi}(\omega_{V,W,\chi}(\omega_{V,W,\chi}(\alpha)\pi^V_{V'}\pi_K(\Phi))(h) = c_\alpha \cdot I_{\chi,triv,\psi}(\pi^V_{V'}\pi_K(\Phi))(h).$$

Then for general Φ, the regularized theta integral is defined as

$$c_\alpha^{-1} I_{\chi,triv,\psi}(\omega_{V,W,\chi}(\alpha)\pi^V_{V'}\pi_K(\Phi))(h) \tag{3.5.4.2}$$

and is independent of the choice of v and α with the required property (of killing support, cf. the proof of [I1, Prop. 2.4]). Since this property is invariant with respect to $Gal(\mathbb{Q}(q_v^{\frac{1}{2}})/\mathbb{Q})$, the regularization is rational over $\mathbb{Q}$.

By analogy with [KR1 (8.12)], if $F \in \pi(2)$ we define

$$I_{\Delta,\chi',REG}(\theta_{\varphi,\chi,\chi'}(F)) = c_\alpha^{-1} I_{\Delta,\chi'}(\theta_{\alpha*\varphi,\chi,\chi'}(F) \tag{3.5.4.3}$$

where we write $\alpha * \varphi$ for $\omega_{V,W,\chi}(\alpha)\pi^V_{V'}\pi_K(\Phi)$, the action of α at the place v. Then the formula (3.5.4.1), with $I_{\Delta,\chi',REG}$ in the place of $I_{\Delta,\chi'}$, is precisely the analogue of Theorem 8.7 of [KR1], and is proved in the same way. Now we are assuming $\theta_{\varphi,\chi,\chi'}(F)$ is cuspidal, hence that the integral $I_{\Delta,\chi'}(\theta_{\varphi,\chi,\chi'}(F))$ over the diagonal converges absolutely. As in [KR1 (8.14)], one then has

$$I_{\Delta,\chi',REG}(\theta_{\varphi,\chi,\chi'}(F)) = I_{\Delta,\chi'}(\theta_{\varphi,\chi,\chi'}(F)) \tag{3.5.4.4}$$

which completes the proof.

We now take $m = n + 1$ but apply Theorem 3.5.4 to the lift in the opposite direction; i.e. from a cuspidal automorphic representation $\pi'(2)$ of $G(G' \times G'^{,-})$ to $G(G \times G^-)$. We make the following hypotheses:

Hypotheses 3.5.5.

(a) *There is a cuspidal automorphic representation π of G such that $\pi' = \Theta(G \to G'; \pi^\vee)$.*
(b) *The archimedean component π_∞ of π belongs to the discrete series, and π'_∞ is of antiholomorphic type.*
(c) *The representation π occurs with multiplicity one in the space of automorphic forms on G, cuspidal or otherwise.*
(d) *The representation π admits a base change to a cuspidal automorphic representation $\pi_{\mathcal{K}}$ of $GL(n, \mathcal{K})$.*
(e) *At all primes w of E dividing 2, the Howe duality conjecture is valid for π_w.*

Hypothesis (a) and (b) are genuine hypotheses that define the problem under consideration. Hypotheses (c) and (e) are unnecessary but allow us to simplify the following arguments by avoiding the proof that the Petersson norm of arithmetic forms of type π'_f are well-defined up to scalars in the field of rationality of the representation. One conjectures that hypothesis (c) is always satisfied if π belongs to a stable L-packet, and they will probably be proved in the near future. If π is cuspidal and tempered then it may already be known that it is not equivalent to a non-cuspidal representation, but we have included this in Hypothesis (c). Hypothesis (d) is proved in [HL] under the assumption that the local component π_v is supercuspidal or Steinberg for some finite prime v of E that splits in $\mathcal{K}$. Hypothesis (e), which I hope to replace in the final draft, is valid if all primes above 2 split in $\mathcal{K}$ (cf. [M]); it implies that the theta lift in (a) is in fact an irreducible representation of $G(\mathbf{A})$. Of course hypothesis (e) is implied by the Howe duality conjecture.

Proposition 3.5.6. *Under Hypotheses 3.5.5, the representation π' has multiplicity one in the space of cusp forms on G'. In particular, $\Theta(G' \to G; \pi'^{,\vee}) = \pi$.*

Proof. We apply an argument due to Rallis. Let π^* be a cuspidal automorphic representation of G' such that π^*_f and π'_f are isomorphic as abstract representations of $G'(\mathbf{A})$. Since π' is a theta lift from G, it follows from the parameters for the unramified correspondence determined in §2.2 (cf. [H4, I. 3.15]) that, for a sufficiently large set S of bad primes,

$$L^S(s, \pi^*, St) = L^S(s, \pi, (\chi'/\chi), St) L^S(1, \chi') \tag{3.5.6.1}$$

In particular, 3.5.5 (d) implies, as in the proof of Corollary 3.4.2, that

$$res_{s=1} L^S(s, \pi^*, (\chi')^{-1}, St) = L^S(1, \pi_{\mathcal{K}}, \chi^{-1}) res_{s=1} \zeta^S_{\mathcal{K}}(s) \neq 0. \tag{3.5.6.2}$$

By the results of [I1], the arguments of [KR1,Theorem 7.2.5] apply in this situation and imply that π^* is a theta lift, with respect to the splitting character χ', from some unitary group $G^* = U(V^*)$ for some hermitian space V^* of dimension n. Now applying Proposition 2.1.7 (iv), the existence of the theta lifting from $U(V')$ to $U(V^*)$ determines the local isomorphism class of V^* at all finite primes, and Theorem 2.3.5 determines the signature of V^* at ∞. Since $\pi^* \cong \pi'$, we even know that $V^*_v \xrightarrow{\sim} V_v$ for all v. By Landherr's theorem, $V \xrightarrow{\sim} V^*$.

It follows that the theta lift, say π^{**}, of $\pi^{*,\vee}$ to G is non-trivial. Since $\pi^* \xrightarrow{\sim} \pi'$, we must have $\pi^{**} \xrightarrow{\sim} \pi$ as abstract representations. By 3.5.5 we must have $\pi^{**} = \pi$. But by adjunction the L_2 pairing on $G'(\mathbb{Q})\backslash G'(\mathbf{A})$ pairs π^* non-trivially with the theta lift of $\pi^{**} = \pi$, hence $\pi^{*,\vee}$ pairs non-trivially with π'.

Corollary 3.5.7. *Under Hypotheses 3.5.5, let $F \in \pi'(2)$. Then the cuspidality hypothesis in Theorem 3.5.4 is automatic, and we can rewrite (3.5.4.1):*

$$c_K I_{\Delta,\chi'}(\theta_{\varphi,\chi,\chi'}(F)) = d^S_{n+1}(\frac{1}{2}, \chi)^{-1} I_{\Delta,\chi}(Z_S(\Phi)(\frac{1}{2})(F)) \cdot L^S(1, \pi_{\mathcal{K}}, \chi^{-1}) res_{s=1} \zeta^S_{\mathcal{K}}(s).$$

In particular, if $E = \mathbb{Q}$, then, up to $\mathcal{K}$-rational multiples, we have

$$c_K I_{\Delta,\chi'}(\theta_{\varphi,\chi,\chi'}(F)) \sim d^S_{n+1}(\frac{1}{2}, \chi)^{-1} I_{\Delta,\chi}(Z_S(\Phi)(\frac{1}{2})(F)) \cdot L^S(1, \pi_{\mathcal{K}}, \chi^{-1}) L(1, \varepsilon_{\mathcal{K}}).$$

Proof. The simplification of the residue (3.5.4.1) follows from (3.5.6.2) and (1.3.6).

In what follows, we make use of general facts about rationality and Petersson norms developed in the following section. We work in the setting of [H4, V.1], to which we refer for notation, with $m = n+1$. Thus $E = \mathbb{Q}$, V is of signature $(n-1, 1)$, V' of signature (r, s), $r' = r - 1$. We assume $\pi_f^{\vee}(2)$ occurs (with multiplicity one, by 3.5.5(c)) in $H_!^{2(r'-1)}(Sh(V^{(2)}, [W_{\Lambda(2)}])$, and that $\pi'_f(2) = \Theta(\pi(2))_f$ contributes to $H^0(Sh(V'^{,(2)}, [E_{\Lambda'(2)}])$. Let

$$V(\pi)(2) \subset H_!^{2(r'-1)}(Sh(V^{(2)}, [W_{\Lambda(2)}]), \text{ resp. } V(\pi')(2) \subset H_!^0(Sh(V'^{,(2)}, [E_{\Lambda'(2)}])$$

be the $\pi_f^{\vee}(2)$ (resp. $\pi'_f(2)$)-isotypic components. Note that we have switched π with $\pi^{\vee}$. The parameters Λ, Λ', etc. are defined in [H4] and do not need to be recalled here. Because local Howe duality has not been established in residue characteristic 2 we actually have to take an irreducible component $\Theta(\pi(2))_0$ in the definition of $\pi'_f(2)$ above, but this can also be ignored.

We let M be a CM field containing the fields of rationality of π, χ, and χ', over which π_f has a model. It's clear from (3.5.6.1) that π'_f ought also to have a model over M without any additional hypothesis, but we will add this to our assumptions about M. Define $V(\pi)(2)_M \subset H_!^{2(r'-1)}(Sh(V^{(2)}, [W_{\Lambda(2)}])$ and $V(\pi')(2)_M$ as in Proposition 3.6.2. We define $Q(\pi(2)), Q(\pi'(2)), Q(\pi), Q(\pi') \in \mathbb{R}^{\times}/M^{+,\times}$ as in Corollary 3.6.4. Define $I_{\Delta,\chi'}$ and $I_{\Delta,\chi}$ as in [H4,V.3]. The characters χ and χ are introduced only for the sake of the theta correspondence, and the function of the twisted inner products $I_{\Delta,\chi}$ and $I_{\Delta,\chi'}$ is precisely to remove the characters. In particular the rationality of elements of $V(\pi)(2)$ is independent of the choice of χ (cf. [H4, V (1.1), (1.2)]). Moreover, bearing in mind the discussion of complex conjugation in [H3,§2.5], it is clear from the definitions that, if $f \in V(\pi)(2)_M$ (resp. $F \in V(\pi')(2)_M$) then

$$I_{\Delta,\chi}(f) \sim_{M^{\times}} Q(\pi),\ I_{\Delta,\chi'}(F) \sim_{M^{\times}} Q(\pi). \tag{3.5.8}$$

Similarly, we can take

$$Q(\pi(2)) = Q(\pi)^2,\ Q(\pi'(2)) = Q(\pi')^2. \tag{3.5.9}$$

Here and in what follows, at the risk of loss of clarity (but in order to spare the reader excessively long formulas) we write $I_{\Delta,\chi}(f)$ to designate the element of $(M \otimes \mathbb{C})^{\times}$ obtained from the restriction of scalars $R_{M/\mathbb{Q}}\pi(2)_f$, as in [H3, (2.6.2), (2.8.1)]; likewise for the other periods.

Lemma 3.5.10. *Assume $F \in V(\pi)_M(2)$. Let Φ_S be as in Corollary 3.5.7. There is a constant $C(\Phi_S) \in \mathbb{C}^{\times}$ such that*

$$I_{\Delta,\chi}(Z_S(\Phi)(s_0)(F)) \in M^{\times} \cdot C(\Phi_S) \cdot Q(\pi).$$

Proof. The composition of the map

$$\pi \otimes \pi \to \pi \otimes \bar{\pi} \to \pi \otimes [\check{\chi} \circ \det \pi^{\vee}],$$

where the first arrow is complex conjugation in the second variable, with the map map $F \mapsto I_{\Delta,\chi}(Z_S(\Phi)(s_0)(F))$ defines a non-degenerate hermitian form $H_{S,\Phi}(\bullet,\bullet)$ on π_f^S. By Schur's Lemma, there is a constant $C(\Phi_S) \in \mathbb{C}^\times$ such that

$$H_{S,\Phi}(\bullet,\bullet) = C(\Phi_S) < \bullet,\bullet >_M .$$

The lemma follows immediately.

Since Φ and F are assumed to be factorizable, the integral (3.5.3) factors over local primes, as in (1.3.3). Let $S = S_f \cup S_\infty$ be the decomposition of S into finite and archimedean primes. Write $F = f \otimes f' \in \pi(2)$ with $f = \otimes_v f_v$, $f' = \otimes_v f'_v$. Setting

$$C(\Phi_{S_f}, F_{S_f}, \psi_{S_f}) = \prod_{v\in S_f} \int_{G_v} (\pi_v(g_v)(f_v), f'_v) \cdot \phi_S(\delta\cdot(g,1),s)dg_v$$

we have
(3.5.11)

$$I_{\Delta,\chi} Z_S(\phi)(s)(F) = C(\Phi_{S_f}, F_{S_f}, \psi_{S_f}) \cdot \int_{G_{S_\infty}} I_{\Delta,\chi}\pi(2)(g,1)F\cdot\phi_\infty(\delta\cdot(g_\infty,1),s)dg_\infty.$$

Lemma 3.5.12. *Assume Φ_{S_f} and F are defined over $\overline{\mathbb{Q}}$. Then $C(\Phi_{S_f}, F_{S_f}, \psi_{S_f})$ is defined over $\overline{\mathbb{Q}}$ and, for any $\sigma \in Gal(\overline{\mathbb{Q}}/\mathcal{K})$,*

$$\sigma(C(\Phi_{S_f}, F_{S_f}, \psi_{S_f})) = C(\sigma(\Phi_{S_f}), \sigma(F_{S_f}), \sigma(\psi_{S_f})).$$

Proof. As explained in §E.2 of the introduction, this follows from [H4, Lemma IV.2.3.2], in view of the relation (E.2.2) that governs the dependence of the local zeta integrals on the choice of additive characters. There is one additional element: the local zeta integral

Several identifications are implicit in the statement of the Lemma. We are assuming F to be an algebraic coherent cohomology class in $V(\pi)(2)$. If $\sigma \in Gal(\overline{\mathbb{Q}}/M)$ then F is again in $V(\pi)(2)$. If σ fixes the field of rationality of the automorphic vector bundle $[W_{\Lambda(2)}]$, then F is again a $\overline{\mathbb{Q}}$-algebraic class in $H_!^{2(r'-1)}(Sh(V^{(2)}),[W_{\Lambda(2)}])$, but in the subspace $V(\sigma(\pi)(2)$, defined in the obvious way. If σ fixes the reflex field of $Sh(V^{(2)})$, then F is a a $\overline{\mathbb{Q}}$ algebraic class in $H_!^{2(r'-1)}(Sh(V^{(2)}),\sigma[W_{\Lambda(2)}])$. Finally, we have been assuming $E = \mathbb{Q}$, so by hypothesis σ fixes the reflex field of $Sh(V^{(2)})$, which is contained in $\mathcal{K}$; however, this part of the argument works for more general totally real fields.

For the following corollary, we write $Z_S(\Phi,\psi_f)(s_0)(F))$ instead of $Z_S(\Phi)(s_0)(F))$ to emphasize the dependence on the additive character.

Corollary 3.5.13. *Under the hypothesis of Lemma 3.5.10, or more generally if $F \in V(\pi)_{\overline{\mathbb{Q}}}(2)$, there is a constant $C(\Phi_\infty) \in \mathbb{C}^\times$ such that,*

$$I_{\Delta,\chi}(Z_S(\Phi,\psi_f)(s_0)(F))/Q(\pi)\cdot C(\Phi_\infty) \in \overline{\mathbb{Q}}$$

and, for all $\sigma \in Gal(\overline{\mathbb{Q}}/\mathcal{K})$,

$$\sigma(I_{\Delta,\chi}(Z_S(\Phi,\psi_f)(s_0)(F))/[Q(\pi)\cdot C(\Phi_\infty))] = I_{\Delta,\sigma(\chi)}(Z_S(\sigma(\Phi),\sigma(\psi_f)/Q(\sigma(\pi)\cdot C(\Phi_\infty)].$$

Here we need only recall that, as explained in §E.2, the archimedean local zeta integral is unaffected by the conjugation of the additive character at finite places. Also the action of σ on χ is defined in terms of the algebraic Hecke character χ^+ obtained by twisting χ by a certain power of the norm.

3.5.14. Application of Rallis' inner product formula.

We continue to assume $E = \mathbb{Q}$. Start now with $f \in V(\pi)_M(2)$. By Theorem E.2.1 of the introduction (the corrected version of the main theorem of [H4]), there exists an explicit

$$\mathbf{p} = \mathbf{p}(\pi, V, V', \chi_0) \in \mathbb{C}^\times$$

defined in [H4, §V.1] such that

$$\mathbf{p}^{-1}\theta_{\phi,\chi,\chi',\psi_f} \in_{\sim_\mathcal{K}} \pi'(2)(M \cdot \mathbb{Q}^{ab}) \tag{3.5.14.1}$$

provided ϕ is M-rational, where the notation $\in_{\sim_\mathcal{K}}$ means homogeneous with respect to the $Gal(\overline{\mathbb{Q}}/\mathcal{K})$-action on χ, χ', ψ_f, etc. We abbreviate $\theta_{\phi,\chi,\chi',\psi_f}(f) = \theta(f)$, and so on, and use the notation $\sim_\mathcal{K}$ to indicate that $Gal(\overline{\mathbb{Q}}/\mathcal{K})$ is acting on the missing arguments.

Using Corollary 2.2.7 one can show that the field of rationality for $\pi'^{,S}$ is contained in $\mathbb{Q}(\pi, \chi^+, \chi'^{,+})$ for an appropriate finite set S of finite places; it suffices to take S the set of primes where the local L-packets are non-trivial (i.e., where $\pi'^{,S}$ can be completed in more than one way to an automorphic representation. Thus, just as in the proof of Lemma 3.5.12, all constructions that follow are homogeneous over $\mathbb{Q}(\pi, \chi^+, \chi'^{,+})$ and even over $\mathcal{K}$, if we allow $Gal(\overline{\mathbb{Q}}/\mathcal{K})$ to act on π, χ^+, $\chi'^{,+}$, and the automorphic vector bundle $[E_{\Lambda(2)}]$ (cf. Theorem E.2.1). Thus we can choose $M' \supset \mathbb{Q}(\pi, \chi^+, \chi'^{,+})$ to be a field of definition for π' in what follows; this choice also disappears in the end.

Let $F \in V(\pi')_{M'}(2) \subset H^0_!(Sh(V'^{,(2)}), [E_{\Lambda(2)}])$, defined by analogy with $V(\pi)_M(2)$. We know

$$I_\Delta(F) \sim_\mathcal{K} Q(\pi'),\ I_\Delta(f) \sim_\mathcal{K} Q(\pi) \tag{3.5.14.2}$$

Combining (3.5.14.1) and (3.5.14.2) we find

$$I_\Delta(\theta(f)) \sim_\mathcal{K} \mathbf{p} \cdot Q(\pi') \tag{3.5.14.3}$$

By Corollaries 3.5.7 and 3.5.13 we have

$$I_\Delta(\theta(F)) \sim_\mathcal{K} Q(\pi')C(\Phi_\infty)L \tag{3.5.14.4}$$

where $L = [c_K d^S_{n+1}(\frac{1}{2}, \chi)]^{-1}L(1, \pi_\mathcal{K} \cdot \chi^{-1})$.

Now (3.5.14.1), together with Corollary 3.6.4 (applied to the doubled Shimura variety $Sh(V^{2)})$ implies

$$(\theta(f), F) \sim_\mathcal{K} \mathbf{p}Q(\pi')^2 \tag{3.5.14.5}$$

But by the adjunction formula

$$(\theta(f), F) = (f, \theta(F)), \tag{3.5.14.6}$$

Letting f vary among M-rational classes, it follows from (3.5.14.5), (3.5.14.6), and Corollary 3.6.5 that

Lemma 3.5.14.7. *If ϕ is M-rational, then $Q(\pi)^2[\mathbf{p}\cdot Q(\pi')^2]^{-1}\theta(F)$ is M-rational as class in $H^{2(r'-1)}(Sh(V^{(2)}),[W_{\Lambda(2)}])$. In particular*

$$I_{\Delta}(\theta(F)) \sim_{\mathcal{K}} \mathbf{p}\cdot Q(\pi')^2 Q(\pi)^{-2}\cdot Q(\pi) \sim_{\mathcal{K}} \mathbf{p}\cdot Q(\pi')^2 Q(\pi)^{-1}. \tag{3.5.14.8}$$

Comparing (3.5.14.8) with (3.5.14.4), we find that

$$\mathbf{p}\cdot Q(\pi') \sim_{\mathcal{K}} Q(\pi)C(\Phi_\infty)L. \tag{3.5.14.9}$$

Finally, (3.5.14.9) and (3.5.14.3) yield

Corollary 3.5.14.10. *If ϕ is M-rational, then*

$$I_{\Delta}(\theta_{\phi,\chi,\chi'}(f)) \sim_{\mathcal{K}} Q(\pi)C(\Phi_\infty)\cdot[c_K d^S_{n+1}(\frac{1}{2},\chi)]^{-1}L(1,\pi_{\mathcal{K}}\cdot\chi^{-1})\cdot L(1,\varepsilon_{\mathcal{K}})$$

We will return to this formula in §4.

Remark 3.5.15. More on rationality of theta kernels. The results of the previous sections are based on Ichino's identification of the residue of an Eisenstein series with an explicit constant multiple of a certain theta lift. The Eisenstein series is attached to a global Schwartz-Bruhat function Φ, whereas the theta lift is attached to $\pi^V_{V'}\pi_K(\Phi)$, which is defined on a smaller space. The above discussion, especially Lemma 3.5.12, relies in an essential way on the rationality of the finite part Φ_f. However, the applications to period relations in §4 refer to rationality properties of the theta lift, and in particular to

$$[\pi^V_{V'}\pi_K(\Phi)]_f = \pi^V_{V'}\pi_K(\Phi_f),$$

where the equality indicates that π_K and $\pi^V_{V'}$ are defined by adelic integrals and thus can be factored as products of their archimedean and finite parts. They are defined respectively by

$$\pi_K(\Phi_f)(v) = \int_{K_f}\Phi_f(kv)dk,\ v\in V^n(\mathbf{A}_f); \tag{3.5.15.1}$$

$$\pi^V_{V'}\Phi_f(v') = \int_{M_{r_0,n}(\mathbf{A}_f)}\Phi\begin{pmatrix}x\\ v'\\ 0\end{pmatrix}dx,\ v'\in V'^{,n}(\mathbf{A}_f). \tag{3.5.15.2}$$

The notation is explained in [I1,§4]. All that needs to be added here is that K_f is a maximal compact subgroup with volume 1, and the action of $k\in K_f$ on the argument preserves rationality, so (3.5.15.1) is consistent with rationality. In (3.5.15.2), the measure dx is assumed self-dual with respect to the choice of additive character ψ_f; thus (3.5.15.2) also preserves rationality, in the sense discussed in (E.2) above.

3.6. Pairings.

We collect some general facts about rational structures on automorphic representations on coherent cohomology. Notation is as in [H3, H4]; we write $H_!$ as in [H4] We let (G', X') be a Shimura datum, $Sh = Sh(G', X')$ the corresponding Shimura variety of dimension d, with reflex field $E' = E(G', X')$. In the previous section we have $G' = GU(V)$ with $\dim_{\mathcal{K}} V = m$; however the present discussion applies to general Shimura varieties. Let Π be a cuspidal automorphic representation of G' that contributes to $H^q_!(Sh, \mathcal{E})$ (cf. [H4, III.3][4]) for some automorphic vector bundle $\mathcal{E}$. Thus $\Pi = \Pi_\infty \otimes \Pi_f$ is an irreducible admissible representation and

$$Hom_{G'(\mathbf{A}_f)}(\Pi_f, H^q_!(Sh, E)) \neq 0 \tag{3.6.1}$$

We make no assumption regarding the dimension of the space in (3.6.1).

Proposition 3.6.2. *(i) The field of rationality of the $G'(\mathbf{A}_f)$ module Π_f, denoted $\mathbb{Q}(\Pi_f)$, viewed as a subfield of $\mathbb{C}$, is a number field contained in a CM field.*

(ii) The representation Π_f has a model over a finite extension M of $\mathbb{Q}(\Pi_f)$. More precisely, for any finite extension $L/\mathbb{Q}(\Pi_f)$, there is a finite abelian extension C of $\mathbb{Q}$, linearly disjoint from L, such that Π_f has a model over $C \cdot \mathbb{Q}(\Pi_f)$. In particular, M can also be taken to be contained in a CM field.

(iii) Let M be as in (ii) – contained in a CM field – and let $\mathcal{V}_M$ be a model of Π_f over M. Let $\mathcal{V}_M^c$ (resp. $\mathcal{V}_M^\vee$ denote the complex conjugate (resp. the contragredient) of $\mathcal{V}_M$, viewed as an M-rational $G'(\mathbf{A}_f)$-module. There is a real-valued algebraic character $\lambda : G'(\mathbf{A}_f) \to \mathbb{R}^\times$, (i.e., a character factoring through an algebraic Hecke character of the abelianization of G'), defined over $\mathbb{Q}(\Pi_f)$, such that $\mathcal{V}_M^c \xrightarrow{\sim} \mathcal{V}_M^\vee \otimes \lambda$. In other words, relative to complex conjugation on M, there is a $G'(\mathbf{A}_f)$-hermitian pairing

$$< \bullet, \bullet >_M : \mathcal{V}_M \otimes \mathcal{V}_M \to M(\lambda)$$

where $G'(\mathbf{A}_f)$ acts on $M(\lambda)$ by the character λ.

Proof. The first assertion is Theorem 4.4.1 of [BHR]. Part (ii) is presumably well-known. If $K \subset G'(\mathbf{A}_f)$ is an open compact subgroup, then the finite-dimensional $H(G'(\mathbf{A}_f), K)$-module Π^K is rational over $\mathbb{Q}(\Pi_f)$ and has a model over any finite extension $C/\mathbb{Q}(\Pi_f)$ that trivializes the Schur index; thus (ii) follows immediately for Π^K. The existence of a model of the representation Π is deduced in [V, II.4.7]. Finally, the first part of (iii) is contained in the proof of Theorem 4.4.1 of [BHR], and the second part is an immediate consequence.

Let Π and M be as in the above proposition, and let $\phi \in Hom_{G'(\mathbf{A}_f)}(\Pi_f, H^q_!(Sh, \mathcal{E}))$ be a non-zero homomorphism, $\mathcal{V}(\phi) = \phi(\Pi_f)(\mathbb{C})$. We identify the elements of $\mathcal{V}(\phi)$ with automorphic forms on $G'(\mathbb{Q})\backslash G'(\mathbf{A})$ with respect to a canonical trivialization (cf. [H3,p. 113]). The canonical trivialization depends on the choice of a CM pair $(T, x) \subset (G', X)$, and we let $E'_x = E' \cdot E(T, x)$. In the applications to unitary similitude groups we will have $E'_x = E'$, as in the proof of [H3, Lemma 2.5.12]. For any $f_1, f_2 \in \mathcal{V}(\phi)$ define the Petersson inner product

$$(f_1, f_2) = \int_{G'(\mathbb{Q})\backslash G'(\mathbf{A})/Z} f_1(g)\bar{f}_2(g)\lambda^{-1}(g)dg \in M^\times \cdot Q(\Pi). \tag{3.6.3}$$

Here $\lambda : G'(\mathbf{A}) \to \mathbb{R}^\times$ is the positive character of 3.6.2(iii) such that $f_1(g)\bar{f}_2(g)\chi^{-1}(g)$ is invariant under translation by Z.

[4] $H_!$ denotes the image of cohomology of $\mathcal{E}^{sub}$ in $\mathcal{E}^{can}$; in [H3, 2.2.6] this is denoted $\bar{H}$.

Corollary 3.6.4. *Let Π, M, and ϕ be as above, and suppose the image of ϕ is a subspace of $H^q_!(Sh,\mathcal{E})$ defined over M. Let $M^+ \subset M$ be the maximal totally real subfield, and let $Z \subset G'(\mathbb{R})$ be the identity component of the center of $G'(\mathbb{R})$. There is a constant $Q(\Pi) \in \mathbb{R}^\times$, well-defined up to scalars in $M^{+,\times}$, such that, for any $f_1, f_2 \in \mathcal{V}(\phi)$ rational over M,*

$$(f_1, f_2) \in M^\times \cdot Q(\Pi).$$

Proof. The non-degenerate pairing (3.6.3) defines an isomorphism $\mathcal{V}^c \xrightarrow{\sim} \mathcal{V}^\vee \otimes \lambda$ of complex vector spaces. Schur's Lemma implies there is a complex constant such that

$$(f_1, f_2) = Q(\Pi) \cdot < f_1, f_2 >_M$$

for any $f_1, f_2 \in \mathcal{V}$. The Corollary follows immediately.

Corollary 3.6.5. *Suppose* $\dim Hom_{G'(\mathbf{A}_f)}(\Pi_f, \bar{H}^q(Sh,\mathcal{E})) = 1$, *and let $\mathcal{V}(\Pi_f)$ denote the Π_f-isotypic subspace of $H^q_!(Sh,\mathcal{E})$, $\mathcal{V}(\Pi_f)(M)$ its M-rational subspace. Let $f_2 \in \mathcal{V}(\Pi_f)$. Then $f_2 \in \mathcal{V}(\Pi_f)(M)$ if and only if, for all $f_1 \in H^q_!(Sh,\mathcal{E})$, $Q(\Pi)^{-1}(f_1, f_2) \in M$.*

Proof. Under the multiplicity one hypothesis, the Π_f-isotypic subspace of $H^q_!(Sh,\mathcal{E})$ is M-rational for any M over which Π_f has a model. Since $\mathcal{V}(\Pi_f)(M)$ pairs trivially with any subspace of $H^q_!(Sh,\mathcal{E})$ which is not Π_f-isotypic, the corollary is then obvious.

This can be interpreted in another way. Let $\mathcal{E}' = \Omega^d_{Sh} \otimes \mathcal{E}^\vee$. Then Serre duality defines a non-degenerate pairing

$$H^q_!(Sh,\mathcal{E}) \otimes H^{d-q}_!(Sh,\mathcal{E}') \to \mathbb{C}$$

which is rational over the reflex field E' with respect to the natural action of $Aut(\mathbb{C}/E')$ on the family of automorphic vector bundles $\mathcal{E}$. Let $\Pi'_f = \Pi^c_f \otimes \lambda^{-1}$, where λ is as in (3.6.3). Then (with respect to a a canonical trivialization) the map on cusp forms

$$f \mapsto \bar{f} \otimes \lambda^{-1} \tag{3.6.6}$$

defines an isomorphism between the Π_f isotypic subspace of $H^q_!(Sh,\mathcal{E})$ and the Π'_f-isotypic subspace of $H^{d-q}_!(Sh,\mathcal{E}')$. In particular, if Π_f occurs with multiplicity one in $H^q_!(Sh,\mathcal{E})$, then Π'_f occurs with multiplicity one in $H^{d-q}_!(Sh,\mathcal{E}')$. Let $\Pi_{f,M}$ be the M-rational structure on Π_f induced by the rational structure of $H^q_!(Sh,\mathcal{E})$, and define $\Pi'_{f,M}$ analogously. Then the image $\bar{\Pi}_{f,M}$ of $\mathcal{V}_M$ under (3.6.6) is a new M-rational structure on the Π'_f-isotypic subspace of $H^{d-q}_!(Sh,\mathcal{E}')$, hence there is a constant $C \in \mathbb{C}^\times$ such that

$$C\Pi'_{f,M} = \bar{\Pi}_{f,M}.$$

One then sees by the arguments in [H3,(2.8)] that $C = Q(\Pi)$.

4. Applications to period relations

The main result of [H4] is Theorem V.1.10, corrected in the introduction to the present paper as Theorem E.2.1, which asserts that the theta lifting from $U(n)$ to $U(n+r)$, $r \geq 0$, preserves rationality up to an explicit period factor. The main application we had in mind for this theorem is to proving the period relations conjectured in [H3, (3.7.3)]. A sketch of a proof of these period relations is presented in section V.3 of [H4]. This sketch is based on a series of hypotheses. Most of these hypotheses have been verified in §3 above and in [H5]. The present section runs through the list of hypotheses and elaborates what remains to be done.

4.1. The hypotheses. Recall that $\mathcal{K}$ is an imaginary quadratic extension of $\mathbb{Q}$. As in section V.3 of [H4], we are given a cohomological automorphic representation π of a unitary similitude group G_1 of signature $(n-1,1)$ relative to $\mathcal{K}/\mathbb{Q}$. We assume π to be of the form $\pi_0 \otimes \beta$ (β replaces the character χ of [H3]) with π_0 self-dual and we are interested in special values of the standard L-function $L^{mot,S}(s, \pi_0 \otimes \beta, St, \alpha^*) = L^{mot,S}(s, \pi_{0,\mathcal{K}}, \tilde{\beta} \cdot \alpha^*)$.

Hypothesis 4.1.1. *The representation π_0 of G_1 admits a base change to a cohomological cuspidal automorphic representation $\pi_{0,\mathcal{K}}$ of $GL(n,\mathcal{K})$.*

This is slightly abusive: the base change of π_0 is to an automorphic representation $\pi_{0,\mathcal{K}} \otimes \Psi$ of $GL(n,\mathcal{K}) \times GL(1,\mathcal{K})$ (cf. [HT, Theorem VI.2.1]), and we are only retaining the first factor $\pi_{0,\mathcal{K}}$.

The article [H3] is motivated by standard conjectures concerning transfer of automorphic representations between inner forms of G_1. These will probably soon be available, at least for cohomological automorphic representations, thanks to the results of Laumon-Ngô and Waldspurger on the Fundamental Lemma for unitary groups. To avoid conjectures we nevertheless assume henceforward that π_0 satisfies the conditions of Theorem 3.1.6 of [HL], namely

Hypothesis 4.1.1 bis. *For at least two distinct places v of $\mathbb{Q}$ split in $\mathcal{K}$ the local factor $\pi_{0,v}$ is supercuspidal.*

This guarantees that π_0 satisfies Hypothesis 4.1.1 [HL,Theorem 2.2.2], hence the name. It also guarantees:

Theorem 4.1.2 [HL, Theorem 3.1.6]. *Assume Hypothesis 4.1.1 bis. Let G be an inner form of G_1. Then π_0 admits a transfer to an antiholomorphic representation $\pi"_0$ of G. More precisely, $\pi"_0$ and π_0 are nearly equivalent in the sense of being isomorphic locally at almost all places, and of course*

$$L^{mot,S}(s, \pi_0 \otimes \beta, St, \alpha^*) = L^{mot,S}(s, \pi"_0 \otimes \beta, St, \alpha^*)$$

if S is big enough.

Moreover,

Theorem 4.1.3 [HL, Theorem 3.1.5]. *Assume Hypothesis 4.1.1 bis. Then $\pi_{0,f}$ is (essentially) tempered at ∞ and at all finite places v such that either (i) v splits in $\mathcal{K}/\mathbb{Q}$ or (ii) v is unramified in $\mathcal{K}/QQ$ and $\pi_{0,v}$ is in the unramified principal series; the same holds for the finite part of any nearly equivalent automorphic representation on any inner form of G*

In fact, Theorem 3.1.5 of [HL] asserts that the base change $\pi_{0,\mathcal{K}}$ of π_0 is everywhere essentially tempered, and even that there is a global character η of $\mathcal{K}_{\mathbf{A}}^{\times}$ such that the twist of $\pi_{0,\mathcal{K}}$ by η is tempered. This almost certainly implies formally that $\pi_{0,\mathcal{K}}$ is tempered at all non-split places, not only at those places satisfying (ii) of Theorem 4.1.3. Indeed, the standard L-function of π_0 equals the principal L-function of $\pi_{0,\mathcal{K}}$ outside a finite set of bad primes. However, the behavior of stable base change has not been worked out in general at ramified places, nor has the proof of the global functional equation of the standard L-functions of unitary groups been written down. Thus we assume the first part of the following hypothesis in order to avoid worrying about spurious poles of local Euler factors:

Hypothesis 4.1.4. *The representation $\pi_{0,f}$ is essentially tempered. Moreover, π_0 occurs with multiplicity one in the space of automorphic forms on G_1, cuspidal or otherwise.*

Since π_0 is cuspidal, hence essentially unitary, the first part of the hypothesis means that π_0 is the twist by some global character of a tempered automorphic representation. See Remark 4.1.14 at the end of this section. The second part of the hypothesis is 3.5.5 (b) which we repeat here because it has been used in the proofs in §3.5; however, it is certainly unnecessary.

We now run through the hypotheses of [H4, V.3]. Notation is as in [*loc. cit.*].

Claim 4.1.5 (= Hypothesis (3.10)(a) of [H4,V]). $L^{mot,S}(s,\pi_0\otimes\beta,St,\alpha^*)\neq 0$ *at* $s=\frac{m}{2}$.

When $m>n$ this is a consequence of the existence of base change, the Jacquet-Shalika theorem (cf. (3.4.2.1)), and the fact that the base change $\pi_{0,\mathcal{K}}$ is essentially tempered [HL,3.1.5].

Hypothesis 4.1.6 (= Hypothesis (3.10)(b) of [H4,V]). *Assume $G(\mathbb{R})=U(r,s)$, as in the statement of Theorem 3.2, above. Theorem 3.5.13 of [H3] remains valid for the L-function $L^{mot,S}(s-\frac{\kappa}{2},\pi"_0\otimes\beta,St,\alpha)$ at the point $s=s_0=\frac{m+\kappa}{2}$.*

It should have been pointed out in [H4] that the normalization has been changed in the course of the proof of Lemma 3.9, since α^* is a splitting character. In fact as pointed out there,

$$L^{mot,S}(s,\pi_0\otimes\beta,St,\alpha^*)=L^{mot,S}(s-\frac{\kappa}{2},\pi_0\otimes\beta,St,\alpha) \tag{4.1.7}$$

(in [H4] $\kappa=1$ when m is odd, $\kappa=0$ when m is even). So although Hypothesis V.3.10 (b) asserts that Theorem 3.5.13 of [H3] remains valid for $L^{mot,S}(s,\pi_0\otimes\beta,St,\alpha^*)$ at the point $s=\frac{m}{2}$, the above formulation is actually correct.

Hypothesis 4.1.4 is that $\pi_{0,f}$ is tempered at all finite places. Thus we can apply Corollary 3.3 and Theorem 3.4. (Actually, for Theorem 3.4 the temperedness assertions in Theorem 4.1.3 are sufficient). When $m-n\geq 2$, Hypothesis 4.1.6 is true unconditionally, by Corollary 3.3. When $m=n+1$, the hypothesis is true for a large class of α; this is in fact sufficient for the applications. Thus for all practical purposes we can consider (3.10)(b) of [H4] to be valid. However, Hypothesis 4.1.6 is only true for α satisfying conditions (a)-(d) of Theorem 3.2. Here we promote the "Hypothesis" to a "Claim:

Claim 4.1.7 (= Hypothesis (3.10)(b) of [H4,V]). *. For all but finitely many finite places v which do not split in $\mathcal{K}$, there is a finite set of splitting characters A_v of $\mathcal{K}_v^\times$ such that Hypothesis 4.1.6 above is valid for α provided $\alpha_v^* \notin A_v$.*

Claim 4.1.8 (= Hypotheses (3.10)(c) and (3.20)(d) of [H4,V]). *There exists a hermitian space V'' over $\mathcal{K}$, of any signature (a,b), with $a+b=n$, and a holomorphic automorphic representation π'' of $GU(V")$ which is nearly equivalent to π_0.*

This follows from Theorem 4.1.2 above.

The results of §3 also apply to the hypotheses (3.20) of [H4]. We have already verified (3.20)(d) in Claim 4.1.8. Hypothesis (3.20)(c) is a mild restriction on $\pi_{0,\infty}$, to which we will return momentarily; for classical modular forms it corresponds to assuming the weight is at least 3. Given Hypothesis 4.1.1, Hypothesis (3.20)(b) is a restatement of Claim 4.1.5.

Thus it remains to verify Hypothesis (3.20)(a). This is the analogue of Hypothesis 4.1.6, with the representation $\pi"_0 \otimes \beta$ of G, an inner form of $U(n)$, replaced by the representation π' of G', an inner form of $U(n+1)$, considered in (3.5), and it is true subject to the same conditions. We let v and ν be as in [H4, 3.23].

Claim 4.1.9 (= Hypothesis (3.20)(a) of [H4,V]). *For all but finitely many finite places v which do not split in $\mathcal{K}$, there is a finite set of splitting characters A_v of $\mathcal{K}_v^\times$ such that Theorem 3.5.13 of [H3] remains valid for the standard L-function $L^{mot,S}(s,\pi',\check{\nu},St)$ of G' at the integer point $s=\frac{n+2-v}{2}$, provided $\check{\nu}^* = \check{\nu}/|\check{\nu}| \notin A_v$.*

The discussion of [H4, 3.23] is rather complicated and it doesn't help matters that many of the formulas include misprints (corrected in the introduction to the present paper). Some remarks are nonetheless in order. The value $\frac{n+2-v}{2}$ above corresponds to the integer μ in [H4, Lemma 3.23.4], since we take $m=n+1$. Then Theorem 3.4 applies directly to (antiholomorphic) cusp forms on G' to yield Claim 4.1.9, provided we take $\check{\nu}^*$ as the splitting character for the theta lift from G' to an appropriate definite $U(n+2)$. This does not satisfy the running assumption in [H4] that $0 \leq \alpha(\check{\nu}) \leq 1$, but in fact this assumption was only made to simplify the formulas and is irrelevant in the setting of Theorem 3.4.

Since it was not mentioned in [H4, V.3], I take this opportunity to point out that the right-hand inequalities (**) in Theorem 3.2, which are the main object of the calculation in [H4, Lemma 3.23.4], correspond to the existence of a non-trivial theta lift of π' (for the splitting characters $\chi' and \check{\nu}^*$) to a definite unitary group; this is explained in [H5, Remark (4.4)(iv)]. The proof of Lemma 3.23.4 of [H4] derives these inequalities from Hypothesis (3.20)(c), which we restate as follows:

Hypothesis 4.1.10 (= Hypothesis (3.20)(c) of [H4,V]). *Let $(p_1 > p_2 > \cdots > p_n)$ be the Hodge types attached to the motive $M(\pi_0)$ in [H3,H4]:*

$$p_i = a_{0,i} + n - i, q_i = p_{n-i+1} = n-1-p_i,$$

where $a_{0,i}$ are defined in [H4,V, (3.2)]. We assume that for all $i=1,2,\ldots,n-1$,

$$p_i - p_{i+1} \geq 2.$$

In [H4,V] this hypothesis is made for $i=r'$, but for the period relations we need to to hold at every step of the induction on r'. The hypothesis guarantees that for every critical interval of $M(\pi_0)$ there is a Hecke character β in that critical interval such that the L-function of π_0 twisted by β has a critical value at $s = 1$ in the unitary normalization. This fails when $p_i - p_{i+1} = 1$ for some i; in that case, the only available critical value is at the center of symmetry, but then it is not known that one can choose β so that the L-function does not vanish, and so we are not sure to obtain non-trivial period relations.

There remain two hypotheses not presented as such in [H4]. The first is the Rallis inner product formula, stated as [H4, V (3.6)]:

$$(4.1.11)\quad I_{\Delta,\chi'}(\theta_{\phi,\chi,\chi'}(f)) = \delta I_{\Delta,\chi}(f)\cdot Z_\infty \cdot Z_S \cdot L^{mot,S}(\frac{m}{2}, \pi_{0,\mathcal{K}}, \tilde{\beta}\cdot\chi)/d_n(*m).$$

Our version of this formula is Corollary 3.5.14.10. The left-hand sides are the same, up to change of notation, and we have removed the rational factor δ and Z_S and replaced equality with $\sim_{\mathcal{K}}$. Our $Q(\pi)$ can be identified with $I_{\Delta,\chi}(f)$. Moreover, with $m = n+1$, the point $s = \frac{m}{2}$ in the motivic normalization corresponds to the point $s = 1$ in the unitary normalization. Since π_0 is self-dual and $\tilde{\beta}$ and χ are conjugate self-dual, we can thus identify $L^{mot,S}(\frac{m}{2}, \pi_{0,\mathcal{K}}, \tilde{\beta}\cdot\chi)$ with $L(1, \pi_{\mathcal{K}}\cdot\chi^{-1})$ up to factors in $\mathbb{Q}(\pi)$ (the missing Euler factors).

Thus it remains to compare the expression

$$(4.1.12)\qquad C(\Phi_\infty)\cdot[c_K d^S_{n+1}(\frac{1}{2},\chi)]^{-1}\cdot L(1,\varepsilon_{\mathcal{K}})$$

from Corollary 3.5.14.10. with $Z_\infty/d_n(*m)$ from [H4, V (3.6)], with $m = n+1$. In fact, taking the normalizations into account, $d_{n+1}(s,\chi)$ is the product of $d_n(*s)$ with an additional shifted Dirichlet L-function. However, we will take the easy way out and simply define Z_∞ by the formula

$$(4.1.13)\qquad Z_\infty = C(\Phi_\infty)\cdot\frac{d_n(*n+1)}{c_K d^S_{n+1}(\frac{1}{2},\chi)}\cdot L(1,\varepsilon_{\mathcal{K}}).$$

With this definition, Corollary 3.5.14.10 can be used in place of [H4,V.3.6] in the arguments of [H4, §V.3]. Note that Z_∞ does not depend on the choice of χ but only on the restriction of χ to the idèles of $\mathbb{Q}$, which depends only on the parity of n.

Our proof of 3.5.14.10 depends on Hypotheses 3.5.5. To avoid complications, we assume

Hypothesis 4.1.14. *The prime* 2 *splits in* $\mathcal{K}$.

This guarantees 3.5.5 (e). Hypotheses 3.5.5 (c) and (d) are 4.1.4 and 4.1.3, respectively, whereas 3.5.5 (a) and (b) are running hypotheses in [H4].

The final hypothesis in [H4] is then the assertion that all the terms Z_∞ that occur as above in the various intermediate steps are actually rational numbers [H4, V.3.28.4]. We will not make this hypothesis here and so the unknown factors will remain in our final result. Note however that the situation is somewhat improved over that in [H4]; the main unknown term in our expression for Z_∞ is the local zeta integral $C(\Phi_\infty)$, which is given in terms of an integral of an automorphy factor against an (anti)-holomorphic matrix coefficient, whereas in [*loc. cit*] the matrix coefficient was obtained from a more general discrete series. Although the automorphy factor in the zeta integral is not nearly holomorphic (because no definite unitary group is involved in the theta lift), the anti-holomorphic matrix coefficients are considerably more elementary than general discrete series matrix coefficients, so there is some hope of calculating the expression Z_∞ explicitly. I hope to return to this matter in a future paper.

4.2. Period relations up to archimedean terms.

Now that all the hypotheses of [H4,§V.3] have been verified, with some minor modifications, we can draw the conclusions. We let G_1 be a unitary similitude group as in 4.1.

Theorem 4.2.1. *Let π_0 be a self-dual cuspidal cohomological tempered automorphic representation of G_1, satisfing Hypotheses 4.1.1, 4.1.4, and 4.1.10. Then for any $r = 1, 2, \ldots, n$, there is a non-zero complex number $Z_r(\pi_{0,\infty})$, depending only on the infinity type of π_0, such that,*

$$P^{(r)}(\pi_0) \sim_{\mathcal{K}} Z_r(\pi_{0,\infty}) \prod_{j=1}^{r} Q_j(\pi_0).$$

This is [H4, V.3.28.2], where we write $Z_r(\pi_{0,\infty})$ for $\prod_{j=1}^{r} Z_{\infty,j}$.

Finally, we restate the last theorems of [H3], in a slightly strengthened form. We are assuming 4.1.14 (i.e. 3.5.5(e)); since the primary object of interest in [H3] is a self-dual automorphic representation of $GL(n,\mathbb{Q})$, we lose no generality by restricting attention to $\mathcal{K}$ in which 2 splits.

Theorem 4.2.2. *Notation is as in [H3], §3.7. Let $\eta = \chi^{(2)} \cdot \alpha$, and let s_0 be an integer satisfying*

$$\frac{n-\kappa}{2} \leq s_0 \leq min(p_{n-r}(\mu) - k, p_r(\mu) + k - \kappa).$$

If $s_0 \leq n - \frac{\kappa}{2}$, we assume in addition that α^* is a splitting character for n and s_0, and if $s_0 = \frac{n-\kappa+1}{2}$ that there is a non-split finite place v such that $\alpha^*_v \notin A_v$, with A_v as in Theorem 3.4. Finally, if n is odd, we assume Hypothesis 3.7.9 on periods of abelian motives.

Then there is a constant $Z_r(\pi_{0,\infty})$ such that

$$L^{mot}(s_0, \pi_0 \otimes \chi, St, \alpha) \sim_{E(\pi_0,\eta),\mathcal{K}} Z_r(\pi_{0,\infty}) \cdot c^+(M(\pi_0) \otimes RM(\eta)(s_0 + k)),$$

where c^+ is the Deligne period.

In other words, Deligne's conjecture is valid for the L-function $L^{mot}(s_0, \pi_0 \otimes \chi, St, \alpha)$, up to uncontrolled factors in $\mathcal{K}^\times$ and the unknown factor $Z_r(\pi_{0,\infty})$ which depends only on the Hodge numbers of $M(\pi_0)$. In particular, if Deligne's conjecture is valid (up to $\mathcal{K}^\times$) for one π_0 with given infinity type, then it is valid for all such π_0.

We

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