TESTING RATIONALITY OF COHERENT COHOMOLOGY OF SHIMURA VARIETIES

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in memory of Ilya Piatetski-Shapiro

ABSTRACT. Let $G' \subset G$ be an inclusion of reductive groups whose real points have a non-trivial discrete series. Combining ergodic methods of Burger-Sarnak and the author with a positivity argument due to Li and the classification of minimal K-types of discrete series, due to Salamanca-Riba, we show that, if π is a cuspidal automorphic representation of G whose archimedean component is a sufficiently general discrete series, then there is a cuspidal automorphic representation of G', of (explicitly determined) discrete series type at infinity, that pairs non-trivially with π. When G and G' are inner forms of $U(n)$ and $U(n-1)$, respectively, this result is used to define rationality criteria for sufficiently general coherent cohomological forms on G.

INTRODUCTION

For years I would meet Ilya Piatetski-Shapiro at conferences and mathematical institutes, but we only spoke a few times, and our conversations were generally brief and to the point. Shortly after I arrived at the IAS in 1983 for the special year on automorphic forms – I later came to suspect that Ilya was in part responsible for my having been invited – he asked me to come to his office to tell him what I had been doing. I explained that I had been working on defining arithmetic models of automorphic vector bundles, to provide a general geometric framework for Shimura's characterization of arithmetic holomorphic automorphic forms – forms rational over number fields. Ilya asked whether I had a similar characterization of arithmetic non-holomorphic coherent cohomology classes. I replied that I had not thought about the question;

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he suggested that I work on the problem; and that was essentially the end of the discussion.[1]

Non-holomorphic coherent cohomology of Shimura varieties is usually defined by automorphic representations whose archimedean components are in the non-holomorphic discrete series. Since Takeyuki Oda had also advised me to look at such automorphic representations, I spent the next few years developing a theory that provides a geometric framework for such cohomology classes, and in many cases I was able to find a characterization of the classes that are rational over number fields. But this characterization, developed in [H90], is based on studying the cup products of the classes in question with holomorphic arithmetic forms on Shimura subvarieties. These cup products are compatible with the rational structure of canonical models of Shimura varieties, and therefore the rationality of these cup products is a necessary condition for arithmeticity of the class being tested. The method of [H90] shows that it is often also a sufficient condition, but provides no information in the typical case, where all such cup products vanish for purely local reasons.

The present paper gives an optimal answer to Piatetski-Shapiro's question for coherent cohomology of Shimura varietes attached to unitary groups, at least when the infinitesimal character is sufficiently regular. As in [H90], the rationality criterion is based on restriction to Shimura subvarieties and integrating against classes on the latter, but this *period integral* in general is purely analytic and has nothing to do with algebraic geometry. Nevertheless, the multiplicity one condition implied by the Gross-Prasad conjecture [GGP] shows the existence of canonical period invariants, called the *Gross-Prasad periods* in [H12], that have the property that the proportionality of the period integral to the corresponding Gross-Prasad period is a necessary and sufficient condition for arithmeticity. When the infinitesimal character is sufficiently regular, it can be shown that the period integrals do not all vanish. Thus the criterion provides a foolproof way to reduce arithmeticity on a Shimura variety attached to an inner form of $U(n)$ to that on an inner form of $U(n-1)$, and by induction we obtain a general criterion.

The remarks of the preceding paragraph need to be qualified, however. The Gross-Prasad periods are defined in [H12], assuming a stronger version of the local Gross-Prasad conjecture at archimedean places

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[1]He did ask one more question: was I married? I was not, and I was surprised to discover that he had advice for me on that matter as well.

than is currently available. It has been proved in [SZ] that the multiplicity one conjecture of Gross-Prasad is true for $U(n-1)$-invariant linear forms on the smooth Frechet completions of moderate growth of discrete series representations of $U(n) \times U(n-1)$, where here for $m = n, n-1$, $U(m)$ denotes any inner form of the compact unitary group $U(m)$. In order to define Gross-Prasad periods, one needs multiplicity one for invariant linear forms on the corresponding Harish-Chandra modules; in other words, one needs to know the *automatic continuity* of such invariant linear forms. This remains an open problem, at least when $U(n-1)$ is non-compact and the representation of $U(n)$ is not holomorphic, so in that sense the results of the present paper are conditional.

The methods of this paper are in large part the same as those of my article [HL] with J. S. Li. The main difference is that the earlier paper only considered rationality criteria based on cohomological cup products. In particular, we make heavy use of Li's idea to apply Flensted-Jensen's formula for the leading matrix coefficients of discrete series representations to show weak inclusion of representations of $U(n-1)$ in restrictions from $U(n)$. We also use the method of Burger-Sarnak, or the more elementary methods of §7 of [H90], to deduce non-vanishing of global pairings from local weak inclusion.

We note that the proof of the full Gross-Prasad conjecture for unitary groups over archimedean fields would allow us to derive the main theorem without reference to the Flensted-Jensen formula. Indeed, the full Gross-Prasad conjecture determines the discrete series representations of (an inner form of) $U(n-1)$ that are weakly contained in any given discrete series representation π of (an inner form of) $U(n)$. In most cases, this weak containment has nothing to do with the leading matrix coefficient of π. The paper is organized in a progression from most general and conditional results – much of the argument is valid for general Shimura varieties – to the most specific and unconditional.

I thank Dipendra Prasad for asking the question answered in this paper and Jeff Adams for directing me to the article [SR] of Salamanca-Riba. I also thank the editors of this volume for inviting me to contribute.

1. RESTRICTIONS OF DISCRETE SERIES REPRESENTATIONS AND GLOBAL CONSEQUENCES

Let G be a connected reductive algebraic group over $\mathbb{Q}$, $G' \subset G$ a reductive subgroup. Let Z_G and $Z_{G'}$ denote their respective centers,

A_G and $A_{G'}$ the maximal $\mathbb{R}$-split tori in Z_G and $Z_{G'}$. We assume in what follows that A_G is of finite index in $A_{G'}$. By a discrete series representation of $G(\mathbb{R})$ we will mean an irreducible Hilbert space representation π of $G(\mathbb{R})$ such that, for some character ξ of $G(\mathbb{R})$, $\pi \otimes \xi$ is trivial on an open subgroup $A^0 \subset A_G(\mathbb{R})$, and such that $\pi \otimes \xi$ defines a square-integrable representation of $G(\mathbb{R})/A^0$.

Suppose π and π' are discrete series representations of $G(\mathbb{R})$ and $G'(\mathbb{R})$, respectively, and assume that their central characters coincide on an open subgroup $A^0 \subset A_G(\mathbb{R})$.

Hypothesis 1.1. *The representation π' is weakly contained in the restriction of π to $G'(\mathbb{R})$; in other words, there is a non-trivial Hilbert space pairing*

$$\pi \otimes \pi'^{,\vee} \to \mathbb{C}$$

invariant under $G'(\mathbb{R})/A^0$.

Suppose π occurs in the L^2-automorphic spectrum of G; in other words, there is an equivariant homomorphism $\lambda : \pi \to C^\infty(\Gamma \backslash G(\mathbb{R}))$, for some congruence subgroup $\Gamma \subset G(\mathbb{Q})$, such that, for some character ξ of $G(\mathbb{R})$ trivial on Γ and for an open subgroup $A^0 \subset A_G(\mathbb{R})$ as above $\lambda(\pi) \otimes \xi \subset L^2(\Gamma \cdot A^0 \backslash G(\mathbb{R}))$. Theorem 1.1(a) of [BS] then asserts that π' is weakly contained in the L^2 automorphic spectrum of G': in other words

Theorem 1.2. *(Burger-Sarnak) Assume the pair (π, π') satisfies hypothesis 1.1. For an appropriate character ξ' of $G'(\mathbb{R})$, $\pi' \otimes \xi'$ is the limit, in the Fell topology of unitary representations of $G'(\mathbb{R})/A^0$, of L^2 automorphic representations of $\Gamma' \cdot A^0 \backslash G'(\mathbb{R})$ as Γ' varies over congruence subgroups of $G'(\mathbb{R})$.*

If α is a Hecke character of Z_G, we let $L^2_{cusp,\alpha}(G(\mathbb{Q}) \backslash G(\mathbf{A}))$ denote the space of complex-valued functions f on $G(\mathbb{Q}) \backslash G(\mathbf{A})$ with central character α such that, for an appropriate Hecke character ξ of $G(\mathbf{A})$, $f \cdot \xi$ is trivial on an open subgroup $A_0 \subset A_G(\mathbb{R})$ and defines a square integrable function on $G(\mathbb{Q}) \cdot A_0 \backslash G(\mathbf{A})$. By a *cuspidal automorphic representation* of G we will mean alternatively an irreducible Hilbert space component of $L^2_{cusp,\alpha}(G(\mathbb{Q}) \backslash G(\mathbf{A}))$ for some α or the corresponding $(Lie(G)_\mathbb{C}, K_\infty) \times G(\mathbf{A}_f))$-module, for some maximal compact subgroup $K_\infty \subset G(\mathbb{R})$. The first part of the following corollary follows by standard arguments. The second part is then obvious, and the third part is a consequence of the well-known fact that integrable discrete series representations are isolated in the Fell topology (or of the Poincaré series argument in [H90], §7). Note that π and π' now denote global automorphic representations.

Corollary 1.3. *Let π be a cuspidal automorphic representation of G whose archimedean component π_∞ is a discrete series representation of $G(\mathbb{R})$. Let π'_∞ by a discrete series representation of $G'(\mathbb{R})$, and assume the pair $(\pi_\infty, \pi'_\infty)$ satisfies hypothesis 1.1.*

(a) *For an appropriate Hecke character ξ' of $G'(\mathbf{A})$, $\pi'_\infty \otimes \xi'_\infty$ is the limit, in the Fell topology of unitary representations of $G'(\mathbb{R})/A^0$, of archimedean components of L^2 automorphic representations of $G'(\mathbf{A})/A^0$.*

(b) *In particular, if π'_∞ is isolated in the automorphic spectrum of G', then there is an L^2-automorphic representation σ of G' with archimedean component $\pi'^{,\vee}_\infty$ and automorphic forms $f \in \pi$ and $f' \in \pi'$ such that*

$$\int_{G'(\mathbb{Q})\cdot A^0 \backslash G'(\mathbf{A})} f(g')f'(g')dg' \neq 0.$$

(c) *If π'_∞ belongs to the integrable discrete series, then there exists σ as in (b).*

2. Restrictions of minimal types

In what follows, G is a connected reductive group over $\mathbb{Q}$ such that $G(\mathbb{R})$ has a discrete series. Our goal in this section is to state a simple algebraic condition on π as in the statement of Hypothesis 1.1 that guarantees that the condition of the hypothesis is verified. We are not striving for maximum generality, and the condition is certainly unnecessarily strong.

Choose maximal compact connected subgroups $K^*_\infty \subset G(\mathbb{R})$ and $K'^{,*}_\infty \subset G'(\mathbb{R})$, with $K'^{,*}_\infty \subset K^*_\infty$. Define $K_\infty = K^*_\infty \cdot Z_G(\mathbb{R})$, $K'_\infty = K'^{,*}_\infty \cdot Z_{G'}(\mathbb{R})$. Let π be a discrete series representation of $G(\mathbb{R})$. It has a minimal, or Blattner K_∞-type τ, an irreducible representation τ of K_∞ with the property that τ occurs in π with multiplicity 1, and is minimal in the sense that its highest weight is shortest among all K_∞-types occurring in π.

In the next section we recall the formula for the highest weight of τ in terms of the Harish-Chandra parameter of π. On the other hand, given an irreducible representation τ of K_∞, the article [SR] provides a necessary and sufficient criterion for τ to be the minimal K_∞-type of a discrete series representation (and more generally, of a cohomologically induced representation of the form $A_{\mathfrak{q}}(\lambda)$). Choose a maximal torus $H \subset K_\infty$. Since $G(\mathbb{R})$ has discrete series, H is also a maximal torus of G. We use lower case gothic letters to denote (complexified real) Lie algebras; thus $\mathfrak{k} = Lie(K_\infty)_\mathbb{C}$, $\mathfrak{h} = Lie(H(\mathbb{C}))$, and so on. Denote by

$\Delta(\mathfrak{k}, \mathfrak{t})$ (resp. $\Delta(\mathfrak{g}, \mathfrak{t})$)the set of roots of $\mathfrak{t}$ in $\mathfrak{k}$ (resp. $\mathfrak{g}$) and let $(\bullet, \bullet)$ be the pairing on $\mathfrak{t}^*$ induced by any non-degenerate $G(\mathbb{R})$-invariant pairing on $\mathfrak{g}$.

Let μ be the highest weight of an irreducible representation τ of K_∞, relative to a system of positive roots of $\mathfrak{t}$. Define

$$(2.1) \qquad \mu' = \mu + \sum_{\alpha \in \Delta(\mathfrak{k}, \mathfrak{t}), (\alpha, \mu) > 0} \alpha.$$

The θ-stable parabolic defined by μ' is the parabolic subalgebra $\mathfrak{q}(\mu')$ spanned by $\mathfrak{t}$ and the root spaces $\mathfrak{g}_\alpha$ for $(\alpha, \mu') \geq 0$. The following observation is clear

Observation 2.2. *For μ sufficiently regular, $\mathfrak{q}(\mu')$ is a Borel subalgebra of $\mathfrak{g}$.*

Denote by $\mathfrak{u}(\mu')$ the unipotent radical of $\mathfrak{q}(\mu')$, and let $2\rho(\mathfrak{u}(\mu'))$ denote the sum of the roots of $\mathfrak{t}$ occurring in $\mathfrak{u}(\mu')$. The following proposition is a special case of a result of Salamanca-Riba.

Proposition 2.3. *(([SR], Proposition 4.1)) Define*

$$\Delta(\mathfrak{u}(\mu'), \mathfrak{t}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) \mid (\alpha, \mu') > 0\}.$$

Then τ is the lowest K_∞-type of a discrete series representation π of $G(\mathbb{R})$ if and only if

(a) *$\mathfrak{q}(\mu')$ is a Borel subalgebra of $\mathfrak{g}$;*
(b)

$$(\mu', \alpha) \geq (2\rho(\mathfrak{u}(\mu')), \alpha)$$

  for all $\alpha \in \Delta(\mathfrak{u}(\mu'), \mathfrak{t})$

In that case, the Harish-Chandra parameter of π is equal to $\mu' - 2\rho(\mathfrak{u}(\mu'))$.

When $\mathfrak{q}(\mu')$ is not a Borel subalgebra, Salamanca-Riba determines the parameters λ for which τ is the lowest K_∞-type of a module $A_{\mathfrak{q}(\mu')}(\lambda)$, under a condition analogous to (b). We let $G' \subset G$ be a reductive subgroup as in the previous section, and we assume $G'(\mathbb{R})$ has a non-trivial discrete series. In the next few propositions, a character λ of $\mathfrak{t}$ is "sufficiently regular" if $|(\lambda, \alpha)| > N(G)$ for all roots α, where $N(G)$ is a positive constant, depending only on $G(\mathbb{R})$, that can be made explicit.

Corollary 2.4. *Let τ be an irreducible representation of K_∞ with highest weight μ. For μ sufficiently regular, τ is the lowest K_∞-type of a discrete series representation π of $G(\mathbb{R})$. Moreover, for μ sufficiently regular, the restriction of τ to K'_∞ contains an irreducible representation τ' that is the lowest K'_∞-type of a discrete series representation π' of $G'(\mathbb{R})$.*

Proof. The set of subsets of $\Delta(\mathfrak{g}, \mathfrak{t})$ is finite, so the term $2\rho(\mathfrak{u}(\mu'))$ that appears in (b) of [SR] belongs to a bounded subset of the vector space $Hom(\mathfrak{t}, \mathbb{R})$. This implies the first part of the Corollary, and the second part follows by applying the first part to G', bearing in mind that the highest weight of τ is an extreme weight of a constituent of its restriction to K'_∞.   □

Proposition 2.5. *Let π be an irreducible discrete series representation of $G(\mathbb{R})$ with Harish-Chandra parameter λ_π. Suppose λ_π is sufficiently regular. Then there is a discrete series representation π' of $G'(\mathbb{R})$ weakly contained in the restriction of π to $G'(\mathbb{R})$. Moreover (possibly under a more stringent regularity condition) we can assume π' to be in the integrable discrete series.*

Proof. The proof is based on the argument of Li used in [L90, HL]. Let π' be the discrete series representation of $G'(\mathbb{R})$ mentioned in the statement of 2.4, and let τ' be its minimal K'_∞-type. Let ψ_π be the matrix coefficient of π occurring in its minimal K-type τ, defined by formula (1.2.6) of [HL]. Proposition 1.2.3 of [HL], based on the calculations in §4 of [L90], asserts that π' is weakly contained in the restriction of π provided three conditions are satisfied. First, the restriction of ψ_π to G' has to be in $L^{2+\epsilon}(H)$; the argument, valid for any tempered representation of G, is reviewed in the course of the proof of Proposition 2.3.4 of [HL]. Next, ψ_π has to satisfy the Flensted-Jensen formula of [FJ]; but this is true for all discrete series representations. The final condition is that τ' be contained in the restriction to K'_∞ of τ; this is our initial hypothesis.

The final claim follows from the characterization of integrable discrete series by Hecht and Schmid [HS]: any discrete series with sufficiently regular Harish-Chandra parameter is integrable.

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We return to the global situation: $G \supset G'$ are connected reductive groups over $\mathbb{Q}$, and both $G(\mathbb{R})$ and $G'(\mathbb{R})$ are assumed to have discrete series. Combining 2.5 with 1.3, we draw the following global conclusion.

Proposition 2.6. *Let π be a cuspidal automorphic representation of $G(\mathbf{A})$, with archimedean component π_∞. Suppose the Harish-Chandra parameter of π_∞ is sufficiently regular. Then there is an integrable discrete series representation π'_∞ of $G'(\mathbb{R})$, a cuspidal automorphic representation π' of G' with archimedean component π'_∞ and automorphic forms $f \in \pi$ and $f' \in \pi'$ such that*

$$\int_{G'(\mathbb{Q}) \cdot A^0 \backslash G'(\mathbf{A})} f(g') f'(g') dg' \neq 0.$$

Moreover, f and f′ can be taken in the minimal K- (resp. K′-) types of the archimedean components of their respective representations.

Proof. We are changing notation: the representation π' here is $\sigma^\vee$ in 1.3, and we have π'_∞ pairing with rather than weakly contained in π_∞. Since 1.3 only claims that σ is an L^2-automorphic representation, we need to show that in fact π' is cuspidal. If not, then π' is a residual square-integrable representation. But since π'_∞ is tempered, this contradicts Wallach's theorem [Wa] that residual representations are non-tempered at all places. □

If we assume that automorphic representations with discrete series components at infinity are isolated in the automorphic spectrum, then we can relax the regularity condition in the last proposition, using the Burger-Sarnak theorem.

3. Coherent cohomology of unitary group Shimura varieties

Preliminary notation. Let F^+ be a totally real field, F a totally imaginary quadratic extension of F^+. The quadratic Hecke character of F^+ attached to the extension F is denoted ε_F. Let W be an n-dimensional F-vector space, endowed with a non-degenerate hermitian form $< \bullet, \bullet >_W$, relative to the extension F/F^+. We let Σ^+, resp. Σ_F, denote the set of complex embeddings of F^+, resp. F, and choose a CM type $\Sigma \subset \Sigma_F$, i.e. a subset which upon restriction to F^+ is identified with Σ^+. Complex conjugation in $Gal(F/F^+)$ is denoted c.

The hermitian pairing $\langle \bullet, \bullet \rangle_W$ defines an involution $\tilde c$ on the algebra $End(W)$ via

(0.1) $\langle a(v), v' \rangle_W = \langle v, a^{\tilde c}(v') \rangle_W,$

and this involution extends to $End(W \otimes_{\mathbb{Q}} R)$ for any $\mathbb{Q}$-algebra R. We define the algebraic group $U(W) = U(W, \langle \bullet, \bullet \rangle_W)$ (resp $GU(W) = GU(W, \langle \bullet, \bullet \rangle_W)$) over F^+ (resp. $\mathbb{Q}$) such that, for any F^+-algebra R (resp. $\mathbb{Q}$-algebra S)

$$U(W)(R) = \{g \in GL(W \otimes_{F^+} R) \mid g \cdot \tilde c(g) = 1\};$$

$$GU(W)(S) = \{g \in GL(W \otimes R) \mid g \cdot \tilde c(g) = \nu(g) \text{ for some } \nu(g) \in R^\times\};$$

The quadratic Hecke character of $\mathbf{A}^\times_{F^+}$ corresponding to the extension F/F^+ is denoted

$$\eta_{F/F^+} : \mathbf{A}^\times_{F^+}/F^{+,\times} N_{F/F^+} \mathbf{A}^\times_F \xrightarrow{\sim} \pm 1.$$

If W is a 1-dimensional hermitian or skew-hermitian space then we write $U(W) = U(1)$. The F^+-rational points of $U(1)$ are given by the group of elements $x \in F^\times$ with $N_{F/F^+}(x) = 1$.

Representations of groups over archimedean local fields are assumed to be admissible smooth Fréchet representations of moderate growth, in the sense of Casselman and Wallach (cf. [C]), and linear forms on such representations are assumed to be continuous with respect to the Fréchet topology. The relation between such linear forms and linear forms on the corresponding Harish-Chandra modules is discussed in the appropriate place. The Harish-Chandra $(Lie(G), K_\infty)$-module attached to a smooth Fréchet representation Π of the group G (and a choice of K_∞) is denoted $\Pi^{(0)}$. Similarly, if Π is a representation of an adelic group $G(\mathbf{A})$ of the form $\Pi_\infty \otimes \Pi_f$, where Π_f is an admissible representation of $G(\mathbf{A}_f)$ and Π_∞ is an admissible smooth Fréchet representation (of moderate growth) of the archimedean part of $G(\mathbf{A})$, then we write $\Pi^{(0)} = \Pi_\infty^{(0)} \otimes \Pi_f$.

3.1. Multiplicity one and Gross-Prasad periods.

Let W be an n-dimensional hermitian space as above, and write W as the orthogonal direct sum $W = W' \oplus W_0$ over F, where $\dim W' = n - 1$. Let $G = GU(W)$ as above, and let $G' = GU(W') \times GU(W_0) \cap G$; it is the subgroup of $(g', g_0) \in GU(W') \times GU(W_0)$ with $\nu(g') = \nu(g_0)$. In this section we apply the results of §2 to the inclusion $G' \subset G$. The new feature in this special case is that when $\pi \otimes \pi'$ is a cuspidal automorphic representation of $G \times G'$, the space of global pairings $\pi \otimes \pi' \to \mathbb{C}$ is of dimension at most 1.

Let X and X' be $G(\mathbb{R})$ and $G'(\mathbb{R})$-homogeneous hermitian symmetric domains, respectively, so that $(G', X') \hookrightarrow (G, X)$ is a morphism of Shimura data. We choose maximal connected compact (mod center) subgroups $K \subset G(\mathbb{R})$ and $K' \subset G'(\mathbb{R}) \cap K$ as in §2. Then $K = Stab_{G(\mathbb{R})}(x)$ for some $x \in X' \subset X$, and we assume x to be a CM point (the fixed point in X' of a $\mathbb{Q}$-rational torus). We are thus able to apply the theory of [H12]. Suppose π and π' are cuspidal automorphic representations of G and G' respectively. As in §4.3 of [H12], there are number fields $E(\pi)$ and $E(\pi')$ (depending on x) over which $\pi^{(0)}$ and $\pi'^{,(0)}$ have rational structures, denoted $\pi_{E(\pi)}^{(0)}$ and $\pi_{E(\pi')}'^{,(0)}$.

As in [H12], we let $H \subset G$ and $H' \subset G'$ denote the unitary subgroups, $\mathfrak{h}' = Lie(H')$. We define

$$L_0(\pi, \pi') = Hom_{(\mathfrak{h}', K') \times H'(\mathbf{A}_f)}(\pi^{(0)} \otimes \pi'^{,(0)}, \mathbb{C}).$$

We assume the *automatic continuity hypothesis* (Conjecture 4.3.1 of [H12]): every element of $L_0(\pi, \pi')$ has a natural continuous extension

to the admissible smooth Fréchet representations of moderate growth associated to π and π'. It then follows from [SZ] that $L_0(\pi, \pi')$ is a space of dimension at most 1 (cf. [H12], Corollary 4.3.2). We also consider the variant

$$L_0^{mot}(\pi, \pi') = L_0^{mot}(\pi, \pi') = Hom_{(\mathfrak{h}', K') \times H'(\mathbf{A}_f)}(\pi^{mot} \otimes \pi'^{,mot}, E(\pi) \otimes E(\pi') \otimes \mathbb{C})$$

where $\pi^{mot} = \pi_{E(\pi)}^{(0)} \otimes_{\mathbb{Q}} \mathbb{C}$, $\pi'^{,mot} = \pi_{E(\pi')}^{',(0)} \otimes_{\mathbb{Q}} \mathbb{C}$. Corollary 4.3.8 of [H12] asserts that $L_0^{mot}(\pi, \pi')$ is either 0 or a free $E(\pi) \otimes E(\pi') \otimes \mathbb{C}$-module of rank 1 (there is a misprint in *loc. cit.*), and it follows as in the beginning of §4.4 that

Proposition 3.1. *Assume $L_0^{mot}(\pi, \pi') \neq 0$. Then the free rank 1 $E(\pi) \otimes E(\pi') \otimes \mathbb{C}$-module $L_0^{mot}(\pi, \pi')$ has a generator $I^{mot}(\pi, \pi')$ such that*

$$I^{mot}(\pi, \pi')(\pi_{E(\pi)}^{(0)} \otimes \pi_{E(\pi')}^{',(0)}) = E(\pi) \otimes E(\pi').$$

(This corrects another misprint in *loc. cit.*.)

On the other hand, the integral (cf. [H12], (4.2.3))

$$(3.2) \qquad\qquad f \otimes f' \mapsto \int_{H'(F) \backslash H'(\mathbf{A})} f(h') f'(h') dh'$$

defines elements of

$$L_0(\pi_{E(\pi)}^{(0)} \otimes_{E(\pi), \alpha} \mathbb{C}, \pi_{E(\pi')}^{',(0)} \otimes_{E(\pi'), \alpha} \mathbb{C})$$

as α varies over $Hom(E(\pi) \otimes E(\pi'), \mathbb{C})$, and thus an element $I^{can}(\pi, \pi') \in L_0^{mot}(\pi, \pi')$ (mislabelled I^{mot} in (4.4.1) of [H12]). The *Gross-Prasad period invariant* is the constant

$$P(\pi, \pi') \in (E(\pi) \otimes E(\pi') \otimes \mathbb{C}),$$

well-defined up to a factor in $(E(\pi) \otimes E(\pi'))^{\times}$, such that

$$I^{can}(\pi, \pi') = P(\pi, \pi') \cdot I^{mot}(\pi, \pi').$$

Again, the definition of the Gross-Prasad period is conditional on the automatic continuity hypothesis.

In *loc. cit.* a conjecture is assumed that implies that either $P(\pi, \pi') = 0$ or $P(\pi, \pi') \in (E(\pi) \otimes E(\pi') \otimes \mathbb{C})^{\times}$. In order to avoid referring to this conjecture – and to avoid creating new misprints – we henceforward fix $\alpha \in Hom(E(\pi) \otimes E(\pi'), \mathbb{C})$ of $E(\pi) \cdot E(\pi')$ and consider $\pi_{E(\pi)}^{(0)}$ and $\pi_{E(\pi')}^{',(0)}$ as subspaces of the respective complex automorphic representations. Then we let $P_{\alpha}(\pi, \pi') \in \mathbb{C}$ be the projection of $P(\pi, \pi')$ on the α-component; $P_{\alpha}(\pi, \pi') = 0$ if and only if $\int_{H'(F) \backslash H'(\mathbf{A})} f(h') f'(h') dh' = 0$ for all $f \in \pi$, $f' \in \pi'$.

We reformulate 2.6 in terms of the Gross-Prasad periods.

Proposition 3.3. *Let π be a cuspidal automorphic representation of $G(\mathbf{A})$, with archimedean component π_∞ in the discrete series. Suppose the Harish-Chandra parameter of π_∞ is sufficiently regular. Then there is an integrable discrete series representation π'_∞ of $G'(\mathbb{R})$, a cuspidal automorphic representation π' of G' with archimedean component π'_∞ such that $I^{can}(\pi, \pi') \neq 0$. Moreover, we can find $f \in \pi$ and $f' \in \pi'$ in the minimal K- (resp. K'-) types of the archimedean components of their respective representations such that $I^{can}(\pi, \pi')(f, f') \neq 0$.*

Corollary 3.4. *Let π and π' be as in the previous proposition and α as in the discussion above. Then $P_\alpha(\pi, \pi')^{-1} \cdot I^{can}(\pi, \pi')$ is a non-zero $\alpha(E(\pi) \otimes E(\pi'))$-bilinear pairing*

$$\pi^{(0)}_{E(\pi)} \otimes \pi'^{,(0)}_{E(\pi')}) \to \alpha(E(\pi) \otimes E(\pi'))$$

that does not vanish on the tensor product of the minimal types at infinity.

Let $\pi^{(0)}_{\overline{\mathbb{Q}}}$ and $\pi'^{,(0)}_{\overline{\mathbb{Q}}}$ be the $\overline{\mathbb{Q}}$-subspaces generated by $\pi^{(0)}_{E(\pi)}$ and $\pi'^{,(0)}_{E(\pi')}$ in the respective complex automorphic representations.

Theorem 3.5. *Assume the automatic continuity hypothesis (Conjecture 4.3.1 of [H12]) Let π be a cuspidal automorphic representation of $G(\mathbf{A})$, with archimedean component π_∞ in the discrete series and minimal K-type τ. Suppose the Harish-Chandra parameter of π_∞ is sufficiently regular. Let $f \in \pi$ be an automorphic form whose archimedean component lies in τ; in particular, $f \in \pi^{(0)}$. Let π'_∞ be a discrete series representation of $G'(\mathbb{R})$ that satisfies the conditions of 3.3. Then $f \in \pi^{(0)}_{\overline{\mathbb{Q}}}$ if and only if, for every $\gamma \in G(\mathbf{A}_f)$, every automorphic representation π' of G' of infinity type π'_∞, and every $f' \in \pi'_{\overline{\mathbb{Q}}}$,*

$$(3.6) \qquad \lambda_{\gamma, f'}(f) := P_\alpha(\pi, \pi')^{-1} \cdot I^{can}(\pi, \pi')(f^\gamma, f') \in \overline{\mathbb{Q}}$$

where $f^\gamma(h) = f(h\gamma)$.

Proof. The proof is exactly analogous to that of Theorem 7.6 of [H90]. We temporarily adopt the notation of that proof: thus π contributes to the coherent cohomology space $\bar{H}^q([\mathcal{V}])$ for some irreducible automorphic vector bundle $[\mathcal{V}]$ on the Shimura variety attached to G. The space $\bar{H}^q$ of [H90] is elsewhere called *interior cohomology*; in [H12] it is denoted $H^q_!$. The regularity hypothesis implies that the π_f-isotypic component of $\bar{H}^q([\mathcal{V}])$ consists entirely of the classes of cusp forms (cf. [H12], 3.3.2); in particular, hypothesis (d) of [H90] 7.6. Hypotheses (a) and (c) hold for similar reasons: the point is that we can take $\mathbf{t}$ to be a Cartan subalgebra of $Lie(G')$ as well as $Lie(G)$, and the walls of a

Weyl chamber for G' form a subset of the walls of a Weyl chamber for G, so if an infinitesimal character in $\mathfrak{t}^*$ is far from the walls for G it is *a fortiori* far from the walls for G'. Condition (b) of [H90], 7.6, which identifies I^{can} as a cup product in coherent cohomology, does not hold; instead we have the condition that $\dim Hom_{(\mathfrak{h}',K')}(\pi_\infty^{(0)} \otimes \pi_\infty'^{,(0)}, \mathbb{C}) = 1$, which requires automatic continuity.

Now the rational structure $\pi_{\overline{\mathbb{Q}}}^{(0)}$ is invariant under the action of $G(\mathbf{A}_f)$, so it is obvious that if $f \in \pi_{\overline{\mathbb{Q}}}^{(0)}$ then (3.6) is satisfied for all π' and all $f' \in \pi'_{\overline{\mathbb{Q}}}$. So it suffices to show that the linear forms $\lambda_{\gamma,f'}$ span the space of linear forms on the (countable-dimensional) subspace $\tau \otimes \pi_f$ of π. This is the content of Theorem 7.4 of [H90]; the condition $0 \neq p(v)$ in the statement of that theorem is implied by the assertion concerning minimal types in 3.3.

$\square$

Remark 3.7. Again, if we know a priori that π'_∞ is isolated in the automorphic spectrum of G', then we can apply [BS] rather than the arguments in [H90].

Remark 3.8. By using the more precise Gross-Prasad periods defined in [H12], we can obtain a criterion for f to be rational over the field $E(\pi)$, and not just over $\overline{\mathbb{Q}}$. The condition is that $Gal(\overline{\mathbb{Q}}/E(\pi))$ acts on the quantities $\lambda_{\gamma,f'}(f)$ by permuting every appearance of f, f', π, π', and α, including in the period invariant appearing on the right-hand side of (3.6).

Theorem 3.5 characterizes $\overline{\mathbb{Q}}$-rational forms on G of (sufficiently regular) coherent cohomology type by their pairings with $\overline{\mathbb{Q}}$-rational forms on G' of coherent cohomology type, but of course one needs to be able to characterize the latter. However, it was noted during the proof that the regularity hypothesis for π_∞ implied a regularity hypothesis for π'_∞. Indeed, it follows from the classical branching laws for the restriction of a representation of (compact) $U(m)$ to $U(m-1)$ that conditions (a) and (b) of 2.3 for the minimal K-type τ imply the corresponding conditions for at least one irreducible constituent τ' of its restriction to K'. Thus by induction on n we can assume that the $\overline{\mathbb{Q}}$-rational f' used to define the $\lambda_{\gamma,f'}$ have already been identified. This gives a complete answer to the question of Piatetski-Shapiro mentioned in the introduction in the case of Shimura varieties attached to unitary groups, at least under a regularity hypothesis. I don't know whether or not the regularity hypothesis can be relaxed.

Remark 3.9. The rationality criterion of Theorem 3.5 makes specific use of the representation π'_∞ that can be proved to pair non-trivially with π_∞, thanks to the positivity property of the Flensted-Jensen minimal matrix coefficients. If we know independently that (the dual of) π'_∞ is weakly contained in π_∞, then the analogue of Theorem 3.5 remains valid. The pairings whose existence is predicted by [GGP] do not restrict in most cases to non-trivial pairings of minimal types; nevertheless, the Gross-Prasad conjecture for real unitary groups implies that each non-trivial pairing gives a new rationality criterion for coherent cohomology.

4. Gross-Prasad periods as cup products

The novel feature of this article is the introduction of a rationality criterion, valid for automorphic representations of sufficiently regular discrete series type, that, unlike the criterion of [H90, HL], is not derived from a cup product in coherent cohomology. In this final section we characterize cup-product pairings among all pairings given by integrals over $H'(\mathbf{A})$, and show that certain π_∞ never admit cup product pairings with coherent cohomology classes on the Shimura variety attached to H'. Thus the rationality criterion of the present paper is substantially more general than that of [H90].

The cup-product pairings are of independent interest, since they can be used to study period relations. This will be explained in forthcoming work.

4.1. **Parameters.** In this section we let H denote the real Lie group $U(r,s)$, the unitary group of the standard hermitian form $I_{r,s} = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix}$. The maximal compact subgroup $K = U(r) \times U(s)$, the subgroup that respects the decomposition of $\mathbb{C}^r$ into positive-definite and negative-definite subspaces as indicated by the form of $I_{r,s}$. For a maximal torus T we take the group of diagonal matrices in H; this is also a maximal torus of K. An irreducible finite-dimensional representation (σ, V) of H is determined by its highest weight a_σ relative to T and the upper-triangular Borel algebra $\mathfrak{b}$ of $\mathfrak{h} = Lie(H)_\mathbb{C}$; a_σ is written in the usual way as a non-increasing n-tuple of integers $(a_1 \geq a_2 \geq \cdots \geq a_n)$.

Let $\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2})$ be the half-sum of positive roots for $\mathfrak{b}$. We let $\tilde{\rho} = \rho + \frac{n-1}{2}(1, 1, \ldots, 1) = (n-1, n-2, \ldots, 1, 0)$. The infinitesimal character of the representation (σ, V) above is the element $\lambda_\sigma = a_\sigma + \rho$ of $\mathfrak{t}^*$; the element $\tilde{\lambda}_\sigma = a_\sigma + \tilde{\rho} = (a_i + n - i, i = 1, \ldots, n)$ will be called the *Hodge parameter*, for reasons that will soon be clear.

We consider the L-packet $\Pi_\sigma = \Pi_\sigma^{r,s}$ of discrete series representations π of H such that $H^i(\mathfrak{k}, K; \pi \otimes V^\vee) \neq 0$ for some i (necessarily $i = rs = \dim H/K$). Then Π_σ contains $\binom{n}{r}$ members, indexed by permutations of the entries of λ_σ or $\tilde{\lambda}_\sigma$, modulo (subsequent) permutations that preserve the first r entries. Write

$$\lambda_\sigma = (\lambda_1 > \lambda_2 > \cdots > \lambda_n); \ \tilde{\lambda}_\sigma = (\tilde{\lambda}_1 > \tilde{\lambda}_2 > \cdots > \tilde{\lambda}_n)$$

It's more convenient to index Π_σ by (r,s)-shuffles of the λ_i, namely (r,s)-tuples of the form $\lambda = (a_1 > \cdots > a_r; b_1 > \cdots > b_s)$ which is a permutation of the entries of λ_σ. If $\pi \in \Pi_\sigma$ corresponds to λ, then λ is the *Harish-Chandra parameter* of π, and we write $\pi = \pi_\lambda$. We also define $\tilde{\lambda} = \lambda + \frac{n-1}{2}(1, 1, \ldots, 1)$.

Let $\Delta^{nc,+}$ be the set of positive non-compact roots – roots of the form

$$\alpha_{i,r+j} : (c_1, c_2, \ldots, c_n) \mapsto c_i - c_{r+j}, 1 \le i \le r, 1 \le j \le s.$$

These are by definition the roots of T acting on $\mathfrak{p}^- \subset \mathfrak{h}$, the Lie algebra of matrices of the form $\begin{pmatrix} 0_r & X \\ 0 & 0_s \end{pmatrix}$, which is also isomorphic to the antiholomorphic tangent space of the hermitian symmetric space H/K at the fixed point of K. In other words, the choice of $\mathfrak{b}$, and therefore ρ, is consistent with a specific choice of complex structure. The *degree* $q(\lambda)$ of λ is the number of pairs (i, j) such that $a_i > b_j$, or equivalently to the cardinality of

$$\Delta^{nc,+}(\lambda) = \{\alpha \in \Delta^{nc,+} \mid \langle \alpha, \lambda \rangle > 0.$$

This is related to the usual notion of length by the formula $q(\lambda) = rs - \ell(w)$, where w is the permutation that takes λ_σ to λ. There are two extreme elements of Π_σ: the *antiholomorphic representation*, with Harish-Chandra parameter λ_σ (and degree rs), and the *holomorphic representation*, corresponding to the longest shuffle, equivalently to the Harish-Chandra parameter $(\lambda_{s+1} > \cdots > \lambda_n; \lambda_1 > \cdots > \lambda_s)$ (with degree 0).

The integer $q(\lambda)$ is the degree of coherent cohomology to which λ contributes. More precisely, to any λ we associate its *coherent parameter* $\Lambda = \lambda - \rho$, viewed as the highest weight of an irreducible representation W_Λ of K. Let $\mathfrak{q} = \mathfrak{p}^- \oplus Lie(K)$. Then

$$(4.1) \qquad\qquad \dim H^{q(\lambda)}(\mathfrak{q}, K; \pi_\lambda \otimes W_\Lambda^\vee) = 1$$

and all other $H^i(\mathfrak{q}, K; \pi_\lambda \otimes W)$ vanish as W runs over irreducible representations of K.

The space in 4.1 is spanned by an element of $(\wedge^{q(\lambda)}(\mathfrak{p}^-)^* \otimes \pi_\lambda \otimes W_\Lambda^\vee)^K$, or equivalently by a homomorphism

$$(4.2) \qquad h_\lambda \in Hom_K(\wedge^{q(\lambda)}(\mathfrak{p}^-) \otimes W_\Lambda, \pi_\lambda).$$

The image of h_λ is an irreducible K-type τ_λ, the minimal (or Blattner) K-type, and its highest weight, also denoted τ, is the *Blattner parameter* of π_λ (or of λ). The formula for τ is given in terms of the chamber $C(\lambda)$ for which λ is positive

$$(4.3) \qquad \tau = \Lambda + \sum_{\alpha \in \Delta^{nc,+}(\lambda)} \alpha.$$

For each $\alpha \in \Delta^{nc,+}$, choose a non-zero basis vector X_α in the corresponding root space.

Lemma 4.4. *The homomorphism h_λ is non-zero on the vector*

$$v_\lambda = \wedge_{\alpha \in \Delta^{nc,+}(\lambda)} X_\alpha \otimes w_\Lambda$$

where w_Λ is a basis vector of the highest weight subspace of W_Λ, and $h(v_\lambda)$ generates the highest weight space of τ.

Proof. This is presumably well-known but I include the one-line proof: if we take a positive root system for K compatible with the chamber $C(\lambda)$, then v_λ, which is a weight vector for the weight 4.3, is an extreme vector in this chamber in the tensor product $\wedge^{q(\lambda)}(\mathfrak{p}^-) \otimes W_\Lambda$ and in particular has multiplicity one. Since the image of h_τ contains a vector of weight 4.3, it can't vanish on v_λ. $\qquad\square$

4.2. **Rationality.** Now we return to the notation of §3, with $G = GU(W)$, $G' = G(U(W') \times U(W_0))$. The elements of the CM type Σ are denoted v, and for each v the space $W_v = W \otimes_{F,v} \mathbb{C}$ is a complex hermitian space with signature (r_v, s_v). We assume that W_v' has signature $(r_v - 1, s_v)$ for each v; although the discussion can be modified if this is not the case, we can always choose W' with this property, and it simplifies the discussion.

A discrete series representation π of $G(\mathbb{R})$ restricts to a sum of discrete series representations of $H(\mathbb{R}) = \prod_v U(r_v, s_v) := \prod_v H_v$. In fact, there is no ambiguity unless $r_v = s_v$ for some v; each such place contributes two summands, and we choose one of the two – which we still call π – and replace H_v at such a point by its (real) identity component H_v^+. We thus obtain a Σ-tuple of Harish-Chandra parameters $(\lambda_v, v \in \Sigma)$. Let $K_v = U(r_v, 0) \times U(0, s_v) \subset U(r_v, s_v)$ be a maximal compact subgroup, as above, and let $K = \prod_v K_v$. As above, the signature determines a holomorphic structure on the the symmetric space

$X(G)^+ = H(\mathbb{R})/K$ – the product over v of the spaces denoted $\mathfrak{p}^-$ above is the space of anti-holomorphic tangent vectors at the fixed point of K in the symmetric space – and therefore we can assign to each v a reference parameter

$$\lambda_{\sigma,v} = (\lambda_{1,v} > \cdots > \lambda_{n,v})$$

which is the infinitesimal character of an irreducible finite-dimensional representation (σ_v, V_v) of H_v. This is a permutation of λ_v, which we write as above

$$\lambda_v = (a_{1,v} > \cdots > a_{r_v,v}; b_{1,v} > \cdots > b_{s_v,v}).$$

Let $\Lambda_v = \lambda_v - \rho_v$ be the v-component of the coherent parameter of π, $\Lambda = (\Lambda_v, v \in \Sigma)$. It defines an automorphic vector bundle $[W_\Lambda]$ on the Shimura variety $Sh(G, X(G))$, where $X(G)$ is the (disconnected) hermitian symmetric space attached to $G(\mathbb{R})$ and $X(G)^+ \subset X(G)$ is a fixed $H(\mathbb{R})$-invariant component (also invariant under the identity component $G(\mathbb{R})^0 \subset G(\mathbb{R})$. As in the proof of 3.5, we follow [H90] and denote by $\bar{H}^*([W_\Lambda]$ the interior cohomology of $[W_\Lambda]$.

Let $H' = U(W') \times U(W_0)$, $K' = \prod_v K'_v$, $K'_v = K_v \cap H'(\mathbb{R}) = U(r_v-1,0) \times U(1,0) \times U(0,s_v)$; we drop the 0's in what follows. Write $\Lambda_v = (\alpha_{1,v} \geq \cdots \geq \alpha_{r_v,v}; \beta_{1,v} \geq \cdots \geq \beta_{s_v,v})$; this is a dominant integral weight of a finite-dimensional representation W_{Λ_v} of K_v. It follows from the classical branching formula that the restriction of W_{Λ_v} to K'_v contains the representation with highest weight

$$\Lambda'_v = (\alpha_{1,v} \geq \cdots \geq \alpha_{r_v-1,v}; \alpha_{r_v,v}; \beta_{1,v} \geq \cdots \geq \beta_{s_v,v})$$

where the semicolons separate the weights for $U(r_v) \times U(1) \times U(s_v)$. Let $\Lambda' = (\Lambda'_v)$ be the corresponding highest weight for K', and let $\lambda' = \Lambda' + \rho'$, where ρ' is by analogy the half-sum of the positive roots of H' relative to T for the chamber containing the chosen positive chamber for (H, T). With the above notation, we have
(4.5)
$$\lambda'_v = (a_{1,v} - \frac{1}{2} > \cdots > a_{r_v-1,v} - -\frac{1}{2}; \alpha_{r_v,v}; b_{1,v} + \frac{1}{2} > \cdots > b_{s_v,v} + \frac{1}{2}).$$

Hypothesis 4.6. *The highest weight $a_{\sigma,v} = \lambda_{\sigma,v} - \rho$ of (σ_v, V_v) is a regular character of T for each v. Equivalently, for each v, $\lambda_{i,v} - \lambda_{i+1,v} \geq 2$, $i = 1, \ldots, n-1$.*

Lemma 4.7. *Under Hypothesis 4.6, λ' is regular for the root system of (H', T), and belongs to $(\frac{n-2}{2} + \mathbb{Z})^{n-1}$. It is therefore the Harish-Chandra parameter for a unique discrete series representation $\pi_{\lambda'}$ of $H'(\mathbb{R})$.*

This is clear from the form of 4.5: under the regularity hypothesis, none of the $a_{i,v} - \frac{1}{2}$ can coincide with any of the $b_{j,v} + \frac{1}{2}$.

We now consider the following hypothesis:

Hypothesis 4.8. *For each v, the parameter $a_{r_v,v}$ equals $\lambda_{n,v}$, the smallest of the $\lambda_{i,v}$.*

Of the $\prod_v \binom{n}{r_v}$ elements $\pi_\lambda \subset \Pi_\sigma$, the set of parameters satisfying 4.8 is

$$\prod_v \frac{\binom{n-1}{s_v}}{\binom{n}{s_v}} = \prod_v \frac{r_v}{n}$$

of the total.

Let $\Delta'^{,nc,+}$ be the set of positive non-compact roots for (H', T), defined as in the previous section. We can view $\Delta'^{,nc,+}$ as the set of $\alpha_{i,r_v+j} \in \Delta^{nc,+}$ with $1 \leq i \leq r_v - 1; 1 \leq j \leq s_v$ for $v \in \Sigma$. Hypothesis 4.8 is equivalent to the hypothesis that

$$(4.9) \qquad \Delta'^{,nc,+}(\lambda') = \Delta^{nc,+}(\lambda).$$

We define $X(G')^+ \subset X(G')$, $\mathfrak{q}'$, and the Shimura variety $Sh(G', X(G'))$, as in the previous paragraph; we also define $\Lambda' = \lambda' - \rho'$, the representation $W_{\Lambda'}$ of K', and the automorphic vector bundle $[W_{\Lambda'}]$ on $Sh(G', X(G'))$. The inclusion of the Shimura data $(G', X(G')) \subset (G, X(G))$ defines an inclusion of Shimura varieties, and by functoriality a restriction map of $\bar{\partial}$-cohomology

$$(4.10) \qquad H^{q(\lambda)}(\mathfrak{q}, K; \pi_\lambda \otimes W_\Lambda^\vee) \to H^{q(\lambda)}(\mathfrak{q}', K'; \pi_{\lambda'} \otimes W_{\Lambda'}^\vee).$$

It follows immediately from Lemma 4.4 and 4.9 that

Proposition 4.11. *Assume Hypothesis 4.6.*
   (a) *If λ satisfies Hypothesis 4.8, then the map (4.10) is an isomorphism of 1-dimensional spaces.*
   (b) *If λ does not satisfy 4.8, then (4.10) is the zero map.*

For global automorphic representations π and π' whose archimedean components satisfy Hypothesis 4.8, the canonical period integral $I^{can}(\pi, \pi')$ can be identified with cup products in coherent cohomology, by the formalism of [H90, HL]. The following theorem is an immediate consequence of Lemma 7.5.5 of [H90] and the definitions:

Theorem 4.12. *Suppose λ satisfies Hypotheses 4.6 and 4.8. Let π and π' be cuspidal automorphic representations of G and G', respectively,*

with archimedean components π_λ and $\pi_{\lambda'}^\vee$. Then the Gross-Prasad period invariants $P_\alpha(\pi, \pi')$ are algebraic numbers.

This can be refined to give vector-valued Gross-Prasad invariants, depending on complex embeddings of the coefficient fields, whose entries belong to specific number fields. Combining Theorem 4.12 with ergodic considerations, along the lines of 1.3, we obtain examples of the rationality criteria provided by Theorem 7.6 and Corollary 7.7.1 of [H90]; the point is that (a) of Proposition 4.11 implies hypothesis (b) of Theorem 7.6 of *loc. cit.* Theorem 4.12 is also of interest in proving period relations, and will be considered at length in a future paper.

On the other hand, (b) of Proposition 4.11, together with Theorem 3.5, gives explicit examples of rationality criteria that have nothing to do with coherent cohomology. It is plausible that when π does not satisfy 4.8 then there is no π' pairing non-trivially with π for which the $P_\alpha(\pi, \pi')$ are algebraic. On the other hand, it is possible that $\pi^\vee$ satisfies 4.8 even though π does not. Since a rationality criterion for $\pi^\vee$ easily gives rise to one for π (using cup products on H rather than on H'), it is natural to wonder whether for all π satisfying 4.6, either π or $\pi^\vee$ also satisfies 4.8. This seems unlikely: in the notation of 4.8, suppose $a_{r_v,v} = \lambda_{n,v}$ for some but not all v. The above reasoning then suggests that, except possibly for some degenerate cases, neither π nor $\pi^\vee$ satisfies 4.8.

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