CHAPTER 2

Symmetric algebras

A. Definition and first properties

A1. Central forms and traces on projective modules

Central forms.

Let A be an R–algebra. A form t defined on A is said to be *central* if it satisfies the property

$$t(aa') = t(a'a) \quad (\forall a, a' \in A)\,.$$

Thus a central form can be identified to a form on the R–module $A/[A, A]$.

Whenever X is an A–module, we denote by $X^* := \mathrm{Hom}_R(X, R)$ its R–dual, viewed as an $E_A X$–module–A.

We denote by $\mathrm{CF}(A,R)$ the R–submodule of A^* consisting of all central forms on A. Then $\mathrm{CF}(A,R)$ is the orthogonal of the submodule $[A,A]$ of A, hence is canonically identified with the R–module $(A/[A,A])^*$.

If $t\colon A \to R$ is a central form on A, we shall still denote by $t\colon A/[A,A] \to R$ the form on $A/[A,A]$ which corresponds to t.

More generally, let M be an A–module–A. A central form on M is a linear form $t\colon M \to R$ such that $t(am) = t(ma)$ for all $a \in A$ and $m \in M$. In other words, the central forms on M are the forms defined by the R dual of $\mathrm{H}^0(A,M) = M/[A,M]$.

Note that the multiplication by elements of the center ZA of A gives the R–module $A/[A,A]$ a natural structure of ZA–module. Thus $\mathrm{CF}(A,R)$ inherits a structure of ZA–module, defined by $zt := t(z\cdot)$ for all $z \in ZA$ and $t \in \mathrm{CF}(A,R)$.

Traces on projective modules, characters.

Let X be an A–module. Then the R–module $X^\vee \otimes_A X$ is naturally equipped with a linear form

$$\begin{cases} X^\vee \otimes_A X \to A/[A,A] \\ y \otimes x \mapsto xy \mod [A,A]\,. \end{cases}$$

In particular, if P is a finitely generated projective A–module, since $P^\vee \otimes_A P \simeq E_A P$, we get an R–linear map (the trace on a projective module)

$$\mathrm{tr}_{P/A}\colon E_A P \to A/[A,A]\,.$$

2.0. Lemma. *Whenever P is a finitely generated projective A–module, the trace*

$$\mathrm{tr}_{P/A}\colon E_A P \to A/[A,A]$$

is central.

Proof of 2.0. In what follows, we identify $P^\vee \otimes_A P$ with $E_A P$. Let $x, x' \in P$ and $y, y' \in P^\vee$. Then we have

$$(y \otimes_A x)(y' \otimes_A x') = y(xy') \otimes_A x' = y \otimes_A (xy')x'\,,$$

from which is follows that

$$\mathrm{tr}_{P/A}((y \otimes_A x)(y' \otimes_A x')) = (xy')(x'y) \mod [A,A]\,,$$

which shows indeed that $\mathrm{tr}_{A/P}$ is central. □

Now if $t\colon A \to R$ is a central form, we deduce by composition a central form

$$t_P\colon E_A P \to R \quad , \quad \varphi \mapsto t(\mathrm{tr}_P(\varphi))\,.$$

In particular, whenever X is a finitely generated projective R–module, we have the trace form

$$\mathrm{tr}_{X/R}\colon E_R X \to R \quad \text{defined by} \quad (y \otimes x) \mapsto xy\ , \forall y \in X^*, x \in X\,.$$

2.1. Definition. *Let X be an A–module which is a finitely generated projective R–module and let $\lambda_X\colon A \to E_R X$ denote the structural morphism. The character of the A–module X (or of the representation of A defined by λ_X) is the central form*

$$\chi_X\colon A \to R \quad , \quad a \mapsto \mathrm{tr}_{X/R}(\lambda_X(a))\,.$$

A2. Symmetric algebras

A central form $t\colon A \to R$ defines a morphism $\widehat{t}$ of A–modules–A as follows :

$$\begin{aligned}&\widehat{t}\colon A \to A^* \\ &a \mapsto \widehat{t}(a)\colon a' \mapsto t(aa')\end{aligned}$$

Indeed, for $a, a', x \in A$, we have

$$\widehat{t}(axa') = t(axa'\cdot) = t(xa'\cdot a) = a\widehat{t}(x)a' \,.$$

Note that the restriction of $\widehat{t}$ to ZA defines a ZA–morphism : $ZA \to \mathrm{CF}(A, R)$.

2.2. Definition. *Let A be an R–algebra. We say that A is a symmetric algebra if the following conditions are fulfilled :*
(S1) *A is a finitely generated projective R–module,*
(S2) *There exists a central form $t\colon A \to R$ such that $\widehat{t}$ is an isomorphism.*
If A is a symmetric algebra and if t is a form like in (S2), we call t a symmetrizing form for A.

Examples.
1. The trace is a symmetrizing form on the algebra $\mathrm{Mat}_n(R)$.
2. If G is a finite group, its group algebra RG is a symmetric algebra. The form

$$t\colon RG \to R \quad , \quad \sum_{g\in G} \lambda_g g \mapsto \lambda_1$$

is called the canonical symmetrizing form on RG.

3. If k is a field, we shall see later that the algebra $A := \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ is not a symmetric algebra.

The following example is singled out as a lemma.

2.3. Lemma. *Let D be a finite dimensional division k–algebra. Then D is a symmetric algebra.*

Proof of 2.3.
First we prove that $[D, D] \neq D$. It is enough to prove it in the case where D is central (indeed, the ZD–vector space generated by $\{ab - ba \mid (a, b \in A)\}$ contains the k–vector space generated by that set). In this case, we know by 1.53 that $\overline{k} \otimes_k D$ is a matrix algebra $\mathrm{Mat}_m(k)$ over $\overline{k}$. If $[D, D] = D$, then every element of $\mathrm{Mat}_m(k)$ has trace zero, a contradiction.

Now choose a nonzero k–linear form t on D whose kernel contains $[D, D]$. Thus t is central. Let us check that t is symmetrizing. To do that, it is enough to prove that $\widehat{t}$ is injective. But if x is a nonzero element of D, the map $y \mapsto xy$ is a permutation of D, hence there exists $y \in D$ such that $t(xy) \neq 0$, proving that $\widehat{t}(x) \neq 0$. $\square$

2.4. Lemma. *Let A be a symmetric algebra, with symmetrizing form t.*
(1) The restriction of $\widehat{t}$ to ZA

$$ZA \to \mathrm{CF}(A, R) \quad , \quad z \mapsto t(z\cdot)$$

is an isomorphism of ZA–modules. In particular, $\mathrm{CF}(A,R)$ *is a free ZA–module of rank 1.*

(2) A central form $\widehat{t}(z)$ corresponding to an element $z \in ZA$ is a symmetrizing form if and only if z is invertible.

Proof of 2.4. Let u be a form on A. By hypothesis, we have $u = t(a\,\cdot\,)$ for some $a \in A$, and u is central if and only if a is central. This shows the surjectivity of the map $\widehat{t}\colon ZA \to \mathrm{CF}(A,R)$. The injectivity results from the injectivity of $\widehat{t}$. Finally, this proves that symmetrizing forms are the elements t of $\mathrm{CF}(A,R)$ such that $\{t\}$ is a basis of $\mathrm{CF}(A,R)$ as a ZA–module. □

Remark. If t is a symmetrizing form on A, its kernel $\ker(t)$ contains no left (or right) non trivial ideal of A.

Annihilators and orthogonals.

Let $\mathfrak{a}$ be a subset of the algebra A. The right annihilator of $\mathfrak{a}$ is defined as

$$\mathrm{Ann}(\mathfrak{a})_A := \{x \in A \mid (\mathfrak{a}.x = 0)\}\,.$$

It is immediate to check that the right annihilator of a subset is a right ideal, and that the right annihilator of a right ideal is a twosided ideal.

Suppose now that A is a symmetric algebra, and choose a symmetrizing form t on A. Whenever $\mathfrak{a}$ is a subset of A, we denote by $\mathfrak{a}^\perp$ its orthogonal for the bilinear form defined by t, *i.e.*,

$$\mathfrak{a}^\perp := \{x \in A \mid t(\mathfrak{a}x) = 0\}\,.$$

Note that if $\mathfrak{a}$ is stable by multiplication by $(ZA)^\times$, then $\mathfrak{a}^\perp$ does not depend on the choice of the symmetrizing form t.

2.5. Proposition. *Assume that A is symmetric.*
(1) We have $[A,A]^\perp = ZA$.
(2) If $\mathfrak{a}$ is a left ideal of A, we have $\mathfrak{a}^\perp = \mathrm{Ann}(\mathfrak{a})_A$.

Proof of 2.5.
(1) We have

$$t(zab) = t(zba) \Leftrightarrow t(bza) = t(zba)\,,$$

which shows that $z \in [A,A]^\perp$ if and only if $z \in ZA$.
(2) We have

$$\mathfrak{a}x = 0 \Leftrightarrow (\forall y \in A)\ \ t(y\mathfrak{a}x) = 0 \Leftrightarrow t(\mathfrak{a}x) = 0\,,$$

which proves (2). □

A3. Characterizations in terms of module categories

Assume that A is an R–algebra which is a finitely generated projective R–module.

Let us first notice a few elementary properties.

1. Any finitely generated projective A–module is also a finitely generated projective R–module.

Indeed, if A is a summand of R^m, any summand of A^n is also a summand of R^{mn}.

2. If X is a finitely generated projective A–module and if Y is a module–A, then $Y \otimes_A X$ is isomorphic to a summand of Y^n for some positive integer n. It follows that if moreover Y is a finitely generated projective R–module, then $Y \otimes_A X$ is also a finitely generated projective R–module.

Let us denote by ${}_A\mathbf{proj}$ the full subcategory of ${}_A\mathbf{Mod}$ whose objects are the finitely generated projective A–modules. We define similarly the notation $\mathbf{proj}_A$ and ${}_R\mathbf{proj}$.

2.6. Proposition. *Let A be an R–algebra, assumed to be a finitely generated projective R–module. The following conditions are equivalent.*

(i) *A is symmetric.*

(ii) *A and A^* are isomorphic as A–modules–A.*

(iii) *As (contravariant) functors ${}_A\mathbf{Mod} \longrightarrow \mathbf{Mod}_A$, we have*

$$\mathrm{Hom}_R(\bullet\,, R) \simeq \mathrm{Hom}_A(\bullet\,, A)\,.$$

(iii') *As (contravariant) functors $\mathbf{Mod}_A \longrightarrow {}_A\mathbf{Mod}$, we have*

$$\mathrm{Hom}_R(\bullet\,, R) \simeq \mathrm{Hom}(\bullet\,, A)_A\,.$$

(iv) *For $P \in {}_A\mathbf{proj}$ and $X \in {}_A\mathbf{Mod} \cap {}_R\mathbf{proj}$ we have natural isomorphisms*

$$\mathrm{Hom}_A(P, X) \simeq \mathrm{Hom}_A(X, P)^*\,.$$

(iv') *For $P \in \mathbf{proj}_A$ and $X \in \mathbf{Mod}_A \cap {}_R\mathbf{proj}$ we have natural isomorphisms*

$$\mathrm{Hom}(P, X)_A \simeq \mathrm{Hom}(X, P)_A^*\,.$$

Proof of 2.6. It is enough to prove (i)$\Leftrightarrow$(ii), and (ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(ii).

(i)$\Rightarrow$(ii) results from the fact, noticed above, that if t is a central form, then $\widehat{t}$ is a morphism of bimodules from A to A^*.

(ii)$\Rightarrow$(i). Assume that $\theta\colon A \xrightarrow{\sim} A^*$ is a bimodule isomorphism. Set $t := \theta(1)$. Then for $a \in A$ we have

$$t(aa') = \theta(1)(aa') = (a'\theta(1))(a) = \theta(a')(a) = (\theta(1)a')(a) = \theta(1)(a'a) = t(a'a)\,,$$

which shows both that t is central and that $\widehat{t} = \theta$.

(ii)$\Rightarrow$(iii). Let X be an A–module. Since $A \simeq A^*$, we have $\mathrm{Hom}_A(X, A) \simeq \mathrm{Hom}_A(X, \mathrm{Hom}_R(A, R))$. By the "isomorphisme cher à Cartan", it follows that $\mathrm{Hom}_A(X, A) \simeq \mathrm{Hom}_R(A \otimes_A X, R)$ hence $\mathrm{Hom}_A(X, A) \simeq \mathrm{Hom}_R(X, R)$.

(iii)$\Rightarrow$(iv). Let P be a finitely generated projective A–module and let X be a finitely generated A–module. Since P is a finitely generated projective R–module, we have $P \simeq \mathrm{Hom}_R(P^*, R)$, and it results from the "isomorphisme cher à Cartan" that $\mathrm{Hom}_A(X, P) \simeq \mathrm{Hom}_R(P^* \otimes_A X, R)$, and since $P^* \simeq P^\vee$, we get $\mathrm{Hom}_A(X, P) \simeq \mathrm{Hom}_R(P^\vee \otimes_A X, R)$.

Since the module–A $P^\vee$ is finitely generated projective and since X is a finitely generated projective R–module, we see that $P^\vee \otimes_A X$ is also a finitely generated projective R–module, hence we have $\mathrm{Hom}_A(X, P)^* \simeq P^\vee \otimes_A X$. Since P is a finitely generated projective A–module, we know that $P^\vee \otimes_A X \simeq \mathrm{Hom}_A(P, X)$. Hence we have proved that $\mathrm{Hom}_A(X, P)^* \simeq \mathrm{Hom}_A(P, X)$.

(iv)$\Rightarrow$(ii). Choose $P = X = A$ (viewed as an A–module). Then the natural isomorphism $\mathrm{Hom}_A(A, A)^* \simeq \mathrm{Hom}_A(A, A)$ is a bimodule isomorphism $A^* \simeq A$. $\square$

Symmetric algebras and projective modules.

2.7. Proposition. *Let A be a symmetric R–algebra, and let P be a finitely generated projective A–module. Then $E_A P$ is a symmetric R–algebra.*

Proof of 2.7. Recall that we have an isomorphism of $E_A P$–modules–$E_A P$

$$P^\vee \otimes_A P \xrightarrow{\sim} E_A P \,.$$

Since P is a finitely generated projective A–module and since $P^\vee$ is a finitely generated R–module, this shows that $E_A P$ is a finitely generated projective R–module.

Moreover, by 2.6, condition (iii), we see that we have a natural isomorphism

$$\mathrm{Hom}_A(P,P)^* \simeq \mathrm{Hom}_A(P,P)\,,$$

i.e., a bimodule isomorphism

$$E_A P^* \simeq E_A P\,,$$

which shows that $E_A P$ is symmetric. □

2.8. Corollary. *An algebra which is Morita equivalent to a symmetric algebra is a symmetric algebra.*

Proof of 2.8. Indeed, we know that an algebra which is Morita equivalent to A is isomorphic to the algebra of endomorphisms of a finitely generated projective A–module. □

Explicit isomorphisms.

We give here explicit formulas for the isomorphisms stated in 2.6. The reader is invited to check the details.

2.9. Proposition.

(1) Whenever X is an A–module, the morphisms t_X^* and u_X defined by

$$t_X^* \colon \begin{cases} \mathrm{Hom}_A(X,A) \to \mathrm{Hom}_R(X,R) \\ \phi \mapsto t\cdot\phi \end{cases} \qquad u_X \colon \begin{cases} u_X \colon \mathrm{Hom}_R(X,R) \to \mathrm{Hom}_A(X,A) \\ \textit{such that } \psi(ax) = t(au_X(\psi)(x)) \\ (\forall a \in A, x\in X, \psi \in \mathrm{Hom}_R(X,R)) \end{cases}$$

are inverse isomorphisms in $\mathbf{Mod}_{E_A X}$.

(2) Whenever X is an A–module which is a finitely generated projective R–module and P is a finitely generated projective A–module, the pairing

$$\begin{cases} \mathrm{Hom}_A(P,X) \times \mathrm{Hom}_A(X,P) \to R \\ (\varphi,\psi) \mapsto t_P(\varphi\psi) \end{cases}$$

is an R–duality.

Let us in particular exhibit a symmetrizing form on $E_A P$ from a symmetrizing form on A.

Recall that the isomorphism $P^\vee \otimes_A P \xrightarrow{\sim} E_A P$ allows us to define the trace of the finitely generated projective A–module P)

$$\mathrm{tr}_{P/A} \colon E_A P \longrightarrow A/[A,A] \quad , \quad y \otimes_A x \mapsto xy \mod [A,A]\,,$$

and that composing this morphism with a central form t on A, we get a central form

$$t_P \colon E_A P \longrightarrow R$$

on $E_A P$.

2.10. PROPOSITION. *If P is a finitely generated projective A–module and if t is a symmetrizing form on A, then the form t_P is a symmetrizing form on $E_A P$.*

EXAMPLE. The identity from R onto R is a symmetrizing form for R. It follows that the trace is a symmetrizing form for the matrix algebra $\mathrm{Mat}_n(R)$.

REMARK. A particular case of projective A–module is given by $P := Ai$ where i is an idempotent of A. We have (see 1.12) $E_A P \simeq iAi$. Through that isomorphism, the form t_P becomes the form

$$iai \mapsto t(iai)\,.$$

Products of symmetric algebras.

The proof of following result is an immediate consequence of the characterizations in 2.6, and its proof is left to the reader.

2.11. PROPOSITION. *Let A_1, A_2, ... , A_n be R–algebras which are finitely generated projective R–modules, and let A be an algebra isomorphic to a product $A_1 \times A_2 \times \cdots \times A_n$. Then A is symmetric if and only if each A_i $(i = 1, 2, \dots, n)$ is symmetric.*

More concretely, we know that an isomorphism $A \simeq A_1 \times A_2 \times \cdots \times A_n$ determines a decomposition of the unit element 1 of A into a sum of mutually orthogonal central idempotents :

$$1 = e_1 + e_2 + \cdots + e_n\,,$$

corresponding to a decomposition of A into a direct sum of twosided ideals :

$$A = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_n \quad \text{with } \mathfrak{a}_i = Ae_i\,.$$

- If $(t_1, t_2, \dots, t_n)$ is a family of symmetrizing forms on $A_1, A_2, \dots, A_n$ respectively, then the form defined on A by $t_1 + t_2 + \cdots + t_n$ is symmetrizing.
- If t is a symmetrizing form on A, its restriction to each $\mathfrak{a}_i = Ae_i$ defines a symmetrizing form in the algebra A_i.

Strongly symmetric algebras.

2.12. PROPOSITION–DEFINITION. *Let A be a symmetric R–algebra, and let t be a symmetrizing form. The following conditions are equivalent.*
(i) *The form $t \colon A \to R$ is onto.*
(ii) *R is isomorphic to a summand of A in ${}_R\mathbf{Mod}$.*
(iii) *As an R–module, A is a progenerator.*
If the preceding conditions are satisfied, we say that the algebra A is strongly symmetric.

PROOF OF 2.12.
(i)⇒(ii) : Since $t \colon A \to R$ is onto and since R is a projective R–module, t splits and R is indeed isomorphic to a direct summand of A as an R–module.
(ii)⇒(iii) : obvious.

(iii)⇒(i) : By 1.18, we see that the ideal of R consisting of all the $\langle a, b\rangle$ (for $a \in A$ and $b \in A^*$) is equal to R. But since t is symmetrizing, this ideal is equal to $t(A)$, which shows that t is onto. □

Examples.

1. If A is strongly symmetric, and if B is an algebra which is Morita equivalent to A, then B is strongly symmetric.

In particular, the algebra $\mathrm{Mat}_m(R)$ is strongly symmetric, and more generally, if X is a progenerator for R, the algebra $E_R X$ is strictly symmetric.

2. If all projective R–modules are free, then all symmetric R–algebras are strongly symmetric.

3. The algebra RG (G a finite group) is strictly symmetric.

4. If $R = R_1 \times R_2$ (a product of two non zero rings), and if $A := R_1$, then A is a symmetric R–algebra which is not strongly symmetric.

B. Symmetric algebras over a field

We shall apply what preceds to finite dimensional algebras over a (commutative) field k.

B1. Dualities for symmetric algebras over a field

Symmetric and semi–simple algebras.

2.13. Proposition. *Let A be a semi–simple k–algebra. Then A is symmetric.*

Proof of 2.13. Since a semi–simple algebra is isomorphic to a direct product of simple algebras, it is enough to prove that a simple algebra is symmetric. Since a simple algebra is Morita equivalent to a division algebra, it is enough to prove that a division algebra is symmetric : this is lemma 2.3. □

2.14. Proposition. *Let k be a commutative field, and let A be a symmetric finite dimensional k–algebra.*

(1) We have

$$\mathrm{Soc}_A(A) = \mathrm{Soc}(A)_A = \mathrm{Soc}_A(A)_A = \mathrm{Rad}(A)^\perp\,.$$

We denote by $\mathrm{Soc}(A)$ *and call the socle of A the ideal described above.*

(2) We have

$$A/\mathrm{Rad}(A) \simeq \mathrm{Soc}(A) \quad \text{in } {}_A\mathbf{mod}_A\,.$$

Proof of 2.14.

(1) We know by 1.39 that $\mathrm{Soc}_A(A) = \mathrm{Ann}(\mathrm{Rad}(A))_A$. It follows then from 2.5 that $\mathrm{Soc}_A(A) = \mathrm{Rad}(A)^\perp$, and so we have $\mathrm{Soc}_A(A) = \mathrm{Soc}(A)_A = \mathrm{Rad}(A)^\perp$. Now by 1.36, we know that

$$\begin{aligned}\mathrm{Soc}_A(A)_A &= \mathrm{Ann}(\mathrm{Rad}(A\otimes_k A^{\mathrm{op}}))_A \\ &= \{x \in A \mid (\forall {\textstyle\sum_i} a_i \otimes b_i \in \mathrm{Rad}(A \otimes_k A^{\mathrm{op}}))({\textstyle\sum_i} a_i x b_i = 0)\}\ .\end{aligned}$$

Let us make the description of $\mathrm{Soc}_A(A)_A$ even more complicated. Let t be a symmetrizing from on A. We have

$$\begin{aligned}
&\mathrm{Soc}_A(A)_A \\
&\quad = \{x \in A \mid (\forall a \in A)(\forall \textstyle\sum_i a_i \otimes b_i \in \mathrm{Rad}(A \otimes_k A^{\mathrm{op}}))(t(\sum_i a a_i x b_i) = 0)\} \\
&\quad = \{x \in A \mid (\forall a \in A)(\forall \textstyle\sum_i a_i \otimes b_i \in \mathrm{Rad}(A \otimes_k A^{\mathrm{op}}))(t(\sum_i x b_i a a_i) = 0)\} \\
&\quad = \{\textstyle\sum_i b_i a a_i \mid (a \in A)(\sum_i a_i \otimes b_i \in \mathrm{Rad}(A \otimes_k A^{\mathrm{op}}))\}^{\perp} .
\end{aligned}$$

We need now the following general property of algebras, whose proof is left to the reader.

2.15. Lemma. *Let A be an R–algebra.*
(1) As R–submodules of A, we have $\mathrm{Rad}(A) = \mathrm{Rad}(A^{\mathrm{op}})$.
(2) We have $(A \otimes_R A^{\mathrm{op}})^{\mathrm{op}} = A^{\mathrm{op}} \otimes_R A$, and the map

$$\iota \colon A \otimes_R A \to A \otimes_R A \quad , \quad a \otimes b \mapsto b \otimes a$$

is an isomorphism of algebras from $A \otimes_R A^{\mathrm{op}}$ onto $A^{\mathrm{op}} \otimes_R A$.
(3) The map ι induces an antiautomorphism of $\mathrm{Rad}(A \otimes_R A^{\mathrm{op}})$.

Then we see that

$$\textstyle\sum_i a_i \otimes b_i \in \mathrm{Rad}(A \otimes_k A^{\mathrm{op}}) \quad \Longleftrightarrow \quad \sum_i b_i \otimes a_i \in \mathrm{Rad}(A \otimes_k A^{\mathrm{op}})$$

and it follows from the formula preceding lemma 2.15 that

$$\mathrm{Soc}_A(A)_A = \mathrm{Rad}(A)^{\perp} ,$$

which proves (1).

(2) Since (by assertion (1)) we have $\mathrm{Soc}(A) = \mathrm{Rad}(A)^{\perp}$, the map $\widehat{t}$ (whenever t is a symmetrizing form on A) induces an isomorphism of A–modules–A :

$$\mathrm{Soc}(A) \xrightarrow{\sim} (A/\mathrm{Rad}(A))^* .$$

Since $A/\mathrm{Rad}(A)$ is symmetric, we have

$$A/\mathrm{Rad}(A) \simeq (A/\mathrm{Rad}(A))^* \quad \text{in} \quad {}_{(A/\mathrm{Rad}(A))}\mathbf{Mod}_{(A/\mathrm{Rad}(A))} ,$$

hence

$$A/\mathrm{Rad}(A) \simeq (A/\mathrm{Rad}(A))^* \quad \text{in} \quad {}_A\mathbf{Mod}_A .$$

Thus we have

$$\mathrm{Soc}(A) \simeq A/\mathrm{Rad}(A) \quad \text{in} \quad {}_A\mathbf{Mod}_A .$$

□

Remark. The preceding proposition shows that the algebra $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ is not symmetric, since its left socle does not coincide with its right socle.

Projective modules for symmetric algebras.

2.16. Proposition. *Let P be a finitely generated projective A–module. We have*

$$\mathrm{Soc}(P) \simeq P/\mathrm{Rad}(P) .$$

Proof of 2.16. We know that

$$\mathrm{Soc}(P) \simeq \bigoplus_{S \in \mathrm{Irr}(A)} S \otimes_{E_A S} \mathrm{Hom}_A(S, P)$$

and also that

$$P/\mathrm{Rad}(P) \simeq \bigoplus_{S \in \mathrm{Irr}(A)} S \otimes_{E_A S} \mathrm{Hom}_A(S, P/\mathrm{Rad}(P))$$

Since $\mathrm{Hom}_A(S, P/\mathrm{Rad}(P)) = \mathrm{Hom}_{A/\mathrm{Rad}(A)}(S, P/\mathrm{Rad}(P))$ and since $A/\mathrm{Rad}(A)$ is semi–simple, hence symmetric, we see that

$$\mathrm{Hom}_{A/\mathrm{Rad}(A)}(S, P/\mathrm{Rad}(P)) \simeq \mathrm{Hom}_{A/\mathrm{Rad}(A)}(P/\mathrm{Rad}(P), S)^* .$$

Now we have

$$\mathrm{Hom}_{A/\mathrm{Rad}(A)}(P/\mathrm{Rad}(P), S) = \mathrm{Hom}_A(P, S)$$

and since A is symmetric we have

$$\mathrm{Hom}_A(P, S) \simeq \mathrm{Hom}_A(S, P)^* .$$

Putting together the preceding isomorphisms, we get

$$P/\mathrm{Rad}(P) \simeq \bigoplus_{S \in \mathrm{Irr}(A)} S \otimes_{E_A S} \mathrm{Hom}_A(S, P) ,$$

and we remark that the right handside of the preceding formula is isomorphic to $\mathrm{Soc}(P)$. □

(!) **Attention** (!)

Notice that along the way we proved the isomorphism

$$\mathrm{Hom}_A(S, P) \xrightarrow{\sim} \mathrm{Hom}_A(S, P/\mathrm{Rad}(P) .$$

Such an isomorphism is *not*, in general, given by the composition of a morphism from S to P by the natural epimorphism from P onto $P/\mathrm{Rad}(P)$: we shall see below that, if P is indecomposable, this composition is nonzero if and only if P is *irreducible*.

The following property will be generalized in §D below.

2.17. Proposition. *Let A be a symmetric k–algebra. A finitely generated A–module is projective if and only if it is injective.*

Proof of 2.17. Since the k–duality is a contravariant isomorphism from ${}_A\mathbf{mod}$ onto $\mathbf{mod}_A$, an A–module P is injective if and only if its k–dual P^* is a projective module–A, *i.e.*, if and only if P^* is isomorphic to a direct summand of A^n (A viewed as an element of $\mathbf{mod}_A$) for some integer n, hence if and only if P is isomorphic to a direct summand of $(A^*)^n$ as an A–module. Since $A \simeq A^*$ in ${}_A\mathbf{mod}$, it follows that this last condition is satisfied if and only if P is isomorphic to a direct summand of A^n. □

For X a finitely generated A–module, let us denote by $\mathrm{Soc}^{\mathrm{pr}}(X)$ the sum if all irreducible A–submodules of X which are also projective A–modules. Since $\mathrm{Soc}^{\mathrm{pr}}(X)$ is injective (by 2.17 above), we see that $\mathrm{Soc}^{\mathrm{pr}}(X)$ is a direct summand of

X. We denote by $\mathrm{Soc}^{\mathrm{npr}}(X)$ the sum of all non projective irreducible submodules of X, and we have

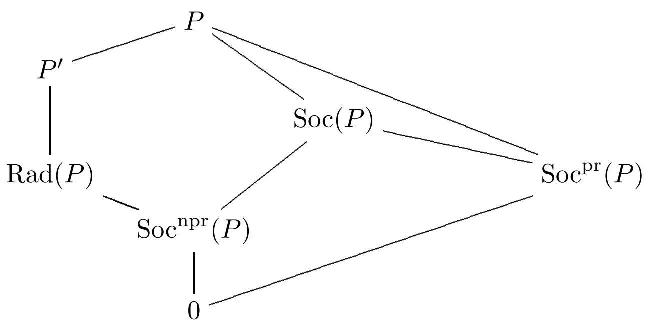
$$\mathrm{Soc}(X) = \mathrm{Soc}^{\mathrm{pr}}(X) \oplus \mathrm{Soc}^{\mathrm{npr}}(P)\,.$$

2.18. *Let P be a finitely generated projective A–module.*
(1) Let S be an irreducible A–module. Then the composition

$$\mathrm{Hom}_A(S,P) \otimes \mathrm{Hom}_A(P,S) \to E_A S$$

is nonzero if and only if S is projective and isomorphic to a summand of $\mathrm{Soc}^{\mathrm{pr}}(P)$.
(2) If P' is a submodule of P such that $P = P' \oplus \mathrm{Soc}^{\mathrm{pr}}(P)$, we have

$$\mathrm{Rad}(P) = \mathrm{Rad}(P') \quad \textit{and} \quad \mathrm{Soc}(P') = \mathrm{Soc}^{\mathrm{npr}}(P) \subset \mathrm{Rad}(P')$$



$$[\,P = P' \oplus \mathrm{Soc}^{\mathrm{pr}}(P)\ ,\ \mathrm{Soc}(P) = \mathrm{Soc}(P') \oplus \mathrm{Soc}^{\mathrm{pr}}(P)\,]$$

Proof of 2.18. Assume that the composition

$$\mathrm{Hom}_A(S,P) \otimes \mathrm{Hom}_A(P,S) \to E_A S$$

is non zero. Since its image is an ideal and $E_A S$ is a division algebra, this is surjective and S is projective. The converse is obvious. □

2.19. Corollary. *If P is an indecomposable non irreducible finitely generated projective A–module, we have*

$$\mathrm{Soc}(P) \subseteq \mathrm{Rad}(P)\,.$$

B2. Projective covers and lifting idempotents

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B3. Blocks

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C. The Casimir element and its applications

Actions on $A \otimes_R A$.

• Let A be an R–algebra. The module $A \otimes_R A$ is naturally endowed with the following structure of $(A \otimes_R A^{\rm op})$–module–$(A \otimes_R A^{\rm op})$:

$$(a \otimes a')(x \otimes y)(b \otimes b') := axb \otimes b'ya' .$$

Remark. That structure should be understood as a particular case of the structure of $(A \otimes_R A^{\rm op})$–module–$(B \otimes_R B^{\rm op})$–module which is defined on $M \otimes_R N$ (for $M \in {}_A\mathbf{Mod}_B$ and $N \in {}_B\mathbf{Mod}_A$) by

$$(a \otimes a')(m \otimes n)(b \otimes b') := amb \otimes b'na' .$$

We define (cf. chap. 1, §A1) the left and right centralizers of A in $A \otimes_R A$:

$$C_A(A \otimes_R A) := \left\{ \sum_i a_i \otimes a'_i \in A \otimes_R A \mid (\forall a) \sum_i aa_i \otimes a'_i = \sum_i a_i \otimes a'_i a \right\}$$
$$C(A \otimes_R A)_A := \left\{ \sum_i a_i \otimes a'_i \in A \otimes_R A \mid (\forall a) \sum_i a_i a \otimes a'_i = \sum_i a_i \otimes aa'_i \right\}$$

We set

$$C_A(A \otimes_R A)_A := C_A(A \otimes_R A) \cap C(A \otimes_R A)_A .$$

Notice that (cf. chap. 1, §A1 for the notation)

$$C_A(A \otimes_R A) = (A \otimes_R A)^A = \{ \xi \in A \otimes_R A \mid (\forall a)\, (a \otimes 1)\xi = (1 \otimes a)\xi \}$$
$$C(A \otimes_R A)_A = {}^A(A \otimes_R A) = \{ \xi \in A \otimes_R A \mid (\forall a)\, \xi(a \otimes 1) = \xi(1 \otimes a) \} .$$

• The algebra $E_R A$ of R–endomorphisms of A has a structure of $(A \otimes_R A^{\rm op})$–module–$(A \otimes_R A^{\rm op})$ inherited from the structure of $(A \otimes_R A^{\rm op})$–module on each of the two factors A as follows :

$$(\forall \alpha \in E_R A\, ,\, a, a', b, b' \in A\ )\ ((a \otimes a').\alpha.(b \otimes b') := [\, \xi \mapsto a\alpha(a'\xi b')b\,]) .$$

Remark. That structure should be understood as a particular case of the structure of $(A \otimes_R A^{\rm op})$–module–$(B \otimes_R B^{\rm op})$–module defined on $\mathrm{Hom}_R(M, M)$ (for $M \in {}_A\mathbf{Mod}_B$) by

$$(a \otimes a').\alpha.(b \otimes b') := [\, \xi \mapsto a\alpha(a'\xi b')b\,] .$$

Case where A is symmetric : the Casimir element.

Now assume that A is symmetric, and let t be a symmetrizing form. Since A is a finitely generated projective R–module, we have an isomorphism

$$A \otimes_R A^* \xrightarrow{\sim} E_R(A) \quad , \quad x \otimes \varphi \mapsto [\, \xi \mapsto \varphi(\xi)x\,] .$$

Composing this isomorphism with the isomorphism

$$A \otimes_R A \xrightarrow{\sim} A \otimes A^* \quad , \quad x \otimes y \mapsto x \otimes \widehat{t}(y) ,$$

we get the isomorphism

$$(*)\begin{cases} A\otimes_R A \xrightarrow{\sim} E_R(A) \\ a\otimes b \mapsto [\,\xi \mapsto t(b\xi)a\,]\ . \end{cases}$$

It is immediate to check that this isomorphism is an isomorphism of $(A\otimes_R A^{\mathrm{op}})$–modules–$(A\otimes_R A^{\mathrm{op}})$.

2.20. Definition. *We denote by $c^{\mathrm{pr}}_{A,t}$ (or simply c^{pr}_A) and we call Casimir element of (A,t) the element of $A\otimes A$ corresponding to the identity Id_A of A through the preceding isomorphism.*

• Notice that, by the formulas above, we have

$$(a\otimes a').\mathrm{Id}_A.(b\otimes b') := [\,\xi\mapsto aa'\xi b'b\,]\ ,$$

or, in other words,

$$(a\otimes a').\mathrm{Id}_A.(b\otimes b') := \lambda(aa')\rho(b'b)\,,$$

where $\lambda(a)$ is the endomorphism of left multiplication by a and $\rho(a)$ is the endomorphism of right multiplication by a. In particular, we see that

$$(a\otimes 1).\mathrm{Id}_A = (1\otimes a).\mathrm{Id}_A = \lambda(a) \quad \text{and} \quad \mathrm{Id}_A.(a\otimes 1) = \mathrm{Id}_A.(1\otimes a) = \rho(a)\,.$$

(!) **Attention** (!)

Notice that the structure of $A\otimes_R A^{\mathrm{op}}$–module on $A\otimes_R A$ defined here does *not* provides a structure of $A\otimes_R A^{\mathrm{op}}$–module on A : the morphism

$$A\otimes_A A^{\mathrm{op}} \to E_R A \quad , \quad a\otimes a' \mapsto \lambda(aa')$$

is not an algebra morphism.

• Moreover, we know that the commutant of $\lambda(A)$ (resp. of $\rho(A)$) in $E_R A$ is $\rho(A)$ (resp. $\lambda(A)$).

Through the isomorphism $A\otimes_A A \xrightarrow{\sim} E_R A$ described above, the preceding properties translate as follows.

2.21. Proposition. *Assume $c^{\mathrm{pr}}_A = \sum_i e_i\otimes e'_i$.*
(1) For all $a,a'\in A$, we have

$$\sum a e_i a' \otimes e'_i = \sum_i e_i \otimes a' e'_i a\,.$$

(2) The map

$$A\to C_A(A\otimes_R A) \quad , \quad a\mapsto \sum_i a e_i\otimes e'_i = \sum_i e_i\otimes e'_i a$$

is an isomorphism of A–modules–A.
(2') The map

$$A\to C(A\otimes_R A)_A \quad , \quad a\mapsto \sum_i e_i a\otimes e'_i = \sum_i e_i\otimes a e'_i$$

is an isomorphism of A–modules–A.

The following lemma is an immediate consequence of the definition of the Casimir element.

2.22. Lemma. *Let I be a finite set, and let $(e_i)_{i\in I}$ and $(e'_i)_{i\in I}$ be two families of elements of A indexed by I. The folllowing properties are equivalent :*

(i) $c_A^{\mathrm{pr}} = \sum_{i\in I} e'_i \otimes e_i$.

(ii) *For all $a \in A$, we have $a = \sum_i t(ae'_i)e_i$.*

Examples.

- If $A = RG$ (G a finite group), we have $c_{RG}^{\mathrm{pr}} = \sum_{g\in G} g^{-1} \otimes g$.
- If $A = \mathrm{Mat}_n(R)$ (and t is the ordinary trace), then $c_A^{\mathrm{pr}} = \sum_{i,j} E_{i,j} \otimes E_{j,i}$ (where $E_{i,j}$ denotes the usual elementary matrix whose all entries are zero except on the i-the row and j-th column where the entry is 1).
- Assume that A is free over R. Let $(e_i)_{i\in I}$ be an R–basis of A, and let $(e'_i)_{i\in I}$ be the dual basis (defined by $t(e_i e'_{i'}) = \delta_{i,i'}$), then $c_A^{\mathrm{pr}} = \sum_{i\in I} e'_i \otimes e_i$.

We also define the *central Casimir element* as the image z_A^{pr} of c_A^{pr} by the multiplication morphism $A \otimes A \longrightarrow A$. Thus, if $c_A^{\mathrm{pr}} = \sum_{i\in I} e'_i \otimes e_i$, we have

$$z_A^{\mathrm{pr}} = \sum_i e'_i e_i \,.$$

Remarks.

- For $A = RG$, the central Casimir element is the scalar $|G|$.
- For $A = \mathrm{Mat}_m(R)$, the central Casimir element is the scalar m.

The existence of an element such as c_A^{pr} is a necessary and sufficient condition for a central form t to be centralizing, as shown by the following lemma (whose proof is left to the reader).

2.23. Lemma. *Let u be a central form on A. Assume that there exists an element $f = \sum_j f'_j \otimes f_j \in A \otimes_R A$ such that $\sum_j u(af'_j)f_j = a$ for all $a \in A$. Then u is symmetrizing, and f is its central element.*

From now on, we assume that I is a finite set and $(e_i)_{i\in I}$, $(e'_i)_{i\in I}$ are two families of elements of A indexed by I, such that

$$c_A^{\mathrm{pr}} = \sum_{i\in I} e'_i \otimes e_i \,.$$

Let us denote by $x \mapsto x^\iota$ the involutive automorphism of $A \otimes A$ defined by $(a \otimes a')^\iota := a' \otimes a$.

2.24. Proposition.

(1) We have

$$(c_A^{\mathrm{pr}})^\iota = c_A^{\mathrm{pr}} \,, \text{ i.e.,} \sum_{i\in I} e'_i \otimes e_i = \sum_{i\in I} e_i \otimes e'_i \,.$$

(2) For all $a \in A$, we have

$$a = \sum_i t(ae'_i)e_i = \sum_i t(ae_i)e'_i = \sum_i t(e'_i)e_i a = \sum_i t(e_i)e'_i a \,.$$

Proof of 2.24. Indeed, by 2.22, we have $e'_i = \sum_j t(e'_i e'_j)e_j$, hence

$$\begin{aligned}\sum_i e'_i \otimes e_i &= \sum_{i,j} t(e'_i e'_j)e_j \otimes e_i = \sum_j e_j \otimes \sum_i t(e'_i e'_j)e_i \\ &= \sum_j e_j \otimes \sum_i t(e'_j e'_i)e_i = \sum_j e_j \otimes e'_j \,.\end{aligned}$$

The assertion (2) is an immediate consequence of (1) and of 2.22. $\square$

We define three maps :

$$\begin{cases} \mathrm{BiTr}^A\colon A\otimes A\to A \quad , \quad a\otimes a'\mapsto \sum_i e_i a e_i' a' \,, \\ \mathrm{Tr}^A\colon A\to A \quad , \quad a\mapsto \sum_i e_i a e_i' = \mathrm{BiTr}^A(a\otimes 1)\,, \\ \mathrm{Tr}_A\colon A\to A \quad , \quad a'\mapsto a' z_A^{\mathrm{pr}} = \sum_i a' e_i e_i' = \mathrm{BiTr}^A(1\otimes a')\,, \end{cases}$$

and we have

- Tr^A is a central morphism of ZA modules :

$$\mathrm{Tr}^A(zaa') = z\mathrm{Tr}^A(a'a) \quad (\forall z\in ZA \text{ and } a,a'\in A)\,,$$

and its image is contained in ZA (hence is an ideal of ZA),

- $\mathrm{BiTr}^A(a\otimes a') = \mathrm{Tr}^A(a)a' = a'\mathrm{Tr}^A(a)\,.$
- Tr_A is a morphism of A–modules–A.

Separably symmetric algebras.

2.25. Proposition. *If z_A^{pr} is invertible in ZA, the multiplication morphism*

$$A\otimes_R A\to A \quad , \quad a\otimes a'\mapsto aa'\,,$$

is split as a morphism of A–modules–A.

Proof of 2.25. Indeed, the composition of the morphism of A–modules–A defined by

$$A\to A\otimes_R A \quad , \quad a\mapsto ac_A^{\mathrm{pr}}$$

by the multiplication $A\otimes_R A\to A$ is equal to the morphism

$$A\to A \quad , \quad a\mapsto az_A^{\mathrm{pr}}\,.$$

In other words, if we view c_A^{pr} as an element of the algebra $A\otimes_R A^{\mathrm{op}}$, then $(c_A^{\mathrm{pr}})^2 = z_A^{\mathrm{pr}}c_A^{\mathrm{pr}}$. Thus we see that if z_A^{pr} is invertible in ZA, the element $(z_A^{\mathrm{pr}})^{-1}c_A^{\mathrm{pr}}$ is a central idempotent in the algebra $A\otimes_A A^{\mathrm{op}}$, and the morphism

$$A\to A\otimes_R A^{\mathrm{op}} \quad , \quad a\mapsto a(z_A^{\mathrm{pr}})^{-1}c_A^{\mathrm{pr}}$$

is a section of the multiplication morphism, identifying A with a direct summand of $A\otimes_R A$ as an A–module–A.

$\square$

Remark. If t is replaced by another symmetrizing form, *i.e.*, by a form $t(z\,\bullet)$ where z is an invertible element of ZA, then z_A^{pr} is replaced by zz_A^{pr}. Hence the inversibility of z_A^{pr} depends only on the algebra A and not on the choice of t.

An algebra A such that the the multiplication morphism

$$A\otimes_R A\to A \quad , \quad a\otimes a'\mapsto aa'\,,$$

is split as a morphism of A–modules–A is called *separable*.

A symmetric algebra A such that z_A^{pr} is invertible in ZA is called *symmetrically separable*.

(!) **Attention** (!)

A symmetrically separable algebra is indeed separable, but the converse is not true. For example, a matrix algebra $\mathrm{Mat}_m(R)$ is separable, but it is symmetrically separable if and only if m is invertible in R.

Note that the previous example shows as well that the property of being symmetrically separable is not stable under a Morita equivalence.

The following fondamental example justifies the notation and the name chosen for the map Tr^A.

2.26. Example. Let us consider the particular case where $A := E_R X$, for X a finitely generated projective R–module. Let us identify A with $X^* \otimes_R X$, and let us set

$$\mathrm{Id}_X = \sum_i f_i \otimes e_i\,.$$

We know that A is symmetric, and that $t := \mathrm{tr}_{X/R}$ is a symmetrizing form.

We leave as an exercise to the reader to check the following properties.

1. $c_A^{\mathrm{pr}} = \sum_{i,j} (f_i \otimes e_i) \otimes (f_j \otimes e_j)\,.$
2. The map $\mathrm{Tr}^A \colon A \to ZA$ coincides with $\mathrm{tr}_{X/R} \colon E_R X \to R$.

C2. Casimir element, trace and characters

For $\tau \colon A \to R$ a linear form, we denote by τ^0 the element of A defined by the condition

$$t(\tau^0 h) = \tau(h) \quad \text{for all } a \in A\,.$$

We know that τ is central if and only if τ^0 is central in A.

It is easy to check the following property.

2.27. Lemma. *We have* $\tau^0 = \sum_i \tau(e_i')e_i = \sum_i \tau(e_i)e_i'$, *and more generally, for all* $a \in A$, *we have* $\tau^0 a = \sum_i \tau(e_i' a)e_i = \sum_i \tau(e_i a)e_i'$.

The biregular representation of A is by definition the morphism

$$A \otimes_R A^{\mathrm{op}} \to E_R A \quad , \quad a \otimes a' \mapsto (x \mapsto axa')\,.$$

defining the structure of A–module–A of A.

Composing this morphism the trace $\mathrm{tr}_{A/R}$, we then get a linear form on $A \otimes_R A^{\mathrm{op}}$, called the biregular character of A, and denoted by χ_A^{bireg}.

2.28. Proposition. *We have*

$$\chi_A^{\mathrm{bireg}}(a \otimes a') = t(\mathrm{BiTr}^A(a \otimes a'))\,,$$

or, in other words

$$\chi_A^{\mathrm{bireg}}(a \otimes a') = \sum_i t(a' e_i a e_i) = t(\mathrm{Tr}^A(a)a') = t(a\mathrm{Tr}^A(a'))\,.$$

PROOF OF 2.28. We know by 2.24, (3), that

$$axa' = \sum_i t(axa'e_i')e_i \,,$$

which shows that the endomorphism of A defined by $a \otimes a'$ correspond to the element

$$\sum_i \widehat{t}(a'e_i a) \otimes e_i \in A^* \otimes A$$

whose trace is

$$\sum_i t(a'e_i a e_i) = t(\mathrm{Tr}^A(a)a') \,.$$

$\square$

Let χ_{reg} denote the character of the (left) regular representation of A, *i.e.*, the linear form on A defined by

$$\chi_{\mathrm{reg}}(a) := \mathrm{tr}_{A/R}(\lambda_A(a))$$

where $\lambda_A(a)\colon A \longrightarrow A \,,\ x \mapsto ax \,,$ is the left multiplication by a.

2.29. COROLLARY. *For all $a \in A$, we have*

$$\chi_{\mathrm{reg}}(a) = t(az_A^{\mathrm{pr}}) \,, \quad \textit{or, in other words,} \quad \chi_{\mathrm{reg}}^0 = z_A^{\mathrm{pr}} \,.$$

2.30. COROLLARY. *Let i be an idempotent of A. Let χ_{Ai} denote the character of the (finitely generated projective) A–module Ai. Then we have*

$$\chi_{Ai}^0 = \mathrm{Tr}^A(i) \,.$$

Indeed, we have

$$\mathrm{tr}_{Ai/R}(a) = \mathrm{tr}_{A/R}(a \otimes i) = t(a\mathrm{Tr}^A(i)) \,.$$

C3. Projective center, Higman's criterion

The projective center of an algebra.

Let A be an R–algebra, and let M be an A–module–A. We know (see chapter 1) that the morphism

$$\mathrm{Hom}_A(A, M)_A \to M^A \quad , \quad \varphi \mapsto \varphi(1)$$

is an isomorphism. In particular, we have

$$\mathrm{Hom}_A(A, A \otimes_R A^{\mathrm{op}})_A = (A \otimes A)^A \,.$$

The module $\mathrm{Hom}_A^{\mathrm{pr}}(A, M)_A$ consisting of projective morphisms from A to M is the image of the map

$$\mathrm{Hom}_A(A, A \otimes_R A^{\mathrm{op}})_A \otimes M \to \mathrm{Hom}_A(A, M)_A \quad , \quad \varphi \otimes m \mapsto (a \mapsto (a\varphi)m) \,.$$

Through the previous isomorphism, this translates to

$$(A \otimes_R A^{\mathrm{op}})^A \otimes M \to M^A \quad , \quad x \otimes m \mapsto xm \,,$$

i.e., we have a natural isomorphism

$$(A \otimes_R A^{\mathrm{op}})^A M = \mathrm{Hom}_A^{\mathrm{pr}}(A, M)_A \,.$$

2.31. Definition–Proposition. *The module*

$$(A \otimes_R A^{\mathrm{op}})^A . A = \left\{ \textstyle\sum_i a_i a a_i' \;\mid\; (a \in A)(\textstyle\sum_i a_i \otimes a_i' \in (A \otimes_R A)^A) \right\}$$

is called the projective center of A and is denoted by $Z^{\mathrm{pr}} A$. This is an ideal in ZA and the map

$$Z^{\mathrm{pr}} A \to \mathrm{Hom}_A(A, A)_A \quad , \quad z \mapsto (a \mapsto az)$$

induces an isomorphism of ZA–modules from $Z^{\mathrm{pr}} A$ onto $\mathrm{Hom}_A^{\mathrm{pr}}(A, A)_A$.

When A is symmetric.

If A is symmetric, and if $c_A^{\mathrm{pr}} = \sum_i e_i' \otimes e_i$, it results from 2.21 that

$$(A \otimes_R A)^A = \{ \textstyle\sum_i e_i' a \otimes e_i \;\mid\; (a \in A) \} \;.$$

Thus we have

$$(A \otimes_R A)^A M = \{ \textstyle\sum_i e_i' m e_i \;\mid\; (m \in M) \} \;,$$

which makes the next result obvious.

2.32. Proposition. *The module* $\mathrm{Hom}_A^{\mathrm{pr}}(A, M)_A$ *is naturally isomorphic to the image of the map*

$$\mathrm{Tr}^A \colon M \to M^A \quad , \quad m \mapsto c_A^{\mathrm{pr}} m = \sum_i e_i' m e_i \,.$$

In particular, Z_A^{pr} is the image of the map $\mathrm{Tr}^A \colon A \to A$.

Notice that since $c_A^{\mathrm{pr}} \in C(A \otimes_R A)_A$, the map Tr^A factorizes through $[A, M]$ and so defines a map

$$\mathrm{Tr}^A \colon H_0(A, M) \to H^0(A, M) \,.$$

Example. If $A = RG$ (G a finite group), then $Z^{\mathrm{pr}} RG$ is the image of

$$\mathrm{Tr}^{RG} \colon RG \to ZRG \quad , \quad x \mapsto \sum_{g \in G} g x g^{-1} \,.$$

Let us denote by $\mathrm{Cl}(G)$ the set of conjugacy classes of G, and for $C \in \mathrm{Cl}(G)$, let us define a central element by

$$\mathcal{S}C := \sum_{g \in C} g \,.$$

Then it is immediate to check that

$$Z^{\mathrm{pr}} RG = \bigoplus_{C \in \mathrm{Cl}(G)} \frac{|G|}{|C|} \mathcal{S}C \,.$$

Higman's criterion.

If X and X' are A–modules, applying what preceds to the case where $M := \mathrm{Hom}_R(X, X')$, we get a map

$$\mathrm{Tr}^A \colon \mathrm{Hom}_R(X, X') \to \mathrm{Hom}_A(X, X') \quad , \quad \alpha \mapsto [\, x \mapsto \sum_i (e_i \alpha(e'_i x))\,] \,.$$

For an A–module X, let us describe in terms of the Casimir element the inverse of the isomorphism (see 2.9)

$$t_X^* \colon \begin{cases} \mathrm{Hom}_A(X, A) \to \mathrm{Hom}_R(X, R) \\ \phi \mapsto t \cdot \phi \,. \end{cases}$$

By the formula given in 2.9, (1), we see that, for all $x \in X$ and $\psi \in \mathrm{Hom}_R(X, R)$, we have

$$u_X(\psi)(x) = \widehat{\psi(\cdot\, x)} \,.$$

By 2.27, we then get the following property.

2.33. Lemma. *For any A–module X, the morphism*

$$\begin{cases} \mathrm{Hom}_R(X, R) \to \mathrm{Hom}_A(X, A) \\ \psi \mapsto [\ x \mapsto \sum_i \psi(e'_i x) e_i = \sum_i \psi(e_i x) e'_i\ ] \end{cases}$$

is the inverse of the isomorphism t_X^*.

Let X and X' be A–modules such that X or X' is a finitely generated projective R–module. It results from 2.33 that the natural morphism $\mathrm{Hom}_A(X, A) \otimes_R X' \to \mathrm{Hom}_A(X, X')$ factorizes as follows :

$$\mathrm{Hom}_A(X, A) \otimes_R X' \xrightarrow{\sim} \mathrm{Hom}_R(X, R) \otimes_R X' \xrightarrow{\sim} \mathrm{Hom}_R(X, X') \xrightarrow{\mathrm{Tr}^A} \mathrm{Hom}_A(X, X') \,.$$

The following proposition is known, in the case where $A = RG$, as the "Higman's criterion". It is an immediate consequence of the characterization of finitely generated projective modules.

2.34. Proposition. *Let X and X' be A–modules such that X is a finitely generated projective R–module.*

(1) The submodule $\mathrm{Hom}_A^{\mathrm{pr}}(X, X')$ of $\mathrm{Hom}_A(X, X')$ consisting of maps factorizing through a finitely generated projective A–module coincides with the image of the map

$$\mathrm{Tr}^A \colon \begin{cases} \mathrm{Hom}_R(X, X') \longrightarrow \mathrm{Hom}_A(X, X') \\ \alpha \mapsto [\, x \mapsto \sum_{i \in I} e'_i \alpha(e_i x)\,] \end{cases}$$

(2) The image of the map

$$\mathrm{Tr}^A \colon E_R X \longrightarrow E_A X$$

is a twosided ideal of $E_A X$, and X is a finitely generated projective A–module if and only if Id_X belongs to this ideal.

C4. Schur elements

Quotients of symmetric algebras.

Let A and B be two symmetric algebras, and let $\lambda\colon A \twoheadrightarrow B$ be a *surjective* algebra morphism. The morphism λ defines a morphism

$$A \otimes_R A^{\mathrm{op}} \to B \otimes_R B^{\mathrm{op}} \quad , \quad a \otimes a' \mapsto \lambda(a) \otimes \lambda(a')\,,$$

hence defines a structure of A–module–A on B.

Remark. We shall apply what follows, for example, to the following context. Let A be a finite dimensional algebra over a (commutative) field k, let X be an irreducible A–module, let $D := E_A X$ (a division algebra), and let $B := EX_D$. We know that B is a symmetric algebra, and (by 1.49) that the structural morphism $\lambda_X : A \to B$ is onto.

Let t be a symmetrizing form on A and let u be a symmetrizing form on B.

Let $c_A^{\mathrm{pr}} = \sum_i e_i \otimes e'_i$ and $c_B^{\mathrm{pr}} = \sum_j f_j \otimes f'_j$ be the corresponding Casimir elements for respectively A and B.

The form $u \cdot \lambda$ is a central form on A, so there exists an element $(u\cdot\lambda)^0 \in ZA$ whose image under $\widehat{t}$ is $u \cdot \lambda$. Since λ is onto, the element $s_\lambda := \lambda((u\cdot\lambda)^0)$ belongs to ZB.

2.35. Definition. *The element s_λ is called the Schur element of the (surjective) morphism λ.*

2.36. Proposition. *We have*

$$(\lambda \otimes \lambda)(c_A^{\mathrm{pr}}) = s_\lambda c_B^{\mathrm{pr}} \quad \textit{and} \quad \lambda(z_A^{\mathrm{pr}}) = s_\lambda z_B^{\mathrm{pr}}\,.$$

Proof of 2.36. Let us set $c_A^{\mathrm{pr}} = \sum_i e'_i \otimes e_i$. We have, for all $a \in A$:

$$(u.\lambda)^0 a = \sum_i t((u.\lambda)^0 a e'_i) e_i\,, \text{ hence } (u.\lambda)^0 a = \sum_i u(\lambda(a)\lambda(e'_i)) e_i\,,$$

from which we deduce

$$s_\lambda \lambda(a) = \sum_i u(\lambda(a e'_i))\lambda(e_i)\,.$$

Since λ is surjective, it follows that for all $b \in B$ we have

$$s_\lambda b = \sum_i u(b\lambda(e'_i))\lambda(e_i)\,,$$

which shows that, through the isomorphism $B \otimes_R B \xrightarrow{\sim} E_R B$ defined by $\widehat{u}$, the element $\sum_i \lambda(e'_i) \otimes \lambda(e_i)$ corresponds to $s_\lambda \mathrm{Id}_B$. This implies that

$$\sum_i \lambda(e'_i) \otimes \lambda(e_i) = s_\lambda c_B^{\mathrm{pr}}\,.$$

$\square$

REMARK. Choose $A = B$ and $\lambda := \mathrm{Id}_A$. Now if t and u are two symmetrizing forms on A, we have $u = t(u^0\bullet)$. The formula of the preceding proposition can be written (with obvious notation) :

$$c^{\mathrm{pr}}_{A,t} = u^0 c^{\mathrm{pr}}_{A,u} .$$

The structure of A–module–A on B defined by λ allows us to define, for N any B–module–B, the trace map

$$\mathrm{Tr}^A \colon N \to N^A \quad , \quad n \mapsto c^{\mathrm{pr}}_A.n = \sum_i \lambda(e_i) n \lambda(e'_i) .$$

The following property is an imediate consequence of 2.36.

2.37. COROLLARY. *Whenever N is a B–module–B, we have*

$$\mathrm{Tr}^A(n) = s_\lambda \mathrm{Tr}^B(n) .$$

We give now a characterisation of the situation where the Schur element is invertible.

2.38. PROPOSITION. *The following properties are equivalent.*
(i) *The Schur element s_λ is invertible in ZB.*
(ii) *The morphism $\lambda \colon A \twoheadrightarrow B$ is split as a morphism of A–modules–A.*
(ii) *B is a projective A–module.*
(iv) *Any projective B–module is a projective A–module.*
If the above properties are fullfilled, then the map

$$\sigma \colon \begin{cases} B \longrightarrow A \\ b \mapsto \sum_i u(s_\lambda^{-1} b \lambda(e'_i)) e_i \end{cases}$$

is a section of λ as a morphism of A–modules–A.

PROOF OF 2.38.
(i)$\Rightarrow$(ii) : Since

$$\sum_i \lambda(e'_i) \otimes \lambda(e_i) = s_\lambda c^{\mathrm{pr}}_B ,$$

and since s_λ is invertible, we have

$$c^{\mathrm{pr}}_B = s_\lambda^{-1} \sum_i \lambda(e'_i) \otimes \lambda(e_i) .$$

It follows that

$$\lambda(\sigma(b)) = \sum_i u(s_\lambda^{-1} b \lambda(e'_i)) \lambda(e_i) = \sum_j u(b f'_j) f_j = b ,$$

which proves that σ is a section of λ.

Let us set $\tilde{s} := (u.\lambda)^0$, and let us choose a primage $\tilde{s}'$ of s_λ^{-1} in A. If we choose a preimage $\tilde{b}$ of b through λ, we have

$$\begin{aligned}\sum_i u(s_\lambda^{-1} b\lambda(e'_i))e_i &= \sum_i u(\lambda(\tilde{s}'\tilde{b}e'_i))e_i = \sum_i t(\tilde{s}\tilde{s}'\tilde{b}e'_i)e_i = \tilde{s}\tilde{s}'\tilde{b} \\ &= \sum_i u(bs_\lambda^{-1}\lambda(e'_i))e_i = \sum_i u(\lambda(\tilde{b}\tilde{s}'e'_i))e_i = \sum_i t(\tilde{b}\tilde{s}\tilde{s}'\tilde{b}e'_i)e_i \\ &= \tilde{b}\tilde{s}\tilde{s}' ,\end{aligned}$$

which makes it obvious that σ commute with the twosided action of A.

(ii)$\Rightarrow$(iii) : Since λ is split as a morphism of A–modules, we see that B is projective as an A–module.

(iii)$\Rightarrow$(iv) : obvious.

(iv)$\Rightarrow$(i) : Since B is a finitely generated projective A–module, Higman's criterion (see 2.34) shows that there is $\beta \in E_R B$ such that $\mathrm{Tr}^A(\beta) = \mathrm{Id}_B$. By 2.37, we then see that

$$s_\lambda \mathrm{Tr}^B(\beta) = \mathrm{Id}_B .$$

Since $\mathrm{Tr}^B(\beta) \in \mathrm{Hom}_B(B, B) = B$, that last equality shows that s_λ is invertible in B, hence is invertible in ZB. $\square$

Remark. Since σ is a morphism of A–modules–A, it follows that, for

$$e_\lambda := \sigma(1) = \sum_i u(s_\lambda^{-1} b\lambda(e'_i))e_i ,$$

we have

$$\sigma(bb') = aea'$$

whenever $\lambda(a) = b$ and $\lambda(a') = b'$, hence in particular e is a central idempotent of A. Thus we may view (B, λ, σ) as :

$$\begin{cases} B = Ae_\lambda \\ \lambda\colon A \to Ae_\lambda \quad , \quad a \mapsto ae_\lambda \\ \sigma\colon Ae_\lambda \to A \quad , \quad ae_\lambda \mapsto ae_\lambda . \end{cases}$$

Schur elements of split irreducible modules.

In the case where $R = k$, a (commutative) field, the next definition coincides with the definition of a *split irreducible module*. The reader may keep this example in mind.

2.39. Definition. *An A–module X is called split quasi irreducible if*

(1) *X is a generator and a finitely generated projective R–module (a "progenerator" for ${}_R\mathbf{Mod}$),*
(2) *the morphism $\lambda_X\colon A \to E_R X$ is onto.*

Note that if X is split quasi irreducible, then X is a Morita module for R and $E_R X$, and so in particular the map

$$R \to E_R X \quad , \quad \lambda \mapsto \lambda \mathrm{Id}_X$$

is an isomorphism from R onto the center $Z(E_R X)$ of $E_R X$. Thus the restriction of λ_X to ZA induces an algebra morphism

$$\omega_X \colon ZA \to R\,.$$

We denote by χ_X the character of the A–module X, *i.e.*, the central form on A defined by

$$\chi_X(a) = \mathrm{tr}_{X/R}(\lambda_X(a))\,.$$

The next result is an immediate application of the definition.

2.40. LEMMA. *Let X be a split quasi irreducible A–module. The Schur element of X is the element $s_X \in R$ defined by*

$$s_X := \omega_X(\chi_X^0)\,.$$

EXAMPLE. Assume $R = \mathbb{C}$ and $A = \mathbb{C}G$ (G a finite group). Let χ be the character of an irreducible $\mathbb{C}G$–module. Then the Schur element of this module is the scalar $s_\chi := |G|/\chi(1)$.

2.41. PROPOSITION. *For X a split quasi irreducible A–module, with character $\chi := \chi_X$, we have*

(1) $s_X\chi(1) = \sum_i \chi(e'_i)\chi(e_i)$,
(2) $s_X\chi(1)^2 = \chi\left(\sum_i e'_i e_i\right)$.

PROOF OF 2.41.

The trace of the central element $s_X = \omega_X(\chi_X^0)$ is $\chi(1)s_X = \chi(1)\chi(\chi_X^0)$, and since $\chi^0 = \sum_i \chi(e'_i)e_i$, we see that $\chi(1)s_X = \sum_i \chi(e'_i)\chi(e_i)$.

The second assertion is a consequence of the following lemma.

2.42. LEMMA. *Whenever $\alpha \in E_R X$, the central element $\mathrm{Tr}^A(\alpha)$ is the scalar multiplication by $s_X \mathrm{tr}_{X/R}(\alpha)$.*

Indeed, this is an immediate application of the results of example 2.26 and of 2.37.

Let us give a "direct" proof as an exercise.

Since for all $a \in A$ we have $a\chi^0 = \sum_i \chi(ae'_i)e_i$, it follows that

$$\lambda_X(a\chi^0) = \sum_i \chi(ae'_i)\lambda_X(e_i)\,,$$

and if $\alpha = \lambda_X(a)$, we get

$$\alpha\lambda_X(\chi^0) = s_X\alpha = \sum_i \mathrm{tr}_{X/R}(\alpha\lambda_X(e'_i))\lambda_X(e_i)\,.$$

Hence, through the isomorphism $E_R X \xrightarrow{\sim} X^* \otimes X$, the action of $s_X\alpha$ on X corresponds to the element

$$\sum_i \mathrm{tr}_{X/R}(\lambda_X(e'_i)\alpha(\bullet)) \otimes \lambda_X(e_i)$$

and its trace is

$$\sigma_X \mathrm{tr}_{X/R}(\alpha) = \sum_i \mathrm{tr}_{X/R}(\lambda_X(e'_i)\alpha(\lambda_X(e_i)))\,.$$

$\square$

Proposition 2.38 has the following important consequence.

2.43. Proposition. *Let X be a split quasi irreducible A–module. The following properties are equivalent.*

(i) *Its Schur element s_X is invertible in R.*
(ii) *The structural morphism $\lambda_X \colon A \twoheadrightarrow E_R X$ is split as morphism of A–modules–A.*
(iii) *X is a projective A–module.*

If the above properties are satisfied, then the map

$$\sigma \colon \begin{cases} E_R X \longrightarrow A \\ \alpha \mapsto \sum_i \mathrm{tr}_{X/R}(s_X^{-1} \alpha e'_i) e_i \end{cases}$$

is a section of λ as a morphism of A–modules–A.

Case of a symmetric algebra over a field.

Let k be a field, and let A be a finite dimensional symmetric k–algebra.

If X is an irreducible A–module, we recall that the algebra $D_X := E_A X$ is a division algebra, that the algebra $B := EX_{D_X}$ is symmetric, and that the structural morphism $\lambda \colon A \to B$ is onto. Thus each irreducible A–module has a Schur element $s_X \in ZD_X$, and since ZD_X is a field, the Schur element s_X is invertible if and only if it is nonzero.

2.44. Proposition. *Let k be a field, and let A be a finite dimensional symmetric k–algebra. The following assertions are equivalent.*

(i) *A is semi–simple.*
(ii) *Whenever $X \in \mathrm{Irr}(A)$, then $s_X \neq 0$.*

Proof of 2.44. This follows from the fact that a finite dimensional k–algebra is semi–simple if and only if all its irreducible modules are projective. $\square$

Now assume that the algebra $\overline{A} := A/\mathrm{Rad}(A)$ is split, *i.e.*, that

$$(\forall S \in \mathrm{Irr}(A)) \ , \ \mathrm{End}_A(S) = k\,\mathrm{Id}_S \,, \quad \text{hence} \quad \overline{A} \xrightarrow{\sim} \prod_{S \in \mathrm{Irr}(A)} \mathrm{End}_k(S)\,.$$

Let us denote by $a \mapsto \overline{a}$ the canonical epimorphism from A onto $\overline{A}$.

Let $S \in \mathrm{Irr}(A)$. By a slight abuse of notation, we consider that the structural morphism defining the structure of A–module of S is defined by the composition :

$$A \to \overline{A} \xrightarrow{\lambda_S} \mathrm{End}_k(S)\,.$$

Let us denote by e_S the corresponding central idempotent of $\overline{A}$, and let us choose an element $\widetilde{e}_S \in A$ whose image modulo $\mathrm{Rad}(A)$ is e_S.

We have

$$\chi_S(a) = t(\chi_S^0 a) = \mathrm{tr}_{S/k}(\lambda_S(\overline{a}))\,.$$

For all $S, T \in \mathrm{Irr}(A)$, it follows that

$$t(\chi_S^0 \widetilde{e}_T a) = \mathrm{tr}_{S/k}(\lambda_S(e_T \overline{a})) = \delta_{S,T} \chi_S(a)\,,$$

and so

$$\chi_S^0 \widetilde{e}_T = \delta_{S,T} \chi_S^0\,.$$

The above formula allows us to prove the following orthogonality relation between characters of absolutely irreducible modules.

2.45. PROPOSITION. *Let A be a symmetric algebra such that $A/\mathrm{Rad}(A)$ is split. Let $\mathrm{c}^{\mathrm{pr}} = \sum_i e_i' \otimes e_i$ be the Casimir element of A. For all $S, T \in \mathrm{Irr}(A)$, we have*

$$\sum_i \chi_S(e_i')\chi_T(e_i) = \begin{cases} s_S \chi_S(1) & \text{if } S = T\,, \\ 0 & \text{if } S \neq T\,. \end{cases}$$

Symmetric split semi–simple algebras.

2.46. PROPOSITION. *Let k be a field, and let A be a finite dimensional symmetric k–algebra. Assume that A is split semi–simple. For each irreducible character χ of A, let e_χ be the primitive idempotent of the center ZA associated with χ, and let s_χ denote its Schur element.*
(1) We have

$$s_\chi \neq 0 \quad \text{and} \quad \chi^0 = s_\chi e_\chi\,.$$

(2) We have

$$t = \sum_{\chi \in \mathrm{Irr}(A)} \frac{1}{s_\chi}\,\chi\,.$$

PROOF OF 2.46.
(1) Since, for all $a \in A$, we have $\chi(e_\chi h) = \chi(h)$, we see that $t(\chi^0 e_\chi h) = t(\chi^0 h)$, which proves that $\chi^0 = \chi^0 e_\chi$. The desired equality results from the fact that, for all $z \in ZA$, we have $z = \sum_{\chi \in \mathrm{Irr}(FA)} \omega_\chi(z) e_\chi$.
(2) Through the isomorphism between A and its dual, the equality

$$t = \sum_{\chi \in \mathrm{Irr}(FA)} \frac{1}{s_\chi}\,\chi$$

is equivalent to

$$1 = \sum_{\chi \in \mathrm{Irr}(FA)} \frac{1}{s_\chi}\,\chi^0\,,$$

which is obvious by (1) above. □

C5. Parabolic subalgebras

Definition and first properties.

The following definition covers the case of subalgebras such as RH (H a subgroup of G) of a group algebra RG, as well as the case of the socalled parabolic subalgebras of Hecke algebras.

2.47. DEFINITION. *Let A be a symmetric R–algebra, and let t be a symmetrizing form on A. A subalgebra B of A is called parabolic (relative to t) if the following two conditions are satisfied*
(Pa1) *Viewed as a B–module through left multiplication, A is projective.*
(Pa2) *The restriction of t to B is a symmetrizing form for B.*

Remarks.

1. Condition (Pa1) is equivalent to :

(Pa1') *Viewed as a module–B through right multiplication, A is projective.*

Indeed, A is a projective B–module if and only if A^* is a projective module–B, hence (since A^* is isomorphic to A) if and only if A is a projective module–B.

2. Condition (Pa2) is equivalent to :

(Pa1') *We have* $B \cap B^{\perp} = 0$.

2.48. Proposition. *Let A be a symmetric algebra with a symmetrizing form t and let B be a subalgebra of A such that A is a projective B–module.*
(1) The subalgebra B is parabolic if and only if $B \oplus B^{\perp} = A$, and then the corresponding projection of A onto B is the morphism of B–modules–B

$$\mathrm{Br}_B^A \colon A \to B \quad \textit{such that} \quad t(\mathrm{Br}_B^A(a)b) = t(ab) \quad \textit{for all } a \in A \textit{ and } b \in B\,.$$

(2) If (1) is satisfied, then $B^{\perp}$ is the B–submodule–B of A characterized by the following two properties :

(a) *We have $A = B \oplus B^{\perp}$ (as B–modules–B),*

(b) $B^{\perp} \subseteq \ker(t)$.

Example. Assume $A = RG$ and $B = RH$ (G a finite group, H a subgroup of G). Then the map Br_{RH}^{RG} is defined as follows :

$$\mathrm{Br}_{RH}^{RG}(g) = \begin{cases} g & \text{if } g \in H\,, \\ 0 & \text{if } g \notin H\,. \end{cases}$$

(!) **Attention** (!)

The subalgebra $R.1$ is not necessarily a parabolic subalgebra.

Indeed, the symmetrizing form on R are the forms τ such that $\tau(1) \in R^{\times}$. Thus $R.1$ is parabolic if and only if $t(1)$ is invertible in R.

This is not always the case, since for $A := \mathrm{Mat}_m(R)$ and $t := \mathrm{tr}$, we have $t(1) = m$. This example shows as well that the property of $R.1$ to be parabolic is not stable under Morita equivalence.

Remarks.

- If $R.1$ is parabolic, we may wish to normalize the form t by assuming that $t(1) = 1$.
- If $R.1$ is parabolic, then A is strongly symmetric (see 2.12).

But an algebra may be strongly symmetric without $R.1$ being parabolic, as shown by the example $A := \mathrm{Mat}_m(R)$ when m is not invertible in R.

D. Exact bimodules and associated functors

D1. Selfdual pairs of exact bimodules

In what follows, we denote by A and B two symmetric R–algebras. We assume chosen two symmetrizing forms t and u on respectively A and B.

2.49. Definition. *An A–module–B M is called exact if M is finitely generated projective both as an A–module and as a module–B.*

If M is exact, the functors

$$M \otimes_B \bullet : {}_B\mathbf{Mod} \to {}_A\mathbf{Mod} \quad \text{and} \quad \bullet \otimes_A M \colon \mathbf{Mod}_A \to \mathbf{Mod}_B$$

defined by M are exact.

Definition. *A selfdual pair of exact bimodules for A and B is a pair (M, N) where M is an exact A–module–B, and N is an exact B–module–A endowed with an R–duality of bimodules*

$$M \times N \to R \quad , \quad (m, n) \mapsto \langle m, n \rangle \,,$$

i.e., an R–bilinear map such that

$$\langle amb, n \rangle = \langle m, bna \rangle \quad (\forall a \in A\,,\, b \in B\,,\, m \in M\,,\, n \in N)\,,$$

which induces (bimodules) isomorphisms

$$M \xrightarrow{\sim} N^* \quad \textit{and} \quad N \xrightarrow{\sim} M^* \,.$$

Examples.

1. Take $B = R$, $M = {}_A A_R$ (*i.e.*, A viewed as an object in ${}_A\mathbf{mod}_R$), $N = {}_R A_A$ (*i.e.*, A viewed as an object in ${}_R\mathbf{mod}_A$, and $\langle a, b \rangle := t(ab)$. Then $({}_A A_R, {}_R A_A)$ is an exact pair of bimodules for A and R, called the *trivial pair* for A.

2. Let G be a finite group, and let U be a subgroup of G whose order is invertible in R. Let $N_G(U)$ denote the normalizer of U in G, and let us set $H := N_G(U)/U$. Then the set G/U is naturally endowed with a left action of G and a right action of H, and the set $U\backslash G$ is naturally endowed with a left action of H and a right action of G.

Take $A := RG$, $B := RH$ (both induced with the canonical symmetrizing forms of group algebras), $M := R[G/U]$ (the R–free module with basis G/U), $N := R[U\backslash G]$, and

$$\langle gU, Ug' \rangle := \begin{cases} 1 & \text{if } Ug' = (gU)^{-1} \\ 0 & \text{if not.} \end{cases}$$

Then the pair $(R[G/U], R[U\backslash G])$ is an exact pair of bimodules for RG and RH.

The functor defined by M is the so–called "Harish–Chandra induction" : take an RH–module Y, view it as an $RN_G(U)$–module, and induce it up to RG.

The adjoint functor defined by N is the "Harish–Chandra restriction (or truncation)" : take an RG–module X, and view its fixed points under U as an RH–module.

3. The following example is a generalization of the previous two examples.

Let B be a parabolic subalgebra of A, let e be a central idempotent of A and let f be a central idempotent of B. Let us choose

$$M := eAf \quad , \quad N := fAe \quad , \quad \langle m, n \rangle := t(mn)\,.$$

Then the functor induced by M is the induction truncated by e :

$$Y \mapsto e.\mathrm{Ind}_B^A Y\,,$$

while the functor induced by N is the restriction truncated by f :

$$X \mapsto f.\mathrm{Res}_B^A X\,.$$

Let (M, N) be a self dual pair of exact bimodules.

1. The isomorphism $N \xrightarrow{\sim} M^*$, composed with the isomorphism $M^* \xrightarrow{\sim} M^\vee = \mathrm{Hom}_A(M, A)$ given by 2.9, gives an isomorphism $N \xrightarrow{\sim} M^\vee$ of B–modules–A , which is described as follows :

2.50. *the element $n \in N$ defines the A–linear form $m \mapsto mn$ on M such that*

$$t(mn) = \langle m, n\rangle\,.$$

Similarly, we have an isomorphism $M \xrightarrow{\sim} N^\vee$ of A–modules–B , which is described as follows :

2.51. *the element $m \in M$ defines the B–linear form $n \mapsto nm$ on N such that*

$$u(nm) = \langle m, n\rangle\,.$$

2. The isomorphism $M \xrightarrow{\sim} N^\vee$ described above induces isomorphisms

$$M \otimes_B N \xrightarrow{\sim} N^\vee \otimes_B N \xrightarrow{\sim} E_B N\,.$$

We know (see 2.10) that there is a symmetrizing form u_N on the algebra $E_B N$. Transporting the algebra structure and the form u_N through the preceding isomorphisms gives the following property.

2.52. Proposition.
(1) The rule

$$(m \otimes_B n)(m' \otimes_B n') := m \otimes_B (nm')n$$

provides $M \otimes_B N$ with a structure of algebra isomorphic to $E_B N$.
(2) The form

$$t_{M,N} \colon M \otimes_B N \to R \quad , \quad m \otimes_B n \mapsto \langle m, n\rangle$$

is a symmetrizing form on the algebra $M \otimes_B N$.

Similarly, we have an algebra structure on $N \otimes_A M$ and a symmetrizing form

$$t_{N,M} \colon N \otimes_A M \to R \quad , \quad n \otimes_A m \mapsto \langle m, n\rangle\,.$$

We denote by $c_{M,N}$ the unity of $M \otimes_B N$ (*i.e.*, the "(M, N)– Casimir element").

Thus, if $c_{M,N} = \sum_i m_i \otimes_B n_i$, for all $m \in M$ and $n \in N$ we have

$$\sum_i m \otimes_B (nm_i)n_i = \sum_i m_i \otimes_B (n_i m)n = nm \otimes_B n\,.$$

Similarly, we denote by $c_{N,M}$ the unity of the algebra $N \otimes_A M$.

The case of the trivial pair.

Let us consider the trivial pair $({}_A A_R, {}_R A_A)$ for A. Then

- The algebra $A \otimes_A A$ is isomorphic to A and its symmetrizing form is the form t.
- The algebra $A \otimes_R A$ is isomorphic to $E_R A$ and its symmetrizing form is defined by $a \otimes a' \mapsto t(aa')$.

(!) **Attention** (!)

The algebra $A \otimes_R A$ mentioned above is not, in general, isomorphic to $A \otimes A^{\mathrm{op}}$.

Notice also that the multiplication in the algebra $A \otimes_R A$ is defined by the rule

$$(a \otimes a')(b \otimes b') := a \otimes t(a'b)b' \,,$$

and that by its very definition, c_A^{pr} is the unity of this algebra.

Adjunctions.

Let (M, N) be a selfdual pair of exact bimodules for A and B. Since $M \simeq N^\vee$ and $N \simeq M^\vee$, the pair $(M \otimes_B \bullet\,, N \otimes_A \bullet)$ is a pair of *biadjoint functors*, *i.e.*, a pair of functors left and right adjoint to each other.

The isomorphisms $N \xrightarrow{\sim} M^\vee$ and $M \xrightarrow{\sim} N^\vee$ described in 2.50 and 2.51, together with the adjunctions defined by the "isomorphisme cher à Cartan", define the following set of four adjunctions (described in terms of morphisms of bimodules) :

$$\varepsilon_{M,N} \colon \begin{cases} M \otimes_B N \to A \\ m \otimes_B n \mapsto mn \end{cases} \quad \text{and} \quad \eta_{M,N} \colon \begin{cases} B \to N \otimes_A M \\ b \mapsto bc_{N,M} \end{cases}$$

$$\varepsilon_{N,M} \colon \begin{cases} N \otimes_A M \to B \\ n \otimes_B m \mapsto nm \end{cases} \quad \text{and} \quad \eta_{N,M} \colon \begin{cases} A \to M \otimes_B N \\ a \mapsto ac_{M,N} \end{cases}$$

2.53. Proposition. *The morphisms*

$$\varepsilon_{M,N} \colon M \otimes_B N \to A \quad \textit{and} \quad \eta_{N,M} \colon A \to M \otimes_B N$$

are adjoint one to the other relatively to the bilinear forms defined on A *and* $M \otimes_B N$ *by respectively* t *and* $t_{M,N}$*, i.e.,*

$$t(\varepsilon_{M,N}(x)a) = t_{M,N}(x\eta_{N,M}(a)) \quad (\forall\, x \in M \otimes_B N \,,\ a \in A) \,.$$

D2. Relative projectivity, relative injectivity

Let us generalize the preceding situation, by replacing ${}_A\mathbf{Mod}$ and ${}_B\mathbf{Mod}$ by two arbitrary R–linear categories, and by considering $M : \mathfrak{B} \to \mathfrak{A}$ and $N : \mathfrak{A} \to \mathfrak{B}$ two functors such that (M, N) is a biadjoint pair.

In addition, like in the "concrete" situation considered above, let $(\varepsilon_{M,N}, \eta_{M,N})$ (resp. $(\varepsilon_{N,M}, \eta_{N,M})$) be a counit and a unit associated with an adjunction for the pair (M, N) (resp. (N, M)).

Convention. For our own purposes, we make the following convention.

We say that an object X' of a category $\mathfrak{A}$ is *isomorphic to a direct summand* of an object X if there exist two morphisms

$$\begin{cases} \iota \colon X' \to X \\ \pi \colon X \to X' \end{cases} \quad \text{such that} \quad \pi \circ \iota = \mathrm{Id}_X \,.$$

If the category $\mathfrak{A}$ is abelian, this is indeed equivalent to the existence of an object X'' and an isomorphism

$$X \xrightarrow{\sim} X' \oplus X'' \,.$$

2.54. Definition. *For X and X' in $\mathfrak{A}$, we denote by $\mathrm{Tr}_N^M(X,X')$, and call relative trace, the map*

$$\mathrm{Tr}_N^M(X,X')\colon \mathrm{Hom}_{\mathfrak{B}}(NX,NX') \longrightarrow \mathrm{Hom}_{\mathfrak{A}}(X,X')$$

defined by

$$\mathrm{Tr}_N^M(X,X')(\beta) := \varepsilon_{M,N}(X') \circ M(\beta) \circ \eta_{N,M}(X) \quad : \quad \begin{tikzcd} X \arrow[d, "\eta_{N,M}"'] & X' \\ MNX \arrow[r, "M\beta"] & MNX' \arrow[u, "\varepsilon_{M,N}"'] \end{tikzcd}$$

If it is clear from the context what the domain and the codomain of β are, we will write $\mathrm{Tr}_N^M(\beta)$ instead of $\mathrm{Tr}_N^M(X,X')(\beta)$. Furthermore, $\mathrm{Tr}_N^M(X)$ stands for $\mathrm{Tr}_N^M(X,X)$. Notice that the map Tr_M^N is defined, as well.

The following example is fundamental.

Example : Induction and restriction from R. Let A be a symmetric R-algebra with symmetrizing form t and Casimir element $c_A^{\mathrm{pr}} = \sum_i e_i \otimes e_i'$. We take $\mathfrak{A} = {}_A\mathbf{Mod}$, $\mathfrak{B} = {}_R\mathbf{Mod}$ and consider the pair of biadjoint functors defined by the module A, considered as an object of ${}_A\mathbf{Mod}_{\mathcal{O}}$, and as an object of ${}_{\mathcal{O}}\mathbf{Mod}_A$. In other words, the functors are the induction Ind_R^A and the restriction Res_R^A. Let us set

$$\mathrm{Tr}_R^A := \mathrm{Tr}_{\mathrm{Res}_R^A}^{\mathrm{Ind}_R^A} \quad \text{and} \quad \mathrm{Tr}_A^R := \mathrm{Tr}_{\mathrm{Ind}_R^A}^{\mathrm{Res}_R^A}\,.$$

The verification of the following two statements is left to the reader.

1. For $X, X' \in {}_A\mathbf{Mod}$, the map

$$\mathrm{Tr}_R^A : \mathrm{Hom}_R(X,X') \to \mathrm{Hom}_A(X,X')$$

is defined by

$$\mathrm{Tr}_R^A(\beta)(x) = \sum_i e_i \beta(e_i' x) = \mathrm{Tr}^A(x)\ ,$$

thus in other words we have

$$\mathrm{Tr}_R^A = \mathrm{Tr}^A\,.$$

2. For $Y, Y' \in {}_R\mathbf{Mod}$, the map

$$\mathrm{Tr}_A^R : \mathrm{Hom}_A(A\otimes_R Y, A\otimes_R Y') \to \mathrm{Hom}_R(Y,Y')$$

is defined in the following way. Let α be an element of $\mathrm{Hom}_A(A\otimes_R Y, A\otimes_R Y')$ and $y \in Y$. If $\alpha(1\otimes y) = \sum_i a_i \otimes y_i$, then the relative trace is given by the formula

$$\mathrm{Tr}_A^R(\alpha)(y) = \sum_i t(a_i) y_i\,.$$

2.55. Proposition. *Whenever we have three morphisms*

$$\beta\colon NX \to NX' \quad , \quad \alpha\colon X_1 \to X \quad , \quad \alpha'\colon X' \to X_1'\,,$$

we have

$$\alpha' \circ \mathrm{Tr}_N^M(\beta) \circ \alpha = \mathrm{Tr}_N^M(N(\alpha') \circ \beta \circ N(\alpha))\ .$$

In particular, the image if Tr_N^M is a two-sided ideal in $\mathrm{Hom}_{\mathfrak{A}}(\,\cdot\,,\,\cdot\,)$.

Proof of 2.55. Since $\eta_{N,M}$ is a natural transformation, the diagram

$$\begin{tikzcd} X_1 \arrow[r, "\eta_{N,M}(X_1)"] \arrow[d, "\alpha"] & MN(X_1) \arrow[d, "MN(\alpha)"] \\ X \arrow[r, "\eta_{N,M}(X)"] & MN(X) \end{tikzcd}$$

commutes, *i.e.*,

$$MN(\alpha) \circ \eta_{N,M}(X_1) = \eta_{N,M}(X) \circ \alpha\,.$$

Similarly, we get

$$\varepsilon_{M,N}(X_1') \circ MN(\alpha') = \alpha' \circ \varepsilon_{M,N}(X')\,.$$

Using these equations, we obtain

$$\begin{tikzcd} X_1 \arrow[r, "\alpha"] \arrow[d, "\eta_{N,M}(X_1)"'] & X \arrow[d, "\eta_{N,M}(X)"'] & X' \arrow[r, "\alpha'"] & X'1 \\ MNX_1 \arrow[r, "MN\alpha"] & MNX \arrow[r, "M\beta"] & MNX' \arrow[u, "\varepsilon_{M,N}(X')"] \arrow[r, "MN\alpha'"] & MNX_1' \arrow[u, "\varepsilon_{M,N}(X_1')"] \end{tikzcd}$$

$$\begin{aligned} \alpha' \circ \mathrm{Tr}_N^M(\beta) \circ \alpha &= \alpha' \circ \varepsilon_{M,N}(X') \circ M(\beta) \circ \eta_{N,M}(X) \circ \alpha \\ &= \varepsilon_{M,N}(X_1') \circ M(N(\alpha') \circ \beta \circ N(\alpha)) \circ \eta_{N,M}(X_1) \\ &= \mathrm{Tr}_N^M(N(\alpha') \circ \beta \circ N(\alpha))\,. \end{aligned}$$

□

2.56. Theorem. *For an object X in $\mathfrak{A}$, the following statements are equivalent:*

(i) *X is isomorphic to a direct summand of $MN(X)$.*

(ii) *X is isomorphic to a direct summand of $M(Y)$, for some object Y in $\mathfrak{B}$.*

(iii) *The morphism Id_X is in the image of $\mathrm{Tr}_N^M(X)$.*

(iv) *The morphism $\eta_{N,M}(X) : X \to MN(X)$ has a left inverse.*

(v) *The morphism $\varepsilon_{M,N}(X) : MN(X) \to X$ has a right inverse.*

(vi) *Relative projectivity of X :*

$$\begin{tikzcd} N(X'') \arrow[r, shift left, "N(\pi)"] & N(X') \arrow[l, shift left, "\beta"] \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[d, "\alpha"] \arrow[dl, dashed, "\widetilde{\alpha}"'] \\ X'' \arrow[r, "\pi"] & X' \end{tikzcd}$$

Given morphisms $\alpha : X \to X'$ and $\pi : X'' \to X'$ such that there exists a morphism $\beta : N(X') \to N(X'')$ with $N(\pi) \circ \beta = \mathrm{Id}_{N(X')}$, then there exists a morphism $\hat{\alpha} : X \to X''$ with $\pi \circ \hat{\alpha} = \alpha$.

(vii) *Relative injectivity of X :*

$$\begin{tikzcd} N(X') \arrow[r, shift left, "N(\iota)"] & N(X'') \arrow[l, shift left, "\beta"] \end{tikzcd} \qquad \begin{tikzcd} X & \\ X' \arrow[u, "\alpha"] \arrow[r, "\iota"] & X'' \arrow[ul, dashed, "\widetilde{\alpha}"'] \end{tikzcd}$$

Given morphisms $\alpha : X' \to X$ and $\iota : X' \to X''$ such that there exists a morphism $\beta : N(X'') \to N(X')$ with $\beta \circ N(\iota) = \mathrm{Id}_{N(X')}$, then there exists a morphism $\hat{\alpha} : X'' \to X$ with $\hat{\alpha} \circ \iota = \alpha$.

To prove the above theorem we need the following lemma.

2.57. Lemma. *We have* $\mathrm{Tr}_N^M(M(Y))(\eta_{M,N}(Y) \circ \varepsilon_{N,M}(Y)) = \mathrm{Id}_{M(Y)}$.

Proof of 2.57. By definition, we have

$$\begin{tikzcd}
MY \arrow[d, "\eta_{N,M}M"'] & & MY \\
MNMY \arrow[dr, "M\varepsilon_{N,M}"'] & & MNMY \arrow[u, "\varepsilon_{M,N}M"'] \\
& MY \arrow[ur, "M\eta_{M,N}"'] &
\end{tikzcd}$$

$$\mathrm{Tr}_N^M(M(Y))(\eta_{M,N}(Y) \circ \varepsilon_{N,M}(Y)) = \\ \varepsilon_{M,N}(M(Y)) \circ M(\eta_{M,N}(Y)) \circ M(\varepsilon_{N,M}(Y)) \circ \eta_{N,M}(M(Y)) \; .$$

It follows from proposition 0.1 that the morphisms $\varepsilon_{M,N}(M(Y)) \circ M(\eta_{M,N}(Y))$ and $M(\varepsilon_{N,M}(Y)) \circ \eta_{N,M}(M(Y))$ are the identity on $M(Y)$. $\square$

Proof of 2.56. We prove the implications

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \begin{cases} \text{(iv)} & \Rightarrow \quad \text{(i)} \\ \text{(v)} & \Rightarrow \quad \text{(i)} \end{cases}$$

and

$$\text{(ii)} \Rightarrow \begin{cases} \text{(vi)} & \Rightarrow \quad \text{(v)} \\ \text{(vii)} & \Rightarrow \quad \text{(iv)} \, . \end{cases}$$

(i)⇒(ii) : trivial.

(ii)⇒(iii) : We may assume that $X = M(Y)$. For if X is a direct summand of $M(Y)$, we have to morphisms $p : M(Y) \to X$ and $i : X \to M(Y)$ such that $p \circ i = \mathrm{Id}_X$. Hence, if $\mathrm{Tr}_N^M(M(Y))(\beta)$ is the identity morphism on $M(Y)$, then the identity morphism on X is given by $p \circ \mathrm{Tr}_N^M(M(Y))(\beta) \circ i$ and using proposition 2.55, we get

$$\mathrm{Id}_X = \mathrm{Tr}_N^M(N(p) \circ \beta \circ N(i)) \; .$$

For $X = M(Y)$ the assertion follows from lemma 2.57.

(iii)⇒(iv) and (iii)⇒(v) : These implications follow from the definition of the relative trace, since we have

$$\mathrm{Id}_X = \mathrm{Tr}_N^M(X)(\beta) = \varepsilon_{M,N}(X) \circ M(\beta) \circ \eta_{N,M}(X) \; .$$

(iv)⇒(i) and (v)⇒(i) : clear.

(ii)⇒(vi) : We may assume that $X = M(Y)$. Let φ be an adjunction for the pair (M, N). Given a morphism $\alpha : M(Y) \to X'$, we must construct a morphism $\hat{\alpha} : M(Y) \to X''$ such that $\pi \circ \hat{\alpha} = \alpha$. Using the adjunction, we get a morphism $\varphi_{Y,X'}(\alpha) : Y \to N(X')$, which we compose with β to obtain a morphism from Y to $N(X'')$. We claim that if we set

$$\hat{\alpha} := \varphi_{Y,X''}^{-1}(\beta \circ \varphi_{Y,X'}(\alpha)) \; ,$$

then $\hat{\alpha}$ has the desired property. Since the adjunction is natural, we have

$$\pi \circ \varphi_{Y,X''}^{-1}(\beta \circ \varphi_{Y,X'}(\alpha)) = \varphi_{Y,X'}^{-1}(N(\pi) \circ \beta \circ \varphi_{Y,X'}(\alpha))\,.$$

By assumption, $N(\pi) \circ \beta = \mathrm{Id}_{N(X')}$, from which it follows that $\pi \circ \hat{\alpha} = \alpha$.

The proof of the implication (ii)⇒(vii) is analogous to the previous one.

(vi)⇒(v) : Let us choose $\alpha := \mathrm{Id}_X$ and $\pi := \varepsilon_{M,N}(X)$. We have to check that the morphism $N(\varepsilon_{M,N}(X))$ splits : this follows from the properties of an adjunction, since $N(\varepsilon_{M,N}(X)) \circ \eta_{M,N}(N(X))$ is the identity on $N(X)$ (see proposition 0.1).

The proof of the implication (vii)⇒(iv) is similar to the previous one. □

2.58. Definition. *An object X of the category $\mathfrak{A}$, satisfying one of the conditions in theorem 2.56, is called M-split (or relatively M-projective, or relatively M-injective).*

Notice that any object isomorphic to $M(Y)$ (for $Y \in \mathfrak{B}$) is M–split.

Example : Induction–restriction with R. Let A be a symmetric algebra over R, and consider the categories

$$\mathfrak{A} = {}_A\mathbf{Mod} \quad \text{and} \quad \mathfrak{B} = {}_R\mathbf{Mod}\,.$$

We have already seen that the functors $M := \mathrm{Ind}_R^A$ and $N := \mathrm{Res}_R^A$ build a biadjoint pair. We shall prove and generalize below the following set of properties.

- The relative trace Tr_R^A is the trace Tr^A defined in the previous paragraph, *i.e.*, the multiplication by the Casimir element.
- The split modules are the relatively R–projective modules.
- For X a finitely generated A–module, the following condition are equivalent.
  (i) X is a projective A–module,
  (ii) X is a projective R–module and a split module (relatively projective R–module).

If $R = k$, a field, the A-split modules are exactly the projective modules and the projective modules coincide with the injective modules.

Relatively projective modules and projective modules.

Consider the following particular situation :

• B is a symmetric subalgebra of A such that A is a projective B–module (hence, as we have already noted, A is a projective module–B). We choose a symmetrizing form t on A and a symmetrizing form u on B.

• We choose $M := A$ (viewed as an object of ${}_A\mathbf{Mod}_B$), $N := A$ (viewed as an object of ${}_B\mathbf{Mod}_A$), and the pairing $A \times A \to R$ is defined by $(a, a') \mapsto t(aa')$.

Thus the functor $M \otimes_B \bullet$ coincides with the induction

$$\mathrm{Ind}_B^A \colon {}_B\mathbf{Mod} \to {}_A\mathbf{Mod}\,,$$

while the functor $N \otimes_A \bullet$ coincides with the restriction

$$\mathrm{Res}_B^A \colon {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}\,.$$

We then say that an A–module X is *relatively B–projective* when it is split for the pair (M, N) just defined.

We construct in this context the analog of the Casimir element.

Since A is a (finitely generated) projective B–module, the natural morphism

$$\mathrm{Hom}_B(A, B) \otimes_B A \to E_B A$$

is an isomorphism. Since B is symmetric, it chosen symmetrizing form u induces a natural isomorphism

$$\mathrm{Hom}_B(A, B) \xrightarrow{\sim} A^*\,,$$

and since A is symmetric, its chosen symmetrizing form t induces an isomorphism $A \xrightarrow{\sim} A^*$. So we get an isomorphism (of $(A \otimes A^{\mathrm{op}})$–modules–$(E_B A \otimes E_B A^{\mathrm{op}})$)

$$A \otimes_B A \xrightarrow{\sim} E_B A\,.$$

We call *relative Casimir element* and we denote by c_B^A the element of $A \otimes_B A$ which corresponds to Id_A through the preceding isomorphism.

Let X be an A–module. The relative trace may be viewed as a morphism

$$\mathrm{Tr}_B^A \colon \mathrm{Hom}_B(X, X') \to \mathrm{Hom}_A(X, X')\,.$$

This morphism is nothing but the multiplication by the relative Casimir element c_B^A : if $c_B^A = \sum_i a_i \otimes_B a_i'$, and if Y is any A–module–A, we have

$$\mathrm{Tr}_B^A \colon \begin{cases} Y^B \to Y^A \\ y \mapsto c_B^A.y = \displaystyle\sum_i a_i y a_i'\,. \end{cases}$$

Example. Assume $A = RG$ and $B = RH$ (G a finite group, H a subgroup of G). Then we have

$$c_{RH}^{RG} = \sum_{g \in [G/H]} g \otimes_{RH} g^{-1}\,,$$

where $[G/H]$ denote a complete set of representatives of the left cosets of G modulo H. Thus, whenever Y is an RG–module–RG and $y \in Y^H$, we have

$$\mathrm{Tr}_{RH}^{RG}(y) = \sum_{g \in [G/H]} g y g^{-1}\,.$$

In such a situation, projectivity and relative projectivity are connected by the following property.

2.59. Proposition. *Let B be a symmetric subalgebra of A such that A is a projective B–module. Let X be a finitely generated A–module. The following conditions are equivalent.*

(i) *X is a projective A–module.*

(ii) *X is relatively B–projective and $\mathrm{Res}_B^A X$ is a projective B–module.*

Proof of 2.59.

(i)⇒(ii) Since A is a projective B–module, any projective A–module is also (by restriction) a projective B–module. Moreover, if a morphism $X'' \to X'$ gets a right inverse after restriction to B, it is onto, and so every morphism $X \to X'$ can be lifted to a suitable morphism $X \to X''$.

$$\mathrm{Res}_B^A(X'') \underset{\beta}{\overset{\mathrm{Res}_B^A(\pi)}{\rightleftarrows}} \mathrm{Res}_B^A(X') \qquad \begin{tikzcd} & X \arrow[dl, dashed, "\widetilde{\alpha}"'] \arrow[d, "\alpha"] \\ X'' \arrow[r, "\pi"] & X' \end{tikzcd}$$

(ii)⇒(i) Since X is relatively projective, we may choose an endomorphism

$$\iota\colon \mathrm{Res}_B^A(X) \to \mathrm{Res}_B^A(X) \quad \text{such that} \quad \mathrm{Tr}_B^A(\iota) = \mathrm{Id}_X\,.$$

Suppose given a surjective morphism $X'' \xrightarrow{\pi} X'$ and a morphism $X \xrightarrow{\alpha} X'$. Since $\mathrm{Res}_B^A X$ is projective, there exists a morphism $\gamma\colon \mathrm{Res}_B^A X \to \mathrm{Res}_B^A X''$ such that the following triangle commutes :

$$\begin{tikzcd} & \mathrm{Res}_B^A X \arrow[dl, dashed, "\gamma"'] \arrow[d, "\alpha\iota"] \\ \mathrm{Res}_B^A X'' \arrow[r, "\pi"] & \mathrm{Res}_B^A X' \end{tikzcd} \qquad i.e., \quad \pi\gamma = \alpha\iota\,.$$

Applying Tr_B^A to this last equality, we get

$$\pi.\mathrm{Tr}_B^A(\gamma) = \alpha\mathrm{Tr}_B^A(\iota) = \alpha\,,$$

and this shows that the morphism α has been indeed lifted to a suitable morphism $X \to X''$. ☐

D3. The M-Stable Category

Generalities.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two categories. We denote by $\mathrm{Hom}_{\mathfrak{A}}^M(X, X')$ the image of $\mathrm{Tr}_N^M(X, X')$ in $\mathrm{Hom}_{\mathfrak{A}}(X, X')$ and call these morphisms the "M-split morphisms".

By definition, the M–split objects are those objects whose identity is M–split (*i.e.*, such that all endomorphisms are M–split).

Since the M–split morphism functor $\mathrm{Hom}_{\mathfrak{A}}^M(\cdot\,,\cdot\,)$ is an ideal (see 2.55), we have the following property.

2.60. Lemma. *A morphism $X \to X'$ in $\mathfrak{A}$ is M-split if and only if it factorizes through an M–split object of $\mathfrak{A}$.*

2.61. Definition. *Let $\mathfrak{A}$ be an abelian category. The category* $\mathbf{Stab}(\mathfrak{A})$*, is defined as follows:*

1. *the objects of* $\mathbf{Stab}(\mathfrak{A})$ *are the objects of* $\mathfrak{A}$*,*
2. *the morphisms in* $\mathbf{Stab}(\mathfrak{A})$*, which we denote by* $\mathrm{Hom}^{\mathrm{st}}_{\mathfrak{A},M}(\cdot,\cdot)$*, are the morphisms in $\mathfrak{A}$ modulo the M-split morphisms, i.e.,*

$$\mathrm{Homst}_{\mathfrak{A},M}(X, X') := \mathrm{Hom}_{\mathfrak{A}}(X, X')/\mathrm{Hom}_{\mathfrak{A}}^M(X, X')\,.$$

Let A be an R-algebra. In the situation where $\mathfrak{A} = {}_A\mathbf{Mod}$, $\mathfrak{B} = {}_R\mathbf{Mod}$ and the biadjoint pair of functors is given by $(\mathrm{Ind}_R^A, \mathrm{Res}_R^A)$, we denote the corresponding stable category by ${}_A\mathbf{Stab}$.

Remarks.

1. If $R = k$, a field, then the category ${}_A\mathbf{Stab}$ coincides with the ususal notion of the stable category, *i.e.*, the module category "modulo the projectives". But in general, our category ${}_A\mathbf{Stab}$ is not the quotient of ${}_A\mathbf{Mod}$ modulo the projective A–modules.

2. $\mathbf{Stab}(\mathfrak{A})$ is an R-linear category.

(!) **Attention** (!)

In general, the category $\mathbf{Stab}(\mathfrak{A})$ is not an abelian category.

Exercices. Let A be a symmetric algebra over a field. Prove that

(1) if X is a non-projective indecomposable A-module, then $\mathrm{St}(M)$ is an indecomposable object in ${}_A\mathbf{Stab}$.

(2) Show that every monomorphism and every epimorphism splits in ${}_A\mathbf{Stab}$.

(3) Give an example of an algebra A, for which ${}_A\mathbf{Stab}$ is not an abelian category. (Hint: Use (1) and (2) to prove that there exist morphisms in ${}_A\mathbf{Stab}$, which do not have a kernel in ${}_A\mathbf{Stab}$).

From the way we defined the M-stable category, it is clear that there is a natural functor $\mathrm{St} : \mathfrak{A} \to \mathbf{Stab}(\mathfrak{A})$.

2.62. Proposition. *If X is an object in $\mathfrak{A}$, then $\mathrm{St}(X) \simeq 0$ if and only if X is M-split.*

Proof of 2.62. If $\mathrm{St}(X) \simeq 0$, then the identity on X is in the image of the relative trace $\mathrm{Tr}_N^M(X)$, which is equivalent to say that X is M-split.

If X is M-split, then the identity on X is an M-split homomorphism and therefore it is zero in $\mathbf{Stab}(\mathfrak{A})$. Thus, we have $\mathrm{St}(X) \simeq 0$. □

The Heller Functor on $\mathbf{Stab}(\mathfrak{A})$.

Whenever $\alpha \in \mathrm{Hom}_{\mathfrak{A}}(X, X')$, we denote by α^{st} its image in $\mathrm{Hom}^{\mathrm{st}}_{\mathfrak{A},M}(X, X')$.

2.63. Proposition. *(Schanuel's lemma) Let $\mathfrak{A}$ and $\mathfrak{B}$ be two abelian categories and let (M, N) be a biadjoint pair of functors on $\mathfrak{A}$ and $\mathfrak{B}$. Assume that*

$$0 \to X_1' \xrightarrow{\iota_1} P_1 \xrightarrow{\pi_1} X_1 \to 0 \quad \textit{and} \quad 0 \to X_2' \xrightarrow{\iota_2} P_2 \xrightarrow{\pi_2} X_2 \to 0$$

are short exact sequences in $\mathfrak{A}$ such that

1. *Their images through N are split,*
2. *P_1 and P_2 are M-split objects.*

Then there exists an isomorphism

$$\begin{array}{ccc} \mathrm{Hom}^{\mathrm{st}}_{\mathfrak{A},M}(X_1, X_2) & \xrightarrow{\sim} & \mathrm{Hom}^{\mathrm{st}}_{\mathfrak{A},M}(X_1', X_2') \\ \alpha^{\mathrm{st}} & \longmapsto & \alpha'^{\mathrm{st}} \end{array}$$

determined, for $\alpha \in \mathrm{Hom}_{\mathfrak{A}}(X_1, X_2)$ and $\alpha' \in \mathrm{Hom}_{\mathfrak{A}}(X_1', X_2')$, by the following condition : there exists $u \in \mathrm{Hom}_{\mathfrak{A}}(P_1, P_2)$ such that the diagram

$$\begin{tikzcd} X_1' \arrow[r, "\iota_1"] \arrow[d, "\alpha'"'] & P_1 \arrow[r, "\pi_1"] \arrow[d, "u"'] & X_1 \arrow[d, "\alpha"'] \\ X_2' \arrow[r, "\iota_1"] & P_2 \arrow[r, "\pi_2"] & X_2 \end{tikzcd}$$

commutes.

Proof of 2.63. We may assume that α is given. Then, since $N(\pi_2)$ splits and P_1 is a M-split object, there exists a map u and a map α' such that the above diagram commutes. It suffices to verify that α^{st} is zero if and only if α'^{st} is zero.

If α^{st} is zero, then α factorizes through the object P_2. Let us say $\alpha = \pi_2 \circ h$, where $h\colon X_1 \to P_2$. The map $u - h \circ \pi_1$ is a map from P_1 to the kernel of π_2. Therefore, if we set $h' = u - h \circ \pi_1$, then $\alpha' = h' \circ \iota_1$, *i.e.*, the map α' factorizes through an M-split object. The converse implication can be verified similarly. $\square$

Remark. It follows from the proof of Schanuel's lemma that (α', u, α) defines a single homotopy class of morphisms from

$$0 \to X_1' \xrightarrow{\iota_1} P_1 \xrightarrow{\pi_1} X_1 \to 0 \quad \text{to} \quad 0 \to X_2' \xrightarrow{\iota_2} P_2 \xrightarrow{\pi_2} X_2 \to 0$$

This is a particular case of a more general lemma which we will prove later on.

2.64. Corollary. *Assume that*

$$0 \to X_1' \to P_1 \to X \to 0 \quad \textit{and} \quad 0 \to X_2' \to P_2 \to X \to 0$$

are short exact sequences in $\mathfrak{A}$ such that

1. *their images through N are split,*
2. *P_1 and P_2 are M-split objects.*

Then there exists an isomorphism

$$\varphi^{\mathrm{st}}\colon X_1' \xrightarrow{\sim} X_2' \quad \textit{in} \quad \mathbf{Stab}(\mathfrak{A})$$

characterized by the following condition : there exists $u \in \mathrm{Hom}_{\mathfrak{A}}(P_1, P_2)$ such that the diagram

$$\begin{tikzcd} X_1' \arrow[r, "\iota_1"] \arrow[d, "\varphi"'] & P_1 \arrow[r, "\pi_1"] \arrow[d, "u"'] & X \arrow[d, "\mathrm{Id}_X"'] \\ X_2' \arrow[r, "\iota_1"] & P_2 \arrow[r, "\pi_2"] & X \end{tikzcd}$$

commutes.

This corollary allows us to define a functor $\Omega_M : \mathbf{Stab}(\mathfrak{A}) \to \mathbf{Stab}(\mathfrak{A})$, the Heller functor. It is given by $\Omega_M(X) := X_1'$.

Similarly, we have a functor $\Omega_M^{-1} : \mathbf{Stab}(\mathfrak{A}) \to \mathbf{Stab}(\mathfrak{A})$. It can be check that the functors Ω_M and Ω_M^{-1} induce reciprocal equivalences of $\mathbf{Stab}(\mathfrak{A})$.

Exercice. Prove that under the assumptions of corollary 2.64, there exists an isomorphism from $X_1' \oplus P_2$ to $X_2' \oplus P_1$.

The case of ${}_A$Stab : the Heller bimodules.

From now on, we assume that $\mathfrak{A} = {}_A\mathbf{Mod}$, $\mathfrak{B} = {}_R\mathbf{Mod}$ and the modules inducing the biadjoint pair of functors are $M \in {}_A\mathbf{Mod}_{\mathcal{O}}$ and $N \in {}_{\mathcal{O}}\mathbf{Mod}_A$. We proceed to give another definition of the Heller functors Ω_A and Ω_A^{-1}.

We call *Heller bimodule* and we denote by Ω_A the kernel of the multiplication morphism

$$A \otimes_R A \to A\,.$$

Thus we have

$$\Omega_A = \left\{ \sum_i a_i \otimes b_i \mid \sum_i a_i b_i = 0 \right\}.$$

• Viewing Ω_A as a left ideal in $A \otimes_R A^{\mathrm{op}}$, we see that if $\sum_i a_i \otimes b_i \in \Omega_A$, we have

$$\sum_i a_i \otimes b_i = \sum_i (a_i \otimes b_i - 1 \otimes a_i b_i) = \sum_i (1 \otimes b_i)(a_i \otimes 1 - 1 \otimes a_i)\,,$$

hence Ω_A is the left ideal of $A \otimes_R A^{\mathrm{op}}$ generated by $\{a \otimes 1 - 1 \otimes a \mid (a \in A)\}$.

• Since $A \otimes_R A)^A$ is by definition the right annihilator in $A \otimes_R A^{\mathrm{op}}$ of the set $\{a \otimes 1 - 1 \otimes a \mid (a \in A)\}$, it follows that

$$(A \otimes_R A)^A = \mathrm{Ann}(\Omega_A)_{(A \otimes_R A^{\mathrm{op}})}\,.$$

• If A is symmetric and t is a symmetrizing form, then the form

$$t^{\mathrm{en}} \colon \begin{cases} A \otimes_R A^{\mathrm{op}} \to R \\ a \otimes a' \mapsto t(a)t(a') \end{cases}$$

is a symmetrizing form on $A \otimes_R A^{\mathrm{op}}$. Then it follows from what preceds that

$$(A \otimes_R A)^A = \Omega_A^{\perp}\,,$$

where the orthogonal is relative to the form t^{en}.

The *inverse Heller bimodule* Ω_A^{-1} is defined as the quotient

$$\Omega_A^{-1} := (A \otimes_R A)/(A \otimes_R A)^A\,.$$

Thus we see that the form t^{en} induces an isomorphism of A–modules–A :

$$\Omega_A^{-1} \xrightarrow{\sim} \Omega_A^*\,.$$

Taking the dual (relative to the forms t and t^{en} of the short exact sequence

$$0 \to \Omega_A \to A \otimes_R A^{\mathrm{op}} \to A \to 0\,,$$

we get the short exact sequence

$$0 \to A \to A \otimes_R A^{\mathrm{op}} \to \Omega_A^{-1} \to 0\,.$$

2.65. Proposition.
(1) The A–modules–A Ω_A and Ω_A^{-1} are exact.
(2) The bimodules $\Omega_A \otimes_A \Omega_A^{-1}$ and $\Omega_A^{-1} \otimes_A \Omega_A$ are both isomorphic to A in the category ${}_A\mathbf{Stab}_A$.

2.66. Corollary. *The functors*

$$\Omega_A\,,\, \Omega_A^{-1} \colon {}_A\mathbf{Mod} \to {}_A\mathbf{Mod}$$

induce reciprocal selfequivalences on ${}_A\mathbf{Stab}$.

PROOF OF 2.65.

(1) Since A is projective on both sides, we see that

$$0 \to \Omega_A \to A \otimes_R A \xrightarrow{\mu} A \to 0$$

is a split short exact sequence in ${}_A\mathbf{Mod}$, as well as in $\mathbf{Mod}_A$. In particular, it is R–split. Taking the dual with respect to the bilinear forms defined above yields the R–split short exact sequence

$$0 \to A \xrightarrow{\mu^*} A \otimes_R A \to \Omega_A^{-1} \to 0 \,.$$

The relative injectivity of A implies that this sequence splits in ${}_A\mathbf{Mod}$ and in $\mathbf{Mod}_A$. Thus, we have shown that Ω_A and Ω_A^{-1} are in ${}_A\mathbf{proj} \cap \mathbf{proj}_A$.

(2) Since we want the isomorphism from $\Omega_A \otimes_A \Omega_A^{-1}$ to A to be in ${}_A\mathbf{Stab}_A$, the symmetric algebra to consider here is $(A \otimes_R A^{\mathrm{op}})$. We shall apply Schanuel's lemma to the short exact sequences

$$0 \longrightarrow \Omega_A \otimes_A \Omega_A^{-1} \longrightarrow A \otimes_R \Omega_A^{-1} \xrightarrow{\mu \otimes \mathrm{Id}_{\Omega_A^{-1}}} \Omega_A^{-1} \longrightarrow 0$$

$$0 \to A \xrightarrow{\mu^*} A \otimes_R A \to \Omega_A^{-1} \to 0$$

These sequences split as sequences in ${}_A\mathbf{Mod}$, since Ω_A^{-1} is an A-projective module. In particular, they split when restricted to R. Thus, by Schanuel's lemma, it is enough to check that $A \otimes_R A$ and $A \otimes_R \Omega_A^{-1}$ are both relatively $(A \otimes_R A^{\mathrm{op}})$-projective, hence it is enough to remark that they are projective $(A \otimes_R A^{\mathrm{op}})$–modules.

Similarly, one shows that $\Omega_A^{-1} \otimes_A \Omega_A$ is isomorphic to A in the category ${}_A\mathbf{Stab}_A$. $\square$

2.67. DEFINITION. *For X , $X' \in {}_A\mathbf{Mod}$ and $n \in \mathbb{N}$, we set*

$$\mathrm{Ext}_A^n(X, X') := \mathrm{Hom}_{{}_A\mathbf{Stab}}(\Omega_A^n(X), X') \,.$$

Note that we have also

$$\mathrm{Ext}_A^n(X, X') = \mathrm{Hom}_{{}_A\mathbf{Stab}}(X, \Omega^{-n}(X')) \,.$$

2.68. PROPOSITION. *Let $M \in {}_A\mathbf{Mod}_B$ be an exact bimodule. Then the functor $M \otimes_B \cdot$ commutes with $\Omega_\cdot$, i.e.,*

$$\Omega_A \otimes_A M \xrightarrow{\sim} M \otimes_B \Omega_B \quad \mathit{in}\, {}_A\mathbf{Stab}_B \,.$$

PROOF OF 2.68. The module M induces a functor on the stable category.

Consider the following two short exact sequences

$$0 \to \Omega_A \to A \otimes_R A \xrightarrow{\mu_A} A \to 0 \quad \text{and} \quad 0 \to \Omega_B \to B \otimes_R B \xrightarrow{\mu_B} B \to 0 \,.$$

If we tensor the first one over A with M and the second one over B with M, we get the two short exact sequences

$$0 \to \Omega_A \otimes_A M \to A \otimes_R M \to M \to 0 \text{ and } 0 \to M \otimes_B \Omega_B \to M \otimes_R B \to M \to 0 \,.$$

Both sequences split as sequences over R. Since M is in $\mathbf{proj}_B$, $A \otimes_R M$ is a projective $A \otimes_R B^{\mathrm{op}}$–module. Similarly, on shows that $M \otimes_R B$ is a projective $A \otimes_R B^{\mathrm{op}}$–module. Thus, we can apply Schanuel's lemma and get an isomorphism

$$\Omega_A \otimes_A M \xrightarrow{\sim} M \otimes_B \Omega_B \ \text{ in } {}_A\mathbf{Stab}_B \,.$$

$\square$

As an application of the previous proposition, we get the following corollary.

2.69. Corollary. *(Schapiro's lemma) Let (M, N) be a self dual pair of exact bimodules for the algebras A and B. Then $(\Omega_A \otimes_A M, \Omega_B^{-1} \otimes_A N)$ is also a self dual pair of exact bimodules for the algebras A and B.*

In particular, for all $n \in \mathbb{N}$, we have

$$\mathrm{Ext}^n_A(M(Y), X) \simeq \mathrm{Ext}^n_B(Y, N(X)) \,.$$

Proof of 2.69. We will only show that $\Omega_A M$ is left adjoint to $\Omega_B^{-1} N$. We know that both, (M, N) and $(\Omega_A, \Omega_A^{-1})$, are biadjoint pairs. Thus the functor $\Omega_A M$ is left adjoint to the functor $N\Omega_A^{-1}$. By proposition 2.68, $N\Omega_A^{-1}$ is naturally equivalent to the functor $\Omega_B^{-1} N$. $\square$

D4. Stable Equivalences of Morita Type

Let (M, N) be a selfdual exact pair of bimodules for A and B. Since the functors $M \otimes_B \centerdot$ and $N \otimes_A \centerdot$ factorize through the functors

$$\mathrm{St}_A : {}_A\mathbf{Mod} \longrightarrow {}_A\mathbf{Stab} \quad \text{and} \quad St_B : {}_B\mathbf{Mod} \longrightarrow {}_B\mathbf{Stab}\,,$$

the bimodules M and N induce two functors

$$M \otimes_B \centerdot : {}_B\mathbf{Stab} \longrightarrow {}_A\mathbf{Stab} \quad \text{and} \quad N \otimes_A \centerdot : {}_A\mathbf{Stab} \longrightarrow {}_B\mathbf{Stab}\,.$$

Since the functors $M \otimes_B \centerdot$ and $N \otimes_A \centerdot$ are biadjoint, the induced functors on the stable categories are biadjoint, as well. The associated adjunctions are the images in the stable categories of the adjunctions of M and N on the module category level.

These preliminaries suggest the following definition of a stable equivalence of Morita type.

2.70. Definition. *Let M and N be bimodules as above. We say that M and N induce a stable equivalence of Morita type between A and B if*

$$M \otimes_B N \simeq A \text{ in } {}_A\mathbf{Stab}_A \quad \text{and} \quad N \otimes_A M \simeq B \text{ in } {}_B\mathbf{Stab}_B \,,$$

through the counits and the units of the adjunctions.

Remark. Notice that by theorem 0.5, we do not need to specify which counits and units provide the above isomorphisms. If one appropriate pair of them are isomorphisms, then all of them will be isomorphisms.

2.71. Definition. *The stable center of the symmetric algebra A, denoted by $Z^{\mathrm{st}}A$, is the quotient $ZA/Z^{\mathrm{pr}}A$.*

REMARK. If we view A as an object in the category ${}_{(A\otimes_R A^{\mathrm{op}})}\mathbf{Mod}$, then the center of A is isomorphic to $\mathrm{End}_{(A\otimes_R A^{\mathrm{op}})}(A)$. It follows from the definition of the stable category ${}_{(A\otimes_R A^{\mathrm{op}})}\mathbf{Stab}$ and the definition of projective endomorphisms of A considered as an $(A \otimes_R A^{\mathrm{op}})$-module, that the stable center of A is isomorphic to $\mathrm{End}_{{}_{(A\otimes_R A^{\mathrm{op}})}\mathbf{Stab}}(A)$.

2.72. PROPOSITION. *A stable equivalence of Morita type between the symmetric algebras A and B induces and algebra isomorphism*

$$Z^{\mathrm{st}}A \simeq Z^{\mathrm{st}}B\,.$$

PROOF OF 2.72. Let ${}_A\mathbf{stab}_A^{\mathrm{pr}}$ denote the full subcategory of ${}_A\mathbf{stab}_A$ whose objects are the exact A–modules–A. Assume that (M, N) induces a stable equivalence of Morita type between A and B. Then the pair $(M \otimes_R N\,,\, N \otimes_R M)$ (where $M \otimes_R N$ is viewed as an $(A \otimes_R A^{\mathrm{op}})$–module–$(B \otimes_R B^{\mathrm{op}})$ and $N \otimes_R M$ is viewed as a $(B \otimes_R B^{\mathrm{op}})$–module–$(A \otimes_R A^{\mathrm{op}})$ as previously) induce inverse equivalences between ${}_A\mathbf{stab}_A^{\mathrm{pr}}$ and ${}_B\mathbf{stab}_B^{\mathrm{pr}}$ which exchange A and B. The assertion follows from the fact that $Z^{\mathrm{st}}(A)$ is the algebra of endomorphisms of A in ${}_A\mathbf{stab}_A^{\mathrm{pr}}$. □

D5. (M, N)–split algebras

We keep the notation introduced in §D1 above. Let us consider the objects (see chapter 1, §A1)

$$\begin{aligned} F &:= M \otimes_R N \,\in\, {}_{(A\otimes_R A^{\mathrm{op}})}\mathbf{Mod}_{(B\otimes_R B^{\mathrm{op}})} \\ G &:= N \otimes_R M \,\in\, {}_{(B\otimes_R B^{\mathrm{op}})}\mathbf{Mod}_{(A\otimes_R A^{\mathrm{op}})}\,. \end{aligned}$$

Then the pair (F, G) with the pairing defined by

$$\begin{cases} F \times G \longrightarrow R \\ (m \otimes n\,,\, n' \otimes m') \mapsto \langle m, n'\rangle\langle m', n\rangle\,, \end{cases}$$

is an exact pair for the algebras $A \otimes_R A^{\mathrm{op}}$ and $B \otimes_R B^{\mathrm{op}}$.

We have

$$\begin{cases} F \otimes_{(B\otimes_R B^{\mathrm{op}})} G \xrightarrow{\sim} (M \otimes_B N) \otimes_R (M \otimes_B N) \\ (m \otimes n) \otimes (n' \otimes m') \mapsto (m \otimes_B n') \otimes_R (m' \otimes_B n)\,, \end{cases}$$

and through that isomorphism the counits are given by

$$\begin{cases} (M \otimes_B N) \otimes_R (M \otimes_B N) \to A \otimes_R A^{\mathrm{op}} \\ (m \otimes n') \otimes (n \otimes m') \mapsto (mn') \otimes_R (m'n)\,, \end{cases}$$

2.73. PROPOSITION.
(1) We have

$$\mathrm{Hom}_B(N \otimes_A M, N \otimes_A M)_B = (M \otimes_B N \otimes_A M \otimes_B N)^A\,,$$

and the relative trace for the pair $\mathrm{Tr}_{GF}^{FG}(A)$ *is given by*

$$\mathrm{Tr}_{GF}^{FG}(A)\colon \begin{cases} (M \otimes_B N \otimes_A M \otimes_B N)^A \to ZA \\ \sum (m \otimes n') \otimes (n \otimes m') \mapsto \sum (mn')(m'n) \end{cases}$$

(2) We also have

$$\mathrm{Hom}_B(N \otimes_A M, N \otimes_A M)_B = (N \otimes_A M \otimes_B N \otimes_A M)^B \,,$$

and the relative trace for the pair $\mathrm{Tr}_{GF}^{FG}(A)$ *is given by*

$$\mathrm{Tr}_{GF}^{FG}(A) \colon \begin{cases} (N \otimes_A M \otimes_B N \otimes_A M)^B \to ZA \\ \sum (n \otimes m') \otimes (n \otimes m') \mapsto \mathrm{Tr}_B^A(\sum (nm')(n'm)) \end{cases}$$

2.74. Proposition–Definition. *The following assertions are equivalent.*

(i) *A is isomorphic to a direct summand of $M \otimes_B N$ in ${}_A\mathbf{Mod}_A$.*

(ii) *$\varepsilon_{M,N}$ is a split epimorphism in ${}_A\mathbf{Mod}_A$.*

(iii) *$\eta_{N,M}$ is a split monomorphism in ${}_A\mathbf{Mod}_A$.*

(iv) *The trace map*

$$\mathrm{Tr}_{GF}^{FG}(A) \colon (N \otimes_A M \otimes_B N \otimes_A M)^B \to ZA$$

is onto.

(v) *Every A–module is M–split.*

If the preceding conditions are satisfied, we say that the algebra A is B–split (through (M,N)).

Example : Induction–Restriction with R. Choose $B := R$, $M :={}_A A_R$, $N :={}_R A_A$ and $\langle a, a' \rangle := t(aa')$.

1. The following conditions are equivalent.
   (i) A is strongly symmetric.
   (ii) R is A–split.
2. The following conditions are equivalent.
   (i) A is separable.
   (ii) A is R–split.

Example : Induction–Restriction with a parabolic subalgebra. Let B be a parabolic subalgebra for A. Choose

$$M :={}_A A_B \ , \ N :={}_B A_A \ , \ \langle a, a' \rangle := t(aa') \,.$$

Then B is always A–split, while A is B–split if and only if A is a summand of $A \otimes_B A$ in ${}_A\mathbf{Mod}_A$.

If $c_B^A = \sum_i e_i' \otimes_B e_i$ is the relative Casimir element, the "double relative trace" is

$$\mathrm{Tr}_B^A \colon \begin{cases} (A \otimes_B A)^B \to ZA \\ \sum_j x_j \otimes y_j \mapsto \sum_i e_i (\sum_j x_j y_j) e_i \end{cases}$$

Notice that the element $1 \otimes_B 1$ belongs to $(A \otimes_B A)^B$. Its image by Tr_B^A is the relative projective central element z_B^A. Thus if z_B^A is invertible in ZA, the algebra A is B–split.

For example if $A = RG$ and $B = RH$, the relative trace is

$$\mathrm{Tr}_{RH}^{RG} \colon \sum a_i \otimes_B a'i \mapsto \sum_{g \in [G/H]} g a_i a_i' g^{-1} \,,$$

and the relative projective central element is $|G : H|$. It follows that if the index $|G : H|$ is invertible in R, then RG is RH–split.

Example : Induction–restriction with idempotents. Let B be a parabolic subalgebra for A. Let e be a central idempotent in A and let f be a central idempotent in B. Choose

$$M := eAf \ , \ N := fAe \ , \ \langle a, a' \rangle := t(aa') \, .$$

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