Complex reflection groups as Weyl groups

Michel Broué

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July 2007
FINITE COMPLEX REFLECTION GROUPS

Let $K$ be a characteristic zero field. A finite reflection group on $K$ is a finite subgroup of $\text{GL}_K(V)$ ($V$ a finite dimensional $K$–vector space) generated by pseudo–reflections, i.e., linear maps represented by

$$
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\zeta & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
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A finite reflection group on $\mathbb{R}$ is called a Coxeter group. A finite reflection group on $\mathbb{Q}$ is called a Weyl group.
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2. The group $G(de, e, r)$ ($d, e$ and $r$ integers) consists of all $r \times r$ monomial matrices with entries in $\mu_{de}$ such that the product of entries belongs to $\mu_d$.

3. We have

   \[
   G(d, 1, r) \simeq C_d \wr S_r \\
   G(e, e, 2) = D_{2e} \quad \text{(dihedral group of order } 2e) \\
   G(2, 2, r) = W(D_r). 
   \]
FINITE REDUCTIVE GROUPS : POLYNOMIAL ORDER

$G$ is a connected reductive algebraic group over $\overline{F}_q$, with Weyl group $W$, endowed with a Frobenius–like endomorphism $F$. The group $G := G^F$ is a finite reductive group.
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Type of \(G\) — The type \(\mathcal{G} = (X, Y, R, R^\vee ; W\phi)\) of \(G\) consists of the root datum of \(G\) endowed with the outer automorphism \(W\phi\) defined by \(F\).
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Example

$GL_n = (X = Y = \mathbb{Z}^n, R = R^\vee = A_n ; \phi = 1)$
Polynomial order — There is a polynomial in $\mathbb{Z}[x]$ such that $|G|(q) = |G|$.
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$$|G|(x) = x^N \prod_{d} \Phi_d(x)^{a(d)}$$

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Example

$$|GL_n|(x) = x^{\binom{n}{2}} \prod_{d=1}^{d=n} (x^d - 1) = x^{\binom{n}{2}} \prod_{d=1}^{d=n} \Phi_d(x)^{[n/d]}$$
Admissible subgroups — The tori of $G$ are the subgroups of the shape $T^F$ where $T$ is an $F$–stable torus (i.e., isomorphic to some $\bar{F} \times \cdots \times \bar{F}$ in $G$).

The Levi subgroups of $G$ are the subgroups of the shape $L^F$ where $L$ is a centralizer of an $F$–stable torus in $G$. 

Example

The split maximal torus $T_1 = (F \times q^n)$ of order $(q^n - 1)^n$

The Coxeter maximal torus $T_c = GL_1(Fq^n)$ of order $q^n - 1$

Levi subgroups have shape $GL_{a_1}(q^{a_1}) \times \cdots \times GL_{a_s}(q^{a_s})$

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- The split maximal torus $T_1 = (\mathbb{F}_q^\times)^n$ of order $(q - 1)^n$
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Levi subgroups and type — For

\[ G = (X, Y, R, R^\vee ; W\phi) \]

a type, a Levi subtype of \( G \) is a type of the shape

\[ L = (X, Y, R', R'^\vee ; W'w\phi) \]

where \( R' \) is a parabolic system of \( R \), with Weyl group \( W' \), and where \( w \in W \) is such that \( w\phi \) stabilizes \( R' \) and \( R'^\vee \).
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- the set of \( G \)–conjugacy classes of Levi subgroups of \( G \),
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- the set of \( W \)–conjugacy classes of Levi subtypes of \( \mathcal{G} \).
For $\Phi(x)$ a cyclotomic polynomial, a $\Phi(x)$–group is a finite reductive group whose (polynomial) order is a power of $\Phi(x)$. Hence such a group is a torus.

Sylow theorem —

1. Maximal $\Phi(x)$–subgroups ("Sylow $\Phi(x)$–subgroups") of $G$ have as (polynomial) order the contribution of $\Phi(x)$ to the (polynomial) order of $G$.

2. Sylow $\Phi(x)$–subgroups are all conjugate by $G$ (i.e., their types are transitively permuted by the Weyl group $W$).

3. The (polynomial) index of the normalizer in $G$ of a Sylow $\Phi(x)$–subgroup is congruent to 1 modulo $\Phi(x)$. 

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**Example**

For each $d$ ($1 \leq d \leq n$), $\text{GL}_n(q)$ contains a subtorus of order $\Phi_d(x)^{\left\lceil \frac{n}{d} \right\rceil}$.
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For each $d$ ($1 \leq d \leq n$), $\text{GL}_n(q)$ contains a subtorus of order $\Phi_d(x)^{\left[\frac{n}{d}\right]}$.

Assume $n = md + r$ with $r < d$. Then a minimal $d$–split Levi subgroup has shape $\text{GL}_1(q^d)^m \times \text{GL}_r(q)$. 
GENERIC AND ORDINARY SYLOW SUBGROUPS

Let $\ell$ be a prime number which does not divide $|W|$. If $\ell$ divides $|G|=G(q)$, there is a unique integer $d$ such that $\ell$ divides $\Phi_d(q)$. Then the Sylow $\ell$–subgroups of $G$ are nothing but the Sylow $\ell$–subgroups $S_\ell$ of $S=F(Sa)$, where $S$ is a Sylow $\Phi_d(x)$–subgroup of $G$. We have $N_G(S_\ell)=N_G(S)$ and $C_G(S_\ell)=C_G(S)$. 

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- We have $N_G(S_\ell) = N_G(S)$ and $C_G(S_\ell) = C_G(S)$. 
Let $L$ (or $L'$, or $L''$) be a minimal $d$–split Levi subgroup, the centralizer of a Sylow $\Phi_d(x)$–subgroup $S$.

(1) We have $N_G(L)/L \cong N_G(S)/C_G(S) \cong N_W(L)/W'$ (where $W'$ is the Weyl group of $L$).

Denote that group by $W_G(L)$.

(2) For $\zeta$ a primitive $d$–th root of the unity, we have $|W_G(L)| = G(\zeta)/L(\zeta)$.

Example

For $n = mr + d$ ($d < r$), we have $W_G(L) \cong C_d \wr S_r$.
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Springer and Springer–Lehrer theorem — The group $W_G(L)$ is a complex reflection group (in its representation over the complex vector space $\mathbb{C} \otimes X((\mathbb{Z}L)_{\phi_d}))$. 

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**Example**

For $n = mr + d$ ($d < r$), we have $W_G(\mathbb{L}) \cong C_d \wr \mathfrak{S}_r$.

The group $W_G(\mathbb{L})$ is called the $d$–cyclotomic Weyl group. If $G$ is split, the 1–cyclotomic Weyl group is nothing but the ordinary Weyl group $W$. 