Complex reflection groups in representations of finite reductive groups

Michel Broué

Institut Henri–Poincaré

January 2008

FINITE COMPLEX REFLECTION GROUPS

FINITE COMPLEX REFLECTION GROUPS

Let K be a characteristic zero field.

FINITE COMPLEX REFLECTION GROUPS

Let K be a characteristic zero field.

A finite reflection group on K is a finite subgroup of $\mathrm{GL}_K(V)$ (V a finite dimensional K–vector space) generated by *reflections*, *i.e.*, linear maps represented by

$$\begin{pmatrix} \zeta & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

FINITE COMPLEX REFLECTION GROUPS

Let K be a characteristic zero field.

A finite reflection group on K is a finite subgroup of $\mathrm{GL}_K(V)$ (V a finite dimensional K–vector space) generated by *reflections*, *i.e.*, linear maps represented by

$$\begin{pmatrix} \zeta & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- A finite reflection group on $\mathbb{R}$ is called a Coxeter group.

FINITE COMPLEX REFLECTION GROUPS

Let K be a characteristic zero field.

A finite reflection group on K is a finite subgroup of $\mathrm{GL}_K(V)$ (V a finite dimensional K–vector space) generated by *reflections*, *i.e.*, linear maps represented by

$$\begin{pmatrix} \zeta & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

- A finite reflection group on $\mathbb{R}$ is called a Coxeter group.
- A finite reflection group on $\mathbb{Q}$ is called a Weyl group.

Main characterisation

Main characterisation

Theorem (Shephard–Todd, Chevalley–Serre)

Let G be a finite subgroup of $\mathrm{GL}(V)$ (V an r–dimensional vector space over a characteristic zero field K). Let $S(V)$ denote the symmetric algebra of V, isomorphic to the polynomial ring $K[X_1, X_2, \ldots, X_r]$.
The following assertions are equivalent.

Main characterisation

Theorem (Shephard–Todd, Chevalley–Serre)

Let G be a finite subgroup of $\mathrm{GL}(V)$ (V an r–dimensional vector space over a characteristic zero field K). Let $S(V)$ denote the symmetric algebra of V, isomorphic to the polynomial ring $K[X_1, X_2, \dots, X_r]$.
The following assertions are equivalent.

1. G is generated by reflections.

Main characterisation

Theorem (Shephard–Todd, Chevalley–Serre)

Let G be a finite subgroup of $\mathrm{GL}(V)$ (V an r–dimensional vector space over a characteristic zero field K). Let $S(V)$ denote the symmetric algebra of V, isomorphic to the polynomial ring $K[X_1, X_2, \ldots, X_r]$.
The following assertions are equivalent.

1. G is generated by reflections.
2. The ring $S(V)^G$ of G–fixed polynomials is a polynomial ring $K[E_1, E_2, \ldots, E_r]$ in r homogeneous algebraically independant elements.

Main characterisation

Theorem (Shephard–Todd, Chevalley–Serre)

Let G be a finite subgroup of $\mathrm{GL}(V)$ (V an r–dimensional vector space over a characteristic zero field K). Let $S(V)$ denote the symmetric algebra of V, isomorphic to the polynomial ring $K[X_1, X_2, \ldots, X_r]$.
The following assertions are equivalent.

1. G is generated by reflections.
2. The ring $S(V)^G$ of G–fixed polynomials is a polynomial ring $K[E_1, E_2, \ldots, E_r]$ in r homogeneous algebraically independant elements.

Example

For $G = \mathfrak{S}_r$, one may choose

$$\begin{cases} E_1 = X_1 + \cdots + X_r \\ E_2 = X_1X_2 + X_1X_3 + \cdots + X_{r-1}X_r \\ \vdots \qquad \vdots \\ E_r = X_1X_2 \cdots X_r \end{cases}$$

Classification

Classification

1. The finite reflection groups on $\mathbb{C}$ have been classified by Coxeter, Shephard and Todd.

Classification

1. The finite reflection groups on $\mathbb{C}$ have been classified by Coxeter, Shephard and Todd.
   - There is one infinite series $G(de, e, r)$ (d ,e and r integers),

Classification

1. The finite reflection groups on $\mathbb{C}$ have been classified by Coxeter, Shephard and Todd.
   - There is one infinite series $G(de, e, r)$ (d ,e and r integers),
   - ...and 34 exceptional groups G_4 , G_5 , . . . , G_{37}.

Classification

1. The finite reflection groups on $\mathbb{C}$ have been classified by Coxeter, Shephard and Todd.
    - There is one infinite series $G(de, e, r)$ (d ,e and r integers),
    - ...and 34 exceptional groups G_4 , G_5 , . . . , G_{37}.
2. The group $G(de, e, r)$ (d ,e and r integers) consists of all $r \times r$ monomial matrices with entries in μ_{de} such that the product of entries belongs to μ_d.

Classification

1. The finite reflection groups on $\mathbb{C}$ have been classified by Coxeter, Shephard and Todd.
   - There is one infinite series $G(de, e, r)$ (d ,e and r integers),
   - ...and 34 exceptional groups G_4 , G_5 , . . . , G_{37}.
2. The group $G(de, e, r)$ (d ,e and r integers) consists of all $r \times r$ monomial matrices with entries in μ_{de} such that the product of entries belongs to μ_d.
3. We have

Classification

1. The finite reflection groups on $\mathbb{C}$ have been classified by Coxeter, Shephard and Todd.
    - There is one infinite series $G(de, e, r)$ (d ,e and r integers),
    - ...and 34 exceptional groups G_4 , G_5 , ... , G_{37}.
2. The group $G(de, e, r)$ (d ,e and r integers) consists of all $r \times r$ monomial matrices with entries in μ_{de} such that the product of entries belongs to μ_d.
3. We have

$$G(d, 1, r) \simeq C_d \wr \mathfrak{S}_r$$

Classification

1. The finite reflection groups on $\mathbb{C}$ have been classified by Coxeter, Shephard and Todd.
   - There is one infinite series $G(de, e, r)$ (d ,e and r integers),
   - ...and 34 exceptional groups G_4 , G_5 , . . . , G_{37}.
2. The group $G(de, e, r)$ (d ,e and r integers) consists of all $r \times r$ monomial matrices with entries in μ_{de} such that the product of entries belongs to μ_d.
3. We have

$$G(d, 1, r) \simeq C_d \wr \mathfrak{S}_r$$
$$G(e, e, 2) = D_{2e} \quad \text{(dihedral group of order } 2e\text{)}$$

Classification

1. The finite reflection groups on $\mathbb{C}$ have been classified by Coxeter, Shephard and Todd.
   - There is one infinite series $G(de, e, r)$ (d ,e and r integers),
   - ...and 34 exceptional groups G_4 , G_5 , ..., G_{37}.
2. The group $G(de, e, r)$ (d ,e and r integers) consists of all $r \times r$ monomial matrices with entries in μ_{de} such that the product of entries belongs to μ_d.
3. We have

$$
\begin{aligned}
&G(d,1,r) \simeq C_d \wr \mathfrak{S}_r \\
&G(e,e,2) = D_{2e} \quad \text{(dihedral group of order } 2e\text{)} \\
&G(2,2,r) = W(\mathrm{D}_r) \\
&G_{23} = H_3 \ , \ G_{28} = F_4 \ , \ G_{30} = H_4 \\
&G_{35,36,37} = E_{6,7,8} \ .
\end{aligned}
$$

FINITE REDUCTIVE GROUPS : POLYNOMIAL ORDER

FINITE REDUCTIVE GROUPS : POLYNOMIAL ORDER

$\mathbf{G}$ is a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, with Weyl group W, endowed with a Frobenius–like endomorphism F. The group $G := \mathbf{G}^F$ is a finite reductive group.

FINITE REDUCTIVE GROUPS : POLYNOMIAL ORDER

$\mathbf{G}$ is a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, with Weyl group W, endowed with a Frobenius–like endomorphism F. The group $G := \mathbf{G}^F$ is a finite reductive group.

Example

$$\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_q)\ ,\ F\ :\ (a_{i,j}) \mapsto (a_{i,j}^q)\ ,\ G = \mathrm{GL}_n(q)$$

FINITE REDUCTIVE GROUPS : POLYNOMIAL ORDER

$\mathbf{G}$ is a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, with Weyl group W, endowed with a Frobenius–like endomorphism F. The group $G := \mathbf{G}^F$ is a finite reductive group.

Example

$$\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_q) \ , \ F \ : \ (a_{i,j}) \mapsto (a_{i,j}^q) \ , \ G = \mathrm{GL}_n(q)$$

- Type of $\mathbf{G}$ — The type $\mathbb{G} = (X, Y, R, R^\vee \ ; \ W\phi)$ of G consists of the root datum of $\mathbf{G}$ endowed with the outer automorphism $W\phi$ defined by F.

FINITE REDUCTIVE GROUPS : POLYNOMIAL ORDER

$\mathbf{G}$ is a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, with Weyl group W, endowed with a Frobenius–like endomorphism F. The group $G := \mathbf{G}^F$ is a finite reductive group.

Example

$$\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_q)\ ,\ F\ :\ (a_{i,j}) \mapsto (a_{i,j}^q)\ ,\ G = \mathrm{GL}_n(q)$$

- Type of $\mathbf{G}$ — The type $\mathbb{G} = (X, Y, R, R^\vee\ ;\ W\phi)$ of G consists of the root datum of $\mathbf{G}$ endowed with the outer automorphism $W\phi$ defined by F.

Examples

FINITE REDUCTIVE GROUPS : POLYNOMIAL ORDER

$\mathbf{G}$ is a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, with Weyl group W, endowed with a Frobenius–like endomorphism F. The group $G := \mathbf{G}^F$ is a finite reductive group.

Example

$$\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_q)\ ,\ F\ :\ (a_{i,j}) \mapsto (a_{i,j}^q)\ ,\ G = \mathrm{GL}_n(q)$$

- Type of $\mathbf{G}$ — The type $\mathbb{G} = (X, Y, R, R^\vee\ ;\ W\phi)$ of G consists of the root datum of $\mathbf{G}$ endowed with the outer automorphism $W\phi$ defined by F.

Examples

$$\mathrm{GL}_n = (X = Y = \mathbb{Z}^n, R = R^\vee = A_n\ ;\ \phi = 1)$$

FINITE REDUCTIVE GROUPS : POLYNOMIAL ORDER

G is a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, with Weyl group W, endowed with a Frobenius–like endomorphism F. The group $G := \mathbf{G}^F$ is a finite reductive group.

Example

$$\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_q)\ ,\ F\ :\ (a_{i,j}) \mapsto (a_{i,j}^q)\ ,\ G = \mathrm{GL}_n(q)$$

- Type of **G** — The type $\mathbb{G} = (X, Y, R, R^\vee\ ;\ W\phi)$ of G consists of the root datum of **G** endowed with the outer automorphism $W\phi$ defined by F.

Examples

$$\mathrm{GL}_n = (X = Y = \mathbb{Z}^n, R = R^\vee = A_n\ ;\ \phi = 1)$$

$$\mathrm{U}_n = (X = Y = \mathbb{Z}^n, R = R^\vee = A_n\ ;\ \phi = -1)$$

- Polynomial order — There is a polynomial in $\mathbb{Z}[x]$

$$|\mathbb{G}|(x) = \frac{\varepsilon_{\mathbb{G}} x^N}{\frac{1}{|W|}\sum_{w\in W} \frac{1}{\det_V(1 - xw\phi)}} = x^N \prod_d \Phi_d(x)^{a(d)}$$

such that $|\mathbb{G}|(q) = |G|$.

- Polynomial order — There is a polynomial in $\mathbb{Z}[x]$

$$|\mathbb{G}|(x) = \frac{\varepsilon_{\mathbb{G}} x^N}{\frac{1}{|W|}\sum_{w\in W} \frac{1}{\det_V(1 - xw\phi)}} = x^N \prod_d \Phi_d(x)^{a(d)}$$

such that $|\mathbb{G}|(q) = |G|$.

Example

- **Polynomial order** — There is a polynomial in $\mathbb{Z}[x]$

$$|\mathbb{G}|(x) = \frac{\varepsilon_{\mathbb{G}} x^N}{\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det_V(1 - xw\phi)}} = x^N \prod_d \Phi_d(x)^{a(d)}$$

such that $|\mathbb{G}|(q) = |G|$.

Example

$$|\mathrm{GL}_n|(x) = x^{\binom{n}{2}} \prod_{d=1}^{d=n}(x^d - 1) = x^{\binom{n}{2}} \prod_{d=1}^{d=n} \Phi_d(x)^{[n/d]}$$

- Polynomial order — There is a polynomial in $\mathbb{Z}[x]$

$$|\mathbb{G}|(x) = \frac{\varepsilon_{\mathbb{G}} x^N}{\frac{1}{|W|} \sum_{w \in W} \frac{1}{\det_V(1 - xw\phi)}} = x^N \prod_d \Phi_d(x)^{a(d)}$$

such that $|\mathbb{G}|(q) = |G|$.

Example

$$|\mathrm{GL}_n|(x) = x^{\binom{n}{2}} \prod_{d=1}^{d=n} (x^d - 1) = x^{\binom{n}{2}} \prod_{d=1}^{d=n} \Phi_d(x)^{[n/d]}$$

$$|\mathrm{GL}_n|(q) = |GL_n(q)|$$

- Polynomial order — There is a polynomial in $\mathbb{Z}[x]$

$$|\mathbb{G}|(x) = \frac{\varepsilon_{\mathbb{G}} x^N}{\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det_V(1-xw\phi)}} = x^N \prod_d \Phi_d(x)^{a(d)}$$

such that $|\mathbb{G}|(q) = |G|$.

Example

$$|\mathrm{GL}_n|(x) = x^{\binom{n}{2}} \prod_{d=1}^{d=n}(x^d - 1) = x^{\binom{n}{2}} \prod_{d=1}^{d=n} \Phi_d(x)^{[n/d]}$$

$$|\mathrm{GL}_n|(q) = |GL_n(q)| \quad \text{and} \quad |\mathrm{GL}_n|(-q) = \pm|\mathrm{U}_n(q)|$$

- Admissible subgroups — The tori of G are the subgroups of the shape $\mathbf{T}^F$ where $\mathbf{T}$ is an F–stable torus (*i.e.*, isomorphic to some $\bar{\mathbb{F}}^\times \times \cdots \times \bar{\mathbb{F}}^\times$ in $\mathbf{G}$).

  The Levi subgroups of G are the subgroups of the shape $\mathbf{L}^F$ where $\mathbf{L}$ is a centralizer of an F–stable torus in $\mathbf{G}$.

- Admissible subgroups — The tori of G are the subgroups of the shape $\mathbf{T}^F$ where $\mathbf{T}$ is an F–stable torus (*i.e.*, isomorphic to some $\bar{\mathbb{F}}^\times \times \cdots \times \bar{\mathbb{F}}^\times$ in $\mathbf{G}$).

  The Levi subgroups of G are the subgroups of the shape $\mathbf{L}^F$ where $\mathbf{L}$ is a centralizer of an F–stable torus in $\mathbf{G}$.

  Examples

- Admissible subgroups — The tori of G are the subgroups of the shape $\mathbf{T}^F$ where $\mathbf{T}$ is an F–stable torus (*i.e.*, isomorphic to some $\bar{\mathbb{F}}^\times \times \cdots \times \bar{\mathbb{F}}^\times$ in $\mathbf{G}$).

  The Levi subgroups of G are the subgroups of the shape $\mathbf{L}^F$ where $\mathbf{L}$ is a centralizer of an F–stable torus in $\mathbf{G}$.

  Examples

  The split maximal torus $T_1 = (\mathbb{F}_q^\times)^n$ of order $(q-1)^n$

- Admissible subgroups — The tori of G are the subgroups of the shape $\mathbf{T}^F$ where $\mathbf{T}$ is an F–stable torus (*i.e.*, isomorphic to some $\bar{\mathbb{F}}^\times \times \cdots \times \bar{\mathbb{F}}^\times$ in $\mathbf{G}$).

  The Levi subgroups of G are the subgroups of the shape $\mathbf{L}^F$ where $\mathbf{L}$ is a centralizer of an F–stable torus in $\mathbf{G}$.

  Examples

  The split maximal torus $T_1 = (\mathbb{F}_q^\times)^n$ of order $(q-1)^n$

  The Coxeter maximal torus $T_c = \mathrm{GL}_1(\mathbb{F}_{q^n})$ of order $q^n - 1$

- Admissible subgroups — The tori of G are the subgroups of the shape $\mathbf{T}^F$ where $\mathbf{T}$ is an F–stable torus (*i.e.*, isomorphic to some $\bar{\mathbb{F}}^\times \times \cdots \times \bar{\mathbb{F}}^\times$ in $\mathbf{G}$).

  The Levi subgroups of G are the subgroups of the shape $\mathbf{L}^F$ where $\mathbf{L}$ is a centralizer of an F–stable torus in $\mathbf{G}$.

  Examples

  The split maximal torus $T_1 = (\mathbb{F}_q^\times)^n$ of order $(q-1)^n$

  The Coxeter maximal torus $T_c = \mathrm{GL}_1(\mathbb{F}_{q^n})$ of order $q^n - 1$

  Levi subgroups have shape $\mathrm{GL}_{n_1}(q^{a_1}) \times \cdots \times \mathrm{GL}_{n_s}(q^{a_s})$

- Admissible subgroups — The tori of G are the subgroups of the shape $\mathbf{T}^F$ where $\mathbf{T}$ is an F–stable torus (*i.e.*, isomorphic to some $\bar{\mathbb{F}}^\times \times \cdots \times \bar{\mathbb{F}}^\times$ in $\mathbf{G}$).

  The Levi subgroups of G are the subgroups of the shape $\mathbf{L}^F$ where $\mathbf{L}$ is a centralizer of an F–stable torus in $\mathbf{G}$.

  Examples

  The split maximal torus $T_1 = (\mathbb{F}_q^\times)^n$ of order $(q-1)^n$

  The Coxeter maximal torus $T_c = \mathrm{GL}_1(\mathbb{F}_{q^n})$ of order $q^n - 1$

  Levi subgroups have shape $\mathrm{GL}_{n_1}(q^{a_1}) \times \cdots \times \mathrm{GL}_{n_s}(q^{a_s})$

Cauchy theorem

The (polynomial) order of an admissible subgroup divides the (polynomial) order of the group.

Levi subgroups and type — Let $\mathbb{G} = (X, Y, R, R^\vee\,;\, W\phi)$ be a type.

Levi subgroups and type — Let $\mathbb{G} = (X, Y, R, R^\vee\,;\, W\phi)$ be a type.

A Levi subtype of $\mathbb{G}$ is a type of the shape

$$\mathbb{L} = (X, Y, R', R'^\vee\,;\, W'w\phi)$$

where

Levi subgroups and type — Let $\mathbb{G} = (X, Y, R, R^\vee\, ;\ W\phi)$ be a type.

A Levi subtype of $\mathbb{G}$ is a type of the shape

$$\mathbb{L} = (X, Y, R', R'^\vee\, ;\ W'w\phi)$$

where

- R' is a parabolic system of R, with Weyl group W',

Levi subgroups and type — Let $\mathbb{G} = (X, Y, R, R^\vee\,;\ W\phi)$ be a type.

A **Levi subtype of $\mathbb{G}$** is a type of the shape

$$\mathbb{L} = (X, Y, R', R'^\vee\,;\ W'w\phi)$$

where

- R' is a parabolic system of R, with Weyl group W',
- $w \in W$ is such that $w\phi$ stablizes R' and $R'^\vee$.

Levi subgroups and type — Let $\mathbb{G} = (X, Y, R, R^\vee\,;\, W\phi)$ be a type.

A Levi subtype of $\mathbb{G}$ is a type of the shape

$$\mathbb{L} = (X, Y, R', R'^\vee\,;\, W'w\phi)$$

where

- R' is a parabolic system of R, with Weyl group W',
- $w \in W$ is such that $w\phi$ stablizes R' and $R'^\vee$.

There is a natural bijection between

Levi subgroups and type — Let $\mathbb{G} = (X, Y, R, R^\vee\,;\, W\phi)$ be a type.

A **Levi subtype of $\mathbb{G}$** is a type of the shape

$$\mathbb{L} = (X, Y, R', R'^\vee\,;\, W'w\phi)$$

where

- R' is a parabolic system of R, with Weyl group W',
- $w \in W$ is such that $w\phi$ stablizes R' and $R'^\vee$.

There is a natural bijection between

- the set of G–conjugacy classes of Levi subgroups of G,

Levi subgroups and type — Let $\mathbb{G} = (X, Y, R, R^\vee\, ;\, W\phi)$ be a type.

A **Levi subtype of $\mathbb{G}$** is a type of the shape

$$\mathbb{L} = (X, Y, R', R'^\vee\, ;\, W'w\phi)$$

where

- R' is a parabolic system of R, with Weyl group W',
- $w \in W$ is such that $w\phi$ stablizes R' and $R'^\vee$.

There is a natural bijection between

- the set of G–conjugacy classes of Levi subgroups of G, and

Levi subgroups and type — Let $\mathbb{G} = (X, Y, R, R^\vee\,;\, W\phi)$ be a type.

A Levi subtype of $\mathbb{G}$ is a type of the shape

$$\mathbb{L} = (X, Y, R', R'^\vee\,;\, W'w\phi)$$

where

- R' is a parabolic system of R, with Weyl group W',
- $w \in W$ is such that $w\phi$ stablizes R' and $R'^\vee$.

There is a natural bijection between

- the set of G–conjugacy classes of Levi subgroups of G, and
- the set of W–conjugacy classes of Levi subtypes of $\mathbb{G}$.

FINITE REDUCTIVE GROUPS : THE SYLOW THEOREMS

FINITE REDUCTIVE GROUPS : THE SYLOW THEOREMS

For $\Phi(x)$ a cyclotomic polynomial, a $\Phi(x)$–group is a finite reductive group whose (polynomial) order is a power of $\Phi(x)$. Hence such a group is a torus.

FINITE REDUCTIVE GROUPS : THE SYLOW THEOREMS

For $\Phi(x)$ a cyclotomic polynomial, a $\Phi(x)$–group is a finite reductive group whose (polynomial) order is a power of $\Phi(x)$. Hence such a group is a torus.

Sylow theorem

FINITE REDUCTIVE GROUPS : THE SYLOW THEOREMS

For $\Phi(x)$ a cyclotomic polynomial, a $\Phi(x)$–group is a finite reductive group whose (polynomial) order is a power of $\Phi(x)$. Hence such a group is a torus.

Sylow theorem

1. Maximal $\Phi(x)$–subgroups ("Sylow $\Phi(x)$–subgroups") of G have as (polynomial) order the contribution of $\Phi(x)$ to the (polynomial) order of G.

FINITE REDUCTIVE GROUPS : THE SYLOW THEOREMS

For $\Phi(x)$ a cyclotomic polynomial, a $\Phi(x)$–group is a finite reductive group whose (polynomial) order is a power of $\Phi(x)$. Hence such a group is a torus.

Sylow theorem

1. Maximal $\Phi(x)$–subgroups ("Sylow $\Phi(x)$–subgroups") of G have as (polynomial) order the contribution of $\Phi(x)$ to the (polynomial) order of G.
2. Sylow $\Phi(x)$–subgroups are all conjugate by G (*i.e.*, their types are transitively permuted by the Weyl group W).

FINITE REDUCTIVE GROUPS : THE SYLOW THEOREMS

For $\Phi(x)$ a cyclotomic polynomial, a $\Phi(x)$–group is a finite reductive group whose (polynomial) order is a power of $\Phi(x)$. Hence such a group is a torus.

Sylow theorem

1. Maximal $\Phi(x)$–subgroups ("Sylow $\Phi(x)$–subgroups") of G have as (polynomial) order the contribution of $\Phi(x)$ to the (polynomial) order of G.
2. Sylow $\Phi(x)$–subgroups are all conjugate by G (*i.e.*, their types are transitively permuted by the Weyl group W).
3. The (polynomial) index of the normalizer in G of a Sylow $\Phi(x)$–subgroup is congruent to 1 modulo $\Phi(x)$.

The centralizers of $\Phi_d(x)$–subgroups are called the *d–split Levi subgroups*.

The centralizers of $\Phi_d(x)$–subgroups are called the *d–split Levi subgroups*.

The *minimal d–split Levi subgroups* are the centralizers of Sylow $\Phi_d(x)$–subgroups. They are all conjugate under G.

The centralizers of $\Phi_d(x)$–subgroups are called the d–split Levi subgroups.

The minimal d–split Levi subgroups are the centralizers of Sylow $\Phi_d(x)$–subgroups. They are all conjugate under G.

Example

The centralizers of $\Phi_d(x)$–subgroups are called the d–split Levi subgroups.

The minimal d–split Levi subgroups are the centralizers of Sylow $\Phi_d(x)$–subgroups. They are all conjugate under G.

Example

For each d $(1 \leq d \leq n)$, $\mathrm{GL}_n(q)$ contains a subtorus of order $\Phi_d(x)^{[\frac{n}{d}]}$

The centralizers of $\Phi_d(x)$–subgroups are called the *d*–split Levi subgroups.

The minimal *d*–split Levi subgroups are the centralizers of Sylow $\Phi_d(x)$–subgroups. They are all conjugate under G.

Example

For each d $(1 \leq d \leq n)$, $\mathrm{GL}_n(q)$ contains a subtorus of order $\Phi_d(x)^{[\frac{n}{d}]}$

Assume $n = md + r$ with $r < d$. Then a minimal d–split Levi subgroup has shape $\mathrm{GL}_1(q^d)^m \times \mathrm{GL}_r(q)$.

GENERIC AND ORDINARY SYLOW SUBGROUPS

GENERIC AND ORDINARY SYLOW SUBGROUPS

Let ℓ be a prime number which does not divide $|W|$.

GENERIC AND ORDINARY SYLOW SUBGROUPS

Let ℓ be a prime number which does not divide $|W|$.

- If ℓ divides $|G| = \mathbb{G}(q)$, there is a unique integer d such that ℓ divides $\Phi_d(q)$.

GENERIC AND ORDINARY SYLOW SUBGROUPS

Let ℓ be a prime number which does not divide $|W|$.

- If ℓ divides $|G| = \mathbb{G}(q)$, there is a unique integer d such that ℓ divides $\Phi_d(q)$.
- Then the Sylow ℓ–subgroups of G are nothing but the Sylow ℓ–subgroups S_ℓ of $S = \mathbf{S}^F$ ($\mathbf{S}$ a Sylow $\Phi_d(x)$–subgroup of $\mathbf{G}$).

GENERIC AND ORDINARY SYLOW SUBGROUPS

Let ℓ be a prime number which does not divide $|W|$.

- If ℓ divides $|G| = \mathbb{G}(q)$, there is a unique integer d such that ℓ divides $\Phi_d(q)$.
- Then the Sylow ℓ–subgroups of G are nothing but the Sylow ℓ–subgroups S_ℓ of $S = \mathbf{S}^F$ ($\mathbf{S}$ a Sylow $\Phi_d(x)$–subgroup of $\mathbf{G}$).
- We have

$$N_G(S_\ell) = N_G(\mathbf{S}) \text{ and } C_G(S_\ell) = C_G(\mathbf{S})\,.$$

THIS IS AN ADVERTISEMENT

THIS IS AN ADVERTISEMENT

- Starting this afternoon (well, actually, it will really start next tuesday)...

THIS IS AN ADVERTISEMENT

- Starting this afternoon (well, actually, it will really start next tuesday)...
- **Tuesday 3.40 pm to 5.00 pm** and **Thursday 3.40 pm to 5.00 pm**

Room 174 Barrows Hall (subject to change)

THIS IS AN ADVERTISEMENT

- Starting this afternoon (well, actually, it will really start next tuesday)...
- **Tuesday 3.40 pm to 5.00 pm** and **Thursday 3.40 pm to 5.00 pm**

Room 174 Barrows Hall (subject to change)

UC Berkeley Graduate Course on

COMPLEX REFLECTION GROUPS AND ASSOCIATED BRAID GROUPS

CYCLOTOMIC WEYL GROUPS AND SPRINGER THEOREM

CYCLOTOMIC WEYL GROUPS AND SPRINGER THEOREM

Let L (or $\mathbf{L}$, or $\mathbb{L}$) be a minimal d–split Levi subgroup, the centralizer of a Sylow $\Phi_d(x)$–subgroup $\mathbf{S}$.

CYCLOTOMIC WEYL GROUPS AND SPRINGER THEOREM

Let L (or $\mathbf{L}$, or $\mathbb{L}$) be a minimal d–split Levi subgroup, the centralizer of a Sylow $\Phi_d(x)$–subgroup $\mathbf{S}$.

- We have

$$N_G(\mathbf{L})/L \simeq N_G(\mathbf{S})/C_G(\mathbf{S}) \simeq N_W(\mathbb{L})/W'$$

(where W' is the Weyl group of $\mathbf{L}$).

Denote that group by $W_{\mathbb{G}}(\mathbb{L})$.

CYCLOTOMIC WEYL GROUPS AND SPRINGER THEOREM

Let L (or $\mathbf{L}$, or $\mathbb{L}$) be a minimal d–split Levi subgroup, the centralizer of a Sylow $\Phi_d(x)$–subgroup $\mathbf{S}$.

- We have

$$N_G(\mathbf{L})/L \simeq N_G(\mathbf{S})/C_G(\mathbf{S}) \simeq N_W(\mathbb{L})/W'$$

(where W' is the Weyl group of $\mathbf{L}$).

Denote that group by $W_{\mathbb{G}}(\mathbb{L})$.

- The "number of Sylow congruence" translates to

For ζ a primitive d-th root of the unity, we have

$$|W_{\mathbb{G}}(\mathbb{L})| = \mathbb{G}(\zeta)/\mathbb{L}(\zeta)\,.$$

The case $d = 1$ — The Sylow $\Phi_1(x)$–subgroups, as well as the minimal d–split subgroups, coincide with the split maximal tori.

The case $d = 1$ — The Sylow $\Phi_1(x)$–subgroups, as well as the minimal d–split subgroups, coincide with the split maximal tori.

In case $\mathbf{G}$ is split (*i.e.*, the automorphism ϕ induced by F is the identity), then the group $W_{\mathbb{G}}(\mathbb{L})$ coincides with W.

The case $d = 1$ — The Sylow $\Phi_1(x)$–subgroups, as well as the minimal d–split subgroups, coincide with the split maximal tori.

In case $\mathbf{G}$ is split (*i.e.*, the automorphism ϕ induced by F is the identity), then the group $W_{\mathbb{G}}(\mathbb{L})$ coincides with W.

Springer and Springer–Lehrer theorem

The group $W_{\mathbb{G}}(\mathbb{L})$ is a complex reflection group (in its representation over the complex vector space $\mathbb{C} \otimes X((Z\mathbf{L})_{\Phi_d})$).

The case $d = 1$ — The Sylow $\Phi_1(x)$–subgroups, as well as the minimal d–split subgroups, coincide with the split maximal tori.

In case $\mathbf{G}$ is split (*i.e.*, the automorphism ϕ induced by F is the identity), then the group $W_{\mathbb{G}}(\mathbb{L})$ coincides with W.

Springer and Springer–Lehrer theorem

The group $W_{\mathbb{G}}(\mathbb{L})$ is a complex reflection group (in its representation over the complex vector space $\mathbb{C} \otimes X((Z\mathbf{L})_{\Phi_d}))$.

Example

For $n = md + r$ $(r < d)$, we have $W_{\mathbb{G}}(\mathbb{L}) \simeq C_d \wr \mathfrak{S}_m$

The case $d = 1$ — The Sylow $\Phi_1(x)$–subgroups, as well as the minimal d–split subgroups, coincide with the split maximal tori.

In case $\mathbf{G}$ is split (*i.e.*, the automorphism ϕ induced by F is the identity), then the group $W_{\mathbb{G}}(\mathbb{L})$ coincides with W.

Springer and Springer–Lehrer theorem

The group $W_{\mathbb{G}}(\mathbb{L})$ is a complex reflection group (in its representation over the complex vector space $\mathbb{C} \otimes X((Z\mathbf{L})_{\Phi_d}))$.

Example

For $n = md + r$ $(r < d)$, we have $W_{\mathbb{G}}(\mathbb{L}) \simeq C_d \wr \mathfrak{S}_m$

The group $W_{\mathbb{G}}(\mathbb{L})$ is called the d–cyclotomic Weyl group.

If G is split, the 1–cyclotomic Weyl group is nothing but the ordinary Weyl group W.

UNIPOTENT CHARACTERS

UNIPOTENT CHARACTERS

Generic degree –

UNIPOTENT CHARACTERS

Generic degree –

- The set $\mathrm{Un}(G)$ of unipotent characters of G is naturally parametrized by a "generic" (*i.e.*, independant of q) set $\mathrm{Un}(\mathbb{G})$. We denote by $\mathrm{Un}(\mathbb{G}) \longrightarrow \mathrm{Un}(G)\ ,\ \gamma \mapsto \gamma_q$ that parametrization.

UNIPOTENT CHARACTERS

Generic degree –

- The set $\mathrm{Un}(G)$ of unipotent characters of G is naturally parametrized by a "generic" (*i.e.*, independant of q) set $\mathrm{Un}(\mathbb{G})$. We denote by $\mathrm{Un}(\mathbb{G}) \longrightarrow \mathrm{Un}(G)\ ,\ \gamma \mapsto \gamma_q$ that parametrization.

  Example for GL_n : $\mathrm{Un}(\mathrm{GL}_n)$ is the set of all partitions of n.

UNIPOTENT CHARACTERS

Generic degree –

- The set $\mathrm{Un}(G)$ of unipotent characters of G is naturally parametrized by a "generic" (*i.e.*, independant of q) set $\mathrm{Un}(\mathbb{G})$. We denote by $\mathrm{Un}(\mathbb{G}) \longrightarrow \mathrm{Un}(G)\ ,\ \gamma \mapsto \gamma_q$ that parametrization.

  Example for GL_n : $\mathrm{Un}(\mathrm{GL}_n)$ is the set of all partitions of n.

- Generic degree : For $\gamma \in \mathrm{Un}(\mathbb{G})$ there is $\mathrm{Deg}_\gamma(x) \in \mathbb{Q}[x]$ such that

$$\mathrm{Deg}_\gamma(x)_{|x=q} = \gamma_q(1)\,.$$

Example for GL_n :

Example for GL_n : For $\lambda = (\lambda_1 \leq \cdots \leq \lambda_m)$ a partition of n, we define

$$\beta_i := \lambda_i + i - 1 .$$

Example for GL_n : For $\lambda = (\lambda_1 \leq \cdots \leq \lambda_m)$ a partition of n, we define

$$\beta_i := \lambda_i + i - 1 \,.$$

Then

$$\mathrm{Deg}_\lambda(x) = \frac{(x-1)\cdots(x^n-1)\prod_{j>i}(x^{\beta_j} - x^{\beta_i})}{x^{\binom{m-1}{2}+\binom{m-2}{2}+\cdots}\prod_i \prod_{j=1}^{\beta_i}(x^j-1)}$$

Example for GL_n : For $\lambda = (\lambda_1 \leq \cdots \leq \lambda_m)$ a partition of n, we define

$$\beta_i := \lambda_i + i - 1 \,.$$

Then

$$\mathrm{Deg}_\lambda(x) = \frac{(x-1)\cdots(x^n-1)\prod_{j>i}(x^{\beta_j} - x^{\beta_i})}{x^{\binom{m-1}{2}+\binom{m-2}{2}+\cdots}\prod_i \prod_{j=1}^{\beta_i}(x^j-1)}$$

- The (polynomial) degree $\mathrm{Deg}_\gamma(x)$ of a unipotent character divides the (polynomial) order $|\mathbb{G}|(x)$ of G.

Example for GL_n : For $\lambda = (\lambda_1 \leq \cdots \leq \lambda_m)$ a partition of n, we define

$$\beta_i := \lambda_i + i - 1\,.$$

Then

$$\mathrm{Deg}_\lambda(x) = \frac{(x-1)\cdots(x^n-1)\prod_{j>i}(x^{\beta_j} - x^{\beta_i})}{x^{\binom{m-1}{2}+\binom{m-2}{2}+\cdots}\prod_i \prod_{j=1}^{\beta_i}(x^j-1)}$$

- The (polynomial) degree $\mathrm{Deg}_\gamma(x)$ of a unipotent character divides the (polynomial) order $|\mathbb{G}|(x)$ of G.

  Note. The polynomial $\dfrac{|\mathbb{G}|(x)}{\mathrm{Deg}_\gamma(x)}$ belongs to $\mathbb{Z}[x]$ and is called the (generic) Schur element of γ.

Deligne–Lusztig induction and restriction –

Deligne–Lusztig induction and restriction –

- Deligne and Lusztig have defined adjoint linear maps

$$R_L^G : \mathbb{Z}\mathrm{Irr}(L) \longrightarrow \mathbb{Z}\mathrm{Irr}(G) \quad \text{and} \quad {}^*R_L^G : \mathbb{Z}\mathrm{Irr}(G) \longrightarrow \mathbb{Z}\mathrm{Irr}(L)\,.$$

Deligne–Lusztig induction and restriction –

- Deligne and Lusztig have defined adjoint linear maps

$$R_L^G : \mathbb{Z}\mathrm{Irr}(L) \longrightarrow \mathbb{Z}\mathrm{Irr}(G) \quad \text{and} \quad {}^*R_L^G : \mathbb{Z}\mathrm{Irr}(G) \longrightarrow \mathbb{Z}\mathrm{Irr}(L)\,.$$

- These maps are generic :

Deligne–Lusztig induction and restriction –

- Deligne and Lusztig have defined adjoint linear maps

$$R_L^G : \mathbb{Z}\mathrm{Irr}(L) \longrightarrow \mathbb{Z}\mathrm{Irr}(G) \quad \text{and} \quad {}^*R_L^G : \mathbb{Z}\mathrm{Irr}(G) \longrightarrow \mathbb{Z}\mathrm{Irr}(L)\,.$$

- These maps are generic :

Theorem

For any generic Levi subgroup $\mathbb{L}$ of $\mathbb{G}$, there exist adjoint linear maps

$$R_{\mathbb{L}}^{\mathbb{G}} : \mathbb{Z}\mathrm{Un}(\mathbb{L}) \longrightarrow \mathbb{Z}\mathrm{Un}(\mathbb{G}) \quad \text{and} \quad {}^*R_{\mathbb{L}}^{\mathbb{G}} : \mathbb{Z}\mathrm{Un}(\mathbb{G}) \longrightarrow \mathbb{Z}\mathrm{Un}(\mathbb{L})\,.$$

which specialize to Deligne–Lusztig maps for $x = q$.

d–Harish–Chandra theories –

d–Harish–Chandra theories –

- Let $\mathcal{S}_d(\mathbb{G})$ denote the set of all pairs $(\mathbb{M}, \mu)$ where

d–Harish–Chandra theories –

- Let $\mathcal{S}_d(\mathbb{G})$ denote the set of all pairs $(\mathbb{M}, \mu)$ where
  - $\mathbb{M}$ is a *d*–split Levi subtype of $\mathbb{G}$,

d–Harish–Chandra theories –

- Let $\mathcal{S}_d(\mathbb{G})$ denote the set of all pairs $(\mathbb{M}, \mu)$ where
  - $\mathbb{M}$ is a d–split Levi subtype of $\mathbb{G}$,
  - $\mu \in \mathrm{Un}(\mathbb{M})$.

d–Harish–Chandra theories –

- Let $\mathcal{S}_d(\mathbb{G})$ denote the set of all pairs $(\mathbb{M}, \mu)$ where
  - $\mathbb{M}$ is a *d*–split Levi subtype of $\mathbb{G}$,
  - $\mu \in \mathrm{Un}(\mathbb{M})$.

  The elements of $\mathcal{S}_d(\mathbb{G})$ are called *d*–split pairs.

d–Harish–Chandra theories –

- Let $\mathcal{S}_d(\mathbb{G})$ denote the set of all pairs $(\mathbb{M}, \mu)$ where
  - $\mathbb{M}$ is a d–split Levi subtype of $\mathbb{G}$,
  - $\mu \in \mathrm{Un}(\mathbb{M})$.

  The elements of $\mathcal{S}_d(\mathbb{G})$ are called d–split pairs.

- A binary relation on $\mathcal{S}_d(\mathbb{G})$ –

d–Harish–Chandra theories –

- Let $\mathcal{S}_d(\mathbb{G})$ denote the set of all pairs $(\mathbb{M}, \mu)$ where
  - $\mathbb{M}$ is a d–split Levi subtype of $\mathbb{G}$,
  - $\mu \in \mathrm{Un}(\mathbb{M})$.

  The elements of $\mathcal{S}_d(\mathbb{G})$ are called d–split pairs.

- A binary relation on $\mathcal{S}_d(\mathbb{G})$ –

Definition :

$$(\mathbb{M}_1, \mu_1) \leq (\mathbb{M}_2, \mu_2)$$

if and only if μ_2 occurs in $R_{\mathbb{M}_1}^{\mathbb{M}_2}(\mu_1)$.

First fundamental theorem

First fundamental theorem

1. The relation $\leq$ is an order relation on $\mathcal{S}_d(\mathbb{G})$.

First fundamental theorem

1. The relation $\leq$ is an order relation on $\mathcal{S}_d(\mathbb{G})$.
2. The minimal d–split pairs contained in a pair $(\mathbb{G}, \gamma)$ are all conjugate under the Weyl group W.

First fundamental theorem

1. The relation $\leq$ is an order relation on $\mathcal{S}_d(\mathbb{G})$.
2. The minimal d–split pairs contained in a pair $(\mathbb{G}, \gamma)$ are all conjugate under the Weyl group W.
    - Such minimal pairs are called d–cuspidal.

First fundamental theorem

1. The relation $\leq$ is an order relation on $\mathcal{S}_d(\mathbb{G})$.
2. The minimal d–split pairs contained in a pair $(\mathbb{G}, \gamma)$ are all conjugate under the Weyl group W.
   - Such minimal pairs are called *d–cuspidal*.
   - For $(\mathbb{L}, \lambda)$ d–cuspidal, define

$$\mathsf{Un}(\mathbb{G}, (\mathbb{L}, \lambda)) := \{\gamma \in \mathsf{Un}(\mathbb{G}) \,|\, (\mathbb{L}, \lambda) \leq (\mathbb{G}, \gamma)\}\,.$$

First fundamental theorem

1. The relation $\leq$ is an order relation on $\mathcal{S}_d(\mathbb{G})$.
2. The minimal d–split pairs contained in a pair $(\mathbb{G}, \gamma)$ are all conjugate under the Weyl group W.
   - Such minimal pairs are called d–cuspidal.
   - For $(\mathbb{L}, \lambda)$ d–cuspidal, define

$$\mathrm{Un}(\mathbb{G}, (\mathbb{L}, \lambda)) := \{\gamma \in \mathrm{Un}(\mathbb{G}) \,|\, (\mathbb{L}, \lambda) \leq (\mathbb{G}, \gamma)\}\,.$$

3. The sets $\mathrm{Un}(\mathbb{G}, (\mathbb{L}, \lambda))$, where $(\mathbb{L}, \lambda)$ runs over a system of representatives of the W–conjugacy classes of d–cuspidal pairs, form a partition of $\mathrm{Un}(\mathbb{G})$.

For $(\mathbb{L}, \lambda)$ a d–cuspidal pair, we set

$$W_{\mathbb{G}}(\mathbb{L}, \lambda) := N_W(\mathbb{L}, \lambda)/W_{\mathbb{L}} = N_G(\mathbf{L}, \lambda_q)/L\,.$$

For $(\mathbb{L}, \lambda)$ a d–cuspidal pair, we set

$$W_{\mathbb{G}}(\mathbb{L}, \lambda) := N_W(\mathbb{L}, \lambda)/W_{\mathbb{L}} = N_G(\mathbf{L}, \lambda_q)/L\,.$$

Second fundamental theorem

Whenever $(\mathbb{L}, \lambda)$ is a d–cuspidal pair, the group $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ is (naturally) a complex reflection group.

For $(\mathbb{L}, \lambda)$ a d–cuspidal pair, we set

$$W_{\mathbb{G}}(\mathbb{L}, \lambda) := N_W(\mathbb{L}, \lambda)/W_{\mathbb{L}} = N_G(\mathbf{L}, \lambda_q)/L\,.$$

Second fundamental theorem

Whenever $(\mathbb{L}, \lambda)$ is a d–cuspidal pair, the group $W_{\mathbb{G}}(\mathbb{L}, \lambda)$ is (naturally) a complex reflection group.

In the case where $\mathbb{L}$ is a minimal d–split Levi subtype, and λ is the trivial character, the above theorem specializes onto Springer–Lehrer theorem.

Third fundamental theorem : description of $R_{\mathbb{L}}^{\mathbb{G}}(\lambda)$

Third fundamental theorem : description of $R_{\mathbb{L}}^{\mathbb{G}}(\lambda)$

There exists a collection of isometries

$$I_{(\mathbb{L},\lambda)}^{\mathbb{M}} : \mathbb{Z}\mathrm{Irr}(W_{\mathbb{M}}(\mathbb{L},\lambda) \xrightarrow{\sim} \mathbb{Z}\mathrm{Un}(\mathbb{M},(\mathbb{L},\lambda))\,,$$

such that

Third fundamental theorem : description of $R_{\mathbb{L}}^{\mathbb{G}}(\lambda)$

There exists a collection of isometries

$$I_{(\mathbb{L},\lambda)}^{\mathbb{M}} : \mathbb{Z}\mathrm{Irr}(W_{\mathbb{M}}(\mathbb{L},\lambda) \xrightarrow{\sim} \mathbb{Z}\mathrm{Un}(\mathbb{M},(\mathbb{L},\lambda))\,,$$

such that

1. The following diagram commute :

$$\begin{array}{ccc} \mathbb{Z}\mathrm{Irr}(W_{\mathbb{G}}(\mathbb{L},\lambda) & \xrightarrow{I_{(\mathbb{L},\lambda)}^{\mathbb{G}}} & \mathbb{Z}\mathrm{Un}(\mathbb{G},(\mathbb{L},\lambda)) \\ \uparrow{\scriptstyle \mathrm{Ind}_{W_{\mathbb{M}}(\mathbb{L},\lambda)}^{W_{\mathbb{G}}(\mathbb{L},\lambda)}} & & \uparrow{\scriptstyle R_{\mathbb{M}}^{\mathbb{G}}} \\ \mathbb{Z}\mathrm{Irr}(W_{\mathbb{M}}(\mathbb{L},\lambda) & \xrightarrow{I_{(\mathbb{L},\lambda)}^{\mathbb{M}}} & \mathbb{Z}\mathrm{Un}(\mathbb{M},(\mathbb{L},\lambda)) \end{array}$$

Third fundamental theorem : description of $R_{\mathbb{L}}^{\mathbb{G}}(\lambda)$

There exists a collection of isometries

$$I_{(\mathbb{L},\lambda)}^{\mathbb{M}} : \mathbb{Z}\mathrm{Irr}(W_{\mathbb{M}}(\mathbb{L},\lambda) \xrightarrow{\sim} \mathbb{Z}\mathrm{Un}(\mathbb{M},(\mathbb{L},\lambda))\,,$$

such that

1. The following diagram commute :

$$\begin{array}{ccc} \mathbb{Z}\mathrm{Irr}(W_{\mathbb{G}}(\mathbb{L},\lambda) & \xrightarrow{I_{(\mathbb{L},\lambda)}^{\mathbb{G}}} & \mathbb{Z}\mathrm{Un}(\mathbb{G},(\mathbb{L},\lambda)) \\ {\scriptstyle \mathrm{Ind}_{W_{\mathbb{M}}(\mathbb{L},\lambda)}^{W_{\mathbb{G}}(\mathbb{L},\lambda)}}\Big\uparrow & & \Big\uparrow{\scriptstyle R_{\mathbb{M}}^{\mathbb{G}}} \\ \mathbb{Z}\mathrm{Irr}(W_{\mathbb{M}}(\mathbb{L},\lambda) & \xrightarrow{I_{(\mathbb{L},\lambda)}^{\mathbb{M}}} & \mathbb{Z}\mathrm{Un}(\mathbb{M},(\mathbb{L},\lambda)) \end{array}$$

2. For all $\chi \in \mathrm{Irr}(W_{\mathbb{G}}(\mathbb{L},\lambda))$, let $\gamma_\chi := \varepsilon_\chi I_{(\mathbb{L},\lambda)}^{\mathbb{G}}(\chi)$. Then if ζ is a primitive d-th root of unity, we have

$$Deg_{\gamma_\chi}(\zeta) = \varepsilon_\chi \chi(1)\,.$$