Complex reflection groups and associated braid groups

Michel Broué

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Let \( K \) be a characteristic zero field and let \( V \) be an \( r \)-dimensional \( K \)-vector space. Let \( S \) be the symmetric algebra of \( V \).

Each choice of a basis \((v_1, v_2, \ldots, v_r)\) of \( V \) determines an identification of \( S \) with a polynomial algebra

\[ S \cong K[v_1, v_2, \ldots, v_r]. \]

Let \( G \) be a finite subgroup of \( \text{GL}(V) \). The group \( G \) acts on the algebra \( S \), and we let \( R := S^G \) denote the subalgebra of \( G \)-fixed polynomials.
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$$S \simeq K[v_1, v_2, \ldots, v_r].$$
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In general $R$ is NOT a polynomial algebra,
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free of rank $m|G|$
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free of rank $m|G|$ \quad not free unless...

$$R = S^G \quad \text{not a polynomial algebra unless...}$$

free of rank $m$

$$P = K[u_1, u_2, \ldots, u_r]$$
Moreover,

As a $PG$–module, we have $S \cong (PG)^m$.

Example. Consider $G = \{ (1 0, 0 1), (-1 0, 0 -1) \} \subset GL_2(K)$.

Denote by $(x, y)$ the canonical basis of $V = K^2$.

Then $S = K[x, y]$ not free

$R = S_G = K[x^2, y^2] \oplus K[x^2, y^2]xy$ free of rank 2

$P = K[x^2, y^2]$ free of rank 4

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Unless...

A finite reflection group on $K$ is a finite subgroup of $\text{GL}_K(V)$ ($V$ a finite dimensional $K$–vector space) generated by reflections, i.e., linear maps represented by:

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\begin{pmatrix}
\zeta & \cdots & 0 \\
\vdots & \ddots & \vdots \\
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A finite reflection group on $\mathbb{R}$ is called a Coxeter group.

A finite reflection group on $\mathbb{Q}$ is called a Weyl group.

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Main characterisation

Theorem (Shephard–Todd, Chevalley–Serre)

Let $G$ be a finite subgroup of $\text{GL}(V)$ ($V$ an $r$–dimensional vector space over a characteristic zero field $K$). Let $S$ denote the symmetric algebra of $V$, isomorphic to the polynomial ring $K[v_1, v_2, \ldots, v_r]$.

The following assertions are equivalent.

1. $G$ is generated by reflections.
2. The ring $R := S^G$ of $G$–fixed polynomials is a polynomial ring $K[u_1, u_2, \ldots, u_r]$ in $r$ homogeneous algebraically independent elements.
3. $S$ is a free $R$–module.

In other words, unless $m = 1$, i.e., $R = P$.

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In other words, unless... $m = 1$, i.e., $R = P$. 
\[ S = K[v_1, v_2, \ldots, v_r] \]

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becomes

\[ \text{free of rank } m|G| \]

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\[ R = S^G = P = K[u_1, u_2, \ldots, u_r] \]
Examples

For $G = S$, one may choose

\[
\begin{align*}
    u_1 &= v_1 + \cdots + v_r \\
    u_2 &= v_1 v_2 + v_1 v_3 + \cdots + v_{r-1} v_r \\
    \vdots \\
    u_r &= v_1 v_2 \cdots v_r
\end{align*}
\]

For $G = \langle e^{2\pi i/d} \rangle$, cyclic group of order $d$ acting by multiplication on $V = \mathbb{C}$, we have $S = K[x]$ and $R = K[x^d]$. 
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- For $G = \mathfrak{S}_r$, one may choose

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$$S = K[x] \quad \text{and} \quad R = K[x^d].$$
Classification

The finite reflection groups on $\mathbb{C}$ have been classified by Coxeter, Shephard and Todd. There is one infinite series $G(d,e,r)$ ($d,e$ and $r$ integers),...and 34 exceptional groups $G_{4}, G_{5}, ..., G_{37}$.

The group $G(d,e,r)$ ($d,e$ and $r$ integers) consists of all $r \times r$ monomial matrices with entries in $\mu^{d}$ such that the product of entries belongs to $\mu^{d}$.

We have $G(d,1,r) \cong S_{r}$

$G(2,2,r) = W(D_r)$

$G_{23} = H_{3}$

$G_{28} = F_4$

$G_{30} = H_4$

$G_{35}, 36, 37 = E_6, 7, 8$. 

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$$G(d, 1, r) \cong C_d \wr \mathfrak{S}_r$$
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   G(d, 1, r) \simeq C_d \wr S_r \\
   G(e, e, 2) = D_{2e} \quad \text{(dihedral group of order $2e$)}
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   \[ G(2, 2, r) = W(D_r) \]
   \[ G_{23} = H_3, \quad G_{28} = F_4, \quad G_{30} = H_4 \]
   \[ G_{35,36,37} = E_{6,7,8}. \]
Let $G$ be a finite subgroup of $\text{GL}(V)$.
A reflection $s$ is associated with

$H := \ker(s - 1)$, $L := \text{im}(s - 1)$, a reflecting pair $(H, L)$.

Properties:

$H \oplus L = V$, $H$ determines $L$, and $L$ determines $H$.
Hence, in terms of normalizers,


The fixator $G_H$ (pointwise stabilizer) of $H$ is a cyclic group consisting of reflections with reflecting hyperplane $H$ and reflecting line $L$. 

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Reflecting hyperplanes, lines, pairs

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Notation

\[ A := \{ H \mid H \text{ reflecting hyperplane of some reflection in } G \} \]

For \( H \in A \), \( e_H := |G_H| \)

\( s_H \) is the generator of \( G_H \) whose nontrivial eigenvalue is \( e^{2 \pi i / e_H} \), called a distinguished reflection.

For a line \( L \) in \( V \), the ideal \( q := SL \) of \( S \) is a height one prime ideal.

In other words, the hypersurface of \( V \) defined by \( q \) is a codimension one irreducible variety.

Now the extension \( S_R = S \downarrow G \uparrow \) (corresponding to the covering \( V \downarrow V / G \)) is ramified at \( q = SL \) if and only if \( L \) is a reflecting line.
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For $L$ a line in $V$, the ideal $q := S_L$ of $S$ is a height one prime ideal. In other words, the hypersurface of $V$ defined by $q$ is a codimension one irreducible variety.

Now the extension

\[
R = S^G
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Now the extension $R = S^G$ (corresponding to the covering $V \rightarrow V/G$).
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For $L$ a line in $V$, the ideal $q := SL$ of $S$ is a height one prime ideal. In other words, the hypersurface of $V$ defined by $q$ is a codimension one irreducible variety.

Now the extension $R = S^G$ is ramified at $q = SL$ if and only if $L$ is a reflecting line.
Thus there are $G$-equivariant bijections $A \leftrightarrow \{\text{reflecting lines}\} \leftrightarrow \{\text{ramified height one prime ideals of } S\}$.

**Ramification and parabolic subgroups**

Steinberg Theorem

Assume $G$ generated by reflections.

1. The ramification locus of $V \rightarrow V/G$ is $\bigcup H \in A H$.

2. Let $X$ be a subset of $V$. Then the fixator of $X$ in $G$ is generated by the reflections which fix $X$.

3. The set $\text{Par}(G)$ of fixators ("parabolic subgroups" of $G$) is in (reverse–order) bijection with the set $I(A)$ of intersections of elements of $A$:

$$I(A) \sim \rightarrow \text{Par}(G), X \mapsto G X.$$
Thus there are $G$–equivariant bijections

$$\mathcal{A} \longleftrightarrow \{\text{reflecting lines}\} \longleftrightarrow \{\text{ramified height one prime ideals of } S\}$$
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Ramification and parabolic subgroups
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$$\text{I}(\mathcal{A}) \sim \text{Par}(G) \quad , \quad X \mapsto G_X .$$
Braid groups

Let $V_{\text{reg}} := V - \bigcup_{H \in A} H$. Since the covering $V_{\text{reg}} \to V_{\text{reg}}/G$ is Galois, it induces a short exact sequence

$$1 \to \Pi_1(V_{\text{reg}}, x_0) \to \Pi_1(V_{\text{reg}}/G, x_0) \to B_G \to 1$$

(Pure braid group) (Braid group)
Braid groups

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$$

$\text{(Pure braid group)}$ $\text{(Braid group)}$

$P_G$ $B_G$
Notation around $H$

Let $H \in A$, with associated line $L$. For $x \in V$, we set $x = x_L + x_H$ (with $x_L \in L$ and $x_H \in H$). Thus, we have $s_H(x) = e^{2\pi i / e_H} x_L + x_H$.

If $t \in \mathbb{R}$, we set $s_t H(x) = e^{2\pi i t / e_H} x_L + x_H$ defining a path $s_H$, $x$ from $x$ to $s_H(x)$.

We have $s_t e_H H(x) = e^{2\pi i t} x_L + x_H$ defining a loop $\pi_H$, $x$ with origin $x$.

In other words, $\pi_H$, $x = s_t e_H H$, $x \in \mathbb{P}G$.
Notation around $H$

- Let $H \in \mathcal{A}$, with associated line $L$. 

\[ s_H(x) = e^{2i\pi/|H|}x_L + x_H \]

Thus, we have $s_{eH}(x) = e^{2i\pi/|H|}x_L + x_H$.

If $t \in \mathbb{R}$, we set:

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- If $t \in \mathbb{R}$, we set:
  \[ s^t_H(x) = e^{2i\pi t/e_H}x_L + x_H \quad \text{defining a path } s_{H,x} \text{ from } x \text{ to } s_H(x). \]
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Notation around $H$

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  We have
  \[ s_{H}^{teH}(x) = e^{2\pi i t}x_L + x_H \quad \text{defining a loop } \pi_{H,x} \text{ with origin } x. \]
  In other words,
  \[ \pi_{H,x} = s_{H,x}^{eH} \in P_G \]
Let $\gamma$ be a path in $V_{reg}$ from $x_0$ to $x_H$. We define:

$$\sigma_{H,\gamma} := s_H(\gamma - 1) \cdot s_H \cdot x_0 \cdot H \cdot x_H \cdot s_H(x_H) \cdot PP \cdot s_H(x_0)$$

**Definition**

We call braid reflections the elements $s_{H,\gamma} \in B$ defined by the paths $\sigma_{H,\gamma}$. 

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Braid reflections

Let $\gamma$ be a path in $V^{\text{reg}}$ from $x_0$ to $x_H$. 

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- $s_{e_{H},\gamma}$ is a loop in $V^{\text{reg}}$:

\[
\begin{array}{c}
\bullet \quad \gamma \quad \cdot x_0
\end{array}
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The variety $V$ (resp. $V/G$) is simply connected, the hyperplanes are irreducible divisors (irreducible closed subvarieties of codimension one), and the braid reflections are “generators of the monodromy” around the irreducible divisors. Then
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1. The braid group \( B_G \) is generated by the braid reflections \((s_{H,\gamma})\) (for all \( H \) and all \( \gamma \)).
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**Theorem**

1. The braid group $B_G$ is generated by the braid reflections $(s_{H,\gamma})$ (for all $H$ and all $\gamma$).
2. The pure braid group $P_G$ is generated by the elements $(s_{eH,\gamma}^H)$.
Linear characters of the reflection groups

For $H \in \mathcal{A}$,
Linear characters of the reflection groups

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Linear characters of the reflection groups

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3. $\text{Hom}(G, \mathbb{C}^\times) \sim \left( \prod_{H \in \mathcal{A}} \text{Hom}(G_H, \mathbb{C}^\times) \right)^G \sim \left( \prod_{H \in \mathcal{A}/G} \text{Hom}(G_H, \mathbb{C}^\times) \right)$
Linear characters of the braid groups

The discriminant at $H \in A$ (or rather $A/G$) is $\Delta_H := \frac{\delta_H}{\delta}$, hence defines a (continuous) map $\Delta_H : V_{\text{reg}} \to C \times V_{\text{reg}}/G$, hence defines a morphism $\Pi_1(\Delta_H) : \Pi_1(V_{\text{reg}}/G) \to \Pi_1(C \times V_{\text{reg}}/G)$. For $H \in A$, $L_H : B_G \to Z$ is a linear character.
Linear characters of the braid groups

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The discriminant at \( H \in \mathcal{A} \) (or rather \( \mathcal{A}/G \)) is \( \Delta_H := j_H^{eH} \). 

\( \Delta_H \in R = S^G \) hence defines a (continuous) map \( \Delta_H : V_{reg} \rightarrow \mathbb{C}^\times \) 

\[ \begin{array}{c}
V_{reg} \\
\downarrow \\
V_{reg}/G \\
\end{array} \xrightarrow{\Delta_H} \mathbb{C}^\times \]
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  \]

- For \( H \in \mathcal{A} \),
  \[
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  \]
  \[
  B_{G_H} \cong \mathbb{Z}
  \]
Proposition 1

\[ \text{Hom}(G, C \times \cdot \cdot \cdot \cdot) \sim - \rightarrow \left( \prod_{H \in A} \text{Hom}(G H, C \times \cdot \cdot \cdot \cdot) \right) \]

\[ \text{Hom}(B G, Z) \sim - \rightarrow \left( \prod_{H \in A} \text{Hom}(B G H, Z) \right) \]

\[ \ell_H \text{ is a length:} \]

\[ \ell_H(s_n H_1, \gamma_1 \cdot \cdot \cdot s_n H_k, \gamma_k) = \sum \left\{ i \mid (H_i = G H) \right\} n_i B G \]

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Proposition
Proposition

1. $\text{Hom}(G, \mathbb{C}^\times) \sim (\prod_{H \in \mathcal{A}} \text{Hom}(G_H, \mathbb{C}^\times))^G$
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Proposition

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\[
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Center of the braid groups

From now on we assume that $G$ is irreducible on $V$. Hence the centre of $G$ is cyclic. Set $z := |\mathbb{Z}G|$ and $\zeta := e^{2i\pi/z}$.

Let $\pi \in \mathbb{P}G$ defined by $\pi : t \mapsto e^{2i\pi t}x_0$

Let $\zeta \in \mathbb{B}G$ defined by $\zeta : t \mapsto e^{2i\pi t/z}x_0$

Theorem 1

$$\mathbb{Z}P_G = \langle \pi \rangle$$ and $$\mathbb{Z}B_G = \langle \zeta \rangle$$

2. We have the short exact sequence

$$1 \rightarrow \mathbb{Z}P_G \rightarrow \mathbb{Z}B_G \rightarrow \mathbb{Z}G \rightarrow 1$$
From now on we assume that $G$ is \textit{irreducible} on $V$. 
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Hence the centre of $G$ is cyclic. Set $z := |ZG|$ and $\zeta := e^{2i\pi/z}$.
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Theorem

1. $ZP_G = \langle \pi \rangle$ and $ZB_G = \langle \zeta \rangle$. 
Center of the braid groups

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Theorem

1. $ZP_G = \langle \pi \rangle$ and $ZB_G = \langle \zeta \rangle$.
2. We have the short exact sequence

$$1 \longrightarrow ZP_G \longrightarrow ZB_G \longrightarrow ZG \longrightarrow 1$$
Case of Coxeter groups

The choice of a Coxeter generating set for $G$ defines a presentation of $B_G$.

Example:

$$\pi = (st_1 t_2 \cdots t_{r-1})^2$$

Let $w_0$ be the longest element of $G$, and let $g_0$ be its lift in $B_G$.

$$\pi = g_2^r$$
Case of Coxeter groups

The choice of a Coxeter generating set for $G$ defines a presentation of $B_G$.
Case of Coxeter groups

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Example:

\[ s \rightarrow t_1 \rightarrow t_2 \rightarrow \cdots \rightarrow t_{r-1} \]

\[ 2 \rightarrow 2 \rightarrow 2 \rightarrow \cdots \rightarrow 2 \]
Case of Coxeter groups

The choice of a Coxeter generating set for $G$ defines a presentation of $B_G$

Example:

\[ \begin{array}{cccc} \circ & \circ & \circ & \cdots & \circ \\
 s & t_1 & t_2 & \cdots & t_{r-1} \\
 \end{array} \]

and a “section” (not a group morphism!) of the map $B_G \rightarrow G$ using reduced decompositions.
Case of Coxeter groups

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Reflection groups and their braids
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\[\pi = g_0^2\]

Example: \[\pi = (st_1t_2 \cdots t_{r-1})^{2r}\]
An Artin–like presentation is
\[ \langle s \in S | \{ v_i = w_i \} \rangle \]
where

\( S \) is a finite set of distinguished braid reflections,
\( I \) is a finite set of relations which are multi–homogeneous.

Theorem (Bessis)

Let \( G \subset GL(V) \) be a complex reflection group. Let
\( d_1 \leq d_2 \leq \cdots \leq d_r \) be the family of its invariant degrees.

The following integers are equal (denoted by \( \Gamma_G \)):

1. The minimal number of reflections needed to generate \( G \)
2. The minimal number of braid reflections needed to generate \( B_G \)

\( \left\lceil \left( N + N_h \right) / d_r \right\rceil \)

Either \( \Gamma_G = r \) or \( \Gamma_G = r + 1 \), and the group \( B_G \) has an Artin–like presentation by \( \Gamma_G \) braid reflections.

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Reflection groups and their braids
Artin–like presentations

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Reflection groups and their braids
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Artin–like presentations

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An Artin–like presentation is

\[ \langle s \in S \mid \{v_i = w_i\}_{i \in I} \rangle \]

where

- **S** is a finite set of distinguished braid reflections,
- **I** is a finite set of relations which are multi–homogeneous, i.e., such that (for each \( i \)) \( v_i \) and \( w_i \) are positive words in elements of **S**
Artin–like presentations

An Artin–like presentation is

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Theorem (Bessis)

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   - $\lceil (N + N_h)/d_r \rceil$ ($N := \text{number of reflections}, N_h := \text{number of hyperplanes}$)
Artin–like presentations

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Theorem (Bessis)

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2. Either $\Gamma_G = r$ or $\Gamma_G = r + 1$, and the group $B_G$ has an Artin–like presentation by $\Gamma_G$ braid reflections.
The braid diagrams

Let $D$ be a diagram like $s \circ a \circ b \circ c$. $D$ represents the relations $stustu \cdots$.

Factors $= tustus \cdots$.

Factors $= ustust \cdots$.

We denote by $D_{br}$ and call braid diagram the diagram $s \circ n \circ t \circ u$ which represents the relations $stustu \cdots$.

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Note that $G_7: s \circ 2 \circ n \circ 3 \circ t \circ 3 \circ u = G_{11}: s \circ 2 \circ n \circ 3 \circ t \circ 4 \circ u = G_{19}: s \circ 2 \circ n \circ 3 \circ t \circ 5 \circ u$ have the same braid diagram.
The braid diagrams

Let $\mathcal{D}$ be a diagram like

\[
\begin{array}{c}
  s \\
  a \\
  e \\
  b \\
  t \\
  c \\
  u \\
\end{array}
\]

We denote by $\mathcal{D}_{br}$ and call braid diagram the diagram $\mathcal{D}$ which represents the relations $stustutu\cdots$.

Note that $G_7$: $s^2n^3t^3u$, $G_{11}$: $s^2n^3t^4u$, $G_{19}$: $s^2n^3t^5u$ have the same braid diagram.

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Reflection groups and their braids
Let $\mathcal{D}$ be a diagram like $s \overset{a}{\circlearrowright} e \overset{b}{\circlearrowleft} t \overset{c}{\circlearrowright} u$. $\mathcal{D}$ represents the relations $stustu\ldots = tustus\ldots = ustust\ldots$.
The braid diagrams

Let $D$ be a diagram like $s\circ a \circ e \circ b \circ t \circ c \circ u$. $D$ represents the relations $s \, t \, u \, s \, t \, u \cdots = t \, u \, s \, t \, u \cdots = u \, s \, t \, u \, s \cdots$ and $s^a = t^b = u^c = 1$.
The braid diagrams

Let $D$ be a diagram like $s \quad e \quad D$ represents the relations

\[ stustu \cdots = tustus \cdots = ustust \cdots \]

and $s^a = t^b = u^c = 1$

We denote by $D_{br}$ and call *braid diagram* the diagram $s \quad e \quad u$
The braid diagrams

Let $\mathcal{D}$ be a diagram like $s \quad e \quad t \quad b \quad a \quad c \quad u$. $\mathcal{D}$ represents the relations

$\underbrace{stustu \cdots} = \underbrace{tustus \cdots} = \underbrace{ustust \cdots}$

e factors  e factors  e factors

and $s^a = t^b = u^c = 1$

We denote by $\mathcal{D}_{br}$ and call braid diagram the diagram $s \quad e \quad t \quad u$ which represents the relations

$\underbrace{stustu \cdots} = \underbrace{tustus \cdots} = \underbrace{ustust \cdots}$

e factors  e factors  e factors
The braid diagrams

Let $D$ be a diagram like $s \ a \ \cdot \ e \ \cdot \ b \ \cdot \ t \ \cdot \ c \ \cdot \ u$ $D$ represents the relations

\[
\underbrace{sstu \cdots} = \underbrace{tustu \cdots} = \underbrace{ustust \cdots} \quad \text{e factors} \quad \text{e factors} \quad \text{e factors}
\]

and $s^a = t^b = u^c = 1$

We denote by $D_{br}$ and call braid diagram the diagram $s \ e \ u$ which represents the relations

\[
\underbrace{sstu \cdots} = \underbrace{tustu \cdots} = \underbrace{ustust \cdots} \quad \text{e factors} \quad \text{e factors} \quad \text{e factors}
\]

Note that

$G_7 : s \ 2 \ 3 \ t \ 3 \ u$  $G_{11} : s \ 2 \ 3 \ t \ 4 \ u$  $G_{19} : s \ 2 \ 3 \ t \ 5 \ u$
The braid diagrams

Let $D$ be a diagram like

$$\begin{array}{c}
\text{Diagram D represents the relations} \\
\text{e factors} = \text{e factors} = \text{e factors}
\end{array}$$

and

$$\begin{array}{c}
s^a = t^b = u^c = 1
\end{array}$$

We denote by $D_{br}$ and call braid diagram the diagram

which represents the relations

$$\begin{array}{c}
\text{e factors} = \text{e factors} = \text{e factors}
\end{array}$$

Note that

$$\begin{array}{c}
G_7 : s \circled{3} t \\
G_{11} : s \circled{3} t \\
G_{19} : s \circled{3} t
\end{array}$$

have same braid diagram.
For each irreducible complex irreducible group $G$, there is a diagram $D$, whose set of nodes $N(D)$ is identified with a set of distinguished reflections in $G$, such that

**Theorem**

For each $s \in N(D)$, there exists a braid reflection $s \in B_G$ above $s$ such that the set $\{s\}_{s \in N(D)}$, together with the braid relations of $D_{br}$, is a presentation of $B_G$.

The groups $G_n$ for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type $\circ s d e \circ t d$, corresponding to the presentation $s d = t d = 1$ and $ststs \cdots \in \circ e$ factors $= tstst \cdots \in \circ e$ factors.
For each irreducible complex irreducible group $G$, there is a diagram $\mathcal{D}$,

Theorem

For each $s \in N(\mathcal{D})$, there exists a braid reflection $s \in B_G$ above $s$ such that the set

\[ \{s\} \cup \{\text{braid relations of } \mathcal{D}\} \]

is a presentation of $B_G$.

The groups $G_n$ for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type $\ast \ast$, corresponding to the presentation $s_d t_d = 1$ and $s_t s_t s_t \cdots = t_t s_t s_t \cdots$ factors $= t_t s_t s_t \cdots$ factors.
For each irreducible complex irreducible group $G$, there is a diagram $D$, whose set of nodes $\mathcal{N}(D)$ is identified with a set of distinguished reflections in $G$,
For each irreducible complex irreducible group $G$, there is a diagram $\mathcal{D}$, whose set of nodes $\mathcal{N}(\mathcal{D})$ is identified with a set of distinguished reflections in $G$, such that

\[\text{Theorem}\]

For each $s \in \mathcal{N}(\mathcal{D})$, there exists a braid reflection $s \in \mathcal{B}_G$ above $s$ such that the set $\{s\} \cup \{s \in \mathcal{N}(\mathcal{D})\}$, together with the braid relations of $\mathcal{D}$, is a presentation of $\mathcal{B}_G$. 

The groups $G_n$ for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type $\mathcal{D}$, corresponding to the presentation $s = t = 1$ and $ststs \cdots$ factors $= tstst \cdots$ factors.
For each irreducible complex irreducible group $G$, there is a diagram $\mathcal{D}$, whose set of nodes $\mathcal{N}(\mathcal{D})$ is identified with a set of distinguished reflections in $G$, such that

**Theorem**

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**Theorem**

For each $s \in \mathcal{N}(\mathcal{D})$, there exists a braid reflection $s \in B_G$ above $s$ such that the set $\{s\}_{s \in \mathcal{N}(\mathcal{D})}$, together with the braid relations of $\mathcal{D}_{br}$, is a presentation of $B_G$.

- The groups $G_n$ for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type $\xymatrix{d & e \ar[r] & d \ar@{-}[r]^s & t}$.
For each irreducible complex irreducible group $G$, there is a diagram $\mathcal{D}$, whose set of nodes $\mathcal{N}(\mathcal{D})$ is identified with a set of distinguished reflections in $G$, such that

**Theorem**

For each $s \in \mathcal{N}(\mathcal{D})$, there exists a braid reflection $s \in B_G$ above $s$ such that the set $\{s\}_{s \in \mathcal{N}(\mathcal{D})}$, together with the braid relations of $\mathcal{D}_{br}$, is a presentation of $B_G$.

- The groups $G_n$ for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type $\mathcal{D}$, corresponding to the presentation...
For each irreducible complex irreducible group $G$, there is a diagram $\mathcal{D}$, whose set of nodes $\mathcal{N}(\mathcal{D})$ is identified with a set of distinguished reflections in $G$, such that

Theorem

For each $s \in \mathcal{N}(\mathcal{D})$, there exists a braid reflection $s \in B_G$ above $s$ such that the set $\{s\}_{s \in \mathcal{N}(\mathcal{D})}$, together with the braid relations of $\mathcal{D}_{br}$, is a presentation of $B_G$.

- The groups $G_n$ for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type $\begin{array}{cccc} & d & & e & d \\ s & & & & t \end{array}$, corresponding to the presentation

$$s^d = t^d = 1$$
For each irreducible complex irreducible group $G$, there is a diagram $D$, whose set of nodes $\mathcal{N}(D)$ is identified with a set of distinguished reflections in $G$, such that

**Theorem**

For each $s \in \mathcal{N}(D)$, there exists a braid reflection $s \in B_G$ above $s$ such that the set $\{s\}_{s \in \mathcal{N}(D)}$, together with the braid relations of $D_{\text{br}}$, is a presentation of $B_G$.

- The groups $G_n$ for $n = 4, 5, 8, 16, 20$, as well as the dihedral groups, have diagrams of type $\circ d \xrightarrow{s} e \xleftarrow{t} \circ d$, corresponding to the presentation

\[
s^d = t^d = 1 \quad \text{and} \quad \overbrace{sts \cdots}^{e \text{ factors}} = \overbrace{tstst \cdots}^{e \text{ factors}}
\]
The group $G_{18}$ has diagram $\circlearrowright s \circlearrowright t \circlearrowright$ corresponding to the presentation $s^5 = t^3 = 1$ and $stst = tsts$.

The group $G_{31}$ has diagram $\circlearrowright v_2 \bigcirclearrowright s_2 \circlearrowright t_2 \bigcirclearrowright w_2 \bigcirclearrowright u_2 \bigcirclearrowleft$ corresponding to the presentation $s_2^2 = t_2^2 = u_2^2 = v_2^2 = w_2^2 = 1$, $uv = vu$, $sw = ws$, $vw = wv$, $sut = utst = tsu$, $svs = vsv$, $tvt = vtv$, $twt = wtw$, $uwu = uwu$. 

Michel Broué
Reflection groups and their braids
The group $G_{18}$ has diagram $\begin{array}{cc} 5 & 3 \\ s & t \end{array}$ corresponding to the presentation $s^5 = t^3 = 1$ and $stst = tsts$. 

Michel Broué

Reflection groups and their braids
The group $G_{18}$ has diagram corresponding to the presentation

$$s^5 = t^3 = 1 \text{ and } stst = tsts.$$ 

The group $G_{31}$ has diagram
• The group $G_{18}$ has diagram corresponding to the presentation

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• The group $G_{31}$ has diagram corresponding to the presentation
• The group $G_{18}$ has diagram \[ \begin{array}{c} 5 \\ s \\ \hline \end{array} \begin{array}{c} 3 \\ t \\ \hline \end{array} \] corresponding to the presentation 

\[ s^5 = t^3 = 1 \text{ and } stst = tsts. \]

• The group $G_{31}$ has diagram \[ \begin{array}{c} 2 \\ \hline \end{array} \begin{array}{c} 2 \\ s \\ \hline \end{array} \begin{array}{c} 2 \\ u \\ \hline \end{array} \begin{array}{c} 2 \\ t \\ \hline \end{array} \begin{array}{c} 2 \\ v \\ \hline \end{array} \begin{array}{c} 2 \\ w \\ \hline \end{array} \] corresponding to the presentation 

\[ s^2 = t^2 = u^2 = v^2 = w^2 = 1, \]
• The group $G_{18}$ has diagram \( \begin{array}{c}
\begin{array}{c}
\text{5} \\
\text{s}
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\text{3} \\
\text{t}
\end{array}
\end{array} \end{array} \) corresponding to the presentation

\[ s^5 = t^3 = 1 \text{ and } stst = tsts. \]

• The group $G_{31}$ has diagram \( \begin{array}{c}
\begin{array}{c}
\text{2} \\
\text{v}
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\text{2} \\
\text{t}
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\text{2} \\
\text{w}
\end{array}
\end{array} \end{array} \) corresponding to the presentation

\[ s^2 = t^2 = u^2 = v^2 = w^2 = 1, \]

\[ uv = vu, \quad sw = ws, \quad vw = wv, \quad sut = uts = tsu, \]
The group $G_{18}$ has diagram $\begin{array}{c} 5 \\ s \\ \hline \end{array} \begin{array}{c} 3 \\ t \\ \hline \end{array}$ corresponding to the presentation

$$s^5 = t^3 = 1 \text{ and } stst = tsts .$$

The group $G_{31}$ has diagram $\begin{array}{c} 2 \\ s \\ \hline \end{array} \begin{array}{c} 2 \\ t \\ \hline \end{array} \begin{array}{c} 2 \\ w \\ \hline \end{array}$ corresponding to the presentation

$$s^2 = t^2 = u^2 = v^2 = w^2 = 1 ,$$

$$uv = vu , sw = ws , vw = wv , \quad sut = uts = tsu ,$$

$$svs = vsv , tvt = vtv , twt = wtw , wuw = uwu .$$
The space $V_{\text{reg}}$ is a $K(\pi, 1)$. Springer's theory of regular elements in complex reflection groups lifts to braid groups.

Let $\zeta_d := e^{2i\pi/d}$. The $\zeta_d$–regular elements in $G$ are the images of the $d$–th roots of $\pi$. All $d$–th roots of $\pi$ are conjugate in $B_G$.

Let $g$ be a $d$–th root of $\pi$, with image $g$ in $G$. Then $C_{B_G}(g)$ is the braid group of $C_G(g)$. Michel Broué

Reflection groups and their braids
More on the work of Bessis

- Solution of an old conjecture

Theorem

The space $V_{reg}$ is a $K(\pi, 1)$.

Springer's theory of regular elements in complex reflections groups lifts to braid groups

Theorem

Let $\zeta_d := e^{2i\pi/d}$. The $\zeta_d$–regular elements in $G$ are the images of the $d$–th roots of $\pi$.

All $d$–th roots of $\pi$ are conjugate in $B_G$.

Let $g$ be a $d$–th root of $\pi$, with image $g$ in $G$. Then $C_{B_G}(g)$ is the braid group of $C_G(g)$.

Michel Broué

Reflection groups and their braids
More on the work of Bessis

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A monodromy representation

(after Knizhnik–Zamolodchikov, Cherednik, Dunkl, Opdam, Kohno, Broué-Malle-Rouquier)

For \( H \in A \), let \( \alpha_H \) be a linear form with kernel \( H \), and \( \omega_H := \frac{1}{2} i \pi d \alpha_H \alpha_H \).

Each family \( (z_H)_H \in A \in \prod_{H \in A} \mathbb{C} G_H \) defines a \( G \)-invariant differential form on \( V_{reg} \) with values in \( \mathbb{C} G \), hence a linear differential equation \( df = \omega f \) for \( f : V_{reg} \to \mathbb{C} G \), i.e.,

\[
\forall v \in V, x \in V_{reg}, df(x)(v) = \frac{1}{2} i \pi \sum_{H \in A} \alpha_H(v) \alpha_H(x) z_H f(x)
\]
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defines a $G$-invariant differential form on $V_{\text{reg}}$ with values in $C^G$. Hence a linear differential equation $df = \omega f$ for $f : V_{\text{reg}} \to C^G$, i.e.,

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For $H \in \mathcal{A}$, let $\alpha_H$ be a linear form with kernel $H$, and

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  \[ \forall v \in V, \ x \in V^{\text{reg}}, \quad df(x)(v) = \frac{1}{2i\pi} \sum_{H \in \mathcal{A}} \frac{\alpha_H(v)}{\alpha_H(x)} z_H f(x) \]
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\[
\begin{align*}
\{ & \\
\end{align*}
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\begin{align*}
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For $H \in \mathcal{A}$, \[ \begin{cases} &\bullet \ G_H^\vee \text{ is the group of characters of } G_H, \\ &\bullet \text{ for } \theta \in G_H^\vee, \ e_{H,\theta} \text{ is the corresponding primitive idempotent in } \mathbb{C}G_H \end{cases} \]
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We set \[ q_H := \exp \left( -2i \pi / e_H \right) z_H \right) =: \sum_{\theta \in G_H^\vee} q_{H,\theta} e_{H,\theta} \]
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\end{cases}
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**Theorem**

1. The form $\omega$ is integrable, hence defines a group morphism

$$
\rho : B_G \longrightarrow (\mathbb{C}G)^\times .
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**Theorem**

1. The form $\omega$ is integrable, hence defines a group morphism

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2. Whenever $s_{H, \gamma}$ is a braid reflection around $H$, there is $u_H \in (\mathbb{C}G)^\times$ such that

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\rho(s_{H, \gamma}) = u_H(q_H s_H) u_H^{-1}
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For $H \in \mathcal{A}$, \begin{align*}
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$$

In particular, we have

$$
\prod_{\theta \in G_H^\vee} (\rho(s_{H,\gamma}) - q_{H,\theta} \theta(s_H)) = 0.
$$
Hecke algebras

Every complex reflection group $G$ has an Artin-like presentation:

- $G_2: \langle s^2 \rangle$,
- $G_4: \langle s^3 \rangle$,

and a field of realization $Q^*_{G} := Q(\{ \text{tr} V(g) | g \in G \})$.

The associated generic Hecke algebra is defined from such a presentation:

- $H(G_2) := \langle S, T; STSTST = TSTSTS(s - q_0)(s - q_1) = 0 \rangle$,
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Every complex reflection group $G$ has an Artin-like presentation:

$G_2 : \begin{array}{ccc}
& \ast & \\
\ast & \ast & \ast \\
& \ast & \\
\end{array}$,  
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& \ast & \\
\ast & \ast & \ast \\
& \ast & \\
\end{array}$
Every complex reflection group $G$ has an Artin-like presentation:

\[ G_2 : \begin{array}{c|c}
  2 & 2 \\
  s & t
\end{array}, \quad G_4 : \begin{array}{c|c}
  3 & 3 \\
  s & t
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  $$
  \mathcal{H}(G_2) := \langle S, T \rangle \left\{ \begin{array}{l}
  STSTST = TSTSTS \\
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  (T - r_0)(T - r_1) = 0
  \end{array} \right.
  $$

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Theorem (G. Malle and al.)

1. The generic Hecke algebra $\mathcal{H}(G)$ is free of rank $|G|$ over the corresponding Laurent polynomial ring $\mathbb{Z}[(q_i^{\pm 1}), (r_j^{\pm 1}), \ldots]$. 
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\[ x_i \rightarrow 1, \quad y_j \rightarrow 1, \quad \ldots \]

The above specialisation defines a bijection $\text{Irr}(G) \sim \text{Irr}(\mathcal{H}(G))$, $\chi \rightarrow \chi_{\mathcal{H}(G)}$. 

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$$\left( x_i^{[\mu(\mathbb{Q}G)]} = \zeta_d^{-i} q_i \right)_{i=0,1,\ldots,d-1} \text{, } \left( y_j^{[\mu(\mathbb{Q}G)]} = \zeta_e^{-j} r_j \right)_{j=0,1,\ldots,e-1}$$
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\text{the algebra } \mathbb{Q}_G((x_i), (y_j), \ldots)) \mathcal{H}(G) \text{ is split semisimple,}
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   \[
   \text{Irr}(G) \rightarrow \text{Irr}(\mathcal{H}(G)) \quad \chi \mapsto \chi\mathcal{H}.
   \]
Theorem–Conjecture

There exists a unique linear form $t_q: H(W, q) \rightarrow \mathbb{Z}[q, q^{-1}]$ with the following properties.

- $t_q$ is a symmetrizing form on the algebra $H(W, q)$.
- $t_q$ specializes to the canonical linear form on the group algebra.
- For all $b \in B$, we have $t_q(b - 1) \lor = t_q(b \pi) t_q(\pi)$. 

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\[ t_q(b^{-1})^\vee = \frac{t_q(b\pi)}{t_q(\pi)}. \]
The form $t_q$ satisfies the following conditions.

As an element of $\mathbb{Z}[q, q^{-1}]$, $t_q(b)$ is multi–homogeneous with degree $\ell H(b)$ in the indeterminates $q_H, \theta$.

If $W'$ is a parabolic subgroup of $W$, the restriction of $t_q$ to a parabolic sub–algebra $H(W', W, q)$ is the corresponding specialization of $t_q(W')$.

The canonical forms $t_q$ are hidden behind Lusztig's theory of characters of finite reductive groups, their generic degrees and Fourier transform matrices.
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