# INTRODUCTION TO COMPLEX REFLECTION GROUPS AND THEIR BRAID GROUPS

## A COURSE AT UC BERKELEY

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## INTRODUCTION

Weyl groups are finite groups acting as reflection groups on rational vector spaces. It is well known that these rational reflection groups appear as "skeletons" of many important mathematical objects : algebraic groups, Hecke algebras, Artin–Tits braid groups, etc.

By extension of the base field, Weyl groups may be viewed as particular cases of finite *complex reflection groups*, *i.e.*, finite subgroups of some $\mathrm{GL}_r(\mathbb{C})$ generated by (pseudo–)reflections. Such groups have been characterized by Shephard–Todd and Chevalley as those finite subgroups of $\mathrm{GL}_r(\mathbb{C})$ whose ring of invariants in the corresponding symmetric algebra $\mathbb{C}[X_1, X_2, \ldots, X_r]$ is a regular graded ring (a polynomial algebra). The irreducible finite complex reflection groups have been classified by Shephard–Todd.

It has been recently discovered that complex reflection groups play also a key role in the structure as well as in the representation theory of finite reductive groups *i.e.*, rational points of algebraic connected reductive groups over a finite field – for a survey on that type of questions, see for example [Bro]. Not only do complex reflection groups appear as "automizers" of peculiar tori (the "cyclotomic Sylow subgroups"), but as much as Weyl groups, they give rise to braid groups and generalized Hecke algebras which govern representation theory of finite reductive groups.

In the meantime, it has been understood that many of the known properties of Weyl groups, and more generally of Coxeter finite groups (reflection groups over $\mathbb{R}$) can be generalized to complex reflection groups – although in most cases new methods have to be found. The most spectacular result in that direction, due to Bessis [Bes3], states that the complement of the hyperplanes arrangement of a complex reflection group is $K(\pi, 1)$. The oldest (but not least important), due to Steinberg [St], states that the subgroup which fixes a subspace is still a complex reflection group (a "parabolic subgroup").

The purpose of this set of Notes is to give a somehow complete treatment of the foundations and basic properties of complex reflection groups (characterization, Steinberg theorem, Gutkin–Opdam matrices, Solomon theorem and applications, etc.) including the basic results of Springer [Sp] and Springer–Lehrer [LeSp] on eigenspaces of elements of complex reflection groups. On our way, we also introduce basic definitions and properties of the associated braid groups, as well as a quick introduction to Bessis lifting of Springer theory to braid groups.

# CHAPTER 0
# PRELIMINARIES

## §1. Reflections and roots

Let $k$ be a commutative field and let $V$ be a finite dimensional $k$-vector space. We set $r := \dim_K V$. We denote by $V^*$ the dual of $V$.

The group $\mathrm{GL}(V)$ acts on $V \times V^*$ : if $r = (v, v^*) \in V \times V^*$ and if $g \in \mathrm{GL}(V)$, we set

$$g \cdot r := (gv, v^* \cdot g^{-1})\,.$$

Let us state a list of remarks, elementary properties, definitions and notation.

**Rank one endomorphisms.**

An element $r \in (V - 0) \times (V^* - 0)$ defines an element $\overline{r}$ of rank 1 of $\mathrm{End}(V)$ as follows. Suppose $r = (v, v^*)$. Then

$$\overline{r} \colon x \mapsto \langle v^*, x\rangle v\,.$$

We denote by $\mathrm{tr}(r)$ the trace of the endomorphism $\overline{r}$. We denote by $H_r$ its kernel and by $L_r$ its image. Thus, for $r = (v, v^*)$, we have

$$\mathrm{tr}(r) = \langle v^*, v\rangle \quad , \quad H_r = \ker v^* \quad , \quad L_r = Kv\,.$$

We see that $L_r \subseteq H_r$ if and only if $\mathrm{tr}(r) = 0$.

**1.1. Lemma.**

*(1) If we view* $\mathrm{End}(V)$ *as acted on by* $\mathrm{GL}(V)$ *through conjugation, the map* $r \mapsto \overline{r}$ *is* $\mathrm{GL}(V)$*-equivariant : the rank one endomorphism attached to* $g \cdot r$ *is* $g\overline{r}g^{-1}$.

*(2) Two elements* $r_1 = (v_1, v_1^*)$ *and* $r_2 = (v_2, v_2^*)$ *of* $V \times V^*$ *define the same rank one endomorphism of* $V$ *if and only if they are in the same orbit under* $k^\times$*, i.e., if there exists* $\lambda \in k^\times$ *such that* $v_2 = \lambda v_1$ *and* $v_2^* = \lambda^{-1} v_1^*$.

**Projections, Transvections, Reflections.**

For $r \in (V - 0) \times (V^* - 0)$, we denote by $s_r$ the endomorphism of $V$ defined by the formula $s_r := 1 - \overline{r}$ , or, in other words, $s_r \colon x \mapsto x - \langle v^*, x\rangle v$ .

Note that, for $g \in \mathrm{GL}(V)$, we have

$$s_{g\cdot r} = g s_r g^{-1}\,.$$

**1.2. Lemma.**
*(1) If* $\mathrm{tr}(r) = 1$*, we have* $H_r \oplus L_r = V$*, and the endomorphism* $s_r$ *is nothing but the projection onto* $H_r$ *parallel to* $L_r$*.*
*(2) If* $\mathrm{tr}(r) = 0$*, we have* $L_r \subseteq H_r$*,* $\overline{r}^2 = 0$*, and the endomorphism* $s_r$ *is a transvection.*

**Definition.**
*A root of* $V$ *is an element* $r$ *of* $V \times V^*$ *such that* $\mathrm{tr}(r) \neq 0, 1$*.*
*A reflection in* $V$ *is an endomorphism of* $V$ *of the shape* $s_r$ *where* $r$ *is a root.*

**Reflections.**

From now on, we assume that $s_r$ is a reflection. Let us set $r = (v, v^*)$.

We set $\zeta_r := 1 - \mathrm{tr}(r)$. Then we have

$$H_r = \ker(s_r - 1)\ ,\ L_r = \ker(s_r - \zeta_r 1) = \mathrm{im}\,(s_r - 1)\ ,\ \zeta_r = \det s_r \text{ and } \zeta_r \neq 0, 1\,.$$

Note that the order of $s_r$ is the order of the element $\zeta_r$ in the group $k^\times$. More generally, for $(x, x^*) \in V \times V^*$ and $n \in \mathbb{N}$, we have

$$s_r^n \cdot (x, x^*) = \left( x - \frac{1 - \zeta_r^n}{1 - \zeta_r} \langle x, v^* \rangle v\ ,\ x^* - \frac{1 - \zeta_r^{-n}}{1 - \zeta_r} \langle v, x^* \rangle v^* \right)\,. \tag{1.3}$$

The following lemma shows that the inverse, the transpose, the contragredient of a reflection are reflections. We omit the proof, which is straightforward.

**1.4. Lemma.**
(1) *The conjugate of a reflection is a reflection, since* $g s_r g^{-1} = s_{g \cdot r}$ *.*
(2) *The inverse of a reflection is a reflection :* $s_r^{-1} = s_{r'}$ *where* $r' := (v, -\zeta_r^{-1} v^*)$*.*
(3) *The transpose of a reflection in* $V$ *is a reflection in* $V^*$ *:* ${}^t s_r = s_{{}^t r}$ *, where* ${}^t r := (v^*, v)$*.*
(4) *The contragredient of a reflection is a reflection :* ${}^t s_r^{-1} = s_{r^\vee}$ *where* $r^\vee := (-\zeta_r^{-1} v^*, v)$*.*

The following lemma (whose proof is also straightforward) gives several ways to index the set of reflections.

**1.5. Lemma.** *The maps*

$$\begin{cases} r \mapsto s_r \\ s \mapsto (\ker(s - 1), \mathrm{im}\,(s - 1), \det s) \end{cases}$$

*define bijections between the following sets*
- *The set of orbits of* $k^\times$ *on roots of* $V$*,*
- *The set of reflections in* $V$*,*
- *The set of triples* $(H, L, \zeta)$*, where* $H$ *is an hyperplane in* $V$ *and* $L$ *is a one-dimensional subspace of* $V$ *such that* $H \oplus L = V$*, and* $\zeta$ *is an element of* $k$ *different from 0 and 1.*

*as well as with the analogous sets obtained by replacing* $V$ *by* $V^*$*.*

A reflection is diagonalisable, hence so is its restriction to a stable subspace. The next lemma follows.

**1.6. Lemma.** *Let $V'$ be a subspace of $V$ stable by a reflection $s_r$. Then*

- *either $V'$ is fixed by $s_r$ (i.e., $V' \subseteq H_r$),*
- *or $V'$ contains $L_r$, and then $V' = L_r \oplus (H_r \cap V')$.*

**Commuting reflections.**

A root $r = (v, v^*)$ is said to be an eigenroot of $g \in \mathrm{GL}(V)$ if there exists $\lambda \in K$ such that $g \cdot r = \lambda \cdot r = (\lambda v, \lambda^{-1} v^*)$.

**1.7. Lemma.** *Let $r_1$ and $r_2$ be two roots in $V$. We have the following four sets of equivalent assertions.*

(i) $s_{r_1} \cdot r_2 = r_2$ ,
(ii) $s_{r_2} \cdot r_1 = r_1$ ,
(iii) $L_{r_1} \subseteq H_{r_2}$ *and* $L_{r_2} \subseteq H_{r_1}$ ,
*in which case we say that $r_1$ and $r_2$ are orthogonal and we write $r_1 \perp r_2$.*

(i) $s_{r_1} \cdot r_2 = \zeta_{r_1} r_2$ ,
(ii) $s_{r_2} \cdot r_1 = \zeta_{r_2} r_1$ ,
(iii) $L_{r_1} = L_{r_2}$ *and* $H_{r_1} = H_{r_2}$ .
*in which case we say that $r_1$ and $r_2$ are parallel and we write $r_1 || r_2$.*

(i) $s_{r_1} s_{r_2} = s_{r_2} s_{r_1}$ ,
(ii) *$r_1$ is en eigenroot of $s_{r_2}$* ,
(iii) *$r_2$ is en eigenroot of $s_{r_1}$* .
(iv) *$r_1$ and $r_2$ are either orthogonal or parallel.*

## §2. Reflection groups

Let $\mathcal{R}$ be a set of reflections on $V$.
We denote by $G_\mathcal{R}$ (or simply $G$) the subgroup of $\mathrm{GL}(V)$ generated by the elements of $\mathcal{R}$.
Notice first that

$$\bigcap_{r \in \mathcal{R}} H_r = V^G \text{ , the set of elements fixed by } G\,.$$

**Definition.** We say that $\mathcal{R}$ is complete if it is stable under $G_\mathcal{R}$-conjugation.

From now on, we assume that $\mathcal{R}$ is complete. Thus $G_\mathcal{R}$ is a (normal) subgroup of the subgroup of $\mathrm{GL}(V)$ which stabilises $\mathcal{R}$.

Let us set

$$V_\mathcal{R} := \sum_{r \in \mathcal{R}} L_r\,.$$

Since $\mathcal{R}$ is complete, the subspace $V_\mathcal{R}$ is stable by the action of $G$.

From now on, we shall assume that

**T***he action of $G$ on $V$ is completely reducible.*

*Remark.* Notice that the preceding hypothesis is satisfied

- when $G$ is finite and $k$ of characteristic zero,
- or when $k$ is a subfield of the field $\mathbb{C}$ of complex numbers and $G$ preserves a positive nondegenerate hermitian form on $V$.

**2.1. Proposition.** *Assume that the action of $G$ on $V$ is completely reducible.*

(1)
$$V = V_{\mathcal{R}} \oplus V^G ,$$

(2) *The restriction from $V$ down to $V_{\mathcal{R}}$ induces an isomorphism from $G$ onto its image in* $\mathrm{GL}(V_{\mathcal{R}})$.

*Proof of 2.1.*

(1) The subspace $V_{\mathcal{R}}$ is $G$-stable, hence there is a supplementary subspace $V'$ which is $G$-stable. Whenever $r \in \mathcal{R}$, the space $V_{\mathcal{R}}$ contains the one-dimensional non trivial eigenspace for $s_r$, hence (by lemma 1.6) $V'$ is contained in $H_r$ ; it follows that $V' \subseteq \bigcap_{r\in\mathcal{R}} H_r$. So it suffices to prove that $V_{\mathcal{R}} \cap V^G = 0$.

(2) Since $V^G$ is stable by $G$, there exists a supplementary subspace $V$" which is stable by $G$. Whenever $s_r \in \mathcal{R}$, we have $L_r \subseteq V$" (otherwise, by lemma 1.6, we have $V$" $\subseteq H_r$, which implies that $s_r$ is trivial since $V = V^G \oplus V$", a contradiction). This shows that $V_{\mathcal{R}} \subseteq V$", and in particular that $V_{\mathcal{R}} \cap V^G = 0$. □

**Orthogonal decomposition.**

We denote by $\sim$ the equivalence relation on the set of roots (or on the set of $k^\times$-orbits on roots, *i.e.*, on the set of reflections) as the transitive closure of the relation "*$r$ and $r'$ are not orthogonal*".

**2.2. Lemma.** *If $r \sim r'$ and if $g \in G$, then $g \cdot r \sim g \cdot r'$. In particular the equivalence classes of $\mathcal{R}$ are $G$-stable.*

*Proof of 2.2.* It suffices to prove that the relation "*being orthogonal*" is stable under $G$, which is obvious. □

Notice that the number of equivalence classes is finite : it is bounded by the dimension of $V$.

Indeed, assume that $r_1 = (v_1, v_1{}^*)\,, r_2 = (v_2, v_2{}^*) \ldots, r_m = (v_m, v_m{}^*)$ are mutually not equivalent (*i.e.*, belong to distinct equivalence classes). Let us check that $(v_1, v_2, \ldots, v_m)$ is linearly independant. Assume $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_m v_m = 0$. The scalar product with $v_i{}^*$ yields $\lambda_i \langle v_i, v_i^* \rangle = 0$, hence $\lambda_i = 0$.

Let $\mathcal{R} = \mathcal{R}_1 \,\dot\cup\, \mathcal{R}_2 \,\dot\cup \ldots \dot\cup\, \mathcal{R}_n$ be the decomposition of $\mathcal{R}$ into equivalence classes.

Let us denote by $G_i$ the subgroup of $G$ generated by the reflections $s_r$ for $r \in \mathcal{R}_i$, and by $V_i$ the subspace of $V$ generated by the spaces $L_r$ for $r \in \mathcal{R}_i$.

The following properties are straightforward.

**2.3. Lemma.**

(1) *The group* $G_i$ *acts trivially on* $\sum_{j\neq i} V_j$.
(2) *For* $1 \leq i \neq j \leq n$, $G_i$ *and* $G_j$ *commute.*
(3) $G = G_1 G_2 \dots G_n$.

**2.4. Proposition.** *Assume that the action of* $G$ *on* $V$ *is completely reducible.*

(1) *For* $1 \leq i \leq n$, *the action of* $G_i$ *on* $V_i$ *is irreducible.*
(2) $V_{\mathcal{R}} = \bigoplus_{i=1}^{i=n} V_i$.
(3) $G = G_1 \times G_2 \times \cdots \times G_n$

*Proof of 2.4.*

(1) The subspace $V_i$ is stable under $G$, and the action of $G$ on a stable subspace is completely reducible. But the image of $G$ in $\mathrm{GL}(V_i)$ is the same as the image of $G_i$. So the action of $G_i$ on $V_i$ is completely reducible.

Assume that $\mathcal{R} = \mathcal{R}_i$ is a single equivalence class, so $V = V_{\mathcal{R}}$ and $G = G_i$. Let us prove that $V$ is irreducible for $G$. Since $V$ is completely reducible for $G$ we may assume that $V = V' \oplus V''$ where $V'$ and $V''$ are stable by $G$, and prove that $V'$ or $V''$ equals $V$. Let us define

$$\mathcal{R}' := \{s_r \in \mathcal{R} \mid L_r \subseteq V'\} \quad \text{and} \quad \mathcal{R}'' := \{s_r \in \mathcal{R} \mid L_r \subseteq V''\}.$$

Then by lemma 1.6, we see that whenever $r \in \mathcal{R}'$, then $V'' \subseteq H_r$, and whenever $r \in \mathcal{R}''$, then $V' \subseteq H_r$, which shows that any two elements of $\mathcal{R}'$ and $\mathcal{R}''$ are mutually orthogonal. Thus one of them has to be all of $\mathcal{R}$.

(2) By 2.3, we have $\sum_{j\neq i} V_j \subset V^{G_i}$. By (1), and by 2.1, we then get $V_i \cap \sum_{j\neq i} V_j = 0$.

(3) An element of $g \in G_i$ which also belongs to $\prod_{j\neq i} G_j$ acts trivially on $V_i$. Since (by (1) and by 2.1) the representation of $G_i$ on $V_i$ is faithful, we see that $g = 1$. □

**The Shephard–Todd classification.**

Here we assume that $k = \mathbb{C}$, the field of complex numbers.

*The general infinite family* $G(de, e, r)$.

Let $d$, $e$ and $r$ be three positive integers.

- Let $D_r(de)$ be the set of diagonal complex matrices with diagonal entries in the group $\boldsymbol{\mu}_{de}$ of all $de$–th roots of unity.
- The $d$–th power of the determinant defines a surjective morphism

$$\det{}^d \colon D_r(de) \twoheadrightarrow \boldsymbol{\mu}_e .$$

  Let $A(de, e, r)$ be the kernel of the above morphism. In particular we have $|A(de, e, r)| = (de)^r/e$.
- Identifying the symmetric group $\mathfrak{S}_r$ with the usual $r \times r$ permutation matrices, we define

$$G(de, e, r) := A(de, e, r) \rtimes \mathfrak{S}_r .$$

We have $|G(de,e,r)| = (de)^r r!/e$, and $G(de,e,r)$ is the group of all monomial $r \times r$ matrices, with entries in $\boldsymbol{\mu}_{de}$, and product of all non-zero entries in $\boldsymbol{\mu}_d$.

Let $(x_1, x_2, \dots, x_r)$ be a basis of $V$. Let us denote by $(\Sigma_j(x_1, x_2, \dots, x_r))_{1 \leq j \leq r}$ the family of fundamental symmetric polynomials. Let us set

$$\begin{cases} f_j := \Sigma_j(x_1^{de}, x_2^{de}, \dots, x_r^{de}) & \text{for } 1 \leq j \leq r-1\,, \\ f_r := (x_1 x_2 \cdots x_r)^d\,. \end{cases}$$

Then we have

$$\mathbb{C}[x_1, x_2, \dots, x_r]^{G(de,e,r)} = \mathbb{C}[f_1, f_2, \dots, f_r]\,.$$

*Examples.*

- $G(e,e,2)$ is the dihedral group of order $2e$.
- $G(d,1,r)$ is isomorphic to the wreath product $\boldsymbol{\mu}_d \wr \mathfrak{S}_r$. For $d = 2$, it is isomorphic to the Weyl group of type $B_r$ (or $C_r$).
- $G(2,2,r)$ is isomorphic to the Weyl group of type $D_r$.

*About the exceptional groups.*

There are 34 exceptional irreducible complex reflection groups, of ranks from 2 to 8, denoted $G_4, G_5, \dots, G_{37}$.

The rank 2 groups are connected with the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ (the *binary polyhedral groups*).

**2.5. Theorem.** (Shephard–Todd) *Let $(V,W)$ be an irreducible complex reflection group. Then one of the following assertions is true :*

- *There exist integers $d, e, r$, with $de \geq 2$, $r \geq 1$ such that $(V,W) \simeq G(de,e,r)$.*
- *There exists an integer $r \geq 1$ such that $(V,W) \simeq (\mathbb{C}^{r-1}, \mathfrak{S}_r)$.*
- *$(V,W)$ is isomorphic to one of the 34 exceptional groups $G_n$ $(n = 4, \dots, 37)$.*

*Field of definition.*

The following theorem has been proved (using a case by case analysis) by Bessis [Bes1] (see also [Ben]), and generalizes a well known result on Weyl groups.

**2.6. Theorem–Definition.**

*Let $(V,W)$ be a reflection group. Let $K$ be the field generated by the traces on $V$ of all elements of $W$. Then all irreducible $KW$–representations are absolutely irreducible.*

*The field $K$ is called the field of definition of the reflection group $W$.*

- If $K \subseteq \mathbb{R}$, the group $W$ is a (finite) *Coxeter group*.
- If $K = \mathbb{Q}$, the group $W$ is a Weyl group.

**2.7. Question.** *Find a "conceptual" proof of theorem 2.6.*

**Reflecting pairs.**

Here we make the following hypothesis

- $V$ is a $k$-vector space of dimension $r$,
- $G$ is a finite subgroup of $\mathrm{GL}(V)$ ; we denote by $\mathrm{Ref}(G)$ the set of all reflections of $G$,
- the order $|G|$ of $G$ is not divisible by the characteristic of $k$ ; in particular the $kG$-module $V$ is completely reducible.

Let $X$ be a subspace of $V$.

We denote by $N_G(X)$, as "normaliser", the stabiliser of $X$ in $G$, *i.e.*, the set of $g \in G$ such that $g(X) = X$.

We denote by $G(X)$ (or $C_G(X)$, as "centraliser") the fixator of $X$, *i.e.*, the set of $g \in G$ such that, for all $x \in X$, $g(x) = x$.

Notice that $G(X) \triangleleft N_G(X)$ and that $N_G(X)/G(X)$ is naturally isomorphic to a subgroup of $\mathrm{GL}(X)$.

**2.8. Definition.**

(1) Let $H$ be a hyperplane of $V$. We say that $H$ is a *reflecting hyperplane* for $G$ if there exists $g \in G$, $g \neq 1$, such that $\ker(g-1) = H$.

(2) Let $L$ be a line in $V$. We say that $L$ is a *reflecting line* for $G$ if there exists $g \in G$, $g \neq 1$, such that $\mathrm{im}\,(g-1) = H$.

For $H$ a reflecting hyperplane, notice that

$$G(H) = \{1\} \bigcup \{g \in G \mid \ker(g-1) = H\}\,.$$

For $L$ a reflecting hyperplane, we set

$$G(V/L) := \{1\} \bigcup \{g \in G \mid \mathrm{im}\,(g-1) = L\}\,.$$

Notice that $G(V/L)$ is a group : this is the group of all elements of $G$ which stabilize $L$ and which act trivially on $V/L$, a normal subgroup of $N_G(L)$.

**2.9. Proposition.**

(1) *Let $H$ be a reflecting hyperplane for $G$. There exists a unique reflecting line $L$ such that $G(H) = G(V/L)$.*

(2) *Let $L$ be a reflecting line for $G$. There exists a unique reflecting hyperplane $H$ such that $G(V/L) = G(H)$.*

*If $L$ and $H$ are as above, we say that $(L, H)$ is a reflecting pair for $G$, and we set $G(H, V/L) := G(H) = G(V/L)$.*

(3) *If $(L, H)$ is a reflecting pair, then*
   *(a) $G(H, V/L)$ consists in the identity and of reflections $s_r$ where $H_r = H$ and $L_r = L$,*
   *(b) $G(H, V/L)$ is a cyclic group, isomorphic to a subgroup of $k^\times$,*
   *(c) we have $N_G(H) = N_G(L)$.*

*Proof of 2.9.*

• Assume $G(H) \neq \{1\}$. Since the action of $G(H)$ on $V$ is completely reducible, there is a line $L$ which is stable by $G(H)$ and such that $H \oplus L = V$. Such a line is obviously the eigenspace (corresponding to an eigenvalue different from 1) for any non trivial element of $G(H)$. This shows that $L$ is uniquely determined, and that $G(H)$ consists of 1 and of reflections with hyperplane $H$ and line $L$. It follows also that $G(H) \subseteq G(V/L)$. Notice that $H$ and $L$ are the isotypic components of $V$ under the action of $G(H)$.

• Assume $G(V/L) \neq \{1\}$. Since the action of $G(V/L)$ on $V$ is completely reducible, there is a hyperplane $H$ which is stable by $G(V/L)$ and such that $L \oplus H = V$. Such an hyperplane is clearly the kernel of any nontrivial element of $G(V/L)$. This shows that $H$ is uniquely determined, and that $G(V/L) \subseteq G(H)$.

We let the reader conclude the proof. $\square$

# CHAPTER I
# PREREQUISITES AND COMPLEMENTS IN COMMUTATIVE ALGEBRA

## §3. Finite ring extensions

**3.1. Proposition.** *Let $A$ be a subring of a ring $B$, and let $x \in B$. The following assertions are equivalent :*

(i) *The element $x$ is integral over $A$, i.e., there exists a monic polynomial $P(t) = t^n + a_1 t^{n-1} + \cdots + a_{n-1}t + a_n \in A[t]$ such that $P(x) = 0$.*
(ii) *The subring $A[x]$ of $B$ generated by $A$ and $x$ is a finitely generated $A$–module.*
(iii) *There exists a subring $A'$ of $B$, containing $A[x]$, which is a finitely generated $A$–module.*

The proof is classical and is left to the reader.

**Properties and definitions.**

**I1.** If $A$ is a subring of $B$, the set of elements of $B$ which are integral over $A$ is a subring of $B$, called the integral closure of $A$ in $B$.

For $S$ a multiplicatively stable subset of $A$, if $\overline{A}$ denotes the integral closure of $A$ in $B$, then $S^{-1}\overline{A}$ is the integral closure of $S^{-1}A$ in $S^{-1}B$

**I2.** One says that $B$ is integral over $A$ if it is equal to the integral closure of $A$.

**I3.** One says that $B$ is finite over $A$ (or that $B/A$ is finite) if $B$ is a finitely generated $A$–module.

The following assertions are equivalent :

(i) The extension $B/A$ is finite.
(ii) $B$ is a finitely generated $A$–algebra and is integral over $A$.
(iii) $B$ is generated as an $A$–algebra by a finite number of elements which are integral over $A$.

**I4.** An integral domain $A$ is said to be integrally closed if it is integrally closed in its field of fractions.

**Examples.**

- A unique factorisation domain, a Dedekind domain are integrally closed.
- The polynomial ring $A[t_1, t_2, \ldots, t_r]$ is integrally closed if and only if $A$ is integrally closed (see for example [Bou2], chap.5, §1, n$^o$3).

**I5.** If $B$ is integral over $A$, then $B$ is a field if and only if $A$ is a field.

**Spectra and finite extensions.**

In all the sequel, we suppose $B/A$ finite.

**3.2. Proposition (Cohen–Seidenberg Theorem).** *The map* $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *is surjective : for each* $\mathfrak{p} \in \operatorname{Spec}(A)$*, there exists* $\mathfrak{q} \in \operatorname{Spec}(B)$ *such that* $\mathfrak{q} \cap A = \mathfrak{p}$ *(we then say that* $\mathfrak{q}$ *"lies above* $\mathfrak{p}$*"). Moreover,*

(1) *If both* $\mathfrak{q}_1$ *and* $\mathfrak{q}_2$ *lie above* $\mathfrak{p}$*, then* $\mathfrak{q}_1 \subset \mathfrak{q}_2$ *implies* $\mathfrak{q}_1 = \mathfrak{q}_2$*,*
(2) *If* $\mathfrak{p}_1, \mathfrak{p} \in \operatorname{Spec}(A)$ *with* $\mathfrak{p}_1 \subset \mathfrak{p}$*, and if* $\mathfrak{q}_1 \in \operatorname{Spec}(B)$ *lies above* $\mathfrak{p}_1$*, then there exists* $\mathfrak{q} \in \operatorname{Spec}(B)$ *which lies above* $\mathfrak{p}$ *and such that* $\mathfrak{q}_1 \subset \mathfrak{q}$*.*
(3) *For each* $\mathfrak{p} \in \operatorname{Spec}(A)$*, there is only a finite number of prime ideals of* $B$ *which lie above* $\mathfrak{p}$*.*

*Proof of 3.2.* We localize at $\mathfrak{p}$ : the extension $B_{\mathfrak{p}}/A_{\mathfrak{p}}$ is finite, and the prime ideals of $B$ which lie above $\mathfrak{p}$ correspond to the prime ideals of $B_{\mathfrak{p}}$ which lie above $\mathfrak{p}A_{\mathfrak{p}}$. Since $\mathfrak{p}A_{\mathfrak{p}}$ is maximal in $A_{\mathfrak{p}}$, the proposition thus follows from the following lemma.

**3.3. Lemma.** *Suppose that* $B/A$ *is finite.*

(1) *The map* $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ *induces a surjective map*

$$\operatorname{MaxSpec}(B) \twoheadrightarrow \operatorname{MaxSpec}(A)\,.$$

(2) *Any prime ideal of* $B$ *which lies above a maximal ideal of* $A$ *is also maximal.*

*Proof of 3.3.*

To prove that $\mathfrak{n}$ is a maximal ideal of $B$ if and only if $\mathfrak{n} \cap A$ is a maximal ideal of $A$, we divide by $\mathfrak{n}$, and we now have to prove that, if $B$ is integral over $A$, with $B$ an integral domain, then $B$ is a field if and only if $A$ is a field (see I5 above).

To prove the surjectivity of the map $\operatorname{MaxSpec}(B) \to \operatorname{MaxSpec}(A)$, it suffices to prove that, for $\mathfrak{m} \in \operatorname{MaxSpec}(A)$, we have $\mathfrak{m}B \neq B$. Now if $\mathfrak{m}B = B$, then there exists $a \in \mathfrak{m}$ such that $(1-a)B = 0$ (it is left to the reader to prove that), whence $1 - a = 0$ and $1 \in \mathfrak{m}$. □

□

**Case of integrally closed rings.**

**3.4. Proposition.** *Let* $A$ *and* $B$ *be integrally closed rings with field of fractions* $K$ *and* $L$ *respectively. Suppose* $B$ *is a finite extension of* $A$*. Suppose the extension* $L/K$ *is normal, and let* $G := \operatorname{Aut}_K(L)$ *be the Galois group of this extension. Then, for each* $\mathfrak{p} \in \operatorname{Spec}(A)$*, the group* $G$ *acts transitively on the set of* $\mathfrak{q} \in \operatorname{Spec}(B)$ *which lie above* $\mathfrak{p}$*.*

*Proof of 3.4.*

We first suppose that the extension $L/K$ is separable, and thus is a Galois extension. Then we have $K = L^G$, so that $A = B^G$ (indeed, every element of $B^G$ is integral over $A$ and thus belongs to $K$, whence to $A$ since $A$ is integrally closed). Let $\mathfrak{q}$ and $\mathfrak{q}'$ be two prime ideals of $B$ which lie above $\mathfrak{p}$. Suppose that $\mathfrak{q}'$ is none of the $g(\mathfrak{q})$'s ($g \in G$). Then $\mathfrak{q}'$ is not contained in any of the $g(\mathfrak{q})$'s ($g \in G$), and there exists $x \in \mathfrak{q}'$ which doesn't belong to any of the $g(\mathfrak{q})$'s ($g \in G$). But then $\prod_{g \in G} g(x)$ is an element of $A \cap \mathfrak{q}'$ which doesn't belong to $A \cap \mathfrak{q}$, which is a contradiction.

We now deal with the general case. Let $p$ be the characteristic of $K$. Let $K' := L^G$. Then $L/K'$ is a Galois extension with Galois group $G$, and the extension $K'/K$ is purely inseparable, *i.e.*, for each $x \in K'$, there exists an integer $n$ such that $x^{p^n} \in K$. Let $A'$ be the integral closure of $A$ in $K'$. Then there is a unique prime ideal of $A'$ which lies above $\mathfrak{p}$, namely $\mathfrak{p}' := \{x \in A' \mid (\exists n \in \mathbb{N})(x^{p^n} \in \mathfrak{p})\}$ . Proposition 3.4 thus follows from the above case $K' = K$. $\square$

**3.5. Proposition.** *Let $A$ be an integrally closed ring and let $K$ be its field of fractions. Let $B$ be an $A$–algebra which is finite over $A$. Suppose $B$ is an integral domain and let $L$ be its field of fractions. Let $\mathfrak{p}, \mathfrak{p}_1 \in \operatorname{Spec}(A)$ be such that $\mathfrak{p} \subset \mathfrak{p}_1$, and let $\mathfrak{q}_1 \in \operatorname{Spec}(B)$ lie above $\mathfrak{p}_1$. Then there exists $\mathfrak{q} \in \operatorname{Spec}(B)$ which lies above $\mathfrak{p}$ and such that $\mathfrak{q} \subset \mathfrak{q}_1$.*

*Proof of 3.5.* Let $M$ be a finite normal extension of $K$ containing $L$ and let $C$ be the normal closure of $A$ in $M$. By 3.2, we know that there exist prime ideals $\mathfrak{r}_1$ and $\mathfrak{r}$ of $C$ which lie above $\mathfrak{q}_1$ and $\mathfrak{p}$ respectively. Since $\mathfrak{r}_1$ lies above $\mathfrak{p}_1$, we also know that there exists $\mathfrak{r}'_1 \in \operatorname{Spec}(C)$ which lies above $\mathfrak{p}_1$, and such that $\mathfrak{r} \subset \mathfrak{r}'_1$. By 3.4, there exists $g \in \operatorname{Gal}(M/K)$ such that $\mathfrak{r}_1 = g(\mathfrak{r}'_1)$. We then set $\mathfrak{q} := g(\mathfrak{r}) \cap B$. $\square$

**Krull dimension : first definitions.**

Let $A$ be a ring. A *chain of length* $n$ of prime ideals of $A$ is a strictly increasing sequence

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

of prime ideals of $A$.

• If the set of lengths of chains of prime ideals of $A$ is bounded, then the greatest of these lengths is called *Krull dimension of* $A$, and written $\operatorname{Krdim}(A)$. Otherwise, $A$ is said to have infinite Krull dimension. The Krull dimension of the ring 0 is, by definition, $-\infty$.

• If $M$ is an $A$–module, then we call *Krull dimension of* $M$ and write $\operatorname{Krdim}_A(M)$ the Krull dimension of the ring $A/\operatorname{Ann}_A(M)$. Note that

$$\operatorname{Krdim}_A(M) \leq \operatorname{Krdim}(A)\,.$$

• For $\mathfrak{p} \in \operatorname{Spec}(A)$, we call *height of* $\mathfrak{p}$ and write $\operatorname{ht}(\mathfrak{p})$ the Krull dimension of the ring $A_{\mathfrak{p}}$. Thus $\operatorname{ht}(\mathfrak{p})$ is the maximal length of chains of prime ideals of $A$ whose greatest element is $\mathfrak{p}$.

The height of $\mathfrak{p}$ is also sometimes called *codimension of* $\mathfrak{p}$.

3.6. *Some properties.*

(1) $\operatorname{Krdim}(A) = \sup\{\operatorname{ht}(\mathfrak{m})\}_{\mathfrak{m} \in \operatorname{MaxSpec}(A)}$ ,
(2) $\operatorname{Krdim}(A/\operatorname{Nilrad}(A)) = \operatorname{Krdim}(A)$ .
(3) If $B$ is an $A$–algebra which is finite over $A$, then $\operatorname{Krdim}(B) = \operatorname{Krdim}(A)$ .

**3.7. Proposition.** *Let $k$ be a field. The Krull dimension of the algebra of polynomials in $r$ indeterminates $k[t_1, t_2, \ldots, t_r]$ over $k$ is $r$.*

*Proof of 3.7.* We first note that there exists a chain of prime ideals of length $r$, namely the sequence $0 \subset (t_1) \subset (t_1, t_2) \subset \cdots \subset (t_1, t_1, \ldots, t_r)$ . It is therefore sufficient to prove that the Krull dimension of $k[t_1, t_2, \ldots, t_r]$ is at most $r$.

If $K/k$ is a field extension, then we denote by $\operatorname{trdeg}_k(K)$ its transcendance degree.

**3.8. Lemma.** *Let $A$ and $B$ be two integral domains which are finitely generated $k$–algebras, with field of fractions $K$ and $L$ respectively. Suppose there exists a surjective $k$–algebra homomorphism $f\colon A \to B$ .*

(1) *We have* $\mathrm{trdeg}_k(L) \leq \mathrm{trdeg}_k(K)$.
(2) *If* $\mathrm{trdeg}_k(L) = \mathrm{trdeg}_k(K)$*, then $f$ is an isomorphism.*

*Proof of 3.8.*

(1) Any generating system for $A$ as $k$–algebra is also a generating system for $K$ over $k$. Thus $K$ has a finite transcendance degree over $k$, and, if this degree is $n$ and if $n \neq 0$, then there exists a system of $n$ algebraically independant elements in $A$ which is a basis of transcendance for $K$ over $k$. The same conclusion applies to $B$ and $L$. Now, by inverse image by $f$, any $k$–algebraically independant system of elements of $B$ can be lifted to a system of $k$–algebraically independant elements of $A$. This proves the first assertion.

(2) Suppose that $\mathrm{trdeg}_k(L) = \mathrm{trdeg}_k(K)$.

If $\mathrm{trdeg}_k(L) = \mathrm{trdeg}_k(K) = 0$, then $K$ and $L$ are algebraic extensions of $k$, and, for each $a \in K$, $a$ and $f(a)$ have the same minimal polynomial over $k$. This proves that the kernel of $f$ is just 0.

Suppose now that $\mathrm{trdeg}_k(L) = \mathrm{trdeg}_k(K) = n > 0$. We know (cf. proof of (1) above) that there exists a basis of transcendance $(a_1, a_2, \dots, a_n)$ for $K$ over $k$ which consists of elements of $A$, and such that $(f(a_1), f(a_2), \dots, f(a_n))$ is a basis of transcendance for $L$ over $k$. In particular, we see that the restriction of $f$ to $k[a_1, a_2, \dots, a_n]$ is an isomorphism onto $k[f(a_1), f(a_2), \dots, f(a_n)]$, and induces an isomorphism

$$k(a_1, a_2, \dots, a_n) \xrightarrow{\sim} k(f(a_1), f(a_2), \dots, f(a_n)) \, .$$

If $a \in A$ has minimal polynomial $P(t)$ over $k(a_1, a_2, \dots, a_n)$, then $f(a)$ has minimal polynomial $f(P(t))$ over $k(f(a_1), f(a_2), \dots, f(a_n))$, which proves that $f$ is injective, whence is an isomorphism. □

Let then $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ be a chain of prime ideals of $k[t_1, t_2, \dots, t_r]$. Applying the above lemma to the sequence of algebras $k[t_1, t_2, \dots, t_r]/\mathfrak{p}_j$, we see that, for each $j$ $(0 \leq j \leq n)$, writing $K_j$ for the field of fractions of $k[t_1, t_2, \dots, t_r]/\mathfrak{p}_j$, we have $\mathrm{trdeg}_k K_j \leq \mathrm{trdeg}_k K_0 - j \leq r - j$ . It follows in particular that $n \leq r$. □

**3.9. Corollary.** *Let $A$ be an integral domain which is a finitely generated algebra over a field $k$. Let $K$ be its field of fractions. Then*

$$\mathrm{Krdim}(A) = \mathrm{trdeg}_k(K) \, .$$

**3.10. Proposition.** *Let $A = k[x_1, x_2, \dots, x_r]$ be a finitely generated algebra over a field $k$, generated by $r$ elements $x_1, x_2, \dots, x_r$. We have* $\mathrm{Krdim}(A) \leq r$ , *and* $\mathrm{Krdim}(A) = r$ *if and only if $x_1, x_2, \dots, x_r$ are algebraically independant.*

*Proof of 3.10.* Consider $r$ indeterminates $t_1, t_2, \dots, t_r$. Let $\mathfrak{A}$ be the kernel of the homomorphism from the polynomial algebra $k[t_1, t_2, \dots, t_r]$ to $A$ such that $t_j \mapsto x_j$. The algebra $A$ is isomorphic to $k[t_1, t_2, \dots, t_r]/\mathfrak{A}$. We thus see that $\mathrm{Krdim}(A) \leq r$. Moreover, if $\mathrm{Krdim}(A) = r$, then we see that

$$\mathrm{Krdim}(k[t_1, t_2, \dots, t_r]) = \mathrm{Krdim}(k[t_1, t_2, \dots, t_r]/\mathfrak{A}) \, ,$$

whence $\mathfrak{A} = 0$ since 0 is a prime ideal of $k[t_1, t_2, \dots, t_r]$. □

## §4. Jacobson Rings and Hilbert's Nullstellensatz

**On maximal ideal of polynomial algebras.**

Let $A$ be a commutative ring (with unity), and let $A[X]$ be a polynomial algebra over $A$.

Whenever $\mathfrak{A}$ is an ideal of $A[X]$, let us denote by $\overline{A}$ and $x$ respectively the images of $A$ and $x$ through the natural epimorphism $A[X] \twoheadrightarrow A[X]/\mathfrak{A}$. Thus we have

$$\overline{A} = A/A \cap \mathfrak{A} \quad \text{and} \quad A[X] = \overline{A}[x]\,.$$

Note that if $\mathfrak{P}$ is a prime ideal of $A[X]$, then $\mathfrak{P} \cap A$ is a prime ideal of $A$. We shall be concerned by the case of *maximal* ideals.

Let us point out two very different behaviour of maximal ideals of $A[X]$ with respect to $A$.

- If $\mathfrak{M}$ is a maximal ideal of $\mathbb{Z}[X]$, then $\mathfrak{M} \cap \mathbb{Z} \neq \{0\}$ (this will be proved below : see 4.1, (3)).
  As a consequence, a maximal ideal $\mathfrak{M}$ of $\mathbb{Z}[X]$ can be described as follows : there is a prime number $p$ and a polynomial $P(X) \in \mathbb{Z}[X]$ which becomes irreducible in $(\mathbb{Z}/p\mathbb{Z})[X]$ such that $\mathfrak{M} = p\mathbb{Z}[X] + P(X)\mathbb{Z}[X]$.
  Thus the quotients of $\mathbb{Z}[X]$ by maximal ideals are the finite fields.
- Let $p$ be a prime number, and let $\mathbb{Z}_p := \{a/b \in \mathbb{Q} \mid p \nmid b\}$. Then $\mathbb{Z}_p[1/p] = \mathbb{Q}$, which shows that $\mathfrak{M} := (1 - pX)\mathbb{Z}_p[X]$ is a maximal ideal of $\mathbb{Z}_p[X]$. Notice that here $\mathfrak{M} \cap \mathbb{Z}_p = \{0\}$.

Let us try to examine these questions through the following proposition.

**4.1. Proposition.**

(1) *If there is* $\mathfrak{M} \in \mathrm{Spec}^{\max}(A[X]$ *such that* $\mathfrak{M} \cap A = \{0\}$, *then there exists* $a \in A^* := A - \{0\}$ *such that* $(1 - aX)A[X] \in \mathrm{Spec}^{\max}(A[X]$.
*In other words : if there exists $x$ in an extension of $A$ such that $A[x]$ is a field, then there is $a \in A^*$ such that $A[1/a]$ is a field.*

(2) *Let* $\mathrm{Spec}^*(A)$ *be the set of all nonzero prime ideals of* $A$. *We have*

$$\bigcap_{\mathfrak{p} \in \mathrm{Spec}^*(A)} \mathfrak{p} = \{0\} \cup \{a \in A^* \mid A[1/a] \text{ is a field}\}\,.$$

(3) *Assume* $\bigcap_{\mathfrak{p} \in \mathrm{Spec}^*(A)} \mathfrak{p} = \{0\}$. *Then for all* $\mathfrak{M} \in \mathrm{Spec}^{\max}(A[X]$ *we have* $\mathfrak{M} \cap A \neq \{0\}$.
*In other words : there is no $x$ such that $A[x]$ is a field.*

*Proof of 4.1.*

(1) Assume that $A[x]$ is a field. Then $A$ is an integral domain, and if $F$ denotes its field of fractions, we have $A[x] = F[x]$. Since $F[x]$ is a field, $x$ is algebraic over $F$, hence a root of a polynomial with coefficients in $A$. If $a$ is the coefficient of the highest degree term of that polynomial, $x$ is integral over $A[1/a]$. Whence $A[x]$ is integral over $A[1/a]$, and since $A[x]$ is a field, it follows that $A[1/a]$ is a field.

(2) Assume first that $a \in \bigcap_{\mathfrak{p} \in \mathrm{Spec}^*(A)} \mathfrak{p}$ and $a \neq 0$. We must show that $A[1/a]$ is a field.

There is a maximal ideal $\mathfrak{M}$ of $A[X]$ containing $(1-aX)A[X]$.

- We then have $\mathfrak{M} \cap A = \{0\}$. Indeed, if it were not the case, we would have $\mathfrak{M} \cap A \in \mathrm{Spec}^*(A)$, hence $a \in \mathfrak{M} \cap A$, then $a \in \mathfrak{M}$, $aX \in \mathfrak{M}$, so $1 \in \mathfrak{M}$.
- Let $x$ be the image of $X$ in $A[X]/\mathfrak{M}$. Thus $A[x]$ is a field. But $1 - ax = 0$, proving that $x = 1/a$ and $A[1/a]$ is a field.

Assume now that $A[1/a]$ is a field, hence that $(1-aX)A[X] \in \mathrm{Spec}^{\max}(A[X])$. Let $\mathfrak{p} \in \mathrm{Spec}^*(A)$. Then $\mathfrak{p} \not\subseteq (1-aX)A[X]$, since $(1-aX)A[X] \cap A = \{0\}$. It follows that $\mathfrak{p}A[X] + (1-aX)A[X] = A$. Interprated in the polyomial ring $(A/\mathfrak{p})[X]$, that equality shows that the polynomial $1 - \overline{a}X$ is invertible, which implies that $\overline{a} = 0$, *i.e.*, $a \in \mathfrak{P}$.

(3) Assume that $A[x]$ is a field. By (1), there is $a \in A^*$ such that $A[1/a]$ is a field. By (2), we know that $a \in \bigcap_{\mathfrak{p} \in \mathrm{Spec}^*(A)} \mathfrak{p}$, a contradiction. $\square$

*Remark.* The assertion (3) of the preceding proposition shows in particular that if $A$ is a principal ideal domain with infinitely many prime ideals (like $\mathbb{Z}$ or $k[X]$ for example), then whenever $\mathfrak{M} \in \mathrm{Spec}^{\max}(A[X])$, we have $\mathfrak{M} \cap A \neq \{0\}$, hence $\mathfrak{M} \cap A \in \mathrm{Spec}^{\max}(A)$.

**4.2. Proposition–Definition.** *The following assertions are equivalent*

(J1) *Whenever* $\mathfrak{p} \in \mathrm{Spec}(A)$, *we have*

$$\mathfrak{p} = \bigcap_{\substack{\mathfrak{m} \in \mathrm{Spec}^{\max}(A) \\ \mathfrak{p} \subseteq \mathfrak{m}}} \mathfrak{m}\,.$$

(J2) *Whenever* $\mathfrak{M} \in \mathrm{Spec}^{\max}(A[X])$, *we have* $\mathfrak{M} \cap A \in \mathrm{Spec}^{\max}(A)$.

*A ring which fulfills the preceding conditions is called a Jacobson ring.*

*Proof of 4.2.*

Let us first notice that both properties (J1) and (J2) transfer to quotients : if $A$ satisfies (J1) (respectively (J2)), and if $\mathfrak{a}$ is an ideal of $A$, then $A/\mathfrak{a}$ satisfies (J1) (respectively (J2) as well.

Let us show (J1) $\Longrightarrow$ (J2). Let $\mathfrak{M} \in \mathrm{Spec}^{\max}(A[X])$. We set $A[X]/\mathfrak{M} = (A/\mathfrak{M} \cap A)[x]$.

We have $\mathfrak{M} \cap A \in \mathrm{Spec}(A)$, hence $\mathfrak{M} \cap A$ is an intersection of maximal ideals of $A$. If $\mathfrak{M} \cap A$ is not maximal, it is an intersection of maximal ideals in which it is properly contained, thus in the ring $A/\mathfrak{M} \cap A$, we have

$$\bigcap_{\mathfrak{p} \in \mathrm{Spec}^*(A/\mathfrak{M} \cap A)} \mathfrak{p} = \{0\}\,,$$

which shows (by 4.1, (3)) that $(A/\mathfrak{M} \cap A)[x]$ cannot be a field, a contradiction.

Let us show (J2) $\Longrightarrow$ (J1). Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Working in $A/\mathfrak{p}$, we see that it suffices to prove that if $A$ is an integral domain which satisfies (J2), then the intersection of maximal ideals is $\{0\}$.

Let $a \in \bigcap_{\mathfrak{m} \in \mathrm{Spec}^{\max}(A)} \mathfrak{m}$. Thus whenever $\mathfrak{M} \in \mathrm{Spec}^{\max}(A[X])$, we have $a \in \mathfrak{M}$, hence $aX \in \mathfrak{M}$, which proves that $1 - aX$ is invertible, hence $a = 0$. $\square$

Let us emphasize the defining property of Jacobson rings, by stating the following proposition (which is nothing but a reformulation of property (J2)).

**4.3. Proposition.** *The following two assertions are equivalent :*

(i) *$A$ is a Jacobson ring.*
(ii) *If $\overline{A}[x]$ is a quotient of $A[X]$ which is a field, then $\overline{A}$ is a field and $x$ is algebraic over $\overline{A}$.*

*Remark.* Let us immediately quote some examples and counterexamples of Jacobson rings :

- Examples of Jacobson rings : fields, principal ideal domains with infinitely many prime ideals, quotients of Jacobson rings.
- Non Jacobson rings : discrete valuation rings.

The next theorem enlarges the set of examples of Jacobson ring to all the finitely generated algebras over a Jacobson ring.

**4.4. Theorem.** *Let $A$ be a Jacobson ring.*

(1) *$A[X]$ is a Jacobson ring.*
(2) *If $B$ is a finitely generated $A$–algebra, then $B$ is a Jacobson ring.*

**4.5. Corollary.**

(1) *Let $A$ be a Jacobson ring. Assume that $\overline{A}[v_1, v_2, \dots, v_r]$ is a finitely generated $A$–algebra which is a field. Then $\overline{A}$ is a field, and $\overline{A}[v_1, v_2, \dots, v_r]$ is an algebraic (hence finite) extension of $\overline{A}$.*
(2) *Let $k$ be a field. If $k[v_1, v_2, \dots, v_r]$ is a finitely generated $k$–algebra which is a field, then it is an algebraic (hence finite) extension of $k$.*
(3) *Let $k$ be an algebraically closed field. If $k[v_1, v_2, \dots, v_r]$ is a finitely generated $k$–algebra which is a field, then it coincides with $k$.*

Assertion (3) of the preceding corollary may be reformulated as Hilbert's Nullstellensatz.

**Hilbert's Nullstellensatz.** *Let $k$ be an algebraically closed field. The map*

$$\begin{aligned} &k^r \longrightarrow \operatorname{Spec}^{\max}(k[v_1, v_2, \dots, v_r]) \\ &(\lambda_1, \lambda_2, \dots, \lambda_r) \mapsto \langle v_1 - \lambda_1, v_2 - \lambda_2, \dots, v_r - \lambda_r \rangle \end{aligned}$$

*is a bijection.*

*Proof of 4.4.*
Let us prove (1).
Let $\mathfrak{M}$ be a maximal ideal of $A[X, Y]$. We set

$$\begin{aligned} &\overline{A} := A/\mathfrak{M} \cap A\,, \\ &\overline{A}[x] := A[X]/\mathfrak{M} \cap A[X] \ \text{ and } \ \overline{A}[y] := A[Y]/\mathfrak{M} \cap A[Y]\,, \\ &\overline{A}[x, y] := A[X, Y]/\mathfrak{M}\,. \end{aligned}$$

We have to prove that $\overline{A}[x]$ is a field.

Since $\overline{A}[x, y]$ is a field, $\overline{A}$ is an integral domain, and if $k$ denotes its field of fractions, we have $\overline{A}[x, y] = k[x, y]$.

Since $k[x,y] = k[x][y]$ is a field, $x$ is not transcendental (by 4.1, (3)) over $k$, hence $k[x]$ is a field. As in the proof of 4.1, (1), we see that there exists $a \in A^*$ such that $x$ is integral over $\overline{A}[1/a]$.

Similarly, there exists $b \in A^*$ such that $y$ is integral over $\overline{A}[1/b]$. It follows that $\overline{A}[x,y]$ is integral over $\overline{A}[1/ab]$. Since $\overline{A}[x,y]$ is a field, it implies that $\overline{A}[1/ab]$ is a field.

Now since $A$ is a Jacobson ring, it follows from Proposition 3 that $\overline{A}$ is a field, *i.e.*, $\overline{A} = k$. We have already seen that $k[x]$ is a field, proving that $\overline{A}[x]$ is a field.

Let us prove (2).

By induction on $r$, it follows from (1) that, for all $r$, $A[v_1, v_2, \ldots, v_r]$ is a Jacobson ring. So are the quotients of these algebras, which are the finitely generated $A$–algebras. □

*Proof of 4.5.*

(1) Assume that $\overline{A}[v_1, v_2, \ldots, v_r]$ is a field. Since $\overline{A}[v_1, v_2, \ldots, v_{r-1}]$ is a Jacobson ring (by theorem 4, (2)), it follows from Proposition 3 that $\overline{A}[v_1, v_2, \ldots, v_{r-1}]$ is a field over which $v_r$ is algebraic. Repeating the argument leads to the required statement.

(2) and (3) are immediate consequences of (1) □

**Radicals and Jacobson rings, application to algebraic varieties.**

**4.6. Proposition–Definition.**

(1) *The Jacobson radical of a ring $A$ is the ideal*

$$\mathrm{Rad}(A) := \bigcap_{\mathfrak{m} \in \mathrm{Spec}^{\max}(A)} \mathfrak{m}\,.$$

*The Jacobson radical coincides with the set of elements $a \in A$ such that, for all $x \in A$, $(1 - ax)$ is invertible.*

(2) *The nilradical of a ring $A$ is the ideal*

$$\mathrm{Nilrad}(A) := \bigcap_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathfrak{p}\,.$$

*The nilradical coincides with the set of nilpotent elements of $A$.*

*Proof of 4.6.* We prove only (2). It is clear that any nilpotent element of $A$ belongs to Nilrad$(A)$. Let us prove the converse.

Whenever $\mathfrak{M}$ is a maximal ideal of $A[X]$, we know that $\mathfrak{M} \cap A$ is a prime ideal of $A$. It implies that $\mathrm{Nilrad}(A) \subset \mathrm{Rad}(A)$, and thus for $a \in \mathrm{Nilrad}(A)$, the polynomial $(1 - aX)$ is invertible, which implies that $a$ is nilpotent. □

Now if $A$ is a Jacobson ring, it follows from 4.2 that

$$\mathrm{Rad}(A) = \mathrm{Nilrad}(A)\,.$$

Applying that remark to a quotient $A/\mathfrak{a}$ of a Jacobson ring, we get the following proposition.

**4.7. Proposition.** *Let $A$ be a Jacobson ring, and let $\mathfrak{a}$ be an ideal of $A$. We have*

$$\bigcap_{\substack{\mathfrak{m}\in\mathrm{Spec}^{\max}(A)\\ \mathfrak{a}\subseteq\mathfrak{m}}} \mathfrak{m} = \{a \in A \mid (\exists n \geq 0)(a^n \in \mathfrak{a})\}\,.$$

Applying the preceding proposition to the case where $A = k[X_1, X_2, \ldots, X_r]$ for $k$ algebraicaly closed gives the “strong form” of Hilbert’s Nullstellensatz.

**4.8. Corollary (Strong Nullstellensatz).** *Let $k$ be an algebraically closed field. For $\mathfrak{A}$ an ideal of $k[X_1, X_2, \ldots, X_r]$, let us set*

$$\mathcal{V}(\mathfrak{A}) := \{(\lambda_1, \lambda_2, \ldots, \lambda_r) \in k^r \mid (\forall P \in \mathfrak{A})(P(\lambda_1, \lambda_2, \ldots, \lambda_r) = 0)\}\,.$$

*If $Q \in k[X_1, X_2, \ldots, X_r]$ is such that*

$$(\forall(\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathcal{V}(\mathfrak{A}))(Q(\lambda_1, \lambda_2, \ldots, \lambda_r)) = 0)\,,$$

*then there exists $n \geq 0$ such that $Q^n \in \mathfrak{A}$.*

*Proof of 4.8.* Translating *via* the dictionary $k^r \longleftrightarrow \mathrm{Spec}^{\max}(k[X_1, X_2, \ldots, X_r])$, we see that

$$\mathcal{V}(\mathfrak{A}) \longleftrightarrow \{\mathfrak{M} \in \mathrm{Spec}^{\max}(k[X_1, X_2, \ldots, X_r]) \mid \mathfrak{A} \subseteq \mathfrak{M}\}\,,$$

while the hypothesis on $Q$ translates to

$$Q \in \bigcap_{\substack{\mathfrak{M}\in\mathrm{Spec}^{\max}(k[X_1,X_2,\ldots,X_r])\\ \mathfrak{A}\subseteq\mathfrak{M}}} \mathfrak{M}\,.$$

□

## §5. Graded algebras and modules

### Graded modules.

Let $k$ be a ring. We call *graded $k$–module* any $k$–module of the form

$$M = \bigoplus_{n=-\infty}^{n=\infty} M_n$$

where, for each $n$, $M_n$ is a finitely generated $k$–module, and $M_n = 0$ whenever $n < N$ for some integer $N$ (*i.e.*, “for $n$ small enough”).

For each integer $n$, the non-zero elements of $M_n$ are said to be *homogeneous of degree $n$*. If $x = \sum_n x_n$ where $x_n \in M_n$, then the element $x_n$ is called the *homogeneous component of degree $n$* of $x$.

A graded module homomorphism $M \to N$ is a linear map $f\colon M \to N$ such that, for each $n \in \mathbb{Z}$, we have $f(M_n) \subset N_n$.

From now on, we suppose that $k$ *is a field*. The graded $k$–modules are then called *graded $k$–vector spaces*.

We set $\mathbb{Z}((q)) := \mathbb{Z}[[q]][q^{-1}]$, the ring of formal Laurent series with coefficients in $\mathbb{Z}$. *The graded dimension* of $M$ is the element of $\mathbb{Z}((q))$ defined by

$$\mathrm{grdim}_k(M) := \sum_{n=-\infty}^{\infty} \dim_k(M_n) q^n\,.$$

**Elementary constructions.**

Direct sum : if $M$ and $N$ are two graded modules, then the graded module $M \oplus N$ is defined by the condition $(M \oplus N)_n := M_n \oplus N_n$. If $k$ is a field, then we have

$$\mathrm{grdim}_k(M \oplus N) = \mathrm{grdim}_k(M) + \mathrm{grdim}_k(N)\,.$$

Tensor product : if $M$ and $N$ are two graded modules, then the graded module $M \otimes N$ is defined by the condition $(M \otimes N)_n := \bigoplus_{i+j=n} M_i \otimes N_j$. If $k$ is a field, then we have

$$\mathrm{grdim}_k(M \otimes N) = \mathrm{grdim}_k(M)\mathrm{grdim}_k(N)\,.$$

Shift : if $M$ is a graded module and $m$ is an integer, then the graded module $M[m]$ is defined by the condition $M[m]_n := M_{m+n}$. If $k$ is a field, then we have

$$\mathrm{grdim}_k(M[m]) = q^{-m}\mathrm{grdim}_k(M)\,.$$

**Examples.** Let $k$ be a field.

- If $t$ is transcendental over $k$ and of degree $d$, then we have $\mathrm{grdim}_k(k[t]) = 1/(1-q^d)$.
- More generally, if $t_1, t_2, \dots, t_r$ are algebraically independant elements over $k$ of degree $d_1, d_2, \dots, d_r$ respectively, then we have $k[t_1, t_2, \dots, t_r] \simeq k[t_1] \otimes k[t_2] \otimes \cdots \otimes k[t_r]$ and
$$\mathrm{grdim}_k(k[t_1, t_2, \dots, t_r]) = \frac{1}{(1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_r})}\,.$$
- If $M$ has dimension 1 and is generated by an element of degree $d$, then we have $M \simeq k[-d]$, and $\mathrm{grdim}_k(M) = q^d$.
- If $V$ is a vector space of finite dimension $r$, then the symmetric algebra $S(V)$ and the exterior algebra $\Lambda(V)$ of $V$ are naturally endowed with structures of graded vector spaces, and we have
$$\mathrm{grdim}_k(S(V)) = \frac{1}{(1-q)^r} \quad \text{and} \quad \mathrm{grdim}_k(\Lambda(V)) = (1+q)^r\,.$$

A linear map $f \colon M \to N$ between two graded vector spaces is said to be of degree $m$ if, for all $n$, we have $f(M_n) \subset N_{n+m}$. Thus, a map of degree $m$ defines a homomorphism from $M$ to $N[m]$.

Suppose then that

$$0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$$

is an exact sequence of $k$–vector spaces, where $M'$, $M$ and $M''$ are graded, and where $\alpha$ and $\beta$ are maps of degree $a$ and $b$ respectively. We then have an exact sequence of graded vector spaces

$$0 \to M' \xrightarrow{\alpha} M[a] \xrightarrow{\beta} M''[a+b] \to 0\,,$$

whence the formula

$$\mathrm{grdim}_k(M'') - q^b\mathrm{grdim}_k(M) + q^{a+b}\mathrm{grdim}_k(M') = 0\,.$$

**Koszul complex.**

Let $V$ be a vector space of dimension $r$. Let $S := S(V)$ and $\Lambda := \Lambda(V)$. The Koszul complex is the complex

$$\begin{array}{ccccccccc} 0 & \to & S\otimes\Lambda^r & \xrightarrow{\delta_r} & S\otimes\Lambda^{r-1} & \xrightarrow{\delta_{r-1}} & \cdots & \xrightarrow{\delta_1} & S\otimes\Lambda^0 \\ &&&&&&&& \downarrow \\ &&&&&&&& k \\ &&&&&&&& \downarrow \\ &&&&&&&& 0 \end{array}$$

where the homomorphism $S\otimes\Lambda^0 \to k$ is the homomorphism defined by $v \mapsto 0$ for all $v \in V$, and where the homomorphism $\delta_j$ is defined in the following way :

$$\delta_j(y\otimes(x_1\wedge\cdots\wedge x_j)) = \sum_i (-1)^{i+1} y x_i \otimes (x_1\wedge\cdots\wedge\widehat{x_i}\wedge\cdots\wedge x_j)\,.$$

If we endow $S\otimes\Lambda^j$ with the graduation of $S$, the homomorphism $\delta_j$ has thus degree 1, and the homomorphism $S\otimes\Lambda^0 \to k$ has degree 0.

One can prove (see for example [Be], lemma 4.2.1) that the Koszul complex is exact. It follows that

$$1 = \sum_{j=0}^{j=r} (-1)^j q^j \dim(\Lambda^j)\mathrm{grdim}_k(S)\,,$$

or, equivalently,

$$1 = \mathrm{grdim}_k(\Lambda)(-q)\mathrm{grdim}_k(S)(q)\,.$$

**Graded algebras and modules.**

Let $k$ be a (noetherian) ring. We call graded $k$–algebra any finitely generated algebra over $k$ of the form $A = \bigoplus_{n=0}^{\infty} A_n$ , with $A_0 = k$, and $A_nA_m \subset A_{n+m}$ for any integers $n$ and $m$. We then write $\mathfrak{M}$ for the maximal ideal of $A$ defined by $\mathfrak{M} := \bigoplus_{n=1}^{\infty} A_n$ .

A graded $A$–module $M$ is then a (finitely generated) $A$–module of the form $M = \bigoplus_{n=-\infty}^{n=\infty} M_n$ where $A_nM_m \subset M_{n+m}$ for all $n$ and $m$, and where $M_n$ is zero if $n < N$ for some integer $N$.

Each homogeneous component $M_n$ is a finitely generated $k$–module.

Indeed, $A$ is a noetherian ring, and we have $M_n \simeq \bigoplus_{m\geq n} M_m / \bigoplus_{m>n} M_m$ , which proves that $M_n$ is finitely generated over $A/\mathfrak{M}$.

A graded $A$–module homomorphism is an $A$–module homomorphism which is a graded $k$–module homomorphism.

A submodule $N$ of a graded $A$–module is an $A$–submodule such that the natural injection is a graded $k$–module homomorphism, *i.e.*, such that $N = \bigoplus_n (N\cap M_n)$ .

A graded (or "homogeneous") ideal of $A$ is a graded submodule of $A$, seen as graded module over itself. If $\mathfrak{a}$ is an ideal of $A$, then the following conditions are equivalent :

(i) $\mathfrak{a}$ is a graded ideal,
(ii) $\mathfrak{a} = \bigoplus_n (\mathfrak{a}\cap A_n)$,
(iii) for all $a \in \mathfrak{a}$, each homogeneous component of $a$ belongs to $\mathfrak{a}$,
(iv) $\mathfrak{a}$ is generated by homogeneous elements.

**The Hilbert–Serre Theorem.**

**5.1. Theorem.** *Let $k$ be a field. Let $A = k[x_1, x_2, \dots, x_r]$ be a graded $k$–algebra, generated by homogeneous elements of degree $d_1, d_2, \dots, d_r$ respectively. Let $M$ be a graded $A$–module. Then there exists $P(q) \in \mathbb{Z}[q, q^{-1}]$ such that the graded dimension of $M$ over $k$ is*

$$\mathrm{grdim}_k(M) = \frac{P(q)}{(1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_r})}\,.$$

*Proof of 5.1.* We use induction on $r$. The theorem is obvious if $r = 0$, so we suppose that $r > 0$. Let $M'$ and $M''$ be the kernel and cokernel of multiplication by $x_r$ respectively. We thus have the following exact sequence of graded $A$–modules :

$$0 \to M' \to M \xrightarrow{x_r} M[d_r] \to M''[d_r] \to 0\,,$$

whence the equality

$$q^{d_r}\mathrm{grdim}_k(M') - q^{d_r}\mathrm{grdim}_k(M) + \mathrm{grdim}_k(M) - \mathrm{grdim}_k(M'') = 0\,.$$

Now $M'$ and $M''$ are both graded modules over $k[x_1, \dots, x_{r-1}]$, so that, by the induction hypothesis, there exist $P'(q), P''(q) \in \mathbb{Z}[q, q^{-1}]$ such that

$$\mathrm{grdim}_k(M') = \frac{P'(q)}{(1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_{r-1}})}$$

and

$$\mathrm{grdim}_k(M'') = \frac{P''(q)}{(1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_{r-1}})}\,.$$

The theorem follows immediately. □

**Nakayama's lemma.**

Let $k$ be a (commutative) field, and let $A$ a graded $k$-algebra.

*Convention.*

We make the convention that

- "ideal of $A$" means "graded ideal of $A$",
- "element of $A$" means "homogeneous element of $A$".

It can be shown that the *"graded Krull dimension"* of $A$, (*i.e.*, the maximal length of chains of (graded) prime ideals of $A$) coincides with its "abstract" Krull dimension (*i.e.*, the maximal length of chains of any prime ideals of $A$).

*Nakayama's lemma.*

With the above conventions, Nakayama's lemma takes the following form.

**5.2. Proposition.** *Let $A$ be a graded $k$–algebra, with maximal ideal $\mathfrak{M}$, and let $M$ be an $A$–module. If $\mathfrak{M}M = M$, then $M = 0$.*

*Proof of 5.2.* Indeed, then we know that there exists $a \in \mathfrak{M}$ such that $(1 - a)M = 0$. If $M \neq 0$, then let $m$ be a non-zero (homogeneous) element of $M$. The equality $m = am$ yields a contradiction. □

**5.3. Corollary.**

(S1) *If $M'$ is a submodule of the $A$–module $M$, then $M' = M$ if and only if $M = M' + \mathfrak{M}M$.*

(S2) *If $f \colon M \to N$ is an $A$–module homomorphism which induces a surjection from $M$ onto $N/\mathfrak{M}N$, then $f$ is surjective.*

(S3) *A system $(x_1, x_2, \dots, x_s)$ of elements of $M$ is a generating system for $M$ if and only if its image in $M/\mathfrak{M}M$ is a generating system of the $k$–vector space $M/\mathfrak{M}M$. In particular, all the minimal generating systems have the same order, which is the dimension of $M/\mathfrak{M}M$ over $k$.*

*Proof of 5.3.*
For (S1), we apply 5.2 to the module $M/M'$.
For (S2), we apply (S1) to the module $N$ and the submodule $f(M)$.
For (S3), we apply $(S2)$ to the module $F := \bigoplus_j A[-\deg(x_j)]$ and the homomorphism $F \to M$ defined by the system we consider. □

If $M$ is an $A$–module, we write $r(M)$ and call *rank of $M$* the dimension of $M/\mathfrak{M}M$ over $k$.

**5.4. Corollary.** *Let $R$ be a graded algebra, with maximal graded ideal $\mathfrak{M}$. Let $(u_1, u_2, \dots, u_n)$ be a family of homogeneous elements of $R$ with positive degrees.*

(1) *The following assertions are equivalent :*
   *(i)* $R = k[u_1, u_2, \dots, u_n]$,
   *(ii)* $\mathfrak{M} = Ru_1 + Ru_2 + \cdots + Ru_n$,
   *(iii)* $\mathfrak{M}/\mathfrak{M}^2 = ku_1 + ku_2 + \cdots + ku_n$.

(2) *Assume moreover that $R$ is a graded polynomial algebra with Krull dimension $r$. Then the following assertions are equivalent :*
   *(i) $n = r$, $(u_1, u_2, \dots, u_r)$ are algebraically independant, and $R = k[u_1, u_2, \dots, u_r]$,*
   *(ii) $(u_1, u_2, \dots, u_n)$ is a minimal set of generators of the $R$–module $\mathfrak{M}$,*
   *(iii) $(u_1, u_2, \dots, u_n)$ is a basis of the $k$–vector space $\mathfrak{M}/\mathfrak{M}^2$.*

*Proof of 5.4.*
(1) The implications (i)⇒(ii)⇒(iii) are clear. The implication (iii)⇒(ii) is a direct application of Nakayama's lemma to the $R$–module $\mathfrak{M}$. Finally if (ii) holds, the image of $k[u_1, u_2, \dots, u_n]$ mudulo $\mathfrak{M}$ is $k$, hence $k[u_1, u_2, \dots, u_n] = R$ again by Nakayama's lemma.

(2) The equivalence between (ii) and (iii) follows from Nakayama's lemma. If (i) holds, then $(u_1, u_2, \dots, u_n)$ generates $\mathfrak{M}$ by (1), and if it contains a proper system of generators of $R$, say $(u_1, u_2, \dots, u_m)$ $(m < r)$ then again by (1) we have $R = k[u_1, u_2, \dots, u_m]$, a contradiction with the hypothesis about the Krull dimension of $R$.

Assume (iii) holds. Since $R$ is a polynomial algebra with Krull dimension $r$, and since (i)⇒(iii), we see that the dimension of $\mathfrak{M}/\mathfrak{M}^2$ is $r$. Hence $n = r$, and since $R = k[u_1, u_2, \dots, u_r]$ (by (1)), we see that $(u_1, u_2, \dots, u_r)$ is algebraically independant (otherwise the Krull dimension of $R$ would be less than $r$). □

**5.5. Proposition.** *Let $A$ be a graded $k$–algebra, and let $M$ be a finitely generated projective $A$–module. Then $M$ is free.*

*Proof of 5.5.* Let $\mathfrak{M} := \sum_{n\geq 1} A_n$ be the unique maximal ideal of $A$. Then $M/\mathfrak{M}M$ is a (left) finite dimensional vector space over the field $k$. Let $d$ denote its dimension. The isomorphism $k^d = (A/\mathfrak{M})^d \xrightarrow{\sim} M/\mathfrak{M}M$ can be lifted (by projectivity of $A^d$) to a morphism $A^d \to M$ , which is onto by Nakayama's lemma. Since $M$ is projective, we get a split short exact sequence

$$0 \to M' \to A^d \to M \to 0\,.$$

Note that $M'$ is then a direct summand of $A^d$, hence is also finitely generated. Tensoring with $k = A/\mathfrak{M}A$, this exact sequence gives (since it is split) the short exact sequence

$$0 \to M'/MM' \to k^d \to M/\mathfrak{M}M \to 0\,,$$

which shows that $M'/\mathfrak{M}M' = 0$, hence again by Nakayama's lemma $M' = 0$. Thus we get that $M$ is isomorphic to $A^d$. $\square$

## §6. Polynomial algebras and parameters subalgebras

**Degrees and Jacobian.**

Let $S = k[v_1, v_2, \dots, v_r]$ be a polynomial graded algebra over the field $k$, where $(v_1, v_2, \dots, v_r)$ is a family of algebraically independant, homogeneous elements, with degrees respectively $e_1, e_2, \dots, e_r$. Assume $e_1 \leq e_2 \leq \cdots \leq e_r$.

Let $(u_1, u_2, \dots, u_r)$ be a family of homogeneous elements with degrees $d_2, d_2, \dots, d_r$ such that $d_1 \leq d_2 \leq \cdots \leq d_r$.

**6.1. Proposition.** *Assume that $(u_1, u_2, \dots, u_r)$ is algebraically free.*

(1) *For all $i$ $(1 \leq i \leq r)$, we have $e_i \leq d_i$.*
(2) *We have $e_i = d_i$ for all $i$ $(1 \leq i \leq r)$ if and only if $S = k[u_1, u_2, \dots, u_r]$.*

*Proof of 6.1.*

(1) Let $i$ such that $1 \leq i \leq r$. The family $(u_1, u_2, \dots, u_i)$ is algebraically free, hence it cannot be contained in $k[v_1, v_2, \dots, v_{i-1}]$. Hence there exist $j \geq i$ and $l \leq i$ such that $v_j$ does appear in $u_l$. It follows that $e_j \leq u_l$, hence $e_i \leq e_j \leq d_l \leq d_i$.

(2) We know that $\mathrm{grdim} R = (\prod_{i=1}^{i=r}(1-q^{e_i}))^{-1}$ . Thus it suffices to prove that $\prod_{i=1}^{i=r}(1-q^{e_i}) = \prod_{i=1}^{i=r}(1-q^{d_i})$ if and only if $e_i = d_i$ for all $i$ $(1 \leq i \leq r)$, which is left as an exercise. $\square$

By 6.1, we see in particular that the family $(e_1, e_2, \dots, e_r)$ (with $e_1 \leq e_2 \leq \cdots \leq e_r$) is uniquely determined by $R$. Such a family is called *the family of degrees* of $R$.

Let us now examine the algebraic independance of the system $(u_1, u_2, \dots, u_r)$.

**Definition.** The Jacobian of $(u_1, u_2, \dots, u_r)$ relative to $(v_1, v_2, \dots, v_r)$ is the homogeneous element of degree $\sum_i (d_i - e_i)$ defined by

$$\mathrm{Jac}((u_1, u_2, \dots, u_r)/(v_1, v_2, \dots, v_r)) := \det(\frac{\partial u_i}{\partial v_j})_{i,j}\,.$$

**6.2. Proposition.**

(1) $\mathrm{Jac}((u_1, u_2, \dots, u_r)/(v_1, v_2, \dots, v_r))$ *is a homogeneous element of* $S$ *with degree* $\sum_i (d_i - e_i)$.

(2) *The family* $(u_1, u_2, \dots, u_r)$ *is algebraically free if and only if*

$$\mathrm{Jac}((u_1, u_2, \dots, u_r)/(v_1, v_2, \dots, v_r)) \neq 0\,.$$

(3) *We have* $k[u_1, u_2, \dots, u_r] = k[v_1, v_2, \dots, v_r]$ *if and only if*

$$\mathrm{Jac}((u_1, u_2, \dots, u_r)/(v_1, v_2, \dots, v_r)) \in k^\times\,.$$

*Proof of 6.2.*

(1) is trivial.

Proof of (2).

(a) Assume that $(u_1, u_2, \dots, u_r)$ is algebraically dependant.

Let $P(t_1, t_2, \dots, t_r) \in k[t_1, t_2, \dots, t_r]$ be a minimal degree polynomial subject to the condition $P(u_1, u_2, \dots, u_r) = 0$. Let us differentiate that equality relatively to $v_j$ :

$$\sum_i \frac{\partial P}{\partial t_i}(u_1, u_2, \dots, u_r)\frac{\partial u_i}{\partial v_j} = 0\,.$$

There is $i$ such that $\dfrac{\partial P}{\partial t_i} \neq 0$, and by minimality of $P$ we have $\dfrac{\partial P}{\partial t_i}(u_1, u_2, \dots, u_r) \neq 0$, which shows that the matrix $(\dfrac{\partial u_i}{\partial v_j})_{i,j}$ is singular and so that

$$\mathrm{Jac}((u_1, u_2, \dots, u_r)/(v_1, v_2, \dots, v_r)) = 0\,.$$

(b) Assume that $(u_1, u_2, \dots, u_r)$ is algebraically free.

For each $i$, let us denote by $P_i(t_0, t_1, \dots, t_r) \in k[t_0, t_1, \dots, t_r]$ a polynomial with minimal degree such that $P_i(v_i, u_1, u_2, \dots, u_r) = 0$. Let us differentiate that equality relatively to $v_j$ :

$$\frac{\partial P_i}{\partial t_0}(v_i, u_1, u_2, \dots, u_r) + \sum_l \frac{\partial P_i}{\partial t_l}(v_i, u_1, u_2, \dots, u_r)\frac{\partial u_l}{\partial v_j} = 0\,,$$

which can be rewritten as an identity between matrices :

$$(\frac{\partial P_i}{\partial t_l}(v_i, u_1, u_2, \dots, u_r))_{i,l} \cdot (\frac{\partial u_l}{\partial v_j})_{l,j} = -D(\frac{\partial P_i}{\partial t_0}(v_i, u_1, u_2, \dots, u_r)_i)\,,$$

where $D((\lambda_i)_i)$ denotes the diagonal matrix with spectrum $(\lambda_i)_i$.

Since, for all $i$, we have $\dfrac{\partial P_i}{\partial t_0}(v_i, u_1, u_2, \dots, u_r) \neq 0$ (by minimality of $P_i$), we see that the matrix $(\dfrac{\partial u_l}{\partial v_j})_{l,j}$ is nonsingular.

(3) follows from 6.1 and from (1). $\square$

**Systems of parameters.**

Let $A$ be a finitely generated graded $k$–algebra.

**Definition.** A system of parameters of $A$ is a family $(x_1, x_2, \ldots, x_r)$ of homogeneous elements in $A$ such that

(P1) $(x_1, x_2, \ldots, x_r)$ is algebraically free,
(P2) $A$ is a finitely generated $k[x_1, x_2, \ldots, x_r]$–module.

We ask the reader to believe, to prove, or to check in the appropriate literature the following fondamental result.

**6.3. Theorem.**

(1) *There exists a system of parameters.*
(2) *All systems of parameters have the same cardinal, equal to* $\mathrm{Krdim}(A)$*.*
(3) *If* $(x_1, x_2, \ldots, x_m)$ *is a system of homogeneous elements of* $A$ *such that* $m \leq \mathrm{Krdim}(A)$ *and if* $A$ *is finitely generated as a* $k[x_1, x_2, \ldots, x_m]$*–module, then* $m = \mathrm{Krdim}(A)$ *and* $(x_1, x_2, \ldots, x_m)$ *is a system of parameters of* $A$*.*
(4) *The following assertions are equivalent.*
*(i) There is a system of parameters* $(x_1, x_2, \ldots, x_r)$ *of* $A$ *such that* $A$ *is a free module over* $k[x_1, x_2, \ldots, x_r]$*.*
*(ii) Whenever* $(x_1, x_2, \ldots, x_r)$ *is a system of parameters of* $A$*,* $A$ *is a free module over* $k[x_1, x_2, \ldots, x_r]$*.*
*In that case we say that* $A$ *is a Cohen-Macaulay algebra.*

We shall now give some characterizations or systems of parameters of a polynomial algebra.

In what follows, we denote by
- $k$ an algebraically closed field,
- $S = k[v_1, v_2, \ldots, v_r]$ a polynomial algebra, where $(v_1, v_2, \ldots, v_r)$ is a family of homogeneous algebraically independant elements with degrees $(e_1, e_2, \ldots, e_r)$,
- $(u_1, u_2, \ldots, u_r)$ is a family of nonconstant homogeneous elements of $S$ with degrees respectively $(d_1, d_2, \ldots, d_r)$
- $R := k[u_1, u_2, \ldots, u_r]$, and $\mathfrak{M}$ the maximal graded ideal of $R$.

**6.4. Proposition.**

(1) *The following assertions are equivalent.*
*(i)* $(x = 0)$ *is the unique solution in* $k^r$ *of the system*

$$u_1(x) = u_2(x) = \cdots = u_r(x) = 0\,.$$

*(ii)* $S/\mathfrak{M}S$ *is a finite dimensional* $k$*–vector space.*
*(iii)* $S$ *is a finitetely generated* $R$*–module.*
*(iv)* $(u_1, u_2, \ldots, u_r)$ *is a system of parameters of* $S$*.*
(2) *If the preceding conditions hold, then*
*(a)* $S$ *is a free* $R$*–module, and its rank is* $\dfrac{\prod_i d_i}{\prod_i e_i}$.

*(b) The map*

$$\begin{cases} k^r \longrightarrow k^r \\ x \mapsto (u_1(x), u_2(x), \dots, u_r(x)) \end{cases}$$

*is onto.*

*Proof of 6.4.*

Let us prove (1)

• (i)⇒(ii). Since $S/\mathfrak{M}S$ is a finitely generated $k$–algebra, it suffices to prove that $S/\mathfrak{M}S$ is algebraic over $k$. Since the set $\mathcal{V}(\mathfrak{M}S)$ of zeros of $\mathfrak{M}S$ reduces to $\{0\}$ by assumption, and since all the indeterminates $v_i$ vanish on that set, it follows from the strong Nullstellensatz that for all $i$ there is an integer $n_i \geq 1$ such that $v_i^{n_i} \in \mathfrak{M}S$, hence $v_i^{n_i} = 0$ in $S/\mathfrak{M}S$, proving that $S/\mathfrak{M}S$ is indeed an algebraic extension of $k$.

• (ii)⇒(iii) results from Nakayama lemma.

• (iii)⇒(iv) results from the general properties of systems of parameters (see 6.3, (3)).

• (iv)⇒(i). Let $\mathcal{V}(\mathfrak{M}S)$ be the set of zeros of $\mathfrak{M}S$. In order to prove that $\mathcal{V}(\mathfrak{M}S) = \{0\}$, it suffices to prove that $\mathcal{V}(\mathfrak{M}S)$ is finite. Indeed, if it contains a nonzero element $x$, it contains $\lambda x$ for all $\lambda \in k$.

Let us prove that $|\mathcal{V}(\mathfrak{M}S)| \leq \dim(S/\mathfrak{M}S)$. Let $x_1, x_2, \dots, x_n \in \mathcal{V}(\mathfrak{M}S)$ be pairwise distinct. Consider the map

$$\begin{cases} S \longrightarrow k^n \\ u \mapsto (u(x_1), u(x_2), \dots, u(x_n)) \end{cases}$$

That map factorizes through $S/\mathfrak{M}S$. But the interpolation theorem shows that it is onto, which proves that $n \leq \dim(S/\mathfrak{M}S)$.

*Remark : The interpolation theorem.* Let $V$ be a $k$–vector space with dimension $r$, and let $S$ be its symmetric algebra, isomorphic to the algebra polynomial in $r$ indeterminates. Let $x_1, x_2, \dots, x_n$ be pairwise distinct elements of $V$. Then the map

$$\begin{cases} S \longrightarrow k^n \\ u \mapsto (u(x_1), u(x_2), \dots, u(x_n)) \end{cases}$$

is onto.

Indeed, for each pair $(i, j)$ with $i \neq j$, let us choose a linear form $t_{i,j} \colon V \to k$ such that $t_{i,j}(x_i) \neq t_{i,j}(x_j)$. Then the polynomial function $u_i$ on $V$ defined by

$$u_i(v) := \prod_{i \neq j} \frac{t_{i,j}(v) - t_{i,j}(x_j)}{t_{i,j}(x_i) - t_{i,j}(x_j)}$$

satisfies $u_i(x_j) = \delta_{i,j}$ .

Let us prove (2)

(a) Since $S$ is free over itself, it is Cohen-Macaulay (see 6.3, (4), hence is free over $R$. Thus we have

$$S \simeq R \otimes_k (S/\mathfrak{M}S)\,, \text{ which implies } \operatorname{grdim}(S) = \operatorname{grdim}(R)\operatorname{grdim}(S/\mathfrak{M}S)\,.$$

It follows that

$$\operatorname{grdim}(S/\mathfrak{M}S) = \frac{\prod_i (1 + q + \cdots + q^{d_i - 1})}{\prod_i (1 + q + \cdots + q^{e_i - 1})}$$

hence

$$\dim(S/\mathfrak{M}S) = \operatorname{grdim}(S/\mathfrak{M}S)_{q=1} = \frac{\prod_i d_i}{\prod_i e_i}\,.$$

(b) Let $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in k^r$. We are looking for $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_r) \in k^r$ such that, for all $i$ ($1 \leq i \leq r$), we have $u_i(\underline{\mu}) = \lambda_i$.

Consider the maximal ideal $\mathfrak{M}_{\underline{\lambda}}$ of $R$ defined by $\underline{\lambda}$, *i.e.*, the kernel of the morphism

$$\varphi_{\underline{\lambda}} \colon \begin{cases} R = k[u_1, u_2, \ldots, u_r] \longrightarrow k \\ u_i \mapsto \lambda_i\,. \end{cases}$$

By Cohen-Seidenberg theorem, there is a maximal ideal $\mathfrak{N}$ of $S$ such that $\mathfrak{N} \cap R = \mathfrak{M}_{\underline{\lambda}}$. By Nullstellensatz, there is $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_r) \in k^r$ such that $\mathfrak{N} = \mathfrak{N}_{\underline{\mu}}$, *i.e.*,, $\mathfrak{N}$ is the kernel of the morphism

$$\psi_{\underline{\mu}} \colon \begin{cases} S = k[v_1, v_1, \ldots, v_r] \longrightarrow k \\ v_i \mapsto \mu_i\,. \end{cases}$$

which, restricted to $R$, is $\varphi_{\underline{\lambda}}$. Thus for all $i$ we have $u_i(\underline{\mu}) = \lambda_i$. $\square$

**The Chevalley Theorem.**

**6.5. Theorem.** *Let $S$ a polynomial algebra : there exist a system $(v_1, v_2, \ldots, v_r)$ of homogeneous algebraically independant elements such that $S = k[v_1, v_2, \ldots, v_r]$. Let $R$ be a graded subalgebra of $S$ such that $S$ is a finitely generated $R$–module.*

*The following assertions are equivalent ;*

(i) *$S$ is a free $R$–module,*

(ii) *$R$ is a polynomial algebra : whenever $(u_1, u_2, \ldots, u_n)$ is a system of homogeneous elements of $R$ which is a generating system for the maximal graded ideal $\mathfrak{M}$ of $R$, and such that $n$ is minimal for that property, then $n = r$, $R = k[u_1, u_2, \ldots, u_r]$, and $(u_1, u_2, \ldots, u_n)$ is algebraically independant.*

*Proof of 6.5.*

The implication (ii)⇒(i) results from the fact that $S$ is Cohen–Macaulay (see [parameters].

The implication (i)⇒(ii) has a natural homological proof (see for example [Se2]) : in order to prove that $R$ is a regular graded algebra, it suffices to prove that it has finite global dimension, which results easily from the same property for $S$ and from the fact that $S$ is free over $R$. We provide below a selfcontained and elementary proof, largely inspired by [Bou1], chap. V, §5, Lemme 1.

Let $(u_1, u_2, \ldots, u_n)$ be a system of homogeneous elements of $R$ which is a generating system for the maximal graded ideal $\mathfrak{M}$ of $R$, and assume that $n$ is minimal for that property. It is clear that $R$ is generated by $(u_1, u_2, \ldots, u_n)$ as a $k$–algebra. We shall prove that $(u_1, u_2, \ldots, u_n)$ is algebraically independant (from which it results that $n = r$).

Assume not. Let $k[t_1, t_2, \ldots, t_n]$ be the polynomial algebra in $n$ indeterminates, graduated by $\deg t_i := \deg u_i$. Let $P(t_1, t_2, \ldots, t_m) \in k[t_1, t_2, \ldots, t_m]$ be a homogeneous polynomial with minimal degree such that

$$P(u_1, u_2, \ldots, u_n) = 0\,.$$

Let us set $\delta_i := \dfrac{\partial P}{\partial t_i}(u_1, u_2, \dots, u_n)$ and let us denote by $\delta\mathfrak{M}$ the (graded) ideal of $R$ generated by $(\delta_1, \delta_2, \dots, \delta_n)$.

Choose $I \subseteq \{1, 2, \dots, n\}$ minimal such that $\delta\mathfrak{M}$ is generated by the family $(\delta_i)_{i\in I}$. So we have

$$(\forall j \notin I)\ \ \delta_j = \sum_{i\in I} a_{i,j}\delta_i \quad \text{with } a_{i,j} \in R\,.$$

Since we have for all $l$

$$0 = \frac{\partial P}{\partial v_l}(u_1, u_2, \dots, u_n) = \sum_{i=1}^{i=n} \delta_i \frac{\partial u_i}{\partial v_l}(u_1, u_2, \dots, u_n)\,,$$

replacing $\delta_j$ (for $j \notin I$) by its value we get

$$(*) \qquad \sum_{i\in I} \delta_i \left( \frac{\partial u_i}{\partial v_l} + \sum_{j\notin I} a_{i,j} \frac{\partial u_j}{\partial v_l} \right) = 0$$

Let us set $x_{i,l} := \dfrac{\partial u_i}{\partial v_l} + \sum_{j\in I} a_{i,j} \dfrac{\partial u_j}{\partial v_l}$ so that the relation (*) becomes

$$(*) \qquad \sum_{i\in I} x_{i,l}\, \delta_i = 0\,.$$

- We shall prove that $x_{i,l} \in \mathfrak{M}S$.

For that purpose, let us remember the hypothesis by introducing a basis $(e_\alpha)_\alpha$ of $S$ as an $R$–module. We have

$$x_{i,l} = \sum_\alpha \lambda_{i,l;\alpha}\, e_\alpha$$

with $\lambda_{i,l;\alpha} \in R$. We want to prove that, for all $i, j, \alpha$, we have $\lambda_{i,l;\alpha} \in \mathfrak{M}$.

The relation (*) implies that, for all $l$ and $\alpha$,

$$\sum_{i\in I} \lambda_{i,l;\alpha}\, \delta_i = 0\,.$$

Assume that for some $i_0, l_0, \alpha_0$, we have $\lambda_{0i,l_0;\alpha_0} \notin \mathfrak{M}$. Let us then consider the projection of the above equality onto the space of elements with degree $\deg \delta_{i_0}$. We get a relation

$$\sum_{i\in I} \lambda'_{i,l_0;\alpha_0}\, \delta_i = 0 \quad \text{where } \lambda'_{i_0,l_0;\alpha_0} \in k^\times\,,$$

*i.e.*, an expression of $\delta_{i_0}$ as linear combination of the $\delta_i$ $(i \neq i_0)$, a contradiction with the minimality of $I$.

- Let us multiply by $v_l$ both sides of the equality $x_{i,l} := \dfrac{\partial u_i}{\partial v_l} + \sum_{j\in I} a_{i,j} \dfrac{\partial u_j}{\partial v_l}$ which defines $x_{i,l}$, and then sum up over $l = 1, 2, \dots, r$. By the Euler relation, we get (for $i \in I$)

$$\deg(u_i)u_i + \sum_{j\notin I} a_{i,j} \deg(u_j)u_j = \sum_l x_{i,l} v_l\,.$$

Since $x_{i,l} \in \mathfrak{M}S$, the above equality shows that (for $i \in I$)

$$\deg(u_i)u_i + \sum_{j \notin I} a_{i,j} \deg(u_j)u_j = \sum_l x_l u_l$$

where, for all $l$, $x_l$ is a positive degree (homogeneous) element of $S$. Projecting onto the space of elements with degree $\deg(u_i)$, we get that, for all $i \in I$, $u_i$ is a linear combination (with coefficients in $S$) of the $u_j$ $(j \neq i)$.

• Since $S$ is free as an $R$–module, it results from Nakayama's lemma that any system of elements of $S$ which defines a $k$–basis of $R/\mathfrak{M}R$ is also an $R$–basis of $S$. In particular there exists a basis of $S$ over $R$ which contains 1, and so there is an $R$–linear projection $\pi\colon S \twoheadrightarrow R$.

Now if $u_i = \sum_{l \neq i} y_l u_l$ with $y_l \in S$, by applying $\pi$ to that equality we get $u_i = \sum_{l \neq i} \pi(y_l) u_l$, an $R$–linear dependance relation on the set of $(u_l)_{1 \leq l \leq n}$, a contradiction with the minimality of $n$. $\square$

# CHAPTER II
# POLYNOMIAL INVARIANTS OF LINEAR FINITE GROUPS

## §7. Finite Groups invariants

**Generalities.**

Let $B$ be an integral domain, with field of fractions $L$. Let $G$ be a finite group of automorphisms of $B$. We set $A := B^G$, the subring of $G$–fixed points of $B$, and we denote by $K$ its field of fractions.

$$\begin{tikzcd}[row sep=small] & L \arrow[dd, dash] \\ B \arrow[ur, hook] \arrow[dd, dash] & \\ & K \\ A \arrow[ur, hook] & \end{tikzcd}$$

**7.1. Proposition.**

(1) *$B$ is integral over $A$.*

(2) *Any element of $L$ can be written $\dfrac{b}{a}$ with $a \in A$ and $b \in B$. We have $K = L^G$ and $L/K$ is a Galois extension, with $G$ as Galois group.*

(3) *If $B$ is integrally closed, $A$ is also integrally closed.*

*Proof of 7.1.*

(1) Every $b \in B$ is aroot of the polynomial $P_b(t) := \prod_{g\in G}(t - g(b))$.

(2) For $b_1, b_2 \in B$ and $b_2 \neq 0$, we have

$$\frac{b_1}{b_2} = \frac{b_1 \prod_{g\in G, g\neq 1} g(b_2)}{\prod_{g\in G} g(b_2)}\,.$$

(3) An element of $K$ which is integral over $A$ is *a fortiori* integral over $B$, whence it belongs to $B$ and so to $B \cap K$. But $B \cap K = B \cap L^G = B^G = A$. $\square$

From now on we assume that $B$ is integrally closed ; hence $A$ is also *integrally closed*.

Let us recall the Cohen–Seidenberg theorems in that context (see for example [Bou2], §2, 2). We recall that the map

$$\mathrm{Spec}(B) \to \mathrm{Spec}(A) \ \text{ is defined by } \ \mathfrak{q} \mapsto A \cap \mathfrak{q}\,.$$

If $\mathfrak{p} = \mathfrak{q} \cap A$ we say that $\mathfrak{q}$ is above $\mathfrak{p}$.

**7.2. Theorem.**

(1) *The map* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is onto. Moreover*

  *(a) If* $\mathfrak{q}, \mathfrak{q}' \in \mathrm{Spec}(B)$ *are both above* $\mathfrak{p}$*, then* $\mathfrak{q} \subseteq \mathfrak{q}'$ *implies* $\mathfrak{q} = \mathfrak{q}'$*.*

  *(b) Given* $\mathfrak{p} \subseteq \mathfrak{p}'$ *in* $\mathrm{Spec}(A)$ *and* $\mathfrak{q} \in \mathrm{Spec}(B)$ *above* $\mathfrak{p}$ *there exists* $\mathfrak{q}' \in \mathrm{Spec}(B)$ *above* $\mathfrak{p}'$ *such that* $\mathfrak{q} \subset \mathfrak{q}'$ *:*

$$\begin{tikzcd} & \mathfrak{q}' \arrow[dd, dash, dashed] \\ \mathfrak{q} \arrow[ur, hook, dashed] \arrow[dd, dash] & \\ & \mathfrak{p}' \\ \mathfrak{p} \arrow[ur, hook] & \end{tikzcd}$$

  *(c) Given* $\mathfrak{p} \subseteq \mathfrak{p}'$ *in* $\mathrm{Spec}(A)$ *and* $\mathfrak{q}' \in \mathrm{Spec}(B)$ *above* $\mathfrak{p}'$ *there exists* $\mathfrak{q} \in \mathrm{Spec}(B)$ *above* $\mathfrak{p}$ *such that* $\mathfrak{q} \subset \mathfrak{q}'$*.*

$$\begin{tikzcd} & \mathfrak{q}' \arrow[dd, dash] \\ \mathfrak{q} \arrow[ur, hook, dashed] \arrow[dd, dash, dashed] & \\ & \mathfrak{p}' \\ \mathfrak{p} \arrow[ur, hook] & \end{tikzcd}$$

(2) *(Transitivity) Given* $\mathfrak{p} \in \mathrm{Spec}(A)$*, the group* $G$ *acts transitively on the set of* $\mathfrak{q} \in \mathrm{Spec}(B)$ *which are over* $\mathfrak{p}$*.*

Let $\mathfrak{q} \in \mathrm{Spec}(B)$, and let $\mathfrak{p} := A \cap \mathfrak{q}$.

• We call *decomposition group of* $\mathfrak{q}$ and we denote by $G^d(\mathfrak{q})$ (or $N_G(\mathfrak{q})$) the subgroup of $G$ consisting in those $g \in G$ such that $g(\mathfrak{q}) = \mathfrak{q}$.

Since $G$ acts transitively on the set of prime ideals of $B$ above $\mathfrak{p}$, the number of such ideals is $|G : G^d(\mathfrak{q})|$. We set $\nu_{\mathfrak{p}} := |G : G^d(\mathfrak{q})|$ .

• We call *inertia group of* $\mathfrak{q}$ and we denote by $G^i(\mathfrak{q})$ the normal subgroup of $G^d(\mathfrak{q})$ consisting in those $g \in G^d(\mathfrak{q})$ which act trivially on $B/\mathfrak{q}$.

The group $G^d(\mathfrak{q})/G^i(\mathfrak{q})$ is identified with a subgroup of the group of automorphisms of $B/\mathfrak{q}$ which act trivially on $A/\mathfrak{p}$.

We denote by $k_A(\mathfrak{p}) := A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ and $k_B(\mathfrak{q}) := B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ the fields of fractions of respectively $A/\mathfrak{p}$ and $B/\mathfrak{q}$. Thus we have $G^d(\mathfrak{q})/G^i(\mathfrak{q}) \hookrightarrow \mathrm{Gal}(k_B(\mathfrak{q})/k_A(\mathfrak{p}))$ .

The degree $[k_B(\mathfrak{q}) : k_A(\mathfrak{p})]$ depends only on $\mathfrak{p}$, and we set $f_{\mathfrak{p}} := [k_B(\mathfrak{q}) : k_A(\mathfrak{p})]$ .

**7.3. Proposition.**

(1) *The field extension* $k_B(\mathfrak{q})/k_A(\mathfrak{p})$ *is normal with Galois group* $G^d(\mathfrak{q})/G^i(\mathfrak{q})$*.*

(2) *The following assertions are equivalent :*

  *(i)* $G^i(\mathfrak{q}) = 1$*.*

  *(ii)* $\mathfrak{p}B_{\mathfrak{q}} = \mathfrak{q}B_{\mathfrak{q}}$*, and the extension* $k_B(\mathfrak{q})/k_A(\mathfrak{p})$ *is Galois.*

If the preceding properties hold, we say that the ideal $\mathfrak{p} \in \mathrm{Spec}(A)$ is unramified on $B$, or that $\mathfrak{q} \in \mathrm{Spec}(B)$ is unramified on $A$.

**Case of height one primes.**

From now on we assume $A$ and $B$ are *normal domains*, *i.e.*, noetherian and integrally closed.

We denote by $\mathrm{Spec}_1(A)$ (resp. $\mathrm{Spec}_1(B)$) the set of height one primes ideals (*i.e.*, minimal nonzero prime ideals) of $A$ (resp. of $B$). By 7.2, if $\mathfrak{q}$ is above $\mathfrak{p} \in \mathrm{Spec}(B)$, $\mathfrak{q}$ has height one if and only if $\mathfrak{p}$ has height one.

For $\mathfrak{p} \in \mathrm{Spec}_1(A)$, the local ring $A_\mathfrak{p}$ is a normal domain, and it has a unique nonzero prime ideal (which is then maximal) ; hence it is a local Dedekind domain, *i.e.*, a local principal ideal domain (discrete valuation ring).

Similarly, for $\mathfrak{q} \in \mathrm{Spec}_1(B)$, $B_\mathfrak{q}$ is a discrete valuation ring.

We then call *ramification index of* $\mathfrak{p}$ *on* $B$ (or of $\mathfrak{q}$ on $A$) and we denote by $e_\mathfrak{p}$ the integer defined by the equality $\mathfrak{p}B_\mathfrak{q} = \mathfrak{q}^{e_\mathfrak{p}} B_\mathfrak{q}$ .

**7.4. Proposition.** *Let* $\mathfrak{p} \in \mathrm{Spec}_1(A)$*, and let* $\mathfrak{q} \in \mathrm{Spec}_1(B)$ *lying over* $\mathfrak{p}$*. We have*

(1) $|G| = \nu_\mathfrak{p} e_\mathfrak{p} f_\mathfrak{p}$ ,
(2) $|G^d(\mathfrak{q})| = e_\mathfrak{p} f_\mathfrak{p}$ ,
(3) $e_\mathfrak{p}$ *divides* $|G^i(\mathfrak{q})$.

*Sketch of proof of 7.4.*

Let us first notice that the spectrum of the ring $B_\mathfrak{p}$ consists in $\{0\}$ and the ideals $g(\mathfrak{q})B_\mathfrak{p}$ for $g \in G$. Hence the ring $B_\mathfrak{p}$ is a Dedekind domain, since it is a normal domain whose all non zero prime ideals are maximal.

Now we have

$$\mathfrak{p}B_\mathfrak{p} = \prod_{g \in G/G^d(\mathfrak{q})} g(\mathfrak{q})^{e_\mathfrak{p}} B_\mathfrak{p} \,,$$

and by the Chinese Remainder theorem it follows that

$$B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} = \prod_{g \in G/G^d(\mathfrak{q})} B_\mathfrak{p}/g(\mathfrak{q})^{e_\mathfrak{p}} B_\mathfrak{p} \,.$$

Let us denote by $[B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} : A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}]$ the dimension of $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ over the field $k_A(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. It is not difficult then to establish that

$$[B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} : A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}] = |G : G^d(\mathfrak{q})| e_\mathfrak{p} f_\mathfrak{p} \,.$$

But since $A_\mathfrak{p}$ is a principal ideal domain, $B_\mathfrak{p}$ is a free $A_\mathfrak{p}$–module, with rank $|G|$ since $[L : K] = |G|$. It implies that $[B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} : A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}] = |G|$, proving (1) and (2).

Since $G^d(\mathfrak{q})/G^i(\mathfrak{q})$ is a subgroup of $\mathrm{Gal}(k_B(\mathfrak{q})/k_A(\mathfrak{p})$, we see that $|G^d(\mathfrak{q}) : G^i(\mathfrak{q})|$ divides $f_\mathfrak{q}$, hence that $e_\mathfrak{p}$ divides $|G^i(\mathfrak{q})|$. $\square$

## §8. Finite linear groups on symmetric algebras

In what follows, we let $S$ be the symmetric algebra of an $r$–dimensional vector space $V$ over the field $k$. We let $\mathfrak{n}$ be its graded maximal ideal, so that $S/\mathfrak{n} = k$. Let $L$ be the field of fractions of $S$. Notice that $V = \mathfrak{n}/\mathfrak{n}^2$ (the tangent space).

Let $G$ be a finite group of automorphisms of $V$.

We denote by $R = S^G$ the subring of fixed points of $G$ on $S$ and we set $\mathfrak{m} := R \cap \mathfrak{n} = \mathfrak{n}^G$ (the maximal graded ideal of $R$. Let $K$ be the field of fractions of $R$.

$$\begin{tikzcd}
& & L \\
\mathfrak{n} \arrow[r, hook, dash] & S \arrow[ur, hook, dash] \arrow[d, dash] \arrow[dr, two heads] & K \arrow[u, dash] \\
\mathfrak{m} \arrow[u, dash] \arrow[r, hook, dash] & R \arrow[ur, hook, dash] \arrow[r, two heads] & k
\end{tikzcd}$$

**8.1. Proposition.**

(1) *$S$ is a finitely generated $R$–module.*
(2) *$R$ is a finitely generated $k$–algebra.*

*Proof of 8.1.*
(1) Since $S$ is a $k$–algebra of finite type, $S$ is *a fortiori* of finite type over $R$. Since $S$ is integral over $R$, $S$ is then a finitely generated $R$–module.
(2) Assume $S = k[x_1, x_2, \dots, x_r]$. Let $P_1(t), P_2(t), \dots, P_r(t) \in R[t]$ be nonzero polynomials having respectively $x_1, x_2, \dots, x_r$ as roots. Let $C_1, C_2, \dots, C_r$ denote respectively the set of coefficients of $P_1(t), P_2(t), \dots, P_r(t)$. Thus, $S$ is integral over $k[C_1, C_2, \dots, C_r]$, and since $S$ is a finitely generated algebra over $k[C_1, C_2, \dots, C_r]$, $S$ is a finitely generated module over $k[C_1, C_2, \dots, C_r]$. Since $k[C_1, C_2, \dots, C_r]$ is noetherian, the submodule $R$ of $S$ is also finitely generated. Hence $R$, a finitely generated module over a finitely generated $k$–algebra, is a finitely generated $k$–algebra. $\square$

Assume $k$ algebraically closed.
It results from Nullstellensatz that there is a ($\mathrm{GL}(V)$–equivariant) bijection

$$\mathrm{Spec}^{\max}(S) \xleftrightarrow{\sim} V \, .$$

Theorem 7.2 implies then that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}^{\max}(S) \arrow[r, "\sim"] \arrow[d, two heads] & V \arrow[d, two heads] \\
\mathrm{Spec}^{\max}(R) \arrow[r, "\sim"] & V/G
\end{tikzcd}$$

From now on, we assume that $k$ *has characteristic* $0$.

**Ramification and reflecting pairs.**

Let us start by studying certain height one prime ideals.
From now on, we denote by $\mathfrak{q}$ be an ideal of $S$ generated by a degree one element of $S$ (hence $\mathfrak{q} = SL$ where $L$ is a line in $V$). The ideal $\mathfrak{q}$ is a prime ideal of height one.
Let us denote by $\mathfrak{p}$ its image in $\mathrm{Spec}(R)$. Thus $\mathfrak{p} = R \cap \mathfrak{q}$.
Let us denote by $\mathrm{Spec}(B, \mathfrak{p})$ the set of prime ideals of $B$ lying over $\mathfrak{p}$. We recall that

$$\mathrm{Spec}(B, \mathfrak{p}) = \{ g(\mathfrak{q}) \mid (g \in G/G^d(\mathfrak{q})) \} \, .$$

**8.2. Lemma.** *For such an ideal $\mathfrak{q} = SL$, we have $G^i(\mathfrak{q}) = G(V/L)$.*

*Proof of 8.2.* Let us prove that $G(V/L) \subseteq G^i(\mathfrak{q})$. Let $g \in G(V/L)$. Whenever $v \in V$, we have $g(v) - v \in L$. The identity $g(xy) - xy = g(x)(g(y) - y) + (g(x) - x)y$ and an easy induction on the degree of $x \in S$ shows then that $g(x) - x \in SL$ for $x \in S$.

The inverse inclusion is obvious. □

The following proposition gives a bijection between the height one prime ideals of $S$ which are ramified over $R$ and and the reflecting pairs of $G$ on $V$ (for the notation used here, the reader may refer to 2.9 above).

**8.3. Proposition.**

(1) *The bijection :*
  - (a) *If $(L, H)$ is a reflecting pair for $G$, then the ideal $\mathfrak{q} = SL$ is a height one prime ideal of $S$ ramified over $R$.*
  - (b) *Reciprocally, if $\mathfrak{q}$ is a height one prime ideal of $S$, ramified over $R$, there exists a a reflecting pair $(L, H)$ for $G$ such that $\mathfrak{q}$ is the principal ideal of $S$ generated by $L$.*

(2) *If $\mathfrak{q}$ and $(L, H)$ are associated as above, then*
  - (a) *$G^i(\mathfrak{q}) = G(H, V/L)$,*
  - (b) *Let $\mathfrak{p} := \mathfrak{q} \cap R$. Then we have*

$$e_{\mathfrak{p}} = |G(H, V/L)| \ \textit{ and } \ f_{\mathfrak{p}} = |N_G(L, H)/G(H, V/L)|\,.$$

*Proof of 8.3.*

(1)(a) Let $(L, H)$ be a reflecting pair of $G$, and let $\mathfrak{q} := SL$. It is clear that $\mathfrak{q}$ is a prime ideal with height one in $S$. Since (see 8.2) $G^i(\mathfrak{q}) = G(H, V/L)$ and since $G(H, V/L) \neq \{1\}$, we see that $\mathfrak{q}$ is ramified over $R$.

(b) Let $\mathfrak{q} \in \mathrm{Spec}_A(B)$, and suppose that $\mathfrak{q}$ is ramified over $R$, *i.e.*, that $G^i(\mathfrak{q}) \neq \{1\}$ : there exists $g \in G$, $g \neq 1$, such that, for all $x \in S$, $g(x) - x \in \mathfrak{q}$.

On the other hand, since the morphism $V \to \mathfrak{n}/\mathfrak{n}^2$ is a $G$–equivariant isomorphism (hence the morphism $G \to \mathrm{GL}(\mathfrak{n}/\mathfrak{n}^2)$ is injective), there is a homogeneous element $x \in S$ such that $g(x) - x \notin \mathfrak{n}^2$. Such an element $x$ is then of degree 1, *i.e.*, $x \in V$.

Let us denote by $L$ the line generated by $g(x) - x$. Thus $L \subseteq \mathfrak{q}$. Since $\mathfrak{q}$ has height one and since it contains the prime ideal generated by $L$, we have $\mathfrak{q} = SL$.

(2) Assertion (a) results from 8.2. Since $k$ has characteristic zero, the extension $k_S(\mathfrak{q})/k_R(\mathfrak{p})$ (see 7.3) is not only normal but also Galois. Since its Galois group is

$$G^d(\mathfrak{q})/G^i(\mathfrak{q}) = N_G(L, H)/G(H, V/L)\,,$$

we see that $f_{\mathfrak{p}} = |N_G(L, H)/G(H, V/L)|$. Now since (see 7.4) $|G^d(\mathfrak{q})| = e_{\mathfrak{p}} f_{\mathfrak{p}}$, it follows that $e_{\mathfrak{p}} = |G(H, V/L)|$. □

We denote by $\mathrm{Spec}_1^{\mathrm{ram}}(S)$ the set of height one prime ideals of $S$ which are ramified over $R$, and we denote by $\mathrm{Spec}_1^{\mathrm{ram}}(R)$ the set of height one prime ideals of $R$ ramified in $S$.

The set $\mathrm{Spec}_1^{\mathrm{ram}}(R)$ is in natural bijection with the set $\mathrm{Spec}_1^{\mathrm{ram}}(S)/G$ of orbits of $G$ on $\mathrm{Spec}_1^{\mathrm{ram}}(S)$.

The set of reflecting pairs is in bijection with the set of reflecting hyperplanes, denoted by $\mathcal{A}$. Thus :

• $\mathrm{Spec}_1^{\mathrm{ram}}(S)$ is in natural bijection with the set $\mathcal{A}$ of reflecting hyperplanes of $G$,

• $\mathrm{Spec}_1^{\mathrm{ram}}(R)$ is in natural bijection with the set $\mathcal{A}/G$ of orbits of $G$ on its reflecting hyperplanes arrangement.

**Linear characters associated with reflecting hyperplanes.**

Let $\mathfrak{p} \in \mathrm{Spec}_1^{\mathrm{ram}}(R)$. For $\mathfrak{q} \in \mathrm{Spec}_1^{\mathrm{ram}}(S)$ lying above $\mathfrak{p}$, associated with the reflecting pair $(L, H)$, we denote by $j_\mathfrak{q}$ a nonzero element of $L$. We define $j_\mathfrak{p} \in S$ by the formula

$$j_\mathfrak{p} := \prod_{\mathfrak{q} \in \mathrm{Spec}(S;\mathfrak{p})} j_\mathfrak{q}\,.$$

Notice that $j_\mathfrak{p}$ is uniquely defined up to multiplication by a nonzero element of $k$, and that it is a homogeneous element of $S$ of degree $\nu_\mathfrak{p}$.

Moreover, $j_\mathfrak{p}$ is an eigenvector for all elements of $G$, so it defines a linear character of $G$, denoted by $\theta_\mathfrak{p}$ : whenever $g \in G$, we have

$$g(j_\mathfrak{p}) = \theta_\mathfrak{p}(g)\, j_\mathfrak{p}\,.$$

**8.4. Lemma.** *Let $\mathfrak{p} \in \mathrm{Spec}_1^{\mathrm{ram}}(R)$. The linear character $\theta_\mathfrak{p}$ takes the following values on a reflection $s \in G$ :*

$$\theta_\mathfrak{p}(s) = \begin{cases} \det_V(s) & \textit{if } s \in G^i(\mathfrak{q}) \textit{ for } \mathfrak{q} \in \mathrm{Spec}(S;\mathfrak{p})\,, \\ 1 & \textit{if } s \in G^i(\mathfrak{q}) \textit{ for } \mathfrak{q} \notin \mathrm{Spec}(S;\mathfrak{p})\,. \end{cases}$$

*Proof of 8.4.* Let $s \in G$ be a reflection . We shall prove that

$$s(j_\mathfrak{p}) = \begin{cases} \det_V(s) j_\mathfrak{p} & \text{if } s \in G^i(\mathfrak{q}) \text{ for } \mathfrak{q} \in \mathrm{Spec}(S;\mathfrak{p})\,, \\ j_\mathfrak{p} & \text{if } s \in G^i(\mathfrak{q}) \text{ for } \mathfrak{q} \notin \mathrm{Spec}(S;\mathfrak{p})\,. \end{cases}$$

For $\mathfrak{q} \in \mathrm{Spec}(S;\mathfrak{p})$, let $n_s(\mathfrak{q})$ denote the cardinal of the orbit of $\mathfrak{q}$ under $s$. Thus (up to multiplication by an element of $k^\times$), we have

$$j_\mathfrak{p} = \prod_{\mathfrak{q} \in (\mathrm{Spec}(S;\mathfrak{p})/\langle s \rangle)} (j_\mathfrak{q} s(j_\mathfrak{q}) \cdots s^{n_s(\mathfrak{q})-1}(j_\mathfrak{q}))\,.$$

By definition of $n_s(\mathfrak{q})$, $j_\mathfrak{q}$ is an eigenvector of $s^{n_s(\mathfrak{q})}$. Let $(L, H)$ be the reflecting pair of $s$. We have $s \in G(H)$, hence $s^{n_s(\mathfrak{q})} \in G(H)$.

• Assume that $G(H) \neq G^i(\mathfrak{q})$. Then we know that $\mathfrak{q} \neq SL$, hence $j_\mathfrak{q} \notin L$. Nevertheless, $j_\mathfrak{q}$ is an eigenvector of $s^{n_s(\mathfrak{q})}$. But, either $s^{n_s(\mathfrak{q})} = 1$, or $s^{n_s(\mathfrak{q})}$ is a reflection with reflecting pair $(L, H)$. Thus, in any case, we see that $j_\mathfrak{q}$ is fixed by $s^{n_s(\mathfrak{q})}$. This proves that if $s \in G^i(\mathfrak{q})$ for some $\mathfrak{q} \notin \mathrm{Spec}(S;\mathfrak{p})$, then $s(j_\mathfrak{p}) = 1$.

• Assume now $G(H) = G^i(\mathfrak{q})$. Then $(L, H)$ is the reflecting pair associated with $\mathfrak{q}$, and $s \in G^i(\mathfrak{q})$. Hence $s(j_\mathfrak{p}) = \det_V(s) j_\mathfrak{p}$ . $\square$

**8.5. Proposition.**

(1) *The restrictions provide a natural morphism*

$$\rho_G\colon \mathrm{Hom}(G,k^\times) \to \left(\prod_{\mathfrak{q}\in\mathrm{Spec}_1^{\mathrm{ram}}(S)} \mathrm{Hom}(G^i(\mathfrak{q}),k^\times)\right)^G = \left(\prod_{H\in\mathcal{A}} \mathrm{Hom}(G(H),k^\times)\right)^G$$

(2) *The morphism $\rho_G$ is onto.*

*Remarks.*

**1.** The homogeneous element

$$j^{\mathrm{ram}} := \prod_{\mathfrak{p}\in\mathrm{Spec}_1^{\mathrm{ram}}(R)} j_{\mathfrak{p}}\,,$$

a monomial with degree

$$\sum_{\mathfrak{p}\in\mathrm{Spec}_1^{\mathrm{ram}}(R)} \nu_{\mathfrak{p}} = |\mathrm{Spec}_1^{\mathrm{ram}}(S)| = |\mathcal{A}|$$

(the number of reflecting hyperplanes of $G$), defines the linear character

$$\theta^{\mathrm{ram}} := \prod_{\mathfrak{p}\in\mathrm{Spec}_1^{\mathrm{ram}}(R)} \theta_{\mathfrak{p}}$$

which coincides with the determinant on the subgroup of $G$ generated by reflections.

In general we have $\theta^{\mathrm{ram}} \neq \det_V$.

Indeed, notice that for any $\zeta\in\mu(k)$, one has $\theta^{\mathrm{ram}}(\zeta\mathrm{Id}_V) = \zeta^N$ where $N$ denotes the number of reflecting hyperplanes of $G$, while $\det_V(\zeta\mathrm{Id}_V) = \zeta^r$.

**2.** Let $(v_1,v_2,\dots,v_r)$ be a basis of $V$, let $(u_1,u_2,\dots,u_r)$ be a system of parameters for the algebra $R$, and let us denote by $J$ the corresponding Jacobian, defined up to a nonzero scalar by

$$J := \det\left(\frac{\partial u_i}{\partial v_j}\right)_{i,j}\,.$$

Then, for all $g\in G$, we have

$$g(J) = \det_V(g^{-1})J\,.$$

We shall see below that, if $G$ is generated by reflections, we can express $J$ as a monomial in the $j_H)_{H\in\mathcal{A}}$.

Now we can describe the ideal $\mathfrak{p}$.

Let us set

$$\Delta_{\mathfrak{p}} := j_{\mathfrak{p}}^{e_{\mathfrak{p}}} = \prod_{g\in G/G^d(\mathfrak{q})} g(j_{\mathfrak{q}})^{e_{\mathfrak{p}}}\,.$$

**8.6. Proposition.**

(1) *We have* $\mathfrak{p} \subseteq R\Delta_{\mathfrak{p}}$ .

(2) *If* $G$ *is generated by its reflections, we have*

$$\mathfrak{p} = R\Delta_{\mathfrak{p}}\,, \text{ hence } S\mathfrak{p} = \prod_{g \in G/G^d(\mathfrak{q})} Sg(j_{\mathfrak{q}})^{e_{\mathfrak{p}}}\,.$$

*Proof of 8.6.*

(1) Let $x \in \mathfrak{p}$. In order to prove that $x$ is divisible by $\Delta_{\mathfrak{p}}$, it suffices to prove that whenever $\mathfrak{q} \in \operatorname{Spec}(B, \mathfrak{p})$, then $x$ is divisible by $j_{\mathfrak{q}}^{e(\mathfrak{q})}$. So let us pick $\mathfrak{q} \in \operatorname{Spec}(B, \mathfrak{p})$, associated with the reflecting pair $(L, H)$. Let us choose a basis $(v_1, v_2, \dots, v_{r-1})$ of $H$, so that $(j_{\mathfrak{q}}, v_1, v_2, \dots, v_{r-1})$ is a basis of $V$. Then $x = P(j_{\mathfrak{q}}, v_1, v_2, \dots, v_{r-1})$, where $P(t_0, t_1, \dots, t_{r-1}) \in k[t_0, t_1, \dots, t_{r-1}]$. Since $x \in \mathfrak{q} = Sj_{\mathfrak{q}}$, there exists a polynomal $Q(t_0, t_1, \dots, t_{r-1}) \in k[t_1, \dots, t_{r-1}]$ such that $P(t_0, t_1, \dots, t_{r-1}) = t_0 Q(t_0, t_1, \dots, t_{r-1})$.

Now let $s$ be a generator of the cyclic group $G(H) = G^i(\mathfrak{q})$, and let us denote by $\zeta_s$ its determinant, a root of the unity of order $|G(H)| = e_{\mathfrak{p}}$. Since $s(x) = x$, we have

$$\begin{aligned} &\zeta_s t_0 Q(\zeta_s t_0, t_1, \dots, t_{r-1}) = t_0 Q(t_0, t_1, \dots, t_{r-1})\,, \text{ i.e.,} \\ &Q(\zeta_s t_0, t_1, \dots, t_{r-1}) = \zeta_s^{e_{\mathfrak{p}}-1} Q(t_0, t_1, \dots, t_{r-1}) \end{aligned}$$

and we apply the following lemma to conclude that $Q(t_0, t_1, \dots, t_{r-1})$ is divisible by $t_0^{e_{\mathfrak{p}}-1}$, hence that $P(t_0, t_1, \dots, t_{r-1})$ is divisible by $t_0^{e_{\mathfrak{p}}}$, *i.e.*, $x$ is divisible by $j_{\mathfrak{q}}^{e_{\mathfrak{p}}}$.

**8.7. Lemma.** *Let $A$ be a commutative domain, let $P(t) \in A[t]$. Assume there exist an integer $m$ and $a \in A$ such that $a^j \neq 1$ for $1 \leq j \leq m-1$, and $P(at) = a^m P(t)$. Then $P(t)$ is divisible by $t^m$.*

*Proof of 8.7.* Set $P(t) = \sum_n b_n t^n$. By hypothesis, we have $\sum_n a^m b_n t^n = \sum_n b_n a^n t^n$ , hence for all $n < m$, $b_n = 0$. ☐

(2) It follows from 8.4 that $\Delta_{\mathfrak{p}}$ is invariant by all reflections in $G$, hence invariant under $G$ if $G$ is generated by reflections. ☐

## §9. Coinvariant algebra and Harmonic polynomials.

**The coinvariant algebra.**

We set

$$S_G := S/\mathfrak{M}S$$

and we call that graded $k$–algebra the *coinvariant algebra* of $G$.

The algebra $S_G$ is a finite dimensional $k$–vector space, whose dimension is the minimal cardinality of a set of generators of $S$ as an $R$–module (by Nakayama's lemma). Thus there is an integer $M$ such that

$$S_G = k \oplus S_G^1 \oplus \cdots \oplus S_G^M\,,$$

and so in particular $\bigoplus_{n>M} S^n \subseteq \mathfrak{M}S$ .

**9.1. Lemma.** *The set of fixed points of $G$ in $S_G$ is $k$.*

*Proof of 9.1.* It suffices to prove that no homogeneous element of $S_G$ of degree $> 0$ can be fixed under $G$. Assume that $x \in S$ is a homogeneous element of $S$ of degree $> 0$ such that, for all $g \in G$, $g(x) - x \in \mathfrak{M}S$. Then we have $x - (1/|G|)\sum_{g \in G} g(x) \in \mathfrak{M}S$ . Since $(1/|G|)\sum_{g \in G} g(x) \in \mathfrak{M}$, we see that $x \in \mathfrak{M}S$. $\square$

Whenever $T$ is a graded subspace of $S$, which is $G$–stable, the multiplication induces a morphism of graded $kG$–modules $R \otimes T \longrightarrow S$ . By complete reducibility of the action of $G$ on each homogeneous component of $S$, there exists a $G$–stable graded subspace $T$ of $S$ such that $\mathfrak{M}S \oplus T = S$. Combined with the isomorphism $S/\mathfrak{M}S \xrightarrow{\sim} T$ which follows from that decomposition, we then get a morphism of graded $kG$–modules

$$\mu_T \colon R \otimes S_G \longrightarrow S \,.$$

**9.2. Lemma.** *The morphism $\mu_T$ is onto.*

*Proof of 9.2.* It follows from Nakayama's lemma. $\square$

*Remarks.*

- In the next chapter we shall prove that $G$ is generated by reflections if and only if $\mu_T$ is an isomorphism.
  Notice that if $\mu_T$ is an isomorphism, then $S$ is a free $R$–module, which implies (by 6.5) that $R$ is a polynomial algebra.
- We shall now define a natural supplement subspace $T$ : the space of harmonic polynomials.

**Galois twisting of a representation.**

*Generalities.*

We shall recall how the group $\mathrm{Aut}(k)$ acts on the set of isomorphism types of the $k$–representations of a finite group $G$.

Let $\rho \colon G \to \mathrm{GL}(X)$ be a representation of $G$ over $k$, and let $\sigma \in \mathrm{Aut}(k)$. We define another $k$–representation $\sigma\rho$, called *the twisting of $\rho$ by $\sigma$*, as follows.

Let $\sigma X$ be the $k$–vector space whose structural abelian group equals $X$, and where the external multiplication by elements of $k$ is defined by

$$\lambda \cdot x := \sigma^{-1}(\lambda)x \quad \text{for } \lambda \in k \text{ and } x \in X \,.$$

Besides, $\sigma$ acts as a ring automorphism on the group algebra $kG$ :

$$\sigma \colon \sum_{g \in G} \lambda_g g \mapsto \sum_{g \in G} \sigma(\lambda_g) g \,.$$

Then the image of the composition map

$$kG \xrightarrow{\sigma^{-1}} kG \xrightarrow{\rho} \mathrm{End}_{\mathbb{Z}}(X)$$

is contained in $\mathrm{End}_k(\sigma X)$ hence defines a $k$–algebra morphism

$$\sigma\rho \colon kG \to \mathrm{End}_k(\sigma X)$$

hence a $k$–representation of $G$.

Let us choose a basis $B$ of $X$. Then it is also a basis of $\sigma X$. It is clear that, for $g \in G$, the matrix of $\sigma\rho(g)$ on $B$ is obtained by applying $\sigma$ to all entries of the matrix of $\rho(g)$ on $B$. Thus in particular

**9.3.**

(1) *the characteristic polynomial* $\det(1 - \sigma\rho(g)q)$ *of* $\sigma\rho(g)$ *is the image under* $\sigma$ *of the characteristic polynomial* $\det(1 - \rho(g)q)$ *of* $\rho(q)$*,*
(2) *whenever* $g \in G$*, we have* $\chi_{\sigma\rho}(g) = \sigma(\chi(g))$ *.*

We shall denote by $\sigma\chi$ the character of the $\sigma$–twisted of a representation with character $\chi$.

It is easy to check that we have canonical isomorphisms

$$\mathrm{Sym}(\sigma X) = \sigma\mathrm{Sym}(X) \text{ and } \Lambda(\sigma X) = \sigma\Lambda(X)\,,$$

and also that

$$\mathrm{Sym}(\sigma X)^{\sigma\rho(G)} = \sigma\mathrm{Sym}(X)^{\rho(G)}\,.$$

More generally, if $\chi$ is an irreducible character of $G$, there is a natural identification

$$\sigma(\mathrm{Sym}(X)_\chi) = (\sigma\mathrm{Sym}(X))_{\sigma\chi}\,. \tag{9.4}$$

Let us fix a basis $(x_1, x_2, \ldots, x_m)$ of $X$. Then it is also a basis of $\sigma X$.

For $\underline{n} = (n_1, n_2, \ldots, n_m) \in \mathbb{N}^m$ , we set $\underline{x}^{\underline{n}} := x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$ . A general element of $\mathrm{Sym}(X)$ has the shape $\sum \lambda_{\underline{n}} \underline{x}^{\underline{n}}$ , where $\lambda_{\underline{n}} \in k$. Then we have

$$\left(\sum \lambda_{\underline{n}} \underline{x}^{\underline{m}} \in \mathrm{Sym}(X)_\chi\right) \Leftrightarrow \left(\sum \sigma(\lambda_{\underline{n}}) \underline{x}^{\underline{n}} \in (\sigma\mathrm{Sym}(X))_{\sigma\chi}\right)\,. \tag{9.5}$$

*Complex conjugation and contragredient representation.*

Assume now that $k = \mathbb{C}$, the field of complex numbers. We denote the complex conjugation by $\lambda \mapsto \lambda^*$.

There exists a positive definite hermitian form on $X$ which is $G$–invariant. Using an orthonormal basis for such a form (as well as its dual basis for the dual space $X^*$), we see that the matrix of the contragredient of $\rho(g)$ is the complex conjugate of the matrix of $\rho(g)$.

**9.6.** *If* $k = \mathbb{C}$*, the contragredient representation of a representation* $\rho$*, denoted by* $\rho^*$*, is the twisting of* $\rho$ *by the complex conjugation.*

*Case of reflection groups.*

For the particular case of reflection groups, we have the following result.

**9.7.** *The group* $\rho(G)$ *is generated by reflections if and only if* $\sigma\rho(G)$ *is generated by reflections.*

**Differential operators, harmonic polynomials.**

*Generalities.*

We denote by $S^*$ the symmetric algebra of the dual space $V^*$.

(!) **Attention** (!)

The space $S^*$ is not the dual of $S$ : the dual is the completion $\widehat{S}^*$ of $S^*$ at its maximal graded ideal.

The proofs of the following two lemmas are left to the reader.

**9.8. Lemma.** *There is a unique $k$–algebra morphism $D\colon S^* \to \mathrm{End}_k S$ satisfying the following properties.*

(1) *For $v^* \in V^*$, $D(v^*)$ is a derivation of $S$, i.e.,*
$$D(v^*)(xy) = D(v^*)(x)y + xD(v^*)(y)\,.$$

(2) *For $v^* \in V^*$ and $v \in V$ we have $D(v^*)(v) =< v^*, v >$ .*

Notice the following properties of $D$.

- For $\lambda \in k$, we have $D(\lambda) = \lambda \mathrm{Id}$.
- For $x^*$ a homogeneous element of $S^*$, $D(x^*)$ is a degree $-\deg(x)$ endomorphism of $S$.
- For $g \in \mathrm{GL}(V)$, $x \in S$ and $x^* \in S^*$, we have
$$g(D(x^*)(x)) = D(g(x^*))(g(x))\,.$$

Let $(v_1, v_2, \dots, v_r)$ be a basis of $V$, and let $(\dfrac{\partial}{\partial v_1}, \dfrac{\partial}{\partial v_2}, \dots, \dfrac{\partial}{\partial v_r})$ be the dual basis of $V^*$. For $\mathbf{m} := (m_1, m_2, \dots, m_r) \in \mathbb{N}^r$, we set
$$\mathbf{m}! := m_1! m_2! \cdots m_r!\ ,\ \mathbf{v}^{\mathbf{m}} := v_1^{m_1} v_2^{m_2} \cdots v_r^{m_r}\ ,\ \frac{\partial}{\partial \mathbf{v}}^{\mathbf{m}} := \frac{\partial}{\partial v_1}^{m_1} \frac{\partial}{\partial v_2}^{m_2} \cdots \frac{\partial}{\partial v_r}^{m_r}\,.$$

**9.9. Lemma.** *There is a natural duality between $S$ and $S^*$ defined by the formula*
$$\langle x^*, x\rangle := D(x^*)(x)(0)\,.$$
*We have*
$$\langle \frac{\partial}{\partial \mathbf{v}}^{\mathbf{m}'}, \mathbf{v}^{\mathbf{m}} \rangle = \begin{cases} \mathbf{m}! & \text{if } \mathbf{m} = \mathbf{m}' \\ 0 & \text{if not.} \end{cases}$$

In what follows, we assume that $k = \mathbb{C}$.

Assume that $V$ is endowed with a positive definite hermitian product stable by $G$, and let us choose an orthonormal basis $(v_1, v_2, \dots, v_r)$ of $V$. Let $(\frac{\partial}{\partial v_i})_{1 \le i \le r}$ denote the dual basis. For
$$x := \sum \lambda_{\mathbf{m}} \mathbf{v}^{\mathbf{m}} \in S$$
(with $\lambda_{\mathbf{m}} \in \mathbb{C}$), let us denote by $x^*$ the element of $S^*$ defined by
$$x^* := \sum \lambda^*_{\mathbf{m}} (\frac{\partial}{\partial \mathbf{v}})^{\mathbf{m}}$$
(where $\lambda^*_{\mathbf{m}}$ denotes the complex conjugate of $\lambda_{\mathbf{m}}$). Let $x = \sum \lambda_{\mathbf{m}} \mathbf{v}^{\mathbf{m}}$ and $y = \sum \mu_{\mathbf{m}} \mathbf{v}^{\mathbf{m}}$ be two elements of $S$. Then we have
$$\langle x^*, y\rangle = \sum_{\mathbf{m}} \lambda^*_{\mathbf{m}} \mu_m\,,$$
and in particular
$$\left( \langle x^*, x\rangle = \sum_{\mathbf{m}} |\lambda_m|^2 = 0 \right) \Leftrightarrow (x = 0)\,.$$
The map $x \mapsto x^*$ is an semi-isomorphism from $S$ onto $S^*$. If $X$ is any graded subspace of $S$, we denote by $X^*$ its image through that semi-isomorphism. We set
$$(X^*)^\perp := \{ y \in S \mid (\forall x \in X)(\langle x^*, y\rangle = 0) \}\,.$$

**9.10. Lemma.** *Assume $k = \mathbb{C}$. Whenever $X$ is any graded subspace of $S$, so is $(X^*)^\perp$, and we have*
$$X \oplus (X^*)^\perp = S\,.$$

*Proof of 9.10.* Indeed, it is enough to prove that the equality holds for each homogeneous component, which is obvious. □

*Harmonic elements.*

We set $R^* := {S^*}^G$, and we denote by $\mathfrak{m}^*$ the maximal graded ideal of $R^*$.

Notice that the notation is consistant with the notation introduced previously about the anti-isomorphism $x \mapsto x^*$ : the image of $R$ and $\mathfrak{m}$ under such an anti-isomorphism are indeed respectively $R^*$ and $\mathfrak{m}^*$.

**Definition.** We call "harmonic elements" of $S$ and we denote by Har the space defined by

$$\mathrm{Har} := (\mathfrak{m}^* S^*)^\perp = \{h \in S \,|\, (\forall x^* \in \mathfrak{m}^*)(\langle x^*, h\rangle = 0)\}\,.$$

**9.11. Proposition.** *We have a $G$–stable decomposition*

$$S = \mathfrak{m}S \oplus \mathrm{Har}\,.$$

*Proof of 9.11.* It is clear that Har is stable by $G$. The assertion is a direct consequence of 9.10. □

Notice that in particular the projection $S \twoheadrightarrow \mathrm{Har}$ parallel to $\mathfrak{m}S$ induces an isomorphism

$$S_G \xrightarrow{\sim} \mathrm{Har}\,.$$

## §10. Graded characters and applications

From now on, we assume that the field $k$ has characteristic zero, and is large enough so that all irreducible representations of $G$ on $k$ are absolutely irreducible.

**Graded characters of graded $kG$–modules.**

A $kG$–module is a finite dimensional $k$–vector space endowed with an operation of $G$.

*A graded $kG$–module* is a graded $k$–vector space $M = \bigoplus_n M_n$ endowed with an operation of $G$ (*i.e.*, for each $n$, $M_n$ is a $kG$–module).

*Th graded character* of $M$ is then the class function

$$\mathrm{grchar}_M : G \to \mathbb{Z}((q))$$

defined by
$$\mathrm{grchar}_M(g) := \sum_n \mathrm{tr}(g\,;\, M_n) q^n\,.$$

In particular, $\mathrm{grchar}_M(1) = \mathrm{grdim}_k(M)$.

**Example.** Let $X$ be a $kG$–module. We have

$$\mathrm{grchar}_{\Lambda(X)}(g) = \det_X(1+gq) \quad \text{and} \quad \mathrm{grchar}_{S(X)}(g) = \frac{1}{\det_X(1-gq)}\,.$$

Note that, since the Koszul complex is exact, we have

$$\mathrm{grchar}_{\Lambda(X)}(-g)\mathrm{grchar}_{S(X)}(g) = 1\,.$$

For all irreducible $kG$–module $X$, since by assumption $X$ is absolutely irreducible, we have

$$\dim_k \mathrm{Hom}_{kG}(X,X) = 1\,.$$

**Definitions and Notation.**

- For all graded $kG$-module $M$, *the graded multiplicity of $X$ in $M$* is the formal series $\mathrm{grmult}(X,M) \in \mathbb{Z}((q))$ defined by

$$\mathrm{grmult}(X,M) := \mathrm{grdim}_k \mathrm{Hom}_{kG}(X,M)\,.$$

- We call *$X$–isotypic component of $M$* and we denote by $M_X$ the direct sum of all $X$–isotypic components of the homogeneous spaces $M_n$ for $-\infty < n < +\infty$. Thus, $M_X$ is a graded $kG$–submodule of $M$.

The proof of the following proposition is easy and left to the reader.

**10.1. Proposition.** *Let $X$ be an irreducible $kG$–module with character $\chi$. We have*

(1) $\mathrm{grchar}_{M_X} = \mathrm{grmult}(X,M)\chi\,.$
(2) $\mathrm{grmult}(X,M) = \frac{1}{|G|}\sum_{g\in G}\mathrm{grchar}_M(g)\chi(g^{-1})\,.$

We also set (using preceding notation) : $\mathrm{grmult}(\chi,M) := \mathrm{grmult}(X,M)\,.$

**Isotypic components of the symmetric algebra.**

As before, let $V$ be an $r$-dimensional $k$–vector space and let $G$ finite subgroup of $\mathrm{GL}(V)$. We denote by $S$ the symmetric algebra of $V$, and we set $R := S^G$.

The algebra $S$ is a graded $kG$–module. For all irreducible character $\chi$ of $G$ on $k$, we let $S^G_\chi$ (or simply $S_\chi$) denote the $\chi$–isotypic component of $S$. Note that if $1_G$ is the trivial character of $G$, then $S^G_{1_G} = R$.

**10.2 Lemma.**

(1) *Each $S^G_\chi$ is a graded $kG$–module, and a graded $R$–submodule of $S$.*
(2) *We have*

$$S = \bigoplus_{\chi\in\mathrm{Irr}_k(G)} S^G_\chi\,.$$

(3) *For $\chi$ an irreducible character of $G$ we have (with obvious notation)*

$$\mathrm{grmult}(\chi,S) = \frac{1}{|G|}\sum_{g\in G}\frac{\chi(g^{-1})}{\det_V(1-gq)} \quad \textit{and} \quad \mathrm{grchar}_{S^G_\chi} = \mathrm{grmult}(\chi,S)\,\chi\,.$$

*Proof of 10.2.* Whenever $S_n$ is a homogeneous component of degree $n$ of $S$, multiplication by a homogeneous element $x \in R$ defines an isomorphism of $kG$–modules from $S_n$ to $xS_n$. Thus multiplication by $x$ sends $S_\chi^G$ into itself, which proves (1).

(2) is immediate. (3) results from 10.1. □

For $\chi$ any class function on $G$, let us set

$$\langle S, \chi \rangle_G := \frac{1}{|G|} \sum_{g \in G} \frac{\chi(g^{-1})}{\det_V(1 - gq)},$$

a linear form on $\chi$.

**Some numerical identities.**

By 10.2, (3), the coefficient of $\dfrac{1}{(1-q)^r}$ in the Laurent series development of $\langle S, \chi \rangle_G$ around $q = 1$ equals $\dfrac{\chi(1)}{|G|}$.

Let $\gamma(\chi)$ be the number such that the coefficient of $\dfrac{1}{(1-q)^{r-1}}$ in the Laurent series development of $\langle S, \chi \rangle_G$ around $q = 1$ equals $\dfrac{\gamma(\chi)}{2|G|}$. Thus

$$\langle S, \chi \rangle_G = \frac{\chi(1)}{|G|} \frac{1}{(1-q)^r} + \frac{\gamma(\chi)}{2|G|} \frac{1}{(1-q)^{r-1}} + \dots, \tag{10.3}$$

or, in other words

$$\begin{aligned} \frac{\chi(1)}{|G|} &= ((1-q)^r(\langle S, \chi \rangle_G)_{|q=1} \\ \frac{\gamma(\chi)}{2|G|} &= -\frac{d}{dq}((1-q)^r(\langle S, \chi \rangle_G)_{|q=1} . \end{aligned}$$

**10.4. Proposition.**

(1) *Let $\mathcal{A}$ the set of reflecting hyperplanes of $G$. We have*

$$\gamma(\chi) = \sum_{H \in \mathcal{A}} \gamma(\mathrm{Res}_{G(H)}^G(\chi)) .$$

(2) *$\gamma(1_G)$ is the number of reflections of $G$.*

*Proof of 10.4.*

(1) The set $\mathrm{Ref}(G)$ of all reflections of $G$ is the disjoint union, for $H \in \mathcal{A}$, of the sets $\mathrm{Ref}(G(H))$ of reflections of the fixator (pointwise stabilizer) $G(H)$ of $H$. For $s \in \mathrm{Ref}(G)$, se wet $\zeta_s := \det_V(s)$ . Since

$$\langle S, \chi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \frac{\chi(g^{-1})}{\det_V(1 - gq)},$$

we see that

$$\begin{aligned}\gamma(\chi) &= 2 \sum_{s\in \mathrm{Ref}(G)} \frac{\chi(s^{-1})}{1-\zeta_s} \\ &= 2 \sum_{H\in\mathcal{A}} \sum_{s\in\mathrm{Ref}(G(H))} \frac{\chi(s^{-1})}{1-\zeta_s} \\ &= \sum_{H\in\mathcal{A}} \gamma(\mathrm{Res}^G_{G_H}(\chi))\,.\end{aligned}$$

(2) By the preceding assertion, it is enough to check (2) for the case $G = G(H)$. Then, following the above computation, we have

$$\begin{aligned}\gamma(1_{G(H)}) &= 2 \sum_{s\in\mathrm{Ref}(G(H))} \frac{1}{1-\zeta_s} \\ &= 2 \sum_{\substack{s\in G(H)\\ s\neq 1, s^2=1}} \frac{1}{2} + \sum_{\substack{s\in G(H)\\ s\neq 1, s^2\neq 1}} \left(\frac{1}{1-\zeta_s} + \frac{1}{1-\zeta_s^{-1}}\right) \\ &= |\mathrm{Ref}(G(H))|\,.\end{aligned}$$

□

**10.5. Corollary.** *The development around $q = 1$ of the graded dimension of $R$ has the shape*

$$\mathrm{grdim}_k R = \frac{1}{|G|}\frac{1}{(1-q)^n} + \frac{|\mathrm{Ref} G|}{2|G|}\frac{1}{(1-q)^{n-1}} + \dots\,.$$

**Isotypic components are Cohen–Macaulay.**

**10.6. Proposition.**

*Whenever $P$ is a parameter algebra of $R$, and whenever $\chi \in \mathrm{Irr}_k(G)$, the isotypic component $S^G_\chi$ is free over $P$.*

*In particular the invariant algebra $R$ is Cohen–Macaulay.*

*Proof of 10.6.* A parameter algebra $P$ of $R$ is a parameter algebra of $S$. Since $S$ is free over itself, $S$ is free over $P$. The proposition follows from 10.2, (2). □

**Computations with power series.**

Let $P$ be a parameter algebra of $R$. We denote by $\mathfrak{m}_P$ the unique maximal graded ideal of $P$.

Let $m$ denote the rank of $R$ over $P$, which is also the dimension of the graded $k$-vector space $R/\mathfrak{m}_P R = k \otimes_P R$.

We call *$P$–exponents* of $R$ the family $(e_1, e_2, \dots, e_m)$ of integers such that

$$\mathrm{grdim}_k(R/\mathfrak{m}_P R) = q^{e_1} + q^{e_2} + \cdots + q^{e_m}\,.$$

Thus there exists a $k$–basis of $R/\mathfrak{m}_P R$ consisting of homogeneous elements of degrees respectively $e_1, e_2, \dots, e_m$.

Let $d_1, d_2, \dots, d_r$ be the characteristic degrees of $P$, so that

$$\mathrm{grdim}_k(P) = \frac{1}{(1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_r})}\,.$$

Since $R$ is free on $P$, we have

$$R \simeq P \otimes_k R/\mathfrak{m}_P R\,.$$

Thus the graded dimension of $R$ is

$$\mathrm{grdim}_k(R) = \frac{q^{e_1}+q^{e_2}+\cdots+q^{e_m}}{(1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_r})}\,.$$

Now let $\chi \in \mathrm{Irr}_k(G)$. Let $\chi(1)m_\chi$ denote the rank of $S_\chi^G$ over $P$, which is also the dimension of the graded $k$-vector space $S_\chi^G/\mathfrak{m}_P S_\chi^G = k \otimes_P S_\chi^G$.

Since each homogeneous component of $S_\chi^G/\mathfrak{m}_P S_\chi^G$ is a direct sum of modules with character $\chi$, the graded dimension of $S_\chi^G/\mathfrak{m}_P S_\chi^G$ has the shape

$$\mathrm{grdim}\, S_\chi^G/\mathfrak{m}_P S_\chi^G = \chi(1)(q^{e_1(\chi)}+q^{e_2(\chi)}+\cdots+q^{e_{m_\chi}(\chi)})\,,$$

from which we deduce

$$\mathrm{grdim}_k(S_\chi^G) = \chi(1)\frac{q^{e_1(\chi)}+q^{e_2(\chi)}+\cdots+q^{e_{m_\chi}(\chi)}}{(1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_r})}\,.$$

Note that

$$\frac{1}{|G|}\sum_{g\in G}\mathrm{grchar}_{(S_\chi^G/\mathfrak{m}_P S_\chi^G)}(g)\chi(g^{-1}) = (q^{e_1(\chi)}+q^{e_2(\chi)}+\cdots+q^{e_{m_\chi}(\chi)})\,. \tag{10.7}$$

We set

$$\mathrm{grmult}(\chi,S) := \langle S,\chi\rangle_G = \frac{q^{e_1(\chi)}+q^{e_2(\chi)}+\cdots+q^{e_{m_\chi}(\chi)}}{(1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_r})}$$
$$E^P(\chi) := e_1(\chi)+e_2(\chi)+\cdots+e_{m_\chi}(\chi) = (\mathrm{grmult}(\chi, S/\mathfrak{M}_P S))_{|q=1}\,.$$

**10.8. Proposition.**

(1) *We have* $(d_1 d_2 \cdots d_r) = m|G|$.

(2) $|\mathrm{Ref}(G)| = \sum_{i=1}^{i=r}(d_i-1) - \dfrac{2}{m}\sum_{j=1}^{j=m} e_j$.

*Moreover, for all* $\chi \in \mathrm{Irr}_k(G)$, *we have*

(3) $m_\chi = m\chi(1)$,

(4) $\gamma(\chi) = \chi(1)|\mathrm{Ref}(G)| + \dfrac{2}{m}\left(\chi(1)\sum_{i=1}^{i=m} e_j - \sum_{i=1}^{i=m_\chi} e_j(\chi)\right)$.

*Proof of 10.8.* We first prove the following lemma.

**10.9. Lemma.** *Let $r$, $n$, $d_1, d_2, \dots, d_r$, $e_1, e_2, \dots, e_n$ be integers, and let $\alpha(q)$ the rational fraction defined by*

$$\alpha(q) := \frac{q^{e_1} + q^{e_2} + \cdots + q^{e_n}}{(1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_r})} \,.$$

*Let*

$$\alpha(q) = \frac{a_r}{(1-q)^r} + \frac{a_{r-1}}{(1-q)^{r-1}} + \dots$$

*be the Laurent development of $\alpha(q)$ around $q=1$. We have*

(1) $a_r(d_1 d_2 \cdots d_r) = n$ ,
(2) $2a_{r-1}(d_1 d_2 \cdots d_r) = n \sum_{j=1}^{j=r}(d_j - 1) - 2\sum_{i=1}^{i=n} e_i$ .

*Proof of 10.9.* Set

$$\alpha_1(q) := (1-q)^r \alpha(q) \,.$$

Then

$$\alpha_1(q) = \frac{q^{e_1} + q^{e_2} + \cdots + q^{e_n}}{(1+q+\cdots+q^{d_1-1})\cdots(1+q+\cdots+q^{d_r-1})} \,.$$

To prove the first assertion, one computes $\alpha_1(q)$ for $q=1$.

To prove the second assertion, one computes the derivative of $\alpha_1(q)$ for $q=1$. We have

$$\alpha_1'(q) = \alpha_1(q)\left(\frac{e_1 q^{e_1-1} + \cdots + e_n q^{e_n-1}}{q^{e_1} + \cdots + q^{e_n}} - \sum_{j=1}^{j=r} \frac{1 + 2q + \dots (d_j-1)q^{d_j-2}}{1+q+\cdots+q^{d_j-1}}\right) \,.$$

It follows that

$$a_{r-1} = -\alpha_1'(1) = a_r\left(\frac{1}{2}\sum_{j=1}^{j=r}(d_j-1) - \frac{e_1+\cdots+e_n}{n}\right) \,,$$

hence the value announced for $a_{r-1}$. $\square$

Let us notice the following particular case of what precedes. Let $P$ be a polynomial algebra with characteristic degrees $d_1, d_2, \dots, d_n$. Then

$$\text{(10.10)} \qquad \mathrm{grdim}_k P = \frac{1}{d_1 d_2 \cdots d_n}\frac{1}{(1-q)^n} + \frac{\sum_{i=1}^n (d_i - 1)}{2 d_1 d_2 \cdots d_n}\frac{1}{(1-q)^{n-1}} + \dots \,.$$

Let us now prove 10.8

We remark that

$$\frac{q^{e_1(\chi)} + q^{e_2(\chi)} + \cdots + q^{e_{m_\chi}(\chi)}}{(1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_r})} = \chi(1)\frac{1}{|G|}\frac{1}{(q-1)^r} + \frac{\gamma(\chi)}{2}\frac{1}{|G|}\frac{1}{(q-1)^{r-1}} + \dots \,,$$

from which, applying the preceding lemma, we get

$$\begin{cases} \chi(1)\dfrac{1}{|G|} d_1 d_2 \cdots d_r = m_\chi \\[2ex] \gamma(\chi)\dfrac{1}{|G|} d_1 d_2 \cdots d_r = m_\chi \displaystyle\sum_{j=1}^{j=r}(d_j - 1) - 2\sum_{i=1}^{i=m_\chi} e_j(\chi) \,. \end{cases}$$

Specializing the above formulae to the case where $\chi = 1$, we get

$$\text{(10.11)} \qquad \begin{cases} \dfrac{1}{|G|} d_1 d_2 \cdots d_r = m \\ |\mathrm{Ref}(G)| m = m \displaystyle\sum_{j=1}^{j=r} (d_j - 1) - 2 \sum_{i=1}^{i=m} e_j \,. \end{cases}$$

Using this in the general formulae gives

$$\text{(10.12)} \qquad \begin{cases} \chi(1) m = m_\chi \\ \gamma(\chi) = \chi(1) |\mathrm{Ref}(G)| + \dfrac{2}{m} \left( \chi(1) \displaystyle\sum_{i=1}^{i=m} e_j - \sum_{i=1}^{i=m_\chi} e_j(\chi) \right) \,. \end{cases}$$

□

**10.13. Theorem.** *Assume that $k$ is a any characteristic zero field. Let $P$ be a parameter algebra for the algebra of invariant $R$, with degrees $(d_i)_{1 \leq i \leq r}$, and let us denote by $m$ the rank of $R$ over $P$.*

1. *the $kG$–module $S/\mathfrak{m}_P S$ is isomorphic to $(kG)^m$,*
2. *the $PG$–module $S$ is isomorphic to $(PG)^m$,*
3. *we have $|G| m = \prod_{i=1}^{=r} d_i$.*

*Proof of 10.13.* It is enough to prove the theorem when $k$ is replaced by an extension. So we may assume that $k$ is a splitting field for the group algebra $kG$, which we do. It suffices to prove that the (ordinary) character $(\mathrm{grchar}_{S/\mathfrak{m}_P S})_{|q=1}$ of the $kG$–module $S/\mathfrak{m}_P S$ equals the character of $(kG)^m$, *i.e.*, for all $\chi \in \mathrm{Irr}_k(G)$, the multiplicity of $\chi$ in $S/\mathfrak{m}_P S$ equals $m\chi(1)$, which is precisely the first formula in 10.12 above. □

**A simple example.**

Let us consider $G := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \subset \mathrm{GL}_2(k)$. As before, we set $S := k[x, y]$ and $R := S^G$.

It is easy to check that

$$\mathrm{grdim}(R) = \sum_{n=0}^{\infty} (2n+1) q^n = \frac{1 - q^4}{(1 - q^2)^3} = \frac{1 + q^2}{(1 - q^2)^2} \,,$$

from which one can deduce the two equalities

$$\begin{cases} R = k[x^2, xy, y^2] \simeq k[u, v, w]/(uw - v^2) \\ R = k[x^2, y^2] \oplus k[x^2, y^2] xy \,. \end{cases}$$

The first equality gives the formula

$$\mathrm{grdim}(R) = \frac{1 - q^4}{(1 - q^2)^3}$$

(see 5.1), while the second equality gives the formula

$$\operatorname{grdim}(R) = \frac{1+q^2}{(1-q^2)^2} \ :$$

in that case, we may choose $P := k[x^2, y^2]$, and then $m = 1$, $e_1 = 0, e_2 = 2$.

# CHAPTER III
# POLYNOMIAL INVARIANTS OF FINITE REFLECTION GROUPS

## §11. The Shephard–Todd/Chevalley–Serre Theorem

From now on we keep the notation previously introduced :

- $V$ is an $r$–dimensional vector space over the characteristic 0 field $k$,
- $S$ is the symmetric algebra of $V$, $\mathfrak{N}$ is its maximal graded ideal,
- $G$ is a finite subgroup of $\mathrm{GL}(V)$, $\mathrm{Ref}(G)$ is the set of reflections in $G$,
- $R = S^G$ is the invariant algebra, $\mathfrak{M}$ is its maximal graded ideal, $S_G := S/\mathfrak{M}S$ is the coinvariant algebra.

**11.1. Theorem.**

*(1) The following assertions are equivalent.*

(i) *$G$ is generated by reflections.*
(ii) *$R$ is a polynomial algebra.*
(iii) *$S$ is a free $R$–module.*
(iv) *$R \otimes S_G \simeq S$ as graded $R$–modules.*

*(2) If this is the case, let us denote by $(d_1, d_2, \dots, d_r)$ the characteristic degrees of $R$. Then*

(a) $|G| = d_1 d_2 \cdots d_r$,
(b) $|\mathrm{Ref}(G)| = d_1 - 1 + d_2 - 1 + \cdots + d_r - 1$,
(c) *As ungraded $RG$–modules (resp. $kG$–modules), we have $S \simeq RG$ (resp. $S_G \simeq kG$).*

*Proof of 11.1.*

We shall prove (i)⇒(iv)⇒(iii)⇒(ii)⇒(i) :

- The proof of (i)⇒(iv) uses the Demazure operators that we introduce below.
- (iv)⇒(iii) is clear, and (iii)⇒(ii) is theorem 6.5.
- We shall then prove that (ii)⇒ (2).
- Finally we shall prove (ii)⇒i) using that (i)⇒(ii) and that (ii)⇒ (2).

So let us start with (i)⇒(iv).

*The Demazure operators*

Let $r = (v, v^*)$ be a root in $V \otimes V^*$, defining the reflection $s_r$, and the reflecting pair $(L_r, H_r)$ (where $L_r = kv$ and $H_r = \ker v^*$). Since $s$ belongs to the inertia group of the ideal $SL_r$, whenever $x \in S$ there is an element $\Delta_r(x) \in S$ such that

$$s_r(x) - x = \Delta_r(x)v\,.$$

It is easy to check the following properties.

($\delta_1$) The operator $\Delta_r \colon S \longrightarrow S$ has degree $-1$, and extends the linear form $v^* \colon V \to k$.
($\delta_2$) We have

$$\Delta_r(xy) = x\Delta_r(y) + \Delta_r(x)y + \Delta_r(x)\Delta_r(y)v \quad \text{and} \quad \Delta_r(x) = 0 \Leftrightarrow x \in R\,,$$

from which it follows that $\Delta_r$ is an $R$–linear endomorphism of $S$. Thus in particular $\Delta_r$ induces a degree $-1$ endomorphism of the coinvariant algebra $S_G$.

**11.2. Lemma.** *Assume $G$ generated by reflections. Let $x \in S_G$.*

(1) *The following assertions are equivalent*
  *(i) $x \in k$,*
  *(ii) for all roots $r$ such that $s_r \in G$, $\Delta_r(x) = 0$.*

(2) *If $x$ is homogeneous of degree $n \geq 1$, there exist roots $r_1, r_2, \dots, r_n$ such that*

$$\Delta_{r_1}\Delta_{r_2}\cdots\Delta_{r_n}(x) \in k^\times .$$

*Proof of 11.2.*

(1) It suffices to prove the assertion for the case where $x$ is homogeneous. Now since $G$ is generated by its reflections, we see that (ii) hold if and only if $x$ is fixed by $G$. We quote 9.1 to conclude.

(2) Follows from (1) by induction on the degree of $x$. $\square$

**11.3. Theorem.** *Assume $G$ is generated by reflections.*

*Then for each choice of a $G$–stable graded submodule $T$ of $S$ such that $\mathfrak{M}S \oplus T = S$, the morphism*

$$\mu_T \colon R \otimes S_G \longrightarrow S$$

*is an isomorphism.*

*Proof of 11.3.* We know that $\mu_T$ is onto (see 9.2). It suffices to prove that if $(x_1, x_2, \dots, x_m)$ is a family of homogeneous elements of $S$ whose image in $S_G$ is $k$–free, then $(x_1, x_2, \dots, x_m)$ is $R$–free.

Assume this is not the case. Choose $m$ minimal such that there is a family of homogeneous elements $(x_1, x_2, \dots, x_m)$ in $S$ which is not $R$–free, while it defines a free family in $S_G$. Assume that $\deg x_1 \leq \deg x_i$. Let

$$t_1x_1 + \cdots + t_mx_m = 0\,,$$

(where $t_i \in R$, $t_i \neq 0$) be a dependance relation.

By 11.2, (2), there is a family (perhaps empty) of roots $(r_1, r_2, \dots, r_n)$ (where $n = \deg x_1$) such that, if $\Delta := \Delta_{r_1}\Delta_{r_2}\cdots\Delta_{r_n}$, then $\Delta(x_1) \in k^\times$. Moreover, since $\Delta$ is an $R$–endomorphism of $S$, we get a dependance relation

$$t_1\Delta(x_1) + \cdots + t_m\Delta(x_m) = 0\,.$$

Let us set $\lambda := -\Delta(x_1)^{-1}$, and $u_i := \lambda\mathrm{pr}_G(\Delta(x_i))$ for $i \geq 2$. We have

$$t_1 = \lambda t_2\Delta(x_2) + \cdots + \lambda t_m\Delta(x_m) = t_2u_2 + \cdots + t_mu_m\,,$$

which implies

$$t_2(x_2 + u_2x_1) + \dots t_m(x_m + u_mx_1) = 0\,.$$

We see that the family $(x_2 + u_2x_1, \dots, x_m + u_mx_1)$ is a family of homogeneous elements of $S$ which defines a free family in $S_G$ : a contradiction with the minimality of $m$. $\square$

Let us prove (ii)$\Rightarrow$(2).

Let us use notation from §9. We may choose $P = R$ hence $m = 1$. So we see that 10.13 does imply (c), and moreover by 10.8 we have

$$\begin{cases} |G| = d_1 d_2 \cdots d_r \\ |\mathrm{Ref}(G)| = \sum_{i=1}^{i=r} (d_i - 1) \end{cases}$$

from which (2) follows.

Let us now prove that (ii)$\Rightarrow$(i).

Let $G^0 := \langle \mathrm{Ref}(G) \rangle$. Let us set $R^0 := S^{G^0}$, and thus $R \subseteq R^0$.

Since (i)$\Rightarrow$(ii), we know that $R^0$ is a polynomial algebra. Let us denote by $(d_1^0 \le d_2^0 \le \cdots \le d_r^0)$ its family of characteristic degrees. If $(d_1 \le d_2 \le \cdots \le d_r)$ denotes the set of charateristic degrees of $R$, we know that, for all $i = 1, \dots, r$, we have $d_i^0 \le d_i$ (see 6.1).

On the other hand, by (2) we know that

$$|\mathrm{Ref}(G)| = \sum_{i=1}^{i=r} (d_i - 1) = \sum_{i=1}^{i=r} (d_i^0 - 1)\,,$$

from which it follows that for all $i$, $d_i = d_i^0$. But since $|G| = d_1 d_2 \cdots d_r$ and $|G^0| = d_1^0 d_2^0 \cdots d_r^0$, we see that $G = G^0$. $\square$

## §12. Steinberg theorem and first applications

**The Jacobian as a monomial.**

We shall compute the Jacobian $\mathrm{Jac}(S/R)$ for $G$ a complex reflection group. Whenever $(H, L)$ is a reflecting pair for $G$, let us choose a nonzero element $j_H \in L$. We recall that we denote by $e_H$ the order of the cyclic group $G(H)$.

**12.1. Proposition.** *We have*

$$\mathrm{Jac}(S/R) = \prod_{H \in \mathcal{A}} j_H^{e_H - 1}\,.$$

*Proof of 12.1.* Let $(u_1, u_2, \dots, u_r)$ be an algebraic basis of $R$. We recall that, for $(v_1, v_2, \dots, v_r)$ a basis of $V$, we have $\mathrm{Jac}(S/R) = \det\left(\frac{\partial u_i}{\partial v_j}\right)_{i,j}$.

If $(d_1, d_2, \dots, d_r)$ are the characteristic degrees of $R$, it follows that $\deg \mathrm{Jac}(S/R) = (d_1 - 1) + (d_2 - 1) + \cdots + (d_r - 1)$, and then (cf. above 11.1) that $\deg \mathrm{Jac}(S/R) = \sum_{H \in \mathcal{A}} e_H - 1$.

It suffices then to prove that for $H \in \mathcal{A}$, $j_H^{e_H - 1}$ divides $\mathrm{Jac}(S/R)$.

Let $(L, H)$ be the reflecting pair corresponding to $H$, let $(v_1 := j_H, v_2, \dots, v_{r-1})$ be a basis such that $(v_2, \dots, v_r)$ is a basis of $H$. For $x \in S$, if $P(t_1, t_2, \dots, t_r) \in k[t_1, t_2, \dots, t_r]$ is the polynomial such that $x = P(j_H, v_2, \dots, v_r)$, we set $\partial_L x := \dfrac{\partial P}{\partial t_1}(j_H, v_2, \dots, v_r)$. It suffices to prove that for $u \in R$, $j_H^{e_H - 1}$ divides $\partial_L u$.

Let $s$ be a generator of $G(H)$, and let $\zeta_s$ denote its determinant. Since $u$ is invariant under $s$, for $u = P(j_H, v_2, \dots, v_r)$, we have

$$P(\zeta_s j_H, v_2, \dots, v_r) = P(j_H, v_2, \dots, v_r)\,,$$

which yields

$$\frac{\partial P}{\partial t_1}(\zeta_s t_1, t_2, \dots, t_r) = \zeta_s^{e_H - 1}\frac{\partial P}{\partial t_1}(t_0, t_2, \dots, t_r)\,.$$

The required property results from 8.7. $\square$

**Action of the normalizer and generalized degrees.**

Let $G \subset \mathrm{GL}(V)$ be a finite group generated by reflections. We set

$$N(G) := N_{\mathrm{GL}(V)}(G) \quad \text{and} \quad \overline{N}(G) := N(G)/G\,.$$

The group $\overline{N}(G)$ acts on $R$, hence on $\mathfrak{M}$ and on the finite dimensional graded $k$–vector space $V_G := \mathfrak{M}/\mathfrak{M}^2$.

The graded dimension of $V_G$ is

$$\mathrm{grdim} V_G = q^{d_1} + q^{d_2} + \dots + q^{d_r} = \sum_d r(d) q^d\,.$$

Let $H$ be a subgroup of $\overline{N}(G)$.

Let us denote by $\xi_d$ the character of the representation of $H$ on the space $V_G^d$ of degree $d$ elements of $V_G$ (a space of dimension $r(d)$). Thus the graded character of the $kH$–module $V_G$ is

$$\mathrm{grchar}_{V_G} = \sum_d \xi_d q^d\,,$$

and we have

$$\mathrm{grchar}_{V_G}(1) = \sum_d \xi_d(1) q^d = \sum_d r(d) q^d\,.$$

For each $d$, we have

$$\xi_d = \sum_{\nu \in \mathrm{Irr}(H)} m_d(\nu)\nu \quad \text{where} \quad m_d(\nu) = \langle \xi_d, \nu \rangle\,.$$

The family

$$(d, \nu)_{d \geq 0, \nu \in \mathrm{Irr}(H)} \quad \text{where } (d,\nu) \text{ is repeated } m_d(\nu) \text{ times}$$

is called the family of *generalized invariant degrees of* $(G, H)$.

For each $d$, let us denote by $X_i(d,\nu)_{\nu, i = 1, \dots, m_d(\nu)}$ a family of $H$–stable subspaces of the space $\mathfrak{M}^d$ of degree $d$ elements of $\mathfrak{M}$, such that

$$\mathfrak{M}^d = (\mathfrak{M}^2)^d \oplus \bigoplus_{\nu, i} X_i(d, \nu)\,.$$

Then we have

$$R = k[X_i(d,\nu)_{d, \nu, i = 1, \dots, m_d(\nu)}]\,.$$

The choice, for all $d, \nu, i$, of a basis of $X_i(d,\nu)$, provides a set of homogeneous algebraically elements $(u_1, u_2, \dots, u_r)$ such that $R = k[u_1, u_2, \dots, u_r]$.

*Action of an element.*

Let $n \in N(G)$, with image $\overline{n}$ in $\overline{N}(G)$. By applying what precedes to the cyclic group $H$ generated by $\overline{n}$, we see that

**12.2. Proposition.** *Let $(\zeta_1, \zeta_2, \ldots, \zeta_r)$ be the spectrum of $\overline{n}$ in its action on $V_G = \mathfrak{M}/\mathfrak{M}^2$.*

(1) *There is an algebraic basis $(u_1, u_2, \ldots, u_r)$ of $R$, with degrees $(d_1, d_2, \ldots, d_r)$, such that, for $1 \leq i \leq r$, we have $n(u_i) = \zeta_i u_i$.*

(2) *We have*
$$(\zeta_1, \zeta_2, \ldots, \zeta_r) = (1, 1, \ldots, 1) \;\Leftrightarrow\; n \in G\,.$$

*The family $((d_1, \zeta_1), (d_1, \zeta_1), , \ldots, (d_1, \zeta_1))$ is called the family of generalized degrees of $(G, \overline{n})$.*

*First application : order of the center.*

**12.3. Proposition.** *Assume that $G$ acts irreducibly on $V$. We have*
$$|ZG| = \gcd\{d_1, d_2, \ldots, d_r\}\,.$$

*Proof of 12.3.* Since $G$ is irreducible, $ZG = \{\zeta \mathrm{Id}_V \mid (\zeta \mathrm{Id}_V \in k^\times)(\zeta \mathrm{Id}_V \in G$.

If $\zeta \mathrm{Id}_V \in G$, then by applying it to the invariant polynomials we see that for each degree $d_i$ we have $\zeta^{d_i} = 1$.

Reciprocally, let $d := \gcd\{d_1, d_2, \ldots, d_r\}$. Let $\zeta$ be a root of unity of order $d$. Since for all $i$ we have $\zeta^{d_i} = 1$, the generalized degrees of the pair $(G, \zeta \mathrm{Id}_V)$ are $(d_1, 1), (d_2, 1), \ldots (d_r, 1)$, proving that $\zeta \mathrm{Id}_V \in G$. $\square$

**Steinberg theorem.**

**12.4. Steinberg theorem.** *Let $G$ be a finite reflection group on $V$. Let $X$ be a subset of $V$. Then the fixator $G(X) = C_G(X)$ of $X$ is a reflection group, generated by those reflections in $G$ whose reflecting hyperplane contains $X$.*

The proof given below is due to Gus Lehrer [Le]. The reader may refer to [St] for the original proof, or to [Bou1], Ch. v, §6, ex. 8, for another proof.

*Proof of 12.4.* It is clear that it is enough to prove that the fixator of an element $v \in V$ is generated by reflections, which we shall prove.

Let $\mathrm{Ref}(G(v))$ be the set of all reflections in $G(v)$, *i.e.*, those reflections of $G$ whose reflecting hyperplane contains $v$. Let $G(v)^0$ be the subgroup of $G(v)$ generated by $\mathrm{Ref}(G(v))$. We shall prove that $G(v)^0 = G(v)$.

Now let us consider $G$ as acting (through the contragredient representation) on the dual $V^*$. Let us denote by $S^*$ the symmetric algebra of $V^*$ (notice that $S^*$ is not the dual vector space of $S$).

Since $G(v)^0$ is a normal subgroup of $G(v)$, whenever $g \in G(v)$, we can consider the generalized characteristic degrees $((d_1(v), \zeta_1), (d_2(v), \zeta_2), \ldots, (d_r(v), \zeta_r))$ of the pair $(G(v)^0, g)$ acting on $S^*$, and it suffices (see 12.2) to prove that $\zeta_i = 1$ for $i = 1, 2, \ldots, r$ to ensure that $g \in G(v)^0$.

Let $(x_1, x_2, \ldots, x_r)$ be a basis of $V^*$.

Let $(u_1^{(v)}, u_2^{(v)}, \ldots, u_r^{(v)})$ be an algebraic basis of $S^{*G(v)^0}$ such that, for $1 \leq i \leq r$, we have $g(u_i^{(v)}) = \zeta_i u_1^{(v)}$. In particular, each $u_i^{(v)}$ is a polynomial in $(x_1, x_2, \ldots, x_r)$.

Let $(u_1, u_2, \dots, u_r)$ be an algebraic basis of $R^* := S^{*G}$. Since $R^* \subseteq S^{*G(v)^0}$, there exist polynomials $P_i(t_1, t_2, \dots, t_r) \in k[t_1, t_2, \dots, t_r]$ such that, for $i = 1, 2, \dots, r$, we have $u_i = P_i(u_1^{(v)}, u_2^{(v)}, \dots, u_r^{(v)})$.

- On one hand, since $u_i$ is fixed by $g$, we have

$$P_i(\zeta_1 t_1, \zeta_2 t_2, \dots, \zeta_r t_r) = P_i(t_1, t_2, \dots, t_r)\,.$$

Taking the partial derivative yields

$$\zeta_j \frac{\partial P_i}{\partial t_j}(\zeta_1 t_1, \zeta_2 t_2, \dots, \zeta_r t_r) = \frac{\partial P_i}{\partial t_j}(t_1, t_2, \dots, t_r)$$

hence

$$g\left(\frac{\partial P_i}{\partial t_j}(t_1, t_2, \dots, t_r)\right) = \zeta_j^{-1} \frac{\partial P_i}{\partial t_j}(t_1, t_2, \dots, t_r)\,.$$

Apply the preceding equality to the vector $v$. Since $g(v) = v$, we get

$$\langle \frac{\partial P_i}{\partial t_j}(u_1^{(v)}, u_2^{(v)}, \dots, u_r^{(v)}), v \rangle = \zeta_j \langle \frac{\partial P_i}{\partial t_j}(u_1^{(v)}, u_2^{(v)}, \dots, u_r^{(v)}), v \rangle\,.$$

Thus we see that if $\zeta_j \neq 1$, for $i = 1, 2, \dots, r$, we have

$$\langle \frac{\partial P_i}{\partial t_j}(u_1^{(v)}, u_2^{(v)}, \dots, u_r^{(v)}), v \rangle = 0\,.$$

- On the other hand, we have

$$\frac{\partial u_i}{\partial x_j} = \sum_{m=1}^{m=r} \frac{\partial P_i}{\partial t_m}(u_1^{(v)}, u_2^{(v)}, \dots, u_r^{(v)}) \frac{\partial u_m^{(v)}}{\partial x_j}\,,$$

hence

$$\det\left(\frac{\partial u_i}{\partial x_j}\right) = \det\left(\frac{\partial P_i}{\partial t_m}(u_1^{(v)}, u_2^{(v)}, \dots, u_r^{(v)})\right) \det\left(\frac{\partial u_m^{(v)}}{\partial x_j}\right)\,.$$

For each reflecting hyperplane $H \in \mathcal{A}$, let us denote by $j_H^*$ a linear form on $V$ with kernel $H$. Then we have (up to a nonzero scalar)

$$\det\left(\frac{\partial u_i}{\partial x_j}\right) = \prod_{(H \in \mathcal{A})} {j_H^*}^{e_H - 1} \quad \text{and} \quad \det\left(\frac{\partial u_m^{(v)}}{\partial x_j}\right) = \prod_{(H \in \mathcal{A})(\langle j_H^*, v\rangle = 0)} {j_H^*}^{e_H - 1}$$

hence

$$\det\left(\frac{\partial P_i}{\partial t_m}(u_1^{(v)}, u_2^{(v)}, \dots, u_r^{(v)})\right) = \prod_{(H \in \mathcal{A})(\langle j_H^*, v\rangle \neq 0)} {j_H^*}^{e_H - 1}\,,$$

and in particular

$$\langle \det\left(\frac{\partial P_i}{\partial t_m}(u_1^{(v)}, u_2^{(v)}, \dots, u_r^{(v)})\right), v \rangle \neq 0\,.$$

Thus we see that, for $j = 1, 2, \dots, r$, we have $\zeta_j = 1$. $\square$

**Fixed points of elements of $G$.**

Let $X$ be a subset of $V$. Let us set

$$\overline{X} := \bigcap_{h\in\mathcal{A}, H\supseteq X} H\,.$$

Then it follows from Steinberg theorem that $G(X) = G(\overline{X})$.

**Definition.** The fixators of subsets of $V$ in $G$ are called the parabolic subgroups of $G$.

Let us denote by $\mathrm{Par}(G)$ the set of parabolic subgroups of $G$, and let us denote by $I(\mathcal{A})$ the set of intersections of elements of $\mathcal{A}$. Then the map

$$I(\mathcal{A}) \to \mathrm{Par}(G)\ ,\ X \mapsto G(X)$$

is a $G$–equivariant inclusion–reversing bijection.

The following result gives another description of $I(\mathcal{A})$.

**12.5. Proposition.** *The set $I(\mathcal{A})$ coincides with the set of fixed points $V^g$ for $g \in G$.*

*Proof of 12.5.*

1. We prove by induction on $|G|$ that for $g \in G$, the space $V^g$ of fixed points of $g$ is a reflecting hyperplanes intersection.

Notice that the property is obvious when $G$ is cyclic.

By Steinberg theorem, we know that $G(V^g)$ is a reflection group. If $G(V^g)$ is a proper subgroup of $G$, since $g \in G(V^g)$, by the induction hypothesis we see that $V^g$ is an intersection of reflecting hyperplanes for $G(V^g)$, which are reflecting hyperplanes for $G$.

2. Conversely, let us prove that whenever $X \in \mathcal{A}$, there exists $g \in G$ such that $X = V^g$. Choose $H_1, H_2, \dots, H_m \in \mathcal{A}$ with $m$ minimal such that $X = H_1 \cap H_2 \cap \dots \cap H_m$. Thus the corresponding lines $L_1, L_2, \dots, L_m$ are linearly independant, *i.e.*, $L_1 + L_2 + \dots + L_m = L_1 \oplus L_2 \oplus \dots, \oplus L_m$. The desired result will follow from the next lemma.

**12.6. Lemma.** *Let $H_1, H_2, \dots, H_m$ be a family of linearly independant reflecting hyperplanes. For all $i = 1, 2, \dots, m$, let us choose a reflection $s_i \in G(H_i)$, and let us set $g := s_1 s_2 \cdots s_m$. Then $V^g = H_1 \cap H_2 \cap \dots \cap H_m$.*

*Proof of 12.6.* We argue by induction on $m$. The case $m = 1$ is trivial. Assume the property holds for $m - 1$ independant hyperplanes. Since obviously $\bigcap_{1\le i\le m} H_i \subseteq V^g$, it is enough to prove that $V^g \subseteq \bigcap_{1\le i\le m} H_i$. Let $v \in V^g$.

Whenever $x \in V$ and $i = 1, 2, \dots, m$, we have $s_i(x) \equiv x \mod L_i$, hence

$$(s_2 s_3 \cdots s_m)(v) \equiv v \mod L_2 \oplus L_3 \oplus \dots \oplus L_m\,.$$

Since $(s_1 s_2 \cdots s_m)(v) = v$, we also have

$$(s_2 s_3 \cdots s_m)(v) = s_1^{-1}(v) \equiv v \mod L_1\,.$$

It follows that

$$(s_2 s_3 \cdots s_m)(v) = s_1^{-1}(v) = v\,,$$

thus $v \in H_1$, and also by the induction hypothesis we have $v \in H_2 \cap H_3 \cap \dots \cap H_m$. $\square$

$\square$

**Braid groups.**

Recall that we set $V^{\mathrm{reg}} := V - \cup_{H\in\mathcal{A}} H$ . We denote by $p\colon V^{\mathrm{reg}} \to V^{\mathrm{reg}}/G$ the canonical surjection.

**Definition.** Let $x_0 \in V^{\mathrm{reg}}$. We introduce the following notation for the fundamental groups:

$$P := \Pi_1(V^{\mathrm{reg}}, x_0) \quad \text{and} \quad B := \Pi_1(V^{\mathrm{reg}}/G, p(x_0))\,,$$

and we call $B$ and $P$ respectively the *braid group* (at $x_0$) and the *pure braid group* (at $x_0$) associated to $G$.

We shall often write $\Pi_1(V^{\mathrm{reg}}/G, x_0)$ for $\Pi_1(V^{\mathrm{reg}}/G, p(x_0))$.

The covering $V^{\mathrm{reg}} \to V^{\mathrm{reg}}/G$ is Galois by Steinberg's theorem, hence the projection $p$ induces a surjective map $B \twoheadrightarrow B$ , $\sigma \mapsto \overline{\sigma}$, as follows :

Let $\tilde{\sigma}\colon [0,1] \to V^{\mathrm{reg}}$ be a path in $V^{\mathrm{reg}}$, such that $\tilde{\sigma}(0) = x_0$, which lifts $\sigma$. Then $\overline{\sigma}$ is defined by the equality $\overline{\sigma}(x_0) = \tilde{\sigma}(1)$ .

We have the following short exact sequence :

$$1 \to P \to B \to G \to 1\,, \tag{12.7}$$

where the map $B \to G$ is defined by $\sigma \mapsto \overline{\sigma}$.

*Remark.* Bessis has recently proved that he spaces $V^{\mathrm{reg}}$ and $V^{\mathrm{reg}}/G$ are $K(\pi,1)$-spaces (see [Bes3]).

*Braid reflections around the hyperplanes.*

For $H \in \mathcal{A}$, we set $\zeta_H := \zeta_{e_H}$, We denote by $s_H$ and call *distinguished reflection* the reflection in $G$ with reflecting hyperplane $H$ and determinant $\zeta_H$. We set

$$L_H := \mathrm{im}\,(s_H - \mathrm{Id}_V)\,.$$

For $x \in V$, we set $x = \mathrm{pr}_H(x) + \mathrm{pr}_H^{\perp}(x)$ with $\mathrm{pr}_H(x) \in H$ and $\mathrm{pr}_H^{\perp}(x) \in L_H$.

Thus, we have $s_H(x) = \zeta_H \mathrm{pr}_H^{\perp}(x) + \mathrm{pr}_H(x)$ .

If $t \in \mathbb{R}$, we set $\zeta_H^t := \exp(2i\pi t/e_H)$, and we denote by $s_H^t$ the element of $\mathrm{GL}(V)$ (a pseudo–reflection if $t \neq 0$) defined by :

$$s_H^t(x) = \zeta_H^t \mathrm{pr}_H^{\perp}(x) + \mathrm{pr}_H(x)\,. \tag{12.8}$$

Notice that, denoting by $s_H^{te_H}$ the $e_H$-th power of the endomorphism $s_H^t$, we have

$$s_H^{te_H}(x) = \exp(2\pi i t)\mathrm{pr}_H^{\perp}(x) + \mathrm{pr}_H(x)\,. \tag{12.9}$$

For $x \in V$, we denote by $\sigma_{H,x}$ the path in $V$ from $x$ to $s_H(x)$, defined by :

$$\sigma_{H,x}\colon [0,1] \to V\ ,\ t \mapsto s_H^t(x)\,.$$

and we denote by $\pi_{H,x}$ the loop in $V$ with initial point $x$ defined by :

$$\pi_{H,x}\colon [0,1] \to V\ ,\ t \mapsto s_H^{te_H}(x)\,.$$

Let $\gamma$ be a path in $V^{\mathrm{reg}}$, with initial point $x_0$ and terminal point $x_H$.

- The path defined by $s_H(\gamma^{-1})\colon t \mapsto s_H(\gamma^{-1}(t))$ is a path in $V^{\mathrm{reg}}$ going from $s_H(x_H)$ to $s_H(x_0)$. We define the path $\sigma_{H,\gamma}$ from $x_0$ to $s_H(x_0)$ as follows :

$$\sigma_{H,\gamma} := s_H(\gamma^{-1}) \cdot \sigma_{H,x_H} \cdot \gamma\,.$$

  It is not difficult to see that, provided $x_H$ is chosen "close to $H$ and far from the other reflecting hyperplanes", the path $\sigma_{H,\gamma}$ is in $V^{\mathrm{reg}}$, and its homotopy class does not depend on the choice of $x_H$.
- We define the loop $\pi_{H,\gamma}$ by the formula

$$\pi_{H,\gamma} := \gamma^{-1} \cdot \pi_{H,x_H} \cdot \gamma\,.$$


**Definition.** We call *braid reflections* the elements $\mathbf{s}_{H,\gamma} \in B$ defined by the paths $\sigma_{H,\gamma}$. If the image of $\mathbf{s}_{H,\gamma}$ in $G$ is $s_H$, we say that $\mathbf{s}_{H,\gamma}$ is an $s_H$–braid reflection, or an $H$–braid reflection.

We still denote by $\pi_{H,\gamma}$ the element of $P$ defined by the loop $\pi_{H,\gamma}$.

The following properties are immediate.

**12.10. Lemma.**

(1) *Whenever $\gamma'$ is a path in $V^{\mathrm{reg}}$, with initial point $x_0$ and terminal point $x_H$, if $\tau$ denotes the loop in $V^{\mathrm{reg}}$ defined by $\tau := {\gamma'}^{-1}\gamma$, one has*

$$\sigma_{H,\gamma'} = \tau \cdot \sigma_{H,\gamma} \cdot \tau^{-1}$$

*and in particular $\mathbf{s}_{H,\gamma}$ and $\mathbf{s}_{H,\gamma'}$ are conjugate in $P$.*

(2) *In the group $B$, we have*

$$\mathbf{s}_{H,\gamma}^{e_H} = \pi_{H,\gamma}\,.$$

The variety $V$ (resp. $V/G$) is simply connected, the hyperplanes (resp. the images of the reflecting hyperplanes in $V/G$) are irreducible divisors (irreducible closed subvarieties of codimension one), and the braid reflections as defined above are "generators of the monodromy" around these irreducible divisors. Then it is not difficult to check the followng fundamental theorem.

**12.11. Theorem.**

(1) *The braid group is generated by the braid reflections $(\mathbf{s}_{H,\gamma})$ (for all $H$ and all $\gamma$).*

(2) *The pure braid group is generated by the elements $(\mathbf{s}_{H,\gamma}^{e_H})$*

## §13. Coinvariant algebra and harmonic polynomials

**On the coinvariant algebra.**

Let us recall our notation.

• $V$ is an $r$–dimensional vector space on the characteristic zero field $k$, and $G$ is a finite subgroup of $\mathrm{GL}(V)$ generated by reflections.

• $S$ is the symmetric algebra of $V$, $R := S^G$ is the subalgebra of fixed points of $G$ on $S$, a polynomial algebra over $k$ with characteristic degrees $(d_1, d_2, \dots, d_r)$ and maximal graded ideal $\mathfrak{M}$.

We know that $S$ is a free $R$-module of rank $|G| = d_1 d_2 \cdots d_r$. We call coinvariant algebra the algebra

$$S_G := k \otimes_R S = S/\mathfrak{M}S\,.$$

We have $S = R \otimes_k S_G$ .

• $\mathrm{Ref}(G)$ is the set of reflections of $G$ and $\mathcal{A}$ is the set of their reflecting hyperplanes. We set $N_h := |\mathcal{A}|$ and $N := |\mathrm{Ref}(G)|$ . For $H \in \mathcal{A}$, $G(H)$ is the fixator of $H$ in $G$, and $e_H$ is its order. We have $N = d_1 - 1 + d_2 - 1 + \cdots + d_r - 1 = \sum_{H \in \mathcal{A}} (e_H - 1)$ .

**13.1. Proposition.** *The maximal degree of an element of the coinvariant algebra $S_G$ is $N$, the number of reflections.*

*Proof of 13.1.* Since

$$\mathrm{grdim}_K(R) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - gx)} = \frac{1}{\prod_{i=1}^{i=r} (1 - x^{d_i})}\,,$$

the graded character of the coinvariant algebra is

$$\chi_{S_G}(g) = \prod_{i=1}^{i=r} (1 - x^{d_i}) \frac{1}{\det(1 - gx)}\,.$$

and its graded dimension is

$$\mathrm{grdim}_K S_G = \prod_{i=1}^{i=r} (1 + x + \cdots + x^{d_i - 1})\,.$$

In particular, we see that the maximal degree occuring in $S_G$ is indeed $N$. Thus we have

$$S_G = \bigoplus_{n=0}^{n=N} S_G^n\,,$$

□

**13.2. Corollary.** *Let $\mathfrak{M}$ be the maximal graded ideal of $R$. Then we have*

$$\bigoplus_{n > N} S^n \subseteq S\mathfrak{M}\,.$$

**Linear characters and their associated polynomials.**

Let us recall notation and results from §8. Whenever $(L, H)$ is a reflecting pair of $G$, we denote by $j_H$ a nonzero element of $L$, and if $\mathfrak{p}$ denotes the orbit of $H$ under $G$, we set

$$j_{\mathfrak{p}} := \prod_{H \in \mathfrak{p}} j_H \,.$$

The linear character $\theta_{\mathfrak{p}} \colon G \to k^\times$ is defined by

$$g(j_{\mathfrak{p}}) = \theta_{\mathfrak{p}}(g) j_{\mathfrak{p}} \,,$$

and for $s \in \mathrm{Ref}(G)$ we have

$$\theta_{\mathfrak{p}}(s) = \begin{cases} \det_V(s) & \text{if } s \in G(H) \text{ for some } H \in \mathfrak{p}\,, \\ 1 \quad \text{if not.} \end{cases}$$

Since $G$ is generated by reflections, we see in particular that for

$$j := \prod_{H \in \mathcal{A}} j_H$$

we have

$$(\forall g \in G)\ g(j) = \det_V(g) j \,.$$

**13.3. Theorem.** *Let $G$ be a finite subgroup of $\mathrm{GL}(V)$ generated by reflections.*

(1) *The restrictions induce an isomorphism*

$$\rho_G \colon \mathrm{Hom}(G, k^\times) \to \left( \prod_{H \in \mathcal{A}} \mathrm{Hom}(G(H), k^\times) \right)^G \,.$$

(2) *Let $\theta \in \mathrm{Hom}(G, k^\times)$. For $H \in \mathcal{A}$, denote by $\mathfrak{p}$ its orbit under $G$. Then there is a unique integer $m_{\mathfrak{p}}(\theta)$ such that*

$$\mathrm{Res}^G_{G(H)}(\theta) = \det_V^{m_{\mathfrak{p}}(\theta)} \quad \textit{and} \quad 0 \leq m_{\mathfrak{p}}(\theta) \leq e_{\mathfrak{p}} - 1 \,.$$

*Set*

$$j_\theta := \prod_{\mathfrak{p} \in \mathcal{A}/G} j_{\mathfrak{p}}^{m_{\mathfrak{p}}(\theta)} \,.$$

*We then have*

$$S^G_\theta = R j_\theta \,.$$

*Proof of 13.3.*

(1) The injectivity of $\rho_G$ is an immediate consequence of the fact that $G$ is generated by its reflections. The surjectivity results from 8.5.

(2) For each $H \in \mathcal{A}$, $\det_V$ generates $\mathrm{Hom}(G(H)), k^\times)$. This implies the existence and unicity of $m_\mathfrak{p}(\theta)$. Moreover, it is clear that $Rj_\theta \subset S_\theta^G$. Let us prove the inverse inclusion.

It suffices to check that for all $H \in \mathcal{A}$ and for all $x \in S_\theta^G$, $x$ is divisible by $j_H^{m_\mathfrak{p}(\theta)}$. We use the same methods as in the proof of 8.6 : for $H \in \mathcal{A}$, we denote $(L, H)$ the associated reflecting pair, and we choose a basis $\{j_1, j_2, \dots, j_{r-1}\}$ of $H$. Let $P(t_0, t_1, \dots, t_{r-1}) \in k[t_0, t_1, \dots, t_{r-1}]$ such that $x = P(j_H, j_1, j_2, \dots, j_{r-1})$. Let $s$ be a generator of $G(H))$, with determinant $\zeta_s$ ($\zeta_s$ is a primitive $e_\mathfrak{p}$–th root of unity).

Since $x \in S_\theta^G$, we have $P(\zeta_s t_0, t_1, \dots, t_{r-1}) = \zeta_s^{m_\mathfrak{p}(\theta)} P(t_0, t_1, \dots, t_{r-1})$. By 8.7, it follows that $t_0^{m_\mathfrak{p}(\theta)}$ divides $P(t_0, t_1, \dots, t_{r-1})$, *i.e.*, that $j_H^{m_\mathfrak{p}(\theta)}$ divides $x$. □

*Remark.* For $\mathfrak{p} \in \mathcal{A}$, we have $j_{\theta_\mathfrak{p}} = j_\mathfrak{p}$ .

We set

$$j'_\theta := j_{\theta^{-1}} = \prod_{\mathfrak{p} \in \mathcal{A}/G} j_\mathfrak{p}^{e_\mathfrak{p} - m_\mathfrak{p}(\theta)} = \prod_{H \in \mathcal{A}} j_H^{e_H - m_H(\theta)} .$$

We also set

$$j = j_{\det_V} \quad \text{and} \quad J := j' := \mathrm{Jac}(S/R) = j_{\det_V^{-1}} .$$

We have

$$j := \prod_{\mathfrak{p} \in \mathcal{A}/G} j_\mathfrak{p} = \prod_{H \in \mathcal{A}} j_H \quad \text{and} \quad J := \prod_{\mathfrak{p} \in \mathcal{A}/G} j'_\mathfrak{p} = \prod_{\mathfrak{p} \in \mathcal{A}/G} j_\mathfrak{p}^{e_\mathfrak{p} - 1} = \prod_{H \in \mathcal{A}} j_H^{e_H - 1} .$$

For $\mathfrak{p} \in \mathcal{A}$, the discriminant at $\mathfrak{p}$ (cf. 8.6 above) is such that

$$\Delta_\mathfrak{p} = j_\mathfrak{p} j'_\mathfrak{p} .$$

We call *discriminant of* $G$ the element of $R$ defined by

$$\Delta := \prod_{\mathfrak{p} \in \mathcal{A}/G} \Delta_\mathfrak{p} = jj' = \prod_{H \in \mathcal{A}} j_H^{e_H} .$$

The following properties result from what precedes.

13.4 *Recall that we set* $N_h = |\mathcal{A}|$ *and* $N = |\mathrm{Ref}(G)|$. *We have*

$$\begin{aligned} \deg j_\mathfrak{p} = \nu_\mathfrak{p} \quad &, \quad \deg j'_\mathfrak{p} = \nu_\mathfrak{p}(e_\mathfrak{p} - 1) , \\ \deg j = \sum_\mathfrak{p} \nu_\mathfrak{p} = N_h \quad &, \quad \deg J = \sum_\mathfrak{p} \nu_\mathfrak{p}(e_\mathfrak{p} - 1) = N , \\ \deg \Delta_\mathfrak{p} = \nu_\mathfrak{p} e_\mathfrak{p} \quad &, \quad \deg \Delta = \sum_\mathfrak{p} \nu_\mathfrak{p} e_\mathfrak{p} = N + N_h . \end{aligned}$$

*The case of cyclic groups.*

Assume that $G \subset \mathrm{GL}(V)$ is a cyclic group of order $e$, consisting of the identity and of $e-1$ reflections with hyperplane $H$ and line $L$. Thus $V = L \oplus H$. Let us denote by $j$ a nonzero element of $L$. Let us summarize in this case the values of all invariants introduced so far.

Let us denote the set of all irreducible characters of $G$ as

$$\mathrm{Irr}(G) = \{1, \det_V, \det_V^2, \dots, \det_V^{e-1}\}\,.$$

We choose a basis $(j, y_1, y_2, \dots, y_{e-1})$ of $V$ such that $(y_1, y_2, \dots, y_{e-1})$ is a basis of $H$. We have

$$S = k[j, y_1, y_2, \dots, y_{e-1}] \quad \text{and} \quad R = k[j^e, y_1, y_2, \dots, y_{e-1}]\,.$$

It is easy to check that for $0 \leq j \leq e$, we have

$$S^G_{\det_V^n} = R\, j^n\,,$$

which shows in particular that the unique exponent of $\det_V^n$ is $n$ : we have

$$\mathrm{grdim}_k S^G_{\det_V^n} = \frac{q^n}{(1-q)^r(1-q^e)} \quad \text{and} \quad \mathrm{grmult}(\det_V^n, S_G) = q^n\,. \tag{13.5}$$

**13.6. Proposition.** *Assume $G$ cyclic of order $e$, consisting of 1 and of reflections with line generated by $j$. Then*

$$\begin{cases} \mathrm{Irr}(G) = \{1, \det_V, \det_V^2, \dots, \det_V^{e-1}\} \\ j_{\det_V^n} = j^n\,, \\ \mathrm{grmult}(\det_V^n, S_G) = q^n\,, \\ \mathrm{Jac}(S/R) = j^{e-1}\,, \\ \Delta = j^e\,. \end{cases}$$

*Duality and isotypic components.*

For $H \in \mathcal{A}$, let $j_H^*$ denote a linear form on $V$ with kernel $H$.

For $\theta\colon G \to k^\times$ a linear character of $G$, recall that we denote by $m_H(\theta)$ the integer such that $0 \leq m_H(\theta) \leq e_H - 1$ and $\mathrm{Res}^G_{G(H)}(\theta) = \det_V^{m_H(\theta)}$. We set

$$E(\theta) := \sum_{H \in \mathcal{A}} m_H(\theta)\,.$$

Let us denote by $j_\theta^*$ the homogeneous element of $S^*$, with degree $E(\theta)$, such that $g(j_\theta^*) = \theta^{-1}(g) j_\theta^*$. Thus we have

$$j_\theta^* = \prod_{H \in \mathcal{A}} {j_H^*}^{m_H(\theta)}\,, \text{ and } S^*_{\theta^{-1}} = R^* j_\theta^*\,.$$

The following lemma will be used later.

**13.7. Lemma.** *Whenever $\theta$ is a linear character of $G$, we have $\langle j_\theta^*, j_\theta\rangle \neq 0$.*

*Proof of 13.7.*

We first prove that whenever $x$ is a homogeneous element of $S$ with degree $E(\theta)$, there is $\lambda \in k$ such that $\langle j_\theta^*, x\rangle = \lambda\langle j_\theta^*, j_\theta\rangle$.

Indeed, let us set $x_\theta := \dfrac{1}{|G|}\sum_{g\in G}\theta^{-1}(g)g(x)$. We have $x_\theta = \lambda j_\theta$ for some $\lambda \in k$. Now, whenever $g \in G$, we have $\langle j_\theta^*, x\rangle = g(\langle j_\theta^*, x\rangle) = \langle j_\theta^*, \theta^{-1}(g)g(x)\rangle$, hence $\langle j_\theta^*, x\rangle = \langle j_\theta^*, x_\theta\rangle = \lambda\langle j_\theta^*, j_\theta\rangle$.

Thus we see that if $\langle j_\theta^*, j_\theta\rangle = 0$, then $j_\theta^*$ is orthogonal to all homogeneous elements of $S$ of its degree, a contradiction. $\square$

The preceding lemma is actually a particular case of a result concerning general isotypic components, whose proof is left to the reader.

**13.8. Proposition.** *For $\chi' \in \mathrm{Irr}(G)$, $\chi' \neq \chi$, we have $\langle S^*_{\chi^*}, S_{\chi'}\rangle = 0$.*

**The harmonic elements of a reflection group and the Poincaré duality.**

The algebra morphism $D\colon S^* \to \mathrm{End}_k(S)$ defines a structure of $S^*$–module on $S$. The next result shows that this module is cyclic.

Recall that we set $\mathrm{Har} := (\mathfrak{M}^* S^*)^\perp$.

We set $S^*G = S^*/\mathfrak{M}^* S^*$, an algebra called the coinvariant dual algebra.

**13.9. Theorem.** *Let $J = \prod_{H\in\mathcal{A}} j_H^{e_H - 1}$ be the jacobian of $G$.*

(1) *The annihilator of $J$ in $S^*$ is $\mathfrak{M}^* S^*$, i.e.,*

$$(x^* \in \mathfrak{M}^* S^*) \Leftrightarrow (D(x^*)(J) = 0)\,.$$

(2) *We have*

$$\mathrm{Har} = S^* J = \{D(x^*)(J) \mid (x^* \in S^*\}\,.$$

(3) *The map*

$$S_G^* \to \mathrm{Har} \quad , \quad x^* \mapsto x^* J$$

*is an isomorphism of $S_G^*$–modules.*

*Remark.* It follows from the assertion (1) above and from Lemma 13.7 that $J^* \notin \mathfrak{M}^* S^*$, hence that $J \notin \mathfrak{M}S$.

In particular, the one dimensional space $S_G^{(N)}$ of elements of maximal degree of the coinvariant algebra $S_G$ is generated by $J$.

*Proof of 13.9.*

(1) For $x^* \in \mathfrak{M}^*$ and $g \in G$ we have

$$g(D(x^*)(J)) = D(g(x^*))(g(J) = \det_V^*(g) D(x^*)(J)\,,$$

and $D(x^*)(J) \in S_{\det_V^*}$, hence $D(x^*)(J)$ must be a multiple of $J$, which shows that $D(x^*)(J) = 0$. This establishes that $\mathfrak{M}^*$ annihilates $J$.

Conversely, assume that $D(x^*)(J) = 0$. In order to prove that $x^* \in \mathfrak{M}^* S^*$, we may assume that $x^*$ is homogeneous. Let us argue by descending induction on the degree of $x^*$. Notice that, since the largest degree of $S^*/\mathfrak{M}^* S^*$ is $N$, all homogeneous elements of $S^*$ with degree strictly larger than $N$ belong to $\mathfrak{M}^* S^*$. Choose $x^* \in \mathfrak{M}^* S^*$ such that $\deg x^* \leq N$, and assume that our desired property is established for elements with degree strictly larger than $\deg(x^*)$.

Let $s$ be a reflection in $G$, and let $j_s^*$ be a non trivial eigenvector of $s$ in $V^*$. Then $D(j_s^* x^*)(J) = 0$, so by induction hypothesis we have $j_s^* x^* \in \mathfrak{M}^* S^*$, *i.e.*, $j_s^* x^* = \sum_j \mu_j^* y_j^*$ with $\mu_j^* \in \mathfrak{M}^*$ and $y_j^* \in S^*$.

Applying $s$ we get $\det_V(s)^* j_s^* s(x^*) = \sum_j \mu_j^* s(y_j^*)$ , which yields

$$j_s^* \left(x^* - \det_V(s)^* s(x^*)\right) = \sum_j \mu_j^* \left(y_j^* - s(y_j^*)\right) .$$

Since each $y_j^* - s(y_j^*)$ is divisible by $j_s^*$, we get

$$x^* - \det_V(s)^* s(x^*) \in \mathfrak{M}^* S^* .$$

Thus $x^*$ belongs to the $\det_V$–isotypic component of $S^*/\mathfrak{M}^* S^*$. That isotypic component is the image modulo $\mathfrak{M}^* S^*$ of $R^* J^*$ (where $J^*$ is the corresponding jacobian), hence $x^*$ is an element of degree at least $N$ of $S^*/\mathfrak{M}^* S^*$. Since by assumption the degree of $x^*$ is smaller than $N$, we must have

$$x^* \equiv \lambda J^* \quad \text{mod } \mathfrak{M}^* S^* \quad \text{for some } \lambda \in k .$$

By the implication already proved above, since $\lambda J^* \in x^* + \mathfrak{M}^* S^*$ , it follows that

$$D(\lambda J^*)(J) = 0 .$$

By lemma 13.7, we conclude that $\lambda = 0$, hence that $x^* \in \mathfrak{M}^* S^*$ as desired.

(2) Let us prove that $S^* J \subseteq \mathrm{Har}$. Since by definition $\mathrm{Har} = (\mathfrak{M}^* S^*)^\perp$, we must prove that $\langle \mathfrak{M}^* S^*, J \rangle = 0$, which results from (1).

Let us now prove that $\mathrm{Har} \subseteq S^* J$. To do that, we prove that $(S^* J)^\perp \subseteq (\mathrm{Har})^\perp = \mathfrak{M}^* S^*$. Assume that $y^* \in (S^* J)^\perp$ *i.e.*,

$$(\forall x^* \in S^*) \, \langle y^*, D(x^*)(J) \rangle = 0 .$$

Since

$$\langle y^*, D(x^*)(J) \rangle = \langle x^*, D(y^*)(J) \rangle ,$$

we see $D(y^*)(J) = 0$ and $y^* \in \mathfrak{M}^* S^*$ by (1).

Assertion (3) is an immediate consequence of (1) and (2). $\square$

**13.10. Corollary.**

(1) *The pairing*

$$(S_G^*)^n \times (S_G^*)^{N-n} \to (S_G^*)^N \quad , \quad (x, y) \mapsto xy$$

*is a duality.*

(2) *The preceding isomorphism provides an isomorphism of graded modules*

$$S_G^* \otimes k_{\det_V^{-1}} \to \mathrm{Hom}_k(S_G^*, k)[N] .$$

*Proof of 13.10.*

(1) Since $\mathrm{Har} = (\mathfrak{M}^* S^*)^\perp$ and $S_G^* = S/\mathfrak{M}^* S^*$, we see that for each $n$ the pairing

$$(S_G^*)^n \times (\mathrm{Har})^n \to k \quad , \quad (x^*, h) \mapsto \langle x^*, h \rangle$$

is a duality.

Since $h = y^* J$ for a well defined $y^*$, and since

$$\langle x^*, y^* J \rangle = \langle x^* y^*, J \rangle \,,$$

that shows that the pairing

$$(S_G^*)^n \times (S_G^*)^{N-n} \to (S_G^*)^N \quad , \quad (x, y) \mapsto xy$$

is a duality.

(2) The map

$$S_G^* \to \mathrm{Hom}_k(S_G^*, k) \quad , \quad a \mapsto [b \mapsto \langle a, bJ \rangle = D(ab)(J)(0)]$$

is indeed an isomorphism of $k$–vector spaces, which sends an homogeneous element $a$ to an homogeneous element with degree $\deg a - N$.

Let compute its behaviour under $G$–action. We have

$$g(a) \mapsto [b \mapsto \langle g(a)b, J \rangle] \,.$$

But

$$\langle g(a)b, J \rangle = \langle g(ag^{-1}(b), J \rangle) = \det_V(g) \langle g(ag^{-1}(b), g(J) \rangle = \det_V(g) \langle ag^{-1}(b), J \rangle$$

proving what we announced. □

A finite dimensional graded $k$–algebra $A$ equipped with an isomorphism of $A$–modules

$$A \xrightarrow{\sim} \mathrm{Hom}_k(A, k)[M]$$

for some integer $M$ is called a *Poincaré duality algebra*.

*Remark.* The isomorphism of $G$–modules described in 13.10, (2) can be detected on the graded character of $S_G^*$. Indeed, we have

$$\mathrm{grchar}_{S_G^*}(g, q) = \frac{\prod_{i-1}^{i=r}(1 - q^{d_i})}{\det_V(1 - g^{-1}q)} \,.$$

hence

$$\mathrm{grchar}_{S_G^*}(g, q))\det_V(g^{-1}) = \frac{\prod_{i-1}^{i=r}(1 - q^{d_i})}{\det_V(g - q)} \,,$$

and since $N = (d_1 + d_2 + \cdots + d_r) - r$, we get

$$\begin{aligned} \mathrm{grchar}_{S_G^*}(g, q)\det_V(g^{-1}) &= q^N \frac{\prod_{i=1}^{i=r}(q^{-d_i} - 1)}{\det_V(gq^{-1} - 1)} = q^N \frac{\prod_{i=1}^{i=r}(1 - q^{-d_i})}{\det_V(1 - gq^{-1})} \\ &= q^N \mathrm{grchar}_{S^*G}(g^{-1}, q^{-1}) = q^N \mathrm{grchar}_{\mathrm{Hom}_k(S^*,k)}(g, q) \,. \end{aligned}$$

## §14. Application to braid groups : Discriminants and length

Let $\mathfrak{p}$ be an orbit of $G$ on $\mathcal{A}$. Recall that we denote by $e_\mathfrak{p}$ the (common) order of the pointwise stabilizer $G(H)$ for $H \in \mathfrak{p}$. We call *discriminant at* $\mathfrak{p}$ and we denote by $\Delta_\mathfrak{p}^*$ the element of the symmetric algebra of $V^*$ defined (up to a non zero scalar multiplication) by

$$\Delta_\mathfrak{p}^* := (\prod_{H\in\mathfrak{p}} j_H^*)^{e_\mathfrak{p}} .$$

Since (see 8.6) $\Delta_\mathfrak{p}^*$ is $G$-invariant, it induces a continuous function $\Delta_\mathfrak{p}^* \colon V^{\mathrm{reg}}/G \to \mathbb{C}^\times$ , hence induces a group homomorphism

$$\Pi_1(\Delta_\mathfrak{p}^*) \colon B \to \mathbb{Z} .$$

**14.1. Proposition.** *For any $H \in \mathcal{A}$, we have*

$$\Pi_1(\Delta_\mathfrak{p}^*)(\mathbf{s}_{H,\gamma}) = \begin{cases} 1 & \textit{if } H \in \mathfrak{p} , \\ 0 & \textit{if } H \notin \mathfrak{p} . \end{cases}$$

What precedes allows us to define length functions on $B$.

• There is a unique length function $\ell \colon B \to \mathbb{Z}$ defined as follows (see [BMR], Prop. 2.19): if $b = \mathbf{s}_1^{n_1} \cdot \mathbf{s}_2^{n_2} \cdots \mathbf{s}_m^{n_m}$ where (for all $j$) $n_j \in \mathbb{Z}$ and $\mathbf{s}_j$ is a distinguished braid reflection around an element of $\mathcal{A}$ in $B$, then

$$\ell(b) = n_1 + n_2 + \cdots + n_m .$$

Indeed, we set $\ell := \Pi_1(\delta)$ . Let $b \in B$. By Theorem 12.11 above, there exists an integer $k$ and for $1 \leq j \leq k$, $H_j \in \mathcal{A}$, a path $\gamma_j$ from $x_0$ to $H_j$ and an integer $n_j$ such that

$$b = \mathbf{s}_{H_1,\gamma_1}^{n_1} \mathbf{s}_{H_2,\gamma_2}^{n_2} \cdots \mathbf{s}_{H_k,\gamma_k}^{n_k} .$$

From Proposition 14.1 above, it then results that we have $\ell(b) = \sum_{j=1}^{j=k} n_j$ .

If $\{\mathbf{s}\}$ is a set of distinguished braid reflections around hyperplanes which generates $B$, let us denote by $B^+$ the sub–monoid of $B$ generated by $\{\mathbf{s}\}$. Then for $b \in B^+$, its length $\ell(b)$ coincide with its length on the distinguished set of generators $\{\mathbf{s}\}$ of the monoid $B^+$.

• More generally, given $\mathfrak{p} \in \mathcal{A}/G$, there is a unique length function $\ell_\mathfrak{p} \colon B \to \mathbb{Z}$ (this is the function denoted by $\Pi_1(\delta_\mathfrak{p})$ in [BMR], see Prop. 2.16 in *loc.cit.*) defined as follows: if $b = \mathbf{s}_1^{n_1} \cdot \mathbf{s}_2^{n_2} \cdots \mathbf{s}_m^{n_m}$ where (for all $j$) $n_j \in \mathbb{Z}$ and $\mathbf{s}_j$ is a distinguished braid reflection around an element of $\mathfrak{p}_j$, then

$$\ell_\mathfrak{p}(b) = \sum_{\{j \mid (\mathfrak{p}_j = \mathfrak{p})\}} n_j .$$

Thus we have, for all $b \in B$,

$$\ell(b) = \sum_{\mathfrak{p}\in\mathcal{A}/G} \ell_\mathfrak{p}(b) .$$

**14.2. Theorem.** *We denote by $B^{\mathrm{ab}}$ the largest abelian quotient of $B$. For $\mathfrak{p} \in \mathcal{A}/G$, we denote by $\mathbf{s}_{\mathfrak{p}}^{\mathrm{ab}}$ the image of $\mathbf{s}_{H,\gamma}$ in $B^{\mathrm{ab}}$ for $H \in \mathfrak{p}$. Then*

$$B^{\mathrm{ab}} = \prod_{\mathfrak{p} \in \mathcal{A}/G} \langle \mathbf{s}_{\mathfrak{p}}^{\mathrm{ab}} \rangle \,,$$

*where each $\langle \mathbf{s}_{\mathfrak{p}}^{\mathrm{ab}} \rangle$ is infinite cyclic.*

*Dually, we have*

$$\mathrm{Hom}(B, \mathbb{Z}) = \prod_{\mathfrak{p} \in \mathcal{A}/G} \langle \Pi_1(\Delta_{\mathfrak{p}}^*) \rangle \,.$$

**Complement : Artin–like presentations of the braid diagrams.**

*General results.*

Following [Op2], 5.2), we say that $B$ has an *Artin–like* presentation if it has a presentation of the form

$$\langle \mathbf{s} \in \mathbf{S} \mid \{\mathbf{v}_i = \mathbf{w}_i\}_{i \in I} \rangle$$

where $\mathbf{S}$ is a finite set of distinguished braid reflections, and $I$ is a finite set of relations which are multi–homogeneous, *i.e.*, such that (for each $i$) $\mathbf{v}_i$ and $\mathbf{w}_i$ are positive words in elements of $\mathbf{S}$ (and hence, for each $\mathfrak{p} \in \mathcal{A}/W$, we have $\ell_{\mathfrak{p}}(\mathbf{v}_i) = \ell_{\mathfrak{p}}(\mathbf{w}_i)$).

The following result is mainly due to Bessis (cf. [Bes3], 4.2 and also [BMR] and [BeMi] for case–by–case results).

**14.3. Theorem.** *Let $G \subset \mathrm{GL}(V)$ be a complex reflection group. Let $(d_1, d_2, \ldots, d_r)$ be the family of its invariant degrees, ordered to that $d_1 \leq d_2 \leq \cdots \leq d_r$.*

(1) *The following integers are equal.*
  - *(a) The minimal number of reflections needed to generate $G$.*
  - *(b) The minimal number of braid reflections needed to generate $B$.*
  - *(c) $\lceil (N + N_h)/d_r \rceil$.*

(2) *If $\Gamma_G$ denotes the integer defined by properties (a) to (c) above, we have either $\Gamma_G = r$ or $\Gamma_G = r + 1$, and the group $B$ has an Artin–like presentation by $\Gamma_G$ braid reflections.*

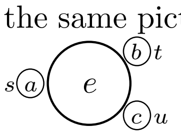
*The braid diagrams.*

Let us first introduce some more notation.

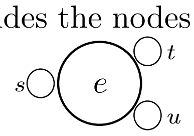
As previously, we set $V^{\mathrm{reg}} := V - \bigcup_{H \in \mathcal{A}} H$, $B := \Pi_1(V^{\mathrm{reg}}/G, x_0)$, and we denote by $\sigma \mapsto \overline{\sigma}$ the morphism $B \twoheadrightarrow G$ defined by the Galois covering $V^{\mathrm{reg}} \twoheadrightarrow V^{\mathrm{reg}}/G$.

Let $\mathcal{D}$ be one of the diagrams given in tables 1 to 4 of the Appendix (see below) symbolizing a set of relations as described in Appendix.

• We denote by $\mathcal{D}_{\mathrm{br}}$ and we call *braid diagram associated to* $\mathcal{D}$ the set of nodes of $\mathcal{D}$ subject to all relations of $\mathcal{D}$ but the orders of the nodes, and we represent the braid diagram $\mathcal{D}_{\mathrm{br}}$ by the same picture as $\mathcal{D}$ where numbers insides the nodes are omitted. Thus, if $\mathcal{D}$ is the diagram

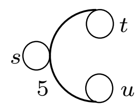



, then $\mathcal{D}_{\mathrm{br}}$ is the diagram




and represents the relations

$$\underbrace{stustu\cdots}_{e\text{ factors}} = \underbrace{tustus\cdots}_{e\text{ factors}} = \underbrace{ustust\cdots}_{e\text{ factors}} \,.$$

Note that this braid diagram for $e = 3$ is the braid diagram associated to $G(2d, 2, 2)$ $(d \geq 2)$, as well as $G_7$, $G_{11}$, $G_{19}$. Also, for $e = 4$, this is the braid diagram associated to $G_{12}$ and for $e = 5$, the braid diagram associated to $G_{22}$. Similarly, the braid diagram 

is associated to the diagrams of both $G_{15}$ and $G(4d, 4, 2)$.

The following statement is well known for Coxeter groups (see for example [Br1] or [De2]). It has been noticed by Orlik and Solomon (see [OrSo3], 3.7) for the case of non real Shephard groups (*i.e.*, non real complex reflection groups whose braid diagram – see above – is a Coxeter diagram). It has been proved for all the infinite series, as well as checked case by case for all the exceptional groups but $G_{24}$, $G_{27}$, $G_{29}$, $G_{31}$, $G_{33}$, $G_{34}$ in [BMM2]. The remaining cases have been treated by Bessis–Michel ([BeMi]) and by Bessis ([Be3]).

**14.4. Theorem.** *Let $G$ be a finite irreducible complex reflection group.*

*Let $\mathcal{N}(\mathcal{D})$ be the set of nodes of the diagram $\mathcal{D}$ for $G$ given in tables 1–4 of the appendix, identified with a set of distinguished reflections in $G$. For each $s \in \mathcal{N}(\mathcal{D})$, there exists an $s$–distinguished braid reflection $\mathbf{s}$ in $B$ such that the set $\{\mathbf{s}\}_{s\in\mathcal{N}(\mathcal{D})}$, together with the braid relations of $\mathcal{D}_{\mathrm{br}}$, is a presentation of $B$.*

## §15. Graded multiplicities and Solomon's theorem

**Preliminary : graded dimension of $(S \otimes V)^G$.**

The $S$–module $S \otimes_k V$ is a free (graded) $S$–module of rank $r$, hence a free $R$–module of rank $|G|r$. It is also endowed with an action of $G$ (defined by $g.(x \otimes v) := gx \otimes gv$).

The graded vector space $(S \otimes V)^G$ of fixed points under $G$ is the image of the projector $(1/|G|)\sum_{g\in G} g$, hence is also a free (graded) $R$–module, and we have

$$(S \otimes V)^G = R \otimes_k (S_G \otimes V)^G ,$$

where $(S_G \otimes V)^G$ is a finite dimensional graded vector space, which we shall describe now.

*The differential $d : S \to S \otimes V$.*

It is easy to check that the map

$$d : S^1 = V \to S \otimes V \quad , \quad v \mapsto dv := 1 \otimes v$$

extends uniquely to a $k$–linear derivation of $S$–modules $d : S \to S \otimes V$, *i.e.*, it satisfies

$$d(xy) = xd(y) + d(x)y \quad \text{for } x, y \in S .$$

That derivation has degree $-1$, and it is such that, whenever $(v_1, v_2, \ldots, v_r)$ is a basis of $V$, we have

$$dx = \sum_{i=1}^{i=r} \frac{\partial x}{\partial v_i} dx_i .$$

The map $d$ in injective. Indeed, let us consider the product

$$\mu : S \otimes V \to S \quad , \quad x \otimes v \mapsto xv\,.$$

That $S$–linear map has degree $+1$, and $\mu \cdot d : S \to S$ is an automorphism of $S$ since, whenever $x$ is a homogeneous element, we have $\mu \cdot d(x) = \deg(x)x$.

Finally the maps $d$ and $\mu$ commute with the action of $G$ hence they define

$$d : R \to (S \otimes V)^G \quad \text{and} \quad \mu : (S \otimes V)^G \to R\,.$$

**15.1. Proposition.**

(1) *The map $d$ induces an isomorphism of graded vector spaces*

$$d : \mathfrak{M}/\mathfrak{M}^2 \xrightarrow{\sim} (S_G \otimes V)^G[1]\,.$$

(2) *We have*

$$\operatorname{grdim}((S_G \otimes V)^G) = q^{d_1 - 1} + q^{d_2 - 1} + \cdots + q^{d_r - 1}\,.$$

*Proof of 15.1.*

(1) Let us first check that the vector spaces $\mathfrak{M}/\mathfrak{M}^2$ and $(S_G \otimes V)^G$ have both dimension $r$.

- We know that the dimension of $\mathfrak{M}/\mathfrak{M}^2$ equals the Krull dimension of $R$, hence equals $r$.
- If we forget the graduation, $(S_G \otimes V)^G$ is isomorphic to $(kG \otimes V)^G$, hence its dimension is $\dfrac{1}{|G|} \sum_{g \in G} \chi_{kG}(g)\chi_V(g) = \chi_V(1) = r$.

Since $d$ is a derivation, it sends $\mathfrak{M}^2$ into $\mathfrak{M}(S \otimes V)^G$, hence induces a morphism of graded vector spaces :

$$d : \mathfrak{M}/\mathfrak{M}^2 \xrightarrow{\sim} (S_G \otimes V)^G[1]\,.$$

Since that map is injective, it is an isomorphism.

(2) is an immediate consequence of (1), since $\operatorname{grdim}(\mathfrak{M}/\mathfrak{M}^2) = q^{d_1} + q^{d_2} + \cdots + q^{d_r}$. $\square$

**Exponents and Gutkin–Opdam matrices.**

To conform ourselves with the usual notation, we switch from the study of the symmetric algebra $S$ of $V$ to the study of the symmetric algebra $S^*$ of $V^*$. We view $S^*$ as the algebra of algebraic functions (polynomials !) on the algebraic variety $V$, hence we set $k[V] := S^*$. We introduce the following complementary notation for the invariant and coinvariant algebras :

$$k[V]^G = R^{*G} \quad \text{and} \quad k[V]_G = S^*_G = S^*/\mathfrak{M}^* S^* = k \otimes_{k[V]^G} k[V]\,.$$

*Multiplicity module, Fake degree, Exponents.*

Let $X$ be any $kG$–module, with dimension denoted by $d_X$.

The $k[V]^G$–module $(k[V] \otimes_k X^*)^G$ is a direct summand of $k[V] \otimes_k X^*$, hence is free. It follows that

$$(k[V] \otimes_k X)^G = k[V]^G \otimes_k \operatorname{Mult}(X)$$

where

$$\operatorname{Mult}(X) := (k[V] \otimes_k X^*)^G / \mathfrak{M}^*(k[V] \otimes_k X^*)^G = (k[V]_G \otimes_k X^*)^G\,.$$

Thus $\operatorname{Mult}(X)$ is a finite dimensional graded vector space.

**Definition.**

(1) The graded dimension of $\mathrm{Mult}(X)$ is called the *fake degree of* $X$ and is denoted by

$$\mathrm{Feg}_X(q) := \mathrm{grdim}\mathrm{Mult}(X)\,.$$

(2) The family of *exponents of* $X$ is the family of integers $(e_i(X))_{1\leq i\leq d_X}$ defined by

$$\mathrm{Feg}_X(q) = q^{e_1(X)} + q^{e_2(X)} + \cdots + q^{e_{d_X}(X)}\,.$$

(3) We set

$$E(X) := \frac{d}{dq}\mathrm{Feg}_X(q)_{|q=1} = e_1(X) + e_2(X) + \cdots + e_{d_X}(X)\,.$$

Let $\chi$ denote the character of the $kG$–module $X$. Then we have

$$\mathrm{Feg}_X(q) = \frac{1}{|G|}\sum_{g\in G}\frac{\chi(g)}{\det_V(1-gq)}\,.$$

The following property shows that $E(X)$ is "local", *i.e.*, may be computed from the fixators of reflecting hyperplanes.

**15.2. Proposition.** *We have*

$$E(X) = \sum_{H\in\mathcal{A}} E(\mathrm{Res}^G_{G(H)}X)\,.$$

*Proof of 15.2.* Let us set

$$\mathrm{Feg}_X(q) = \frac{\chi(1)}{|G|}\frac{1}{(q-1)^r} + \frac{\gamma(X)}{2|G|}\frac{1}{(q-1)^{r-1}} + \cdots\,.$$

We know from 10.4, (1), that $\gamma(X)$ is local. But it also results from 10.8, 4), that

$$\gamma(X) = \chi(1)|\mathrm{Ref}(G)| - 2E(X)\,,$$

which shows that $E(X)$ is local. $\square$

*$E(X)$ and $E(\det_X)$.*

Let us compute $E(\mathrm{Res}^G_{G(H)}X)$.

Since the degree of $X$ is $d_X$, and since the irreducible characters of $G(H)$ are powers of $\det_{V^*}$, there is a family $(m_{H,1}(X), m_{H,2}(X), \ldots, m_{H,d_X}(X))$ of integers such that $0 \leq m_{H,n}(X) \leq e_H - 1$ and

$$\mathrm{Res}^G_{G(H)}\chi = \sum_{n=1}^{n=d_X}\det{}_{V^*}^{m_{H,n}(X)}\,.$$

It follows from 13.6 that

$$\mathrm{Feg}_{\det_{V^*}}(q) = q \quad , \text{hence} \quad \mathrm{Feg}_{\det_{V^*}^{m_{H,n}(X)}}(q) = q^{m_{H,n}(X)} \, ,$$

and it follows that

$$\mathrm{Feg}_{\mathrm{Res}^G_{G(H)} X}(q) = \sum_{n=1}^{n=d_X} q^{m_{H,n}(X)} \, . \tag{15.3}$$

Hence we have

$$E(\mathrm{Res}^G_{G(H)} X) = \sum_{n=1}^{n=d_X} m_{H,n}(X) \quad \text{and} \quad E(X) = \sum_{H \in \mathcal{A}} \sum_{n=1}^{n=d_X} m_{H,n}(X) \, .$$

Notice that the preceding notation means that there exists a basis of the $k$-vector space $X^*$ on which the matrix of $s_H$ is the diagonal matrix with spectrum

$$(\zeta_H^{m_{H,1}(X)}, \zeta_H^{m_{H,2}(X)}, \dots, \zeta_H^{m_{H,d_X}(X)}) \, .$$

In particular we have $\det_X(s_H) = \zeta_H^{E(X)}$ .

The following proposition follows from what precedes.

**15.4. Proposition.**

(1) *The integer $E(\mathrm{Res}^G_{G(H)} X)$ depends only on the orbit $\mathfrak{p}$ of $H$ under $G$, and for all $H \in \mathfrak{p}$ we have*

$$E(\mathrm{Res}^G_{G(H)} X) = e_{\mathfrak{p}} m_{\mathfrak{p}} + E(\mathrm{Res}^G_{G(H)} \det_X)$$
$$\textit{where} \quad 0 \leq E(\mathrm{Res}^G_{G(H)} \det_X) < e_{\mathfrak{p}} \quad \textit{and} \quad m_{\mathfrak{p}} \in \mathbb{N} \, .$$

(2) *If the reflections of $G$ acts trivially or as reflections on $X$, we have*

$$E(\det_X) = E(X) \, .$$

(3) *We have*

$$E(V) = \sum_{H \in \mathcal{A}} e_H - 1 = N \quad \textit{and} \quad E(V^*) = \sum_{H \in \mathcal{A}} 1 = N_h \, .$$

*The Gutkin–Opdam matrix.*

Let us go on along the lines of the preceding analysis.

Let us choose a basis $(\mu_1, \mu_2, \dots, \mu_{d_X})$ of $\mathrm{Mult}(X)$ consisting of homogeneous elements of degrees respectively $e_1(X), e_2(X), \dots, e_{d_X}(X)$. We may view them as elements of $k[V]^G \otimes_k \mathrm{Mult}(X) = (k[V] \otimes_k X^*)^G$ with the same degrees.

Thus these elements belong to $k[V]\otimes_k X^*$, and if $(\xi_1,\xi_2,\dots,\xi_{d_X})$ is a basis of $X^*$, we get a matrix $J_X=(j_{\alpha,\beta}(X))_{1\leq\alpha,\beta\leq d_X}$, with entries in $k[V]$, each element $j_{\alpha,\beta}(X)$ being homogeneous of degree $e_\beta(X)$, defined by

$$\mu_\beta=\sum_{\alpha=1}^{\alpha=d_X} j_{\alpha,\beta}(X)\xi_\alpha\,,$$

*i.e.*, we have the identity between matrices

$$(\mu_1,\mu_2,\dots,\mu_{d_X})=(\xi_1,\xi_2,\dots,\xi_{d_X}).J_X\,.$$

In other words, $J_X$ is the matrix (written over the basis $(\xi_1,\xi_2,\dots,\xi_{d_X})$) of the endomorphism of the $k[V]$–module which sends the basis $(\xi_1,\xi_2,\dots,\xi_{d_X})$ onto the system $(\mu_1,\mu_2,\dots,\mu_{d_X})$.

Let us denote by $\rho_{X^*}(g)$ the matrix of $g$ written on the basis $(\xi_1,\xi_2,\dots,\xi_{d_X})$, *i.e.*,

$$\begin{pmatrix} g(\xi_1)\\ g(\xi_2)\\ \vdots\\ g(\xi_{d_X})\end{pmatrix}=\rho_{X^*}(g)\begin{pmatrix}\xi_1\\ \xi_2\\ \vdots\\ \xi_{d_X}\end{pmatrix}\,.$$

Since $\mu_\alpha\in(k[V]\otimes_k X^*)^G$, we see that

$$(\forall g\in G)\,,\ g(J_X)={}^t\rho_{X^*}(g^{-1})J_X\,,$$

hence

$$(\forall g\in G)\,,\ g(J_X)=\rho_X(g)J_X\,,$$

where $\rho_X(g)$ is the matrix (computed on the dual basis of $(\xi_1,\xi_2,\dots,\xi_{d_X})$) of the operation of $g$ on $X$.

**15.5. Theorem.**

(1) *We have (up to a nonzero scalar)*

$$\det(J_X)=\prod_{H\in\mathcal{A}} {j_H^*}^{E(\mathrm{Res}^G_{G(H)}X)}\,.$$

(2) *We have* $\det(J_X)\in k[V]^G j^*_{\det_{X^*}}$, *and more precisely*

$$\det J_X=\left(\prod_{\mathfrak{p}\in\mathcal{A}/G}\Delta_{\mathfrak{p}}^{m_{\mathfrak{p}}}\right)j^*_{\det_{X^*}}\quad \textit{for some integers } m_{\mathfrak{p}}\,.$$

(3) *For $g\in G$, let $\rho_X(g)$ denote the matrix of the operation of $g$ on $X$ computed on the dual basis of $(\xi_1,\xi_2,\dots,\xi_{d_X})$. Let $M\in\mathrm{Mat}_{d_X}(k[V])$. The following two conditions are equivalent :*
   *(i)* $\forall g\in G\,,\ g(M)=\rho_X(g)M\,,$
   *(ii)* $M\in J_X\mathrm{Mat}_{d_X}(k[V]^G)\,.$

(4) *A matrix $J\in\mathrm{Mat}_{d_X}(k[V]^G)$ satisfies assertion (3) if and only if there exists an element $\Phi\in\mathrm{GL}_{d_X}(k)$ such that $J=J_X\Phi$.*

(5) *If $X$ is absolutely irreducible, the $X$–isotypic component $k[V]_X$ of $k[V]$ is isomorphic to $J_X\mathrm{Mat}_{d_X}(R)$.*

*Proof of 15.5.*

*Proof of (1)*

(a) Let us first prove that $\det(J_X) \neq 0$.

Choose a regular element $v \in V^{\mathrm{reg}}$, and choose a $G$–stable subspace $H$ of $k[V]$ complementary to $\mathfrak{M}^* k[V]$, thus $H \oplus \mathfrak{M}^* k[V] = k[V]$, $H$ is a finite dimensional graded vector space, isomorphic to $k[V]_G$, and the multiplication induces an isomorphism of graded vector spaces

$$k[V]^G \otimes_k H \xrightarrow{\sim} k[V] \,.$$

**15.6. Lemma.** *The $k$–linear map*

$$\iota_v : \begin{cases} H \otimes X^* \to X^* \\ h \otimes \xi \mapsto h(v)\xi \end{cases}$$

*induces a $k$–linear isomorphism*

$$\iota_v : (H \otimes X^*)^G \xrightarrow{\sim} X^* \,.$$

*Proof of 15.6.* The vector space $(H \otimes X^*)^G$ has the same dimension as $X^*$ since $H$ is isomorphic to the regular module $kG$. Hence it suffices to prove that $\iota_v$ is onto.

Let $G.v$ denote the orbit of $v$ under $G$. Since $v$ is regular, that orbit has cardinality $|G|$. By Lagrange interpolation theorem, the any $k$–valued function on the finite set $G.v$ is the restriction of a polynomial function on $V$, *i.e.*, of the form $x \mapsto f(x)$ for $f \in k[V]$. Since $k[V] = k[V]^G \otimes_k H$ and since $k[V]^G$ defines the constant functions on $G.v$, it follows that any function on $G.v$ is of the shape $x \mapsto h(x)$ for some $h \in H$.

In particular, there exists $h_v \in H$ such that $h_v(g(v)) = \delta_{g,1}$.

Let $\xi \in X^*$. Consider the element $\sum_{g \in G} g(h_v) \otimes g(\xi) \in (H \otimes X^*)^G$. Its image under $\iota_v$ is $\xi$, proving the desired surjectivity. $\square$

Consider now the $k$–linear automorphism of $X^*$ defined as the composition of the two $k$–linear isomorphisms

$$\begin{cases} X^* \to (H \otimes_k X^*)^G \\ \xi_\alpha \mapsto \mu_\alpha \end{cases} \quad \text{and} \quad \iota_v : (H \otimes_k X^*)^G \to X^* \,,$$

thus defined by

$$\xi_\beta \mapsto \sum_{\alpha=1}^{\alpha=d_X} j_{\alpha,\beta}(v)\xi_\alpha \,.$$

The determinant of that automorphism is

$$\det(j_{\alpha,\beta}(v))_{\alpha,\beta} = (\det J_X)(v) \,,$$

which shows indeed that $\det J_X \neq 0$.

(b) Let us now prove (1).

Since for all $\alpha$, $j_{\alpha,\beta}(X)$ is homogeneous of degree $e_\beta(X)$, we see that $\det J_X$ has degree $\sum_\beta e_\beta(X) = E(X)$. By 15.2, we see that $\det J_X$ and $\prod_{H\in\mathcal{A}}(j_H^*)^{E(\mathrm{Res}^G_{G(H)}X)}$ have the same degree. Thus it suffices to prove that, whenever $H \in \mathcal{A}$, then $(j_H^*)^{E(\mathrm{Res}^G_{G(H)}X)}$ divides $\det J_X$.

Choose $H \in \mathcal{A}$. Since

$$\mathrm{Res}^G_{G(H)}X \simeq \bigoplus_{n=1}^{n=d_X} \det_{V^*}^{m_{H,n}(X)}$$

we have

$$(k[V]\otimes_k X^*)^{G(H)} = \bigoplus_{n=1}^{n=d_X} k[V]_{\det_{V^*}^{m_{H,n}(X)}} = \bigoplus_{n=1}^{n=d_X} k[V]{j_H^*}^{m_{H,n}(X)}\,.$$

Thus every $\mu_\alpha$ has the shape

$$\mu_\alpha = \sum_{n=1}^{n=d_X} f_{\alpha,n}{j_H^*}^{m_{H,n}(X)}$$

with $f_{\alpha,n} \in k[V]$, which shows that the determinant of the matrix $J_X$ is indeed divisible by

$$\prod_{n=1}^{n=d_X} {j_H^*}^{m_{H,n}(X)} = {j_H^*}^{E(X)}\,.$$

*Proof of (2)*

*Proof of (3)*

In order to prove (3), we establish the following isomophisms of graded vector spaces :

$$\begin{aligned}
&\left\{M \in \mathrm{Mat}_{d_X}(k[V]^G) \;\mid\; g(M) = \rho_X(g)M\right\} \\
&\qquad \simeq (k[V]\otimes_k X^*)^G \otimes_k X \\
&\qquad \simeq \mathrm{Hom}_{k[V]^G}(k[V]^G\otimes_k X^*, k[V]^G\otimes_k \mathrm{Mult}(X)) \\
&\qquad \simeq J_X\,\mathrm{Hom}_{k[V]^G}(k[V]^G\otimes_k X^*, k[V]^G\otimes_k X^*) \\
&\qquad = J_X \mathrm{Mat}_{d_X}(k[V]^G)\,.
\end{aligned}$$

*Proof of (4)*

*Proof of (5)*

□

**Solomon theorem.**

*Exterior algebra and bigrading.*

Let $A$ be a graded $k$–algebra, and let $M$ be a free graded $A$–module. Then the $A$–module $\Lambda_A(M)$ is naturally bigraded with the following rule : for $(x_1, x_2, \dots, x_n)$ homogeneous elements of $M$ with degrees respectively $(d_1, d_2, \dots, d_n)$, we set

$$\mathrm{bideg}(x_1\wedge x_2\wedge\cdots\wedge x_n) = (\sum_{i=1}^{i=n} d_i\,,\, n)\,.$$

The bigraded dimension of $\Lambda_A(M)$ is the power series in two indeterminates $x$ and $y$ defined by the formula

$$\mathrm{bigrdim}\Lambda_A(M) := \sum_{m,n} \dim\Lambda_A(M)^{(m,n)} x^m y^n$$

(so $x$ counts the grading of $M$ while $y$ counts the "exterior power grading"). More generally, if $M$ is endowed with an action of a finite group $G$, we define the bigraded character of $M$ by

$$\mathrm{bigrchar}_M(g) := \sum_{m,n} \mathrm{tr}(g, \Lambda_A(M)^{(m,n)}) x^m y^n \,.$$

In particular,

- if $M$ is a finite dimensional graded $k$–module with $\mathrm{grdim}M = q^{e_1} + \cdots + q^{e_d}$ , then
$$\mathrm{bigrdim}\Lambda(M) = (1 + yx^{e_1})(1 + yx^{e_2}) \cdots (1 + yx^{e_d}) \,.$$
- If $A$ is a graded $k$–algebra, and if we view $A \otimes_k M$ as graded by the product, then $\Lambda_A(A \otimes M) = A \otimes \Lambda(M)$ is bigraded as follows : for $a$ an homogeneous element of $A$ and $(x_1, x_2, \dots, x_n)$ homogeneous elements of $M$, we set
$$\mathrm{bideg}(a(x_1 \wedge x_2 \wedge \cdots \wedge x_n)) = (\deg a + \sum_{i=1}^{i=n} \deg(x_i) \,,\, n) \,.$$

**15.7. Theorem.** *Let $X$ be a $kG$–module such that $E(X) = E(\det_X)$, i.e., such that $\det J_X = j^*_{\det_{X^*}}$ .*

(1) *The identity endomorphism of $(k[V] \otimes_k X^*)^G$, viewed as an isomorphism*
$$\Lambda^1_{k[V]^G}(k[V]^G \otimes_k \mathrm{Mult}X) \xrightarrow{\sim} (k[V] \otimes_k \Lambda^1(X)^*)^G \,,$$
*extends uniquely to an isomorphism of bigraded $k[V]^G$–algebras*
$$\Lambda_{k[V]^G}(k[V]^G \otimes_k \mathrm{Mult}X) \xrightarrow{\sim} (k[V] \otimes_k \Lambda(X)^*)^G) \,,$$
*thus defining isomorphisms of bigraded algebras*
$$\Lambda_{k[V]^G}(k[V]^G \otimes_k \mathrm{Mult}X) \xrightarrow{\sim} (k[V]^G \otimes_k \mathrm{Mult}\Lambda X) \quad \textit{and} \quad \Lambda\mathrm{Mult}X \simeq \mathrm{Mult}\Lambda X$$

(2) *We have the following identities between power series*
$$\frac{1}{|G|} \sum_{g \in G} \frac{\det_X(1 + gy)}{\det_V(1 - gx)} = \frac{(1 + yx^{e_1(X)})(1 + yx^{e_2(X)}) \cdots (1 + yx^{e_{d_X}(X)})}{(1 - x^{d_1})(1 - x^{d_2}) \cdots (1 - x^{d_r})} \,,$$
$$(1 + yx^{e_1(X)})(1 + yx^{e_2(X)}) \cdots (1 + yx^{e_{d_X}(X)}) = \sum_{n=1}^{n=d_X} \mathrm{grdimMult}(\Lambda^n(X))(x) y^n$$
*In particular, we have*
$$\mathrm{Feg}_{\Lambda^m(X)}(q) = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq d_X} q^{e_{i_1}(X) + e_{i_2}(X) + \cdots + e_{i_m}(X)} \,.$$

*Proof of 15.7.*

(1) We keep using the notation introduced above. If $(\mu_1, \mu_2, \dots, \mu_{d_X})$ is an homogeneous basis of $\mathrm{Mult}(X)$, hence a homogeneous $k[V]^G$–basis of $(k[V] \otimes X^*)^G$, it suffices to prove that, if for $I = (i_1 < i_2 < \cdots < i_n)$ a subset of $(1, 2, \dots, d_X)$ we set $\mu_I := \mu_{i_1} \wedge \mu_{i_2} \wedge \cdots \wedge \mu_{i_{d_X}}$, then the family $(\mu_I)_I$ (a $k[V]^G$–basis of $\Lambda_{k[V]^G}(k[V]^G \otimes_k \mathrm{Mult}X)$), defines a $k[V]^G$–basis of $(k[V] \otimes_k \Lambda(X^*))^G$.

For $I$ as above, let us denote by $I'$ its complementary subset in $(1, 2, \dots, d_X)$, and let us set $\boldsymbol{\mu} := \mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_{d_X}$, so that $\mu_I \mu_{I'} = \pm\boldsymbol{\mu}$. Notice that $\boldsymbol{\mu} \neq 0$ since $\boldsymbol{\mu} = \det J_X(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{d_X})$ and $\det J_X \neq 0$.

Let us denote by $K$ the field of fractions of $k[V]^G$ and by $L$ the field of fractions of $k[V]$.

• Let us first check that the family $(\mu_I)_I$ is free over $k[V]$.

Indeed, given a linear combination $\sum_J f_I \mu_I$ where $f_J \in k[V]$, we can pick an $I$, multiply that linear combination by $\mu_{I'}$, which gives $f_I \boldsymbol{\mu} = 0$, whence $f_I = 0$.

• Let us now check that the family $(\mu_I)_I$ generates $(k[V] \otimes_k \Lambda(X)^*)^G)$ as a $k[V]^G$–module.

The family $(\mu_I)_I$ is a basis of $\Lambda_L(L \otimes X^*)$, hence whenever $\alpha \in (k[V] \otimes_k \Lambda(X)^*)^G)$, there are elements $\alpha_I \in K$ such that $\alpha = \sum_I \alpha_I \mu_I$ . Applying the projector $1/|G| \sum_{g \in G} g$ shows that $\alpha_I \in L^G = K$.

Picking an $I$ and mulpiplyng by $\mu_{I'}$ gives

$$\alpha \mu_{I'} = \alpha_I \boldsymbol{\mu} = \alpha_I \det J_X(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_{d_X}) \,.$$

We see that the element $\beta_I := \alpha_I \det J_X$ belongs to $k[V]$, and that $g(\beta_I) = \det_{X^*}(g)\beta_I$. It follows that $\beta_I \in k[V]^G j^*_{\det_{X^*}}$.

By hypothesis we have $\det J_X = j^*_{\det_{X^*}}$. This shows that $\alpha_I \in k[V]^G$.

(2) Let us express the preceding isomorphism as

$$k[V]^G \otimes_k \Lambda\mathrm{Mult}X \simeq (k[V] \otimes_k \Lambda(X^*))^G \,.$$

• The bigraded dimension of the left handside is

$$\begin{aligned} \mathrm{bigrdim}(k[V]^G \otimes_k \Lambda\mathrm{Mult}X) &= \mathrm{bigrdim}(k[V]^G) \cdot \mathrm{bigrdim}(\Lambda\mathrm{Mult}X) \\ &= \frac{(1 + yx^{e_1(X)})(1 + yx^{e_2(X)}) \cdots (1 + yx^{e_{d_X}(X)})}{(1 - x^{d_1})(1 - x^{d_2}) \cdots (1 - x^{d_r})} \,. \end{aligned}$$

• The bigraded dimension of the right handside is

$$\frac{1}{|G|} \sum_{g \in G} \mathrm{bigrchar}_{k[V]}(g) \cdot \mathrm{bigrchar}_{\Lambda(X^*)}(g) = \frac{1}{|G|} \sum_{g \in G} \frac{\det_X(1 + g^{-1}y)}{\det_V(1 - g^{-1}x)} \,.$$

This shows the first identity of (2).

The second one is an immediate consequence of the isomorphism

$$\Lambda\mathrm{Mult}X \simeq \mathrm{Mult}\Lambda X \,.$$

□

**Derivations and Differential forms on $V$.**

*Here we follow closely Orlik and Solomon* [OrSo2].

Let us denote by $\Delta_1$ the $k[V]$–module of derivations of the $k$–algebra $k[V]$, and by $\Omega^1$ the $k[V]$–dual of $\Delta_1$ ("module of 1–forms"). We have

$$\Delta_1 = k[V] \otimes V \quad \text{and} \quad \Omega^1 = k[V] \otimes V^*\,,$$

and there is an obvious duality

$$\langle\ ,\ \rangle \colon \Omega^1 \times \Delta_1 \longrightarrow k[V]\,.$$

Let $(x_1, x_2, \dots, x_r)$ be a basis of $V^*$. Note that the family $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_r})$ of elements of $\Delta_1$ is the dual basis. Denote by $d \colon \Omega^1 \longrightarrow \Omega^1$ the derivation of the $k[V]$–module $\Omega^1 = k[V] \otimes V^*$ defined by $d(x \otimes 1) := 1 \otimes x \quad$ for all $x \in V^*$. Then

$$\begin{cases} (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_r}) \text{ is a basis of the } k[V]\text{–module } \Delta_1\,. \\ (dx_1\,,\, dx_2\,,\, \dots\,,\, dx_r) \text{ is a basis of the } k[V]\text{–module } \Omega^1\,. \end{cases}$$

Let us endow $\Delta_1$ and $\Omega^1$ respectively with the graduations defined by

$$\begin{cases} \Delta_1 = \bigoplus_{n=-1}^{\infty} \Delta_1^{(n)} \text{ where } \Delta_1^{(n)} := k[V]^{n+1} \otimes V \\ \\ \Omega^1 = \bigoplus_{n=1}^{\infty} \Omega^{1,(n)} \text{ where } \Omega^{1,(n)} := k[V]^{n-1} \otimes V^* \end{cases}$$

In other words, we have

$$\begin{cases} (\delta \in \Delta_1^{(n)}) \iff (\delta(k[V]^m) \subseteq k[V]^{n+m})\,, \\ (\omega \in \Omega^{1,(n)}) \iff (\langle \omega, \Delta_1^{(m)} \rangle \subseteq k[V]^{n+m})\,, \end{cases}$$

and so (with previous notation)

$$\begin{cases} \text{the elements } (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_r}) \text{ have degree } -1, \\ \text{while the elements } (dx_1\,,\, dx_2\,,\, \dots\,,\, dx_r) \text{ have degree } 1. \end{cases}$$

We denote by $\Delta$ and $\Omega$ the exterior algebras of respectively the $k[V]$–modules $\Delta_1$ and $\Omega^1$. We have

$$\Delta = k[V] \otimes \Lambda(V) \quad \text{and} \quad \Omega = k[V] \otimes \Lambda(V^*)\,.$$

We endow $\Delta$ and $\Omega$ respectively with the bi–graduations extending the graduations of $\Delta_1$ and $\Omega^1$ :

$$\begin{cases} \Delta^{(m,n)} := k[V]^m \otimes \Lambda^{-n}(V) \\ \Omega^{(m,n)} := k[V]^m \otimes \Lambda^n(V^*) \end{cases}$$

The Poincaré series of the preceding bigraded modules are defined as follows :

$$\begin{cases} \operatorname{grdim} \Delta(t,u) := \displaystyle\sum_{m=0}^{\infty} \sum_{n=0}^{n=r} \dim \Delta^{(m,-n)} t^m (-u)^{-n} \\ \operatorname{grdim} \Omega(t,u) := \displaystyle\sum_{m=0}^{\infty} \sum_{n=0}^{n=r} \dim \Omega^{(m,-n)} t^m (-u)^{n} \,. \end{cases}$$

The following assertion is easy to check.

**15.8. Proposition.** *The map*

$$d : k[V] \to \Omega^1 \quad , \quad d(x) = \sum_i \frac{\partial x}{\partial x_i} \otimes dx_i$$

*extends uniquely to a $k$-linear endomorphism of $\Omega$ which satisfies*

- $d(\omega\eta) = d\omega.\eta + (-1)^{\deg \omega}\omega.\mathrm{d}\eta$
- $d^2 = 0$
- $d(x \otimes \lambda) = dx.\lambda$ *for* $x \in k[V]$ *and* $\lambda \in \Lambda(V^*)$.

*Fixed points under $G$.*

By our previous notation, we have

$$(\Omega^1)^G = \mathrm{Mult} V \quad \text{and} \quad (\Delta_1)^G = \mathrm{Mult} V^* \,.$$

- *Degrees again* : If $f_1, \ldots, f_r$ is a family of algebraically independant homogeneous elements of $k[V]$ such that $k[V]^G = k[f_1, f_2, \ldots, f_r]$, then $df_1, \ldots, df_r$ is a basis of $(\Omega^1)^G$ over $k[V]^G$ (thus consisting in a family of homogeneous elements with degrees respectively $(d_1, d_2, \ldots, d_r)$).
- *Codegrees* : If $\delta_1, \delta_2, \ldots, \delta_r$ is a basis of $(\Delta_1)^G$ over $k[V]^G$ consisting of homogeneous elements of degrees $(d_1^\vee, d_2^\vee, \ldots, d_r^\vee)$, the family $(d_1^\vee, d_2^\vee, \ldots, d_r^\vee)$ is called the family of codegrees of $G$.

The Poincaré series of $(\Omega^1)^G$ and $(\Delta_1)^G$ are respectively

$$\begin{cases} \operatorname{grdim} (\Omega^1)^G(q) := \mathrm{Feg}_V(q) = \dfrac{q^{-1}\sum_{j=1}^{j=r} q^{d_j}}{\prod_{j=1}^{j=r}(1-q^{d_j})} \\ \operatorname{grdim} (\Delta_1)^G(q) := \mathrm{Feg}_{V^*}(q) = \dfrac{q\sum_{j=1}^{j=r} q^{d_j^\vee}}{\prod_{j=1}^{j=r}(1-q^{d_j})} \,. \end{cases}$$

Let us denote by $\Delta^G := (k[V] \otimes \Lambda(V^*))^G$ and $\Omega^G := (k[V] \otimes \Lambda(V))^G$ respectively the subspaces of fixed points under the action of $G$.

Solomon's theorem (see 15.7 stays in particular that the $k[V]^G$–modules $\Delta^G$ and $\Omega^G$ are the exterior algebras of respectively the $k[V]^G$–modules $(\Delta_1)^G$ and $(\Omega^1)^G$ :

$$\begin{cases} \Delta^G = \Lambda_{k[V]^G}((\Delta_1)^G) \\ \Omega^G = \Lambda_{k[V]^G}((\Omega^1)^G) \end{cases}$$

Let us denote by $\operatorname{grdim} \Delta(t,u)$ and $\operatorname{grdim} \Omega(t,u)$ respectively the generalized Poincaré series of these modules defined by

$$\begin{cases} \operatorname{grdim} \Delta^G(t,u) := \displaystyle\sum_{m=0}^{\infty} \sum_{n=0}^{n=r} \dim{(\Delta^{(m,-n)})^G} t^m (-u)^{-n} \\ \operatorname{grdim} \Omega^G(t,u) := \displaystyle\sum_{m=0}^{\infty} \sum_{n=0}^{n=r} \dim{(\Omega^{(m,-n)})^G} t^m (-u)^{n} \,. \end{cases}$$

Then the identities following from Solomon's theorem may be rewritten as

$$\begin{cases} \operatorname{grdim} \Delta^G(t,u) = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} \dfrac{\det(1 - g^{-1}u^{-1})}{\det(1-gt)} = \prod_{j=1}^{j=r} \dfrac{1 - u^{-1} t^{d_j^\vee + 1}}{1 - t^{d_j}} \\ \operatorname{grdim} \Omega^G(t,u) = \dfrac{1}{|G|} \displaystyle\sum_{g \in G} \dfrac{\det(1 - gu)}{\det(1-gt)} = \prod_{j=1}^{j=r} \dfrac{1 - u t^{d_j - 1}}{1 - t^{d_j}} \end{cases}$$

**First applications of Solomon's theorem.**

*A multiplicative average.*

**15.9. Lemma.** *We have*

$$\left( \prod_{g \in G} \det_V(1 - gq) \right)^{1/|G|} = \prod_{j=1}^{j=r} (1 - q^{d_j})^{1/d_j} \,.$$

*Proof of 15.9.* By 15.7, (2), we have

$$\frac{1}{|G|} \sum_{g \in G} \frac{\det(1-gy)}{\det(1-gq)} = \prod_{j=1}^{j=r} \frac{1 - y q^{d_j - 1}}{1 - q^{d_j}} \,.$$

Let us differentiate with respect to $y$ both sides of the preceding equality. We get

$$\frac{1}{|G|} \sum_{g \in G} \frac{\frac{d}{dy} \det_V(1-gy)}{\det_V(1-gq)} = \sum_{j=1}^{j=r} -q^{d_j - 1} \left( \frac{\prod_{k \neq j} (1 - y q^{d_k - 1})}{\prod_{k=1}^{k=r} (1 - q^{d_k})} \right) \,.$$

Now specialize the preceding equality at $y = q$. We get

$$\frac{d}{dq} \operatorname{Log} \left( \prod_{g \in G} \det_V(1 - gq) \right)^{1/|G|} = \frac{d}{dq} \operatorname{Log} \left( \prod_{j=1}^{j=r} (1 - q^{d_j})^{1/d_j} \right) \,,$$

thus proving the identity announced in 15.9. □

*Degrees, Codegrees, Hyperplane intersections.*

Whenever $g \in G$, we denote by $V^g$ the set of fixed points of $V$ under $g$. Thus we have $V^g = \ker(g-1)$. We recall that the family of fixed points of elements of $G$ coincides with the family $I(\mathcal{A})$ of intersections of reflecting hyperplanes (see 12.5).

The following identities are consequences of Solomon's theorem.

**15.10. Proposition.**

(1) $\sum_{g \in G} q^{\dim V^g} = \prod_{i=1}^{i=r} (q + d_i - 1)$,

(2) $\sum_{g \in G} (-1)^{\mathrm{codim} V^g} \det_V(g) q^{\dim V^g} = \prod_{i=1}^{i=r} (q + d_i^\vee + 1)$,

*Proof of 15.10.*

(1) We apply 15.7, (2), to the case where $X = V$. We get

$$\text{(O-V)} \qquad \frac{1}{|G|} \sum_{g \in G} \frac{\det_V(1+gy)}{\det_V(1-gx)} = \prod_{1 \leq i \leq r} \frac{1 + yx^{d_i - 1}}{1 - x^{d_i}}.$$

Let $g \in G$. Assume $\dim V^g = n$, and assume that the nontrivial eigenvalues of $g$ on $V$ are $\zeta_1, \ldots \zeta_{r-n}$. Then we have

$$\frac{\det_V(1+gy)}{\det_V(1-gx)} = \frac{(1+y)^n}{(1-x)^n} \frac{\prod_i (1 + \zeta_i y)}{\prod_i (1 - \zeta_i x)}.$$

Let us define the indeterminate $q$ by the formula

$$1 + y = q(1-x).$$

The preceding equality becomes

$$\frac{\det_V(1+gy)}{\det_V(1-gx)} = q^n \frac{\prod_i (1 - \zeta_i y) + \zeta_i q(1-x)}{\prod_i (1 - \zeta_i x)},$$

Now let $x$ tend to 1. The left hand side of (O-V) becomes $\dfrac{1}{|G|} \sum_{g \in G} q^{\dim V^g}$, while each $i$-th factor of the right hand side of (O-V) becomes

$$\frac{(1 - x^{d_i - 1}) + qx^{d_i - 1}(1-x)}{1 - x^{d_i}}$$

which tends to $(q + d_i - 1)/d_i$ when $x$ tends to 1.

The desired formula comes now from the fact that $|G| = d_1 d_2 \cdots d_r$.

(2) We apply now 15.7, (2) to the case where $X = V^*$. We get

$$\text{(O-V}^*\text{)} \qquad \frac{1}{|G|} \sum_{g \in G} \frac{\det_V(1+g^{-1}y)}{\det_V(1-gx)} = \prod_{1 \leq i \leq r} \frac{1 + yx^{d_i^* + 1}}{1 - x^{d_i}}.$$

Let $g \in G$, and assume $\dim V^g = n$. Then we have (with previous notation)

$$\frac{\det_V(1+g^{-1}y)}{\det_V(1-gx)} = \frac{(1+y)^n}{(1-x)^n}\frac{\prod_i(1+\zeta_i^{-1}y)}{\prod_i(1-\zeta_i x)}\,.$$

As previously, with $1+y = q(1-x)$ and then $x=1$, we get

$$\frac{\det_V(1+g^{-1}y)}{\det_V(1-gx)} = q^n\frac{\prod_i(1-\zeta_i^{-1}y)+\zeta_i^{-1}q(1-x)}{\prod_i(1-\zeta_i x)}\,,$$

which tends to

$$q^n\frac{\prod_i(1-\zeta_i^{-1})}{\prod_i(1-\zeta_i)}\,.$$

Since $\dfrac{(1+\zeta_i^{-1})}{(1-\zeta_i)} = -\zeta_i^{-1}$, we get

$$\frac{\det_V(1+g^{-1}y)}{\det_V(1-gx)} = q^n(-1)^{r-n}\det_V(g^{-1})\,,$$

so the left hand side of (O-V$^*$) becomes

$$\frac{1}{|G|}\sum_{g\in G}(-1)^{\operatorname{codim}V^g}\det_V(g)q^{\dim V^g}\,.$$

As in the proof of (1), the right hand side of (O-V$^*$) becomes

$$\prod_{1\le i\le r}(q+d_i^\vee+1)\,,$$

which proves (2). $\square$

*Remark.*

- The relation

$$E(V) = (d_1-1)+(d_2-1)+\cdots+(d_r-1) = N = |\mathrm{Ref}(G)|$$

is a consequence of the first equality of the preceding proposition.

- The second equality provides another known identity :

$$E(V^*) = (d_1^\vee+1)+(d_1^\vee+1)+\cdots+(d_1^\vee+1) = N_h = |\mathcal{A}|\,.$$

Indeed, if we identify the coefficients of $q^{r-1}$ in both sides of the equality (2), we get $\sum_{s\in\mathrm{Ref}(G)} -\det(s) = \sum_{1\le i\le r}(d_i^\vee+1)$. But

$$\sum_{s\in\mathrm{Ref}(G)} -\det(s) = -\sum_{H\in\mathcal{A}}\sum_{i=1}^{i=e_H-1}\zeta_H^i = -\sum_{H\in\mathcal{A}}(-1) = |\mathcal{A}|\,.$$

## §16. Eigenspaces

**Pianzola–Weiss formula.**

Let $\phi \in N_{\mathrm{GL}(V)}(G)$ be an element of finite order. Let $(d_1,\zeta_1),(d_2,\zeta_2),\dots,(d_r,\zeta_r)$ be the family of generalized characteristic degrees of $(G,\phi)$ : there exists an algebraic basis $(u_i)_{1\leq i\leq r}$ of $R$ such that, for $1\leq i\leq r$ we have $\deg u_i = d_i$ and $\phi(u_i)=\zeta_i u_i$ .

We shall study eigenspaces of elements $g\phi$ for $g\in G$. Whenever $\zeta$ is a root of unity, we set

$$V(g\phi,\zeta) = \ker(g\phi - \mathrm{Id}_V)\,,$$

and we call such a subspace of $V$ a *$\zeta$–eigenspace of an element of $G\phi$*.

Define the family $\mathrm{Deg}(\phi,\zeta)$ as

$$\mathrm{Deg}(\phi,\zeta) := \big((d_i,\zeta_i) \mid (\zeta^{d_i}=\zeta_i)\big)\ ,$$

and the set $I(\phi,\zeta)\subseteq\{1,2\dots,r\}$ by

$$I(\phi,\zeta) := \big\{ i \mid (1\leq i\leq r)((\zeta^{d_i}=\zeta_i)\big\}\ .$$

Thus we have

$$\mathrm{Deg}(\phi,\zeta) = ((d_i,\zeta_i) \mid (i\in I(\phi,\zeta)))\ .$$

The folllowing formula generalizes 15.10.

**16.1. Proposition.**

(1) *Whenever $\phi$ is an element of finite order of $N_{\mathrm{GL}(V)}(G)$, we have*

$$\frac{1}{|G|}\sum_{g\in G}\frac{\det_V(1+g\phi y)}{\det_V(1-g\phi x)} = \prod_{1\leq i\leq r}\frac{1+\zeta_i y x^{d_i-1}}{1-\zeta_i x^{d_i}}$$

(2) *Whenever $\zeta$ is a root of unity, then*

$$\sum_{g\in G} q^{\dim V(g\phi,\zeta)} = (\prod_{i\notin I(\phi,\zeta)} d_i)\prod_{i\in I(\phi,\zeta)}(q+d_i-1)\,.$$

*Proof of 16.1.*

(1) From theorem 15.7, (1), we deduce an isomorphism of bigraded $N_{\mathrm{GL}(V)}(G)$–modules

$$(S\otimes_k \Lambda(V))^G \xrightarrow{\sim} R\otimes_k \Lambda((S_G\otimes_k V)^G)\,,$$

which implies

$$\mathrm{bigrchar}(\phi,(S\otimes_k\Lambda(V))^G) = \mathrm{bigrchar}(\phi, R\otimes_k \Lambda_R((S_G\otimes_k V)^G))\,.$$

Expanding the above equation provides assertion (1).

(2) Let $(\xi_1, \xi_2, \dots, \xi_r)$ denote the spectrum of $g\phi$ on $V$. Since

$$\frac{\det_V(1+g\phi y)}{\det_V(1-g\phi x)} = \prod_{i=1}^{i=r} \frac{1+\xi_i y}{1-\xi_i x}\,,$$

we see that this formal series has a pole of order $\dim V(g\phi,\zeta)$ at $x=\zeta^{-1}$. Let us define the indeterminate $q$ by the formula

$$1+\zeta y = q(1-\zeta x)\,.$$

Then

$$\frac{\det_V(1+g\phi y)}{\det_V(1-g\phi x)} = q^{\dim V(g\phi,\zeta)} \prod_{\xi_i\neq\zeta} \frac{1-\xi_i\zeta^{-1}+q\xi_i\zeta^{-1}(1-\zeta x)}{1-\xi_i x}\,,$$

which tends to $q^{\dim V(g\phi,\zeta)}$ when $x$ tends to $\zeta^{-1}$.

On the other hand, we have

$$\frac{1+\zeta_i y x^{d_i-1}}{1-\zeta_i x^{d_i}} = \frac{1-\zeta_i\zeta^{-1}x^{d_i-1}+\zeta_i\zeta^{-1}x^{d_i-1}(1-\zeta x)q}{1-\zeta_i x^{d_i}}$$

which tends to

$$\begin{cases} \dfrac{1}{d_i}(q+d_i-1) & \text{if } \zeta^{d_i}=\zeta_i \\ 1 \quad \text{if} & \text{if } \zeta^{d_i}\neq\zeta_i \end{cases}$$

Since $|G| = d_1 d_2 \cdots d_r$, the formula of (1) becomes

$$\sum_{g\in G} q^{\dim V(g\phi,\zeta)} = (\prod_{i\notin I(\phi,\zeta)} d_i) \prod_{i\in I(\phi,\zeta)} (q+d_i-1)\,.$$

□

**16.2. Corollary.** *The maximal dimension of the $\zeta$–eigenspaces of elements of $G\phi$ is $|I(\phi,\zeta)|$.*

*Proof of 16.2.* This results from the value of the degree of the polynomial described in 16.1, (2). □

**16.3. Corollary.** *The lcm of orders of elements of $G$ is equal to* $\mathrm{lcm}(d_1, d_2, \dots, d_r)$.

*Proof of 16.2.* Applying 16.1 to the case where $\phi = 1$, we see that $\zeta$ is an eigenvalue of an element of $G$ if and only if there exists an invariant degree $d$ of $G$ such that $\zeta^d = 1$, *i.e.*, if and only if the order of $\zeta$ divides one of the invariant degrees. This proves the statement :

- If $g \in G$, its order is the lcm of the orders of its eigenvalues, so the order of $g$ divides the lcm of the invariant degrees.
- Conversely, if $d$ is an invariant degree and if $\zeta$ is a root of the unity with order $d$, there is $g \in G$ with $\zeta$ as an eigenvalue, hence the order of such a $g$ is a multiple of $d$.

□

**Maximal eigenspaces : Lehrer–Springer theory.**

Here we follow mainly [Sp] and [SpLe].

*More generalities.*

Let us start by some preliminary results.

Recall that we view $k[V] = \mathrm{Sym}(V^*)$ as acting on $V$ :

$$\left(\sum \lambda_{m_1,\dots,m_n} {v_1^*}^{m_1} \cdots {v_n^*}^{m_n}\right)(v) := \sum \lambda_{m_1,\dots,m_n} \langle v_1^*, v\rangle^{m_1} \cdots \langle v_n^*, v\rangle^{m_n} .$$

For $v \in V$, we denote by $\mathrm{e}_v : k[V] \to k$ the algebra morphism "evaluation at $v$", such that $\mathrm{e}_v(x^*) := x^*(v)$ and we denote by $\mathfrak{m}_v^*$ its kernel, a maximal ideal of $k[V]$. The Hilbert NullstellenSatz tells us that the map $v \mapsto \mathfrak{m}_v^*$ is a bijection from $V$ onto the maximal spectrum of $k[V]$.

It is clear that, for $g \in G$, we have $g(\mathfrak{m}_v^*) = \mathfrak{m}_{g(v)}^*$ .

**16.4. Lemma.** *Let $v, v' \in G$. The following properties are equivalent :*

(i) *Whenever $u^* \in k[V]^G$, we have $u^*(v) = u^*(v')$.*

(ii) *We have*

$$k[V]^G \cap \mathfrak{m}_v^* = k[V]^G \cap \mathfrak{m}_{v'}^* ,$$

(iii) *There exists $g \in G$ such that $v' = g(v)$,*

*Proof of 16.4.*

(i)⇒(ii) : If (i) holds, we see that the restrictions to $k[V]^G$ of both $\mathrm{e}_v$ and $\mathrm{e}_{v'}$ coincide. Hence they have the same kernel, *i.e.*, $k[V]^G \cap \mathfrak{m}_v^* = k[V]^G \cap \mathfrak{m}_{v'}^*$.

(ii)⇒(iii) : If (iii) holds, both prime (maximal) ideals $\mathfrak{m}_v^*$ and $\mathfrak{m}_{v'}^*$ lie over the same prime ideal of $k[V]^G = k[V]$. Hence they have to be conjugate by $G$, which implies (iii).

(iii)⇒(i) : clear. □

The preceding lemma shows, as already noticed before, that the set of orbits $G\backslash V$ of $V$ under $G$ may be viewed as the maximal spectrum of $k[V]^G$. Thus, viewing $V$ as an algebraic affine variety whose functions algebra is $k[V]$, we see that $G\backslash V$ is an algebraic affine variety whose functions algebra is $k[V]^G$.

More generally, the spectrum $\mathrm{Spec}(k[V]^G)$ of $k[V]^G$ (in bijection with the set of irreducible subvarieties of $G\backslash V$) is naturally identified with the set $G\backslash\mathrm{Spec}(k[V])$ of orbits of $G$ on the spectrum of $k[V]$.

**16.5. Lemma.** *Let $X_1$ and $X_2$ be two irreducible subvarieties of $V$. The following assertions are equivalent :*

(i) *Whenever $u^* \in k[V]^G$, $u^*(X_1) = u^*(X_2)$,*

(ii) *There exists $g \in G$ such that $X_2 = g(X_1)$.*

*Proof of 16.5.* Let $\mathfrak{q}_1^* := \{u^* \in k[V] \,|\, (u^*(X_1) = 0)\}$ and $\mathfrak{q}_2^* := \{u^* \in k[V] \,|\, (u^*(X_2) = 0)\}$ be the prime ideals of $k[V]$ attached to $X_1$ and $X_2$ respectively.

(i)⇒(ii) : If (i) holds, we see that $\mathfrak{q}_1^* \cap k[V]^G = \mathfrak{q}_2^* \cap k[V]^G$, hence there exists $g \in G$ such that $\mathfrak{q}_2^* = g(\mathfrak{q}_1^*)$. Since $X_i = \{x \in V \,|\, (\forall u^* \in \mathfrak{q}_i^*)(u^*(x) = 0)\}$ , we see that $X_2 = g(X_1)$.

(ii)⇒(i) is clear. □

We recall that $\phi \in N_{\mathrm{GL}(V)}(G)$ is an element of finite order, and $(d_1, \zeta_1), (d_2, \zeta_2), \ldots, (d_r, \zeta_r)$ is the family of generalized characteristic degrees of $(G, \phi)$.

For $\zeta$ a root of unity, we recall that we set

$$\begin{aligned} &V(g\phi, \zeta) = \ker(g\phi - \mathrm{Id}_V) \\ &\mathrm{Deg}(\phi, \zeta) := \big((d_i, \zeta_i) \mid (\zeta^{d_i} = \zeta_i)\big) \\ &I(\phi, \zeta) := \{i \mid (1 \leq i \leq r)(\zeta^{d_i} = \zeta_i)\}\,. \end{aligned}$$

We shall prove the following theorem, essentially due to Springer and Springer–Lehrer (see [Sp] and [LeSp]).

**16.6. Theorem.** *Let $\phi$ be an element of finite order of the normalizer of $G$ in* $\mathrm{GL}(V)$*, with generalised characterisitic degrees $(d_1, \zeta_1), (d_2, \zeta_2), \ldots, (d_r, \zeta_r)$.*

(1) *The maximal $\zeta$–eigenspaces of elements of $G\phi$ are all conjugate by $G$.*
(2) *The maximal $\zeta$–eigenspaces of elements of $G\phi$ have dimension $|I(\phi, \zeta)|$.*
(3) *Assume that $V(g\phi, \zeta)$ is such a maximal $\zeta$–eigenspace. Then the group*

$$G(g\phi, \zeta) := N_G(V(g\phi, \zeta))/C_G(V(g\phi, \zeta))$$

*is a reflection group in its action on $V(g\phi, \zeta)$.*

*Moreover, if $V^G = 0$ and $V(g\phi, \zeta) \neq 0$, then $G(g\phi, \zeta)$ is nontrivial.*

(4) *The family $(\mathrm{Res}^V_{V(g\phi,\zeta)}(u_i^*))_{i \in I(\phi,\zeta)}$ is an algebraic basis of $\mathrm{Sym}(V(g\phi, \zeta)^*)^{G(g\phi,\zeta)}$, and* $\mathrm{Deg}(\phi, \zeta)$ *is the family of generalized characteristic degrees of $(G(g\phi, \zeta), g\phi)$.*
(5) *The set $\mathcal{A}(g\phi, \zeta)$ of reflecting hyperplanes of $G(g\phi, \zeta)$ is the set of traces of reflecting hyperplanes of $G$ on $V(g\phi, \zeta)$, i.e.,*

$$\mathcal{A}(g\phi, \zeta) = \{\, H \cap V(g\phi, \zeta) \mid (H \in \mathcal{A})(V(g\phi, \zeta) \not\subseteq H)\,\}\,.$$

*Proof of 16.6.*

Note that the generalized characteristic degrees of $(G, \phi)$ in its action on the dual space $V^*$ are $(d_1, \zeta_1^{-1}), (d_2, \zeta_2^{-1}), \ldots, (d_r, \zeta_r^{-1})$. Let $(u_i^*)_{1 \leq i \leq r}$ be an algebraic basis of $k[V]^G$ such that $\deg u_i^* = d_i$ and $\phi(u_i^*) = \zeta_i^{-1} u_i^*$.

For $1 \leq i \leq r$, let us set

$$H(u_i^*) := \{v \in V \mid (u_i^*(v) = 0)\}\,.$$

**16.7. Lemma.** *We have*

$$\bigcup_{g \in G} V(g\phi, \zeta) = \bigcap_{i \notin I(\phi,\zeta)} H(u_i^*)\,.$$

*Proof of 16.7.* Indeed, the set described in the left hand side is the set of all vectors $v \in V$ such that $\zeta v$ and $\phi(v)$ are in the same orbit under $G$. By 16.4 we see that it is also the set of vectors $v \in V$ such that $u_i^*(\zeta v) = u_i^*(\phi(v))$, *i.e.*, $\zeta^{d_i} u_i^*(v) = \zeta_i u_i^*(v)$, which is indeed $\cap_{i \notin I(\phi,\zeta)} H(u_i^*)$. $\square$

We prove now (2) : the maximal $\zeta$–eigenspaces have all dimension $|I(\phi, \zeta)|$. These maximal $\zeta$–eigenspaces are the irreducible components of $\cap_{i \notin I(\phi,\zeta)} H(u_i^*)$. The codimension of an irreducible component of the intersection of $n$ hypersurfaces in $V$ is at least $r-n$, hence in this case that codimension is at least $|I(\phi,\zeta)|$. But we know by 16.1, (2), that it is at most $I(\phi,\zeta)|$, whence the desired equality.

We prove now (1), (3) and (4). Consider the system of equations

$$(u_i^*(v)=0) \quad \text{for } i\in I(\phi,\zeta) \text{ and } v\in V(g\phi,\zeta)\,.$$

The only solution of that system is $v=0$. Indeed, by lemma 16.7, a solution $v$ of the system satisfies $(u_i^*(v)=0)$ for all $i$, and it results from lemma 16.4 that $v=0$. The following lemma (a reformulation of 6.4) shows then that the family

$$(\mathrm{Res}^V_{V(g\phi,\zeta)}(u_i^*))_{i\in I(\phi,\zeta)}$$

is algebraically independant in $k[V(g\phi,\zeta)]^{G(g\phi,\zeta)}$.

**16.8. Lemma.** *Let $X$ be a complex vector space of dimension $d$. Let $(\mu_i)_{1\le i\le d}$ be a family of homogeneous elements of $k[X]$ with strictly positive degrees. Assume that $x=0$ is the only solution in $X$ of the system of equations*

$$(\mu_i(x)=0)_{1\le i\le d}\,.$$

*Then*

(1) *the family $(\mu_i)_{1\le i\le d}$ is algebraically independant,*
(2) *the map*
$$X\to\mathbb{C}^d\ ,\ x\mapsto(\mu_1(x),\mu_2(x),\dots,\mu_r(x))$$
*is onto.*

We prove now assertion (1) of 16.6.

Let $V(g\phi,\zeta)$ and $V(g'\phi,\zeta)$ be two maximal $\zeta$-eigenspaces.

- For $i\notin I(\phi,\zeta)$, we have $u_i^*(V(g\phi,\zeta))=u_i^*(V(g'\phi,\zeta))=0$.
- It results from 16.8 that the maps
$$V(g\phi,\zeta)\to\mathbb{C}^{|I(\phi,\zeta)|}\ ,\ x\mapsto(u_i^*(x))_{i\in I(\phi,\zeta)}$$
$$V(g'\phi,\zeta)\to\mathbb{C}^{|I(\phi,\zeta)|}\ ,\ x\mapsto(u_i^*(x))_{i\in I(\phi,\zeta)}$$
are onto.

Thus for all $u^*\in k[V]^G$, we have

$$u^*(V(g\phi,\zeta))=u^*(V(g'\phi,\zeta))\,.$$

It follows then from 16.5 that there exists $h\in G$ such that $V(g'\phi,\zeta)=h((V(g\phi,\zeta))$.

We prove (3) and (4).

The family $(\mathrm{Res}^V_{V(g\phi,\zeta)}(u_i^*))_{i\in I(\phi,\zeta)}$ is is a family of parameters for $k[V(g\phi,\zeta)]^{G(g\phi,\zeta)}$. Thus it follows from 10.13 that

- $k[V(g\phi,\zeta)]^{G(g\phi,\zeta)}$ is free of finite rank, say $m$, on $k[(\mathrm{Res}^V_{V(g\phi,\zeta)}(u_i^*))_{i\in I(\phi,\zeta)}]$,
- $k[V(g\phi,\zeta)]$ is free of rank $m|G(g\phi,\zeta)|=\prod_{i\in I(\phi,\zeta)}d_i$ on that same polynomial algebra $k[(\mathrm{Res}^V_{V(g\phi,\zeta)}(u_i^*))_{i\in I(\phi,\zeta)}]$.

Hence by 11.1 it suffices to prove that $|G(g\phi,\zeta)| = \prod_{i\in I(\phi,\zeta)} d_i$ . Consider again 16.1, (2). The coefficient of $q^{\dim V(g\phi,\zeta)}$ in the left hand side equals

$$|C_G(V(g\phi,\zeta))|.|G : N_G(V(g\phi,\zeta))| = |G|/|G(g\phi,\zeta)|\,.$$

Comparison with the coefficient of $q$ in the right hand side (and remembering that $|G| = \prod_{1\le i\le r} d_i$) gives the result.

If $V^G = 0$, none of the characteristic degrees is equal to 1. We have $\dim V(g\phi,\zeta) = |I(\phi,\zeta)|$, hence $V(g\phi,\zeta) \neq 0$ implies $I(\phi,\zeta) \neq \emptyset$, from which it follows that

$$|G(g\phi,\zeta)| = \prod_{i\in I(\phi,\zeta)} d_i \neq 1\,.$$

We prove (5).

Let $X$ be a reflecting hyperplane for $G(g\phi,\zeta)$, an hyperplane of $V(g\phi,\zeta)$. Let us show that there exists $H\in\mathcal{A}$ such that $X = H\cap V(g\phi,\zeta)$. We know by Steinberg theorem that $C_G(X)$ is a reflection group, whose set of fixed points is $\bigcap_{H\in\mathcal{A}\,,\,H\supseteq X} H$. That set of fixed points does not contain $V(g\phi,\zeta)$ since there is at least an element in $C_G(X)\cap N_G(V(g\phi,\zeta))$ which induces a reflection on $V(g\phi,\zeta)$. Hence we have

$$\left( V(g\phi,\zeta) \cap \bigcap_{H\in\mathcal{A}\,,\,H\supseteq X} H \right) \neq V(g\phi,\zeta)\,,$$

from which it follows that there exists $H\in\mathcal{A}$ such that $X = H\cap V(g\phi,\zeta)$.

Conversely, assume that $H_0\in\mathcal{A}$ is such that $X := H_0\cap V(g\phi,\zeta) \neq V(g\phi,\zeta)$.

By Steinberg theorem, the group $C_G(X)$ is generated by its reflections, namely those reflections of $G$ whose hyperplane contains $X$. It follows that the set of fixed points $V^{C_G(X)}$ of $C_G(X)$ is the intersection of all reflecting hyperplanes of $G$ which contain $X$, and so we have $V(g\phi,\zeta) \not\subseteq V^{C_G(X)}$

Notice that $g\phi$ normalizes $X$ (since it acts on $X$ as $\zeta\mathrm{Id}_X$), hence normalizes $C_G(X)$, and that $V(g\phi,\zeta))$ is a maximal $\zeta$–eigenspace for $C_G(X)g\phi$.

Now consider the pair $(C_G(X),g\phi)$ in its action on the vector space $V/V^{C_G(X)}$. The last assertion of 16.6, (3), applies here, if we replace $V$ by $V/V^{C_G(X)}$ and $(G,\phi)$ by $(C_G(X),g\phi))$ : we get

$$N_G(V(g\phi,\zeta))\cap C_G(X)/C_G(V(g\phi,\zeta)) \neq 1\,,$$

proving the existence of a reflection in $G(g\phi,\zeta)$ with hyperplane $X$. $\square$

## §17. Regular elements

**First properties.**

The content of this paragraph is essentially due to Springer ([Sp]).

We keep using the same notation as in previous paragraph.

In particular, $\phi$ is a finite order element of $N_{\mathrm{GL}(V)}(G)$, $((d_1,\zeta_1),(d_1,\zeta_1),\dots,(d_1,\zeta_1))$ is the corresponding family of generalized degrees for the pair $(G,\phi)$, and $(u_1^*,u_2^*,\dots,u_r^*)$ is an algebraic basis of $k[V]^G$ such that, for all $i$, we have

$$\deg(u_i^*) = d_i \quad\text{and}\quad \phi(u_i^*) = \zeta_i^{-1}u_i^*\,.$$

**Definition.** Let $\zeta$ be a root of unity. We say that an element $g\phi$ in the coset $W\phi$ is $\zeta$–regular if

$$V(g\phi, \zeta) \cap V^{\mathrm{reg}} \neq \emptyset\,.$$

We say that the element $g\phi$ is regular if it is $\zeta$–regular for some root of unity $\zeta$.

Notice that by Steinberg's theorem (see 12.4), and element $g\phi$ is $\zeta$–regular if and only if

$$C_G(V(g\phi, \zeta)) = \{1\}\,.$$

Let us also remark that the regularity is invariant $G$–conjugation : the coset $G\phi$ is stable under $G$–conjugation (for $h, g \in G$, we have $h(g\phi)h^{-1} = hg\phi h\phi^{-1}\phi$) and $h(V(g\phi, \zeta)) = V(h(g\phi)h^{-1}, \zeta)$ ,

The following result relates the regularity to Lehrer–Springer theory.

**17.1. Proposition.** *Assume that $g\phi$ is $\zeta$–regular, where $\zeta$ has order $d$.*

(1) *The space $V(g\phi, \zeta)$ is a maximal $\zeta$ eigenspace.*
(2) *We have $N_G(V(g\phi, \zeta)) = G(g\phi, \zeta) = C_G(g\phi)$ .*

The following corollary is then an immediate consequence of Springer–Lehrer theorem 16.6.

**17.2. Corollary.** *Assume $g\phi$ is $\zeta$–regular.*

(1) $\dim V(g\phi, \zeta) = |I(\phi, \zeta)|$ .
(2) *$C_G(g\phi)$ is a reflection group in its action on $V(g\phi, \zeta)$.*
(3) *$(\mathrm{Res}^V_{V(g\phi,\zeta)}(u_i^*))_{i\in I(\phi,\zeta)}$ is an algebraic basis of $\mathrm{Sym}(V(g\phi, \zeta)^*)^{C_G(g\phi)}$, and $\mathrm{Deg}(\phi, \zeta)$ is the family of generalized degrees of $(C_G(g\phi), g\phi)$.*
(4) *The set $\mathcal{A}(g\phi, \zeta)$ of reflecting hyperplanes of $C_G(g\phi)$ is the set of traces of reflecting hyperplanes of $G$ on $V(g\phi, \zeta)$, i.e.,*

$$\mathcal{A}(g\phi, \zeta) = \{\, H \cap V(g\phi, \zeta) \mid (H \in \mathcal{A})(V(g\phi, \zeta) \not\subseteq H) \,\}\,.$$

*Proof of 17.1.* We use the following lemma.

**17.3. Lemma.** *If $g\phi$ is $\zeta$–regular and if $V(g\phi, \zeta) \subseteq V(g'\phi, \zeta)$ for some $g' \in G$, then $g'\phi = g\phi$.*

*Proof of 17.3.* The element $(g'\phi)^{-1}g\phi = {g'}^{-1}g$ centralizes $V(g\phi, \zeta)$ which implies $g' = g$. $\square$

(1) follows immediately from the preceding lemma. Let us prove (2). Since $C_G(g\zeta)$ stabilizes all the eigenspaces of $g\phi$, it suffices to prove that $N_G(V(g\phi, \zeta)) \subseteq C_G(g\phi)$. Now if $h \in N_G(V(g\phi, \zeta))$, we have $V(h(g\phi)h^{-1}, \zeta) = V(g\phi, \zeta)$, hence $h(g\phi)h^{-1} = g\phi$. $\square$

**17.4. Proposition.**

(1) *The group $G$ acts transitively on the set of $\zeta$–regular elements of $G\phi$.*
(2) *Let $m$ be the order of the image of $\phi$ in $N_{\mathrm{GL}(V)}(G)/G$, and let $d$ be the order of $\zeta$. Then the (common) order of the $\zeta$–regular elements of $G\phi$ is $\mathrm{lcm}(d, m)$.*

*Proof of 17.4.*

(1) Assume that $g\phi$ and $g'\phi$ are $\zeta$–regular. Since $V(g\phi, \zeta)$ and $V(g'\phi, \zeta)$ are maximal $\zeta$–eigenspaces by 17.1, it follows from 16.6 that they are conjugate : there is $h \in G$ such that

$V(g'\phi, \zeta) = h(V(gp, \zeta))$, hence $V(g'\phi, \zeta) = V(h(g\phi)h^{-1}, \zeta)$ and lemma 17.3 implies that $g'\phi = h(g\phi)h^{-1}$.

(2) Let us first assume that $\phi = 1$, hence $m = 1$. In that case we see that $g^d$ induces the identity on $V(g, \zeta)$, hence is trivial. This shows that the order of $g$ is $d$.

Let us now treat the general case. The element $(g\phi)^m$ belongs to $G$, and is $\zeta^m$–regular. By what precedes, we know that the order of $(g\phi)^m$ is the order of $\zeta^m$, *i.e.*, $d/\gcd(d,m)$. Assume the order of $g\phi$ is $dd'$. Notice that $m$ divides that order. It follows from what precedes that $dd'/m = d/\gcd(d,m)$, hence $d' = m/\gcd(d,m)$, and $dd' = dm/\gcd(d,m) = \mathrm{lcm}(d,m)$. $\square$

**Exponents and eigenvalues of regular elements.**

**17.5. Proposition.** *Let $g\phi$ be $\zeta$–regular, and let $X$ be a $k[G\langle\phi\rangle]$–module.*
*Let $(\zeta_1^{(X)}, \zeta_2^{(X)}, \ldots, \zeta_{d_X}^{(X)})$ be the spectrum of $\phi$ on $\mathrm{Mult}(X) = k[V]_G \otimes X^*$.*

(1) *The spectrum of $g\phi$ in its action on $X^*$ is*

$$\mathrm{Spec}(g\phi, X^*) = (\zeta_\alpha^{(X)}\zeta^{e_\alpha})_{1\leq\alpha\leq d_X}\,.$$

(2) *In particular, we have*

$$\det_{X^*}(g\phi) = \zeta^{E(X)}\det_{\mathrm{Mult}(X)}(\phi)\,.$$

*Proof of 17.5.*

Let $(\lambda_1, \lambda_2, \ldots, \lambda_{d_X})$ be the spectrum of $g\phi$ on $X^*$. We choose a basis $(\xi_1, \xi_2, \ldots, \xi_{d_X})$ of $X^*$ over which $g\phi$ is diagonal.

We choose a basis $(\mu_1, \mu_2, \ldots, \mu_{d_X})$ of $\mathrm{Mult}(X)$ consisting of homogeneous elements of degrees $(e_1(X), e_2(X), \ldots, e_{d_X}(X))$ which are eigenvectors of $\phi$ with eigenvalues $(\zeta_1^{(X)}, \zeta_2^{(X)}, \ldots, \zeta_{d_X}^{(X)})$.

Finally, we choose a basis $(v_1, v_2, \ldots, v_r)$ of $V$ over which $g\phi$ is diagonal, and such that $v_1$ is a regular $\zeta$–eigenvector of $g\phi$. We denote by $(x_1, x_2, \ldots, x_r)$ its dual basis, and we view elements of $k[V]$ as elements of $k[x_1, x_2, \ldots, x_r]$.

We recall that the matrix $J_X = (j_{\alpha,\beta}(X))_{1\leq\alpha,\beta\leq d_X}$, with entries in $k[V]$, where $j_{\alpha,\beta}(X)$ is homogeneous of degree $e_\beta(X)$, is defined by

$$\mu_\beta = \sum_{\alpha=1}^{\alpha=d_X} j_{\alpha,\beta}(X)\xi_\alpha\,.$$

Applying $g\phi$ to both sides of the preceding equality gives

$$\zeta_\beta^{(X)}\mu_\beta = \sum_{\alpha=1}^{\alpha=d_X} g\phi(j_{\alpha,\beta}(X))\lambda_\alpha\xi_\alpha\,,$$

from which we deduce

$$\zeta_\beta^{(X)} j_{\alpha,\beta}(X) = g\phi(j_{\alpha,\beta}(X))\lambda_\alpha\,.$$

We know that $\det(J_X)(v) \neq 0$, which implies the existence of a permutation $\sigma \in \mathfrak{S}_{d_X}$ such that $j_{\sigma(\beta),\beta}(X)(v) \neq 0$, which in turn implies that, as a polynomial in $(x_1, x_2, \ldots, x_r)$, $j_{\sigma(\beta),\beta}(X)$ must involve a monomial $x_1^{e_\beta(X)}$.

Projecting the previous equality onto that monomial gives

$$\zeta_\beta^{(X)} = \zeta^{-e_\beta(X)} \lambda_{\sigma(\beta)} \,,$$

which gives the announced equality. $\square$

*Remark.* By assumption, we have

$$\mathrm{grchar}_{\mathrm{Mult}(X)}(\phi) = \zeta_1^{(X)} q^{e_1(X)} + \zeta_2^{(X)} q^{e_2(X)} + \cdots + \zeta_{d_X}^{(X)} q^{e_{d_X}(X)} \,,$$

from which we deduce that

$$\chi_{X^*}(g\phi) = \mathrm{grchar}_{\mathrm{Mult}(X)}(\phi)(\zeta q)_{|q=1} \,.$$

Let us draw some consequences of 17.5 in some particular cases.

We say that the integer $d$ is regular of $G$ if there exists a root of unity $\zeta$ of order $d$ and a $\zeta$–regular element $g \in G$.

**17.6. Corollary.** *Assume that $d$ is regular for $G$.*

(1) *Whenever $X$ is a $kG$–module, $d$ divides $E(X) + E(X^*)$.*
(2) *Whenever $\mathfrak{p} \in \mathcal{A}/G$, $d$ divides $\omega_\mathfrak{p} e_\mathfrak{p}$.*
(3) *$d$ divides $N + N_h$.*

*Proof of 17.6.*

(1) Let $g \in G$ be $\zeta$–regular where $\zeta$ has order $d$. By 17.5 we have

$$\det{}_X(g) = \zeta^{E(X^*)} \text{ and } \det{}_{X^*}(g) = \zeta^{E(X)} \,,$$

which implies

$$\zeta^{E(X)+E(X^*)} = 1 \,,$$

proving (1).

(2) and (3) are particular cases of (1), when applied successively to $X = \theta_\mathfrak{p}$ (see 8.4) and to $X = V$. $\square$

Let $(\zeta_1^\vee, \zeta_2^\vee, \ldots, \zeta_r^\vee)$ be the spectrum of $\phi^{-1}$ in its action on $\mathrm{Mult}(V^*)$. We set

$$I^\vee(\phi, \zeta) := \{\, i \mid (1 \le i \le r)(\zeta^{d_i^\vee} = \zeta_i^\vee) \} \,.$$

We recall that

$$I(\phi, \zeta) := \{\, i \mid (1 \le i \le r)(\zeta^{d_i} = \zeta_i) \} \,.$$

**17.7. Corollary.**

(1) *Assume that $g\phi$ is $\zeta$–regular. Then we have*

$$\mathrm{Spec}(g\phi, V) = \{(\zeta_i^\vee)^{-1} \zeta^{d_i^\vee + 1} \mid (1 \le i \le r)\}$$
$$\mathrm{Spec}(g\phi, V^*) = \{\zeta_i^{-1} \zeta^{d_i - 1} \mid (1 \le i \le r)\}$$

(2) *We have $|I(\phi, \zeta)| = |I^\vee(\phi, \zeta)|$.*

*Proof of 17.7.*

The first assertion is an immediate consequence of 17.5. The second follows then from the fact that the multiplicity of $\zeta$ as an eigenvalue of $g\phi$ in its action of both $V$ and $V^*$ is $\dim V(g\phi,\zeta) = |I(\phi,\zeta)|$. $\square$

*A characterisation of regularity.*

We prove now yet another consequence of Solomon's theorem (for the first three assertions, see [LeMi] and [BLM]).

**17.8. Proposition.**

(1) *We always have* $I(\phi,\zeta) \subseteq I^\vee(\phi,\zeta)$.

(2) *The following assertions are equivalent :*

*(i) There is a* $\zeta$ *regular element in* $G\phi$.

*(ii) We have* $|I(\phi,\zeta)| = |I^\vee(\phi,\zeta)|$.

(3) *If this is the case, denoting by* $g_0\phi$ *a* $\zeta$*–regular element, we have*

$$\sum_{g\in G} \det_V(g\phi) q^{\dim V(g\phi,\zeta)} = \det_V(g_0\phi) \prod_{i\notin I(\phi,\zeta)} d_i \prod_{i\in I(\phi,\zeta)} (q - d_i^\vee - 1)$$

(4) *and the set of codegrees of the reflection group* $C_G(g\phi)$ *is* $\{d_i^\vee \mid (i \in I(\phi,\zeta))\}$.

*Proof of 17.8.* The proof depends on the following consequence of Solomon's theorem, an analog of Pianzola–Weiss formula.

**17.9. Lemma.**

(1) $\dfrac{1}{|G|}\sum_{g\in G} \dfrac{\det_{V^*}(1+g\phi y)}{\det_V(1-g\phi x)} = \prod_{i=1}^{i=r} \dfrac{1+\zeta_i^\vee y x^{d_i^\vee+1}}{1-\zeta_i x^{d_i}}$ .

(2) $(-\zeta)^r \sum_{g\in G} \det_V(g\phi)^{-1}(-q)^{\dim V(g\phi,\zeta)} =$

$$\begin{cases} \displaystyle\prod_{i\in I^\vee(\phi,\zeta)} (q+d_i^\vee+1) \prod_{i\notin I^\vee(\phi,\zeta)} (1-\zeta_i^\vee) \prod_{i\notin I(\phi,\zeta)} \frac{d_i}{1-\zeta_i} & \text{if } I(\phi,\zeta) = I^\vee(\phi,\zeta) \\ 0 \quad \text{otherwise.} \end{cases}$$

*Proof of 17.9.*

(1) As previously, the formula expresses the equality of bigraded traces of $\phi$ acting on both sides of the isomorphism

$$(S\otimes_k \Lambda(V^*))^G \simeq R\otimes_k \Lambda((S_G\otimes_k V^*)^G)\,.$$

Then comparing the order of pole at $x = \zeta^{-1}$ of the two sides of the equation (1) of 17.9 gives the inclusion $I(\phi,\zeta) \subseteq I^\vee(\phi,\zeta)$.

(2) We replace the pair of indeterminates $(x,y)$ in (1) by the pair $(x,q)$ where $1+\zeta x = q(1-\zeta y)$, and we compute the limits of the two sides when $x \to \zeta^{-1}$. $\square$

Let us now prove 17.8.

(1) was already noticed in the proof of 17.9.

(2) We have already seen above (see 17.7, (2), that (i) implies (ii). Let us prove the converse. Assume that no element of $G\phi$ is $\zeta$–regular. Let us then prove that the coefficient of $q^{|I(\phi,\zeta)|}$ in the polynomial $\sum_{g\in G}\det_V(g\phi)^{-1}(-q)^{\dim V(g\phi,\zeta)}$ is zero. Denote by $c$ that coefficient. We have

$$c = \pm \sum_{V' \text{ max. eigens.}} \sum_{\substack{g\in G \\ V(g\phi,\zeta)=V'}} \det_V(g\phi)^{-1}\,.$$

If $V(g\phi,\zeta)$ is maximal, we have $V(g\phi,\zeta)=V(g'\phi,\zeta)$ if and only if there exists $z\in C_G(V(g\phi,\zeta))$ such that $g'\phi = zg\phi$. Hence denoting by $g_{V'}$ an element such that $V(g_{V'}\phi,\zeta)=V'$, we have

$$c = \pm \sum_{V' \text{ max. eigens.}} \det_V(g_{V'}\phi)^{-1} \sum_{z\in C_G(V')} \det_V(z)^{-1}\,.$$

Since $C_G(V')$ is a reflection group, the character det is nontrivial, hence

$$\sum_{z\in C_G(V')} \det_V(z)^{-1} = 0\,,$$

proving that $c=0$.

(3) Applying 17.9, (2), we see that

$$\sum_{g\in G}\det_V(g\phi)q^{\dim V(g\phi,\zeta)} = a(\zeta)\prod_{i\in I(\phi,\zeta)}(q-d_i^\vee-1)$$

for some scalar $a(\zeta)$, which we compute by computing the coefficient of $q^{|I(\phi,\zeta)|}$.

*Remark.* Combining (3) and 17.9, (2), we see that

$$\det_V(g_0\phi) = \zeta^r \prod_{i\notin I(\phi,\zeta)} \frac{1-\zeta_i^\vee}{1-\zeta_i}\,.$$

Let us now prove assertion (4) of 17.8.

It relies first on a remark about “control of fusion”, quite analogous to Burnside’s theorem for Sylow subgroups of a finite group.

**17.10. Lemma.** *Let $E_0 := V(g_0\phi)$ be a maximal $\zeta$–eigenspace for $G\phi$.*

(1) *The group $G_0 := N_G(E_0)$ controls the fusion of $\zeta$–eigenspaces for $G\phi$, i.e., whenever $E$ is a $\zeta$–eigenspace for $G\phi$ and $g\in G$ are such that $E, g(E)\subseteq E_0$, then there exist $g_0\in G_0$ and $z\in C_G(E)$ such that $g=g_0z$.*

(2) *In particular, if $E$ is a $\zeta$–eigenspace contained in $E_0$, we have $N_G(E)=N_{G_0}(E)C_G(E)$.*

*Proof of 17.10.* It suffices to apply Springer–Lehrer theorem to the reflection group $C_G(E)$ endowed with the automorphism $g_0\phi$. Indeed, $E, g(E)\subseteq E_0$ implies that $E_0$ and $g^{-1}(E_0)$ are both maximal $\zeta$–eigenspaces for $C_G(E)g_0\phi$, hence are conjugate under $C_G(E)$. $\square$

We choose a maximal $\zeta$-eigenspace $E_0 = V(g_0\phi,\zeta)$, and we set $G_0 := N_G(E_0) = C_G(g_0\phi)$.

Let $(d_{0,j}^\vee)_{j\in J}$ be the family of codegrees of $G_0$.

Assertion (3) of 17.8 implies that

$$\begin{cases} \displaystyle\sum_{g\in G} \det_V(g\phi) q^{\dim V(g\phi,\zeta)} = \frac{|G|}{|G_0|}\det_V(g_0\phi) \prod_{i\in I(\phi,\zeta)} (q - d_i^\vee - 1) \\ \displaystyle\sum_{g\in G_0} \det_{E_0}(g\phi) q^{\dim E_0(g\phi,\zeta)} = \det_{E_0}(g_0\phi) \prod_{j\in J} (q - d_{0,j}^\vee - 1)\,. \end{cases}$$

Thus we see that in order to prove (4), it suffices to prove that

(0) $$\frac{1}{|G|}\sum_{g\in G} \det_V(gg_0^{-1}) q^{\dim V(g\phi,\zeta)} = \frac{1}{|G_0|}\sum_{g\in G_0} \det_{E_0}(gg_0^{-1}) q^{\dim E_0(g\phi,\zeta)}\,.$$

Let us reorder the lefthand side of the above desired identity as follows.
For each $\zeta$–eigenspace $E$, we denote by $G_E$ the set of $g\in G$ such that $V(g\phi,\zeta) = E$. Then

$$\sum_{g\in G} \det_V(gg_0^{-1}) q^{\dim V(g\phi,\zeta)} = \sum_E q^{\dim E} \sum_{g\in G_E} \det_V(gg_0^{-1})\,.$$

• Choose a complete set of representatives for the orbits of $G$ on the set of $\zeta$–eigenspaces, chosen as a complete set of representatives for the orbits of $G_0$ on its set of $\zeta$–eigenspaces in $E_0 = V(g_0\phi,\zeta)$. We have

$$\sum_{g\in G} \det_V(gg_0^{-1}) q^{\dim V(g\phi,\zeta)} = \sum_{(E\subseteq E_0)/G_0} |G : N_G(E)| q^{\dim E} \sum_{g\in G_E} \det_V(gg_0^{-1})\,.$$

Since $N_G(E) = N_{G_0}(E)C_G(E)$, we have

$$|G : N_G(E)| = |G : G_0)||G_0 : N_{G_0(E)}|/|C_G(E) : C_{G_0}(E)|$$

so

$$\frac{1}{|G|}\sum_{g\in G} \det_V(gg_0^{-1}) q^{\dim V(g\phi,\zeta)} = \frac{1}{|G_0|}\sum_{(E\subseteq E_0)/G_0} \frac{|G_0 : N_{G_0}(E)|}{|C_G(E) : C_{G_0}(E)|} q^{\dim E} \sum_{g\in G_E} \det_V(gg_0^{-1})\,.$$

Hence in order to prove (0) it suffices to prove

(0') $$\frac{1}{|C_G(E)|}\sum_{g\in G_E} \det_V(gg_0^{-1}) = \frac{1}{|C_{G_0}(E)|}\sum_{g\in (G_0)_E} \det_{E_0}(gg_0^{-1})\,.$$

• Let us define the two functions $\alpha$ and $\alpha_0$ on the set of $\zeta$–eigenspaces contained in $E_0$ by the formulae

$$\alpha(E) := \frac{1}{|C_{G_0}(E)|}\sum_{g\in G_E} \det_V(gg_0^{-1}) \ \text{ and }\ \alpha_0(E) := \frac{1}{|C_{G_0}(E)|}\sum_{g\in (G_0)_E} \det_{E_0}(gg_0^{-1})\,.$$

We want to prove that $\alpha = \alpha_0$. Notice that

$$\alpha(E_0) = \alpha_0(E_0) = 1\,.$$

- The sets $G_E$ and $(G_0)_E$ may be described through the formulae

$$C_G(E)g_0 = \dot{\bigcup}_{E\subseteq E'} G_{E'} \quad \text{and} \quad C_{G_0}(E)g_0 = \dot{\bigcup}_{E\subseteq E'\subseteq E_0} (G_0)_{E'}\,.$$

Since (for $E$ strictly contained in $E_0$) $C_G(E)$ is a nontrivial reflection group, this implies that, for $E \subset E_0$,

$$\sum_{\{E'|(E\subseteq E'\}} |C_G(E')|\alpha(E') = \sum_{\{E'|(E\subseteq E'\subseteq E_0)\}} |C_{G_0}(E')|\alpha_0(E') = 0\,.$$

- For the left handside of the previous formula, let us sum over a set of representatives for the $C_G(E)$–orbits of eigenspaces $E'$ containing $E$. Since $C_{G_0}(E)$ controls the fusion of $C_G(E)$ onto the set of such eigenspaces, we can sum over a set of representatives of $C_{G_0}(E)$–orbits of eigenspaces $E'$ such that $E \subseteq E' \subseteq E_0$. We get

$$\sum_{\{E'|(E\subseteq E'\}} |C_G(E')|\alpha(E') = \sum_{\{E'|(E\subseteq E'\subseteq E_0)\}/C_{G_0}(E)} \frac{|C_G(E)|}{|N_{C_G(E)}(E')|}|C_G(E')|\alpha(E')\,.$$

Since

$$N_{C_G(E)}(E') = N_{C_{G_0}(E)}(E')C_G(E')\,,$$

we have

$$\frac{|C_G(E)|}{|N_{C_G(E)}(E')|}|C_G(E')| = |C_G(E)|\,\frac{|C_{G_0}(E)|}{|N_{C_{G_0}(E)}(E')|}\,,$$

from which it follows that

$$\begin{aligned}\sum_{\{E'|(E\subseteq E'\}} |C_G(E')|\alpha(E') &= |C_G(E)| \sum_{\{E'|(E\subseteq E'\subseteq E_0)\}/C_{G_0}(E)} \frac{|C_{G_0}(E')|}{|N_{C_{G_0}(E)}(E')|}\alpha(E') \\ &= |C_G(E)| \sum_{\{E'|(E\subseteq E'\subseteq E_0)\}} \alpha(E')\end{aligned}$$

hence

$$\sum_{\{E'|(E\subseteq E'\subseteq E_0)\}} \alpha(E') = 0\,.$$

This shows that the function $\alpha$ is recursively determined by its value on $E_0$.

The same holds of $\alpha_0$, and since both $\alpha$ and $\alpha_0$ take the same value on $E_0$ we see that $\alpha = \alpha_0$. $\square$

## §18. Regular braid automorphisms

**Lifting the regular automorphisms.**

*When the base point is an eigenvector.*

Let $x_0 \in V^{\mathrm{reg}}$. We set $B := \Pi^1(V/V^{\mathrm{reg}}, x_0)\,.$

Let us denote by $N_{\mathrm{GL}(V)}(G)(x_0)$ the fixator (centralizer) of $x_0$ in $N_{\mathrm{GL}(V)}(G)$.
Then we define the injective group morphism

$$\begin{cases} \mathbf{a} : N_{\mathrm{GL}(V)}(G)(x_0) \longrightarrow \mathrm{Aut}(B) \\ \phi \mapsto \mathbf{a}(\phi)\,, \end{cases}$$

as follows.

Let $\gamma : t \mapsto \gamma(t)$ be a path from $x_0$ to $gx_0$. We denote by $\mathbf{a}(\phi)(\gamma)$ the path from $x_0$ to ${}^{\phi}gx_0$ defined by

$$\mathbf{a}(\phi)(\gamma) : \begin{cases} [0,1] \to V^{\mathrm{reg}} \\ t \mapsto \phi\gamma(t)\,. \end{cases}$$

Thus $\mathbf{a}(\phi)$ defines an automorphism of $B$, and we have

- $\mathbf{a}$ is a group morphism,
- the natural epimorphism $B \twoheadrightarrow G$ is $N_{\mathrm{GL}(V)}(G)(x_0)$–equivariant,
- $\ker \mathbf{a} \subseteq C_{\mathrm{GL}(V)}(G)(x_0)$.

Moreover, if $G$ is irreducible in its action on $V$, then

$$C_{\mathrm{GL}(V)}(G) = \mathbb{C}^{\times}\mathrm{Id}_V \quad \text{hence} \quad C_{\mathrm{GL}(V)}(G)(x_0) = \{1\} \quad \text{and} \quad \ker \mathbf{a} = \{1\}\,.$$

More generally, let $L = \mathbb{C}x_0$ be the line generated by $x_0$, let $N_{\mathrm{GL}(V)}(G, L)$ be the subgroup of $N_{\mathrm{GL}(V)}(G)$ which stabilizes (normalizes) $L$. The elements of finite order of $N_{\mathrm{GL}(V)}(G, L)$ are regular automorphisms.

Whenever $\phi \in N_{\mathrm{GL}(V)}(G, L)$, let $\zeta_\phi \in \mathbb{C}^{\times}$ be such that $\phi x_0 = \zeta_\phi x_0$. Then the map

$$\begin{cases} N_{\mathrm{GL}(V)}(G, L) \to N_{\mathrm{GL}(V)}(G)(x_0) \\ \phi \mapsto \zeta_\phi^{-1}\phi \end{cases}$$

is a group morphism, and we extend the group morphism $\mathbf{a}$ to

$$\begin{cases} \mathbf{a} : N_{\mathrm{GL}(V)}(G, L) \longrightarrow \mathrm{Aut}(B) \\ \phi \mapsto \mathbf{a}(\phi)\,, \end{cases}$$

by the formula

$$\mathbf{a}(\phi) := \mathbf{a}(\zeta_\phi^{-1}\phi)\,.$$

**18.1. Lemma.** *The group morphism* $\mathbf{a} : N_{\mathrm{GL}(V)}(G, L) \longrightarrow \mathrm{Aut}(B)$ *has the following properties.*

(1) *The natural epimorphism* $B \twoheadrightarrow G$ *is* $N_{\mathrm{GL}(V)}(G, L)$*–equivariant.*
(2) *If* $G$ *is irreducible on* $V$*, then* $\ker \mathbf{a} = \mathbb{C}^\times \mathrm{Id}_V$*.*
(3) *If* $G$ *is irreducible on* $V$*, the order of* $\mathbf{a}(\phi)$ *is equal to the order of the automorphism* $\mathrm{Ad}(\phi)$ *of* $G$ *defined by* $\phi$*.*

*Proof of 18.1.* Only (3) requires a proof. Assume that $\phi x_0 = \zeta x_0$. The order of $\mathbf{a}(\phi)$ is equal to the order of $\zeta^{-1}\phi$. Now $(\zeta^{-1}\phi)^m = 1$ if and only if $\phi^m = \zeta^m \mathrm{Id}_V$, hence if and only if $\phi^m \in \mathbb{C}^\times \mathrm{Id}_V$. □

Let $\zeta := \exp(2\pi i m/d)$ be a primitive $d$–th root of 1 (thus $m$ is prime to $d$). We recall that, for $x \in V^{\mathrm{reg}}$, we denote by $\pi_{\zeta,x}$ the path from $x$ to $\zeta x$ defined by

$$\pi_{\zeta,x} : \begin{cases} [0,1] \to V^{\mathrm{reg}} \\ t \mapsto \exp(2\pi i m t/d) x \,. \end{cases}$$

**18.2. Lemma.** *Let* $\zeta = \exp(2\pi i m/d)$ *with order* $d$*. Assume that* $\phi \in N_{\mathrm{GL}(V)}(G, L)$ *is such that* $\phi x_0 = \zeta x_0$*. Then we have the following identity between paths in* $V^{\mathrm{reg}}$ *:*

$$\pi_{\zeta,x_0} \cdot \phi(\pi_{\zeta,x_0}) \cdots \phi^{d-1}(\pi_{\zeta,x_0}) = \pi^m \,.$$

*Proof of 18.2.* This is immediate. □

*The general case.*

Let now $x_1$ be another element of $V^{\mathrm{reg}}$. We set

$$B(x_i) := \Pi^1(V^{\mathrm{reg}}/G, x_i) \quad \text{for } i = 0, 1 \,.$$

Given a path $\gamma$ from $x_0$ to $x_1$, we denote by

$$\tau_\gamma : B(x_0) \xrightarrow{\sim} B(x_1)$$

the isomorphism induced by the formula (with an admissible abuse of notation)

$$\tau_\gamma(\mathbf{g}) := \gamma^{-1} \cdot \mathbf{g} \cdot g\gamma$$

whenever $\mathbf{g}$ is a path from $x_0$ to $gx_0$.

Now if $\phi_1$ is an element of $N_{\mathrm{GL}(V)}(G)$ which fixes $x_1$ we denote by $\mathbf{a}(\phi_1)$ the automorphism of $B(x_1)$ defined as above. Through the isomorphism $\tau_\gamma$, that automorphism becomes an automorphism $\mathbf{a}_\gamma(\phi_1)$, induced by the formula

$$\begin{aligned} \mathbf{a}_\gamma(\phi_1)(\mathbf{g}) : &= \gamma \cdot \phi_1 \tau_\gamma(\mathbf{g}) \cdot ({}^{\phi_1}g)\gamma^{-1} \\ &= (\gamma \cdot \phi_1\gamma^{-1}) \cdot \phi_1\mathbf{g} \cdot ({}^{\phi_1}g)(\phi_1\gamma \cdot \gamma^{-1}) \,. \end{aligned}$$

Notice that if $p : B \twoheadrightarrow G$ denotes the natural surjection, we have

$$p \cdot \mathbf{a}_\gamma(\phi_1) = \mathrm{Ad}(\phi_1) \cdot p \,.$$

**Definition.** *A $\zeta$–regular braid automorphism is an automorphism of the braid group $B(x_0)$ of the form*

$$\mathbf{a}_\gamma(\phi_1) : \mathbf{g} \mapsto (\gamma \cdot \phi_1\gamma^{-1}) \cdot \phi_1\mathbf{g} \cdot ({}^{\phi_1}g)(\phi_1\gamma \cdot \gamma^{-1})$$

*where*

- *$\phi_1$ is an element of $N_{\mathrm{GL}(V)}(G)$ such that $\phi x_1 = \zeta x_1$ for some element $x_1 \in V^{\mathrm{reg}}$,*
- *$\gamma$ is a path from $x_0$ to $x_1$.*

*A regular braid automorphism is an automorphism which is $\zeta$–regular for some $\zeta$.*

A regular braid automorphism $\alpha = \mathbf{a}_\gamma(\phi_1)$ has a well–defined image $\overline{\alpha} := \mathrm{Ad}(\phi_1)$ in $N_{\mathrm{GL}(V)}(G)/C_{\mathrm{GL}(V)}(G)$, hence *a fortiori* it has a well–defined image in $N_{\mathrm{GL}(V)}(G)/G$.

In the case where $\phi$ and $\phi_1$ have the same image in $N_{\mathrm{GL}(V)}(G)/G$, it results from 17.4 that $\phi$ and $\phi_1$ are conjugate under $G$.

**18.3. Proposition.** *The set of regular braid automorphisms with given image in $N_{\mathrm{GL}(V)}(G)/G$ is a single orbit under $B$.*

*Proof of 18.3.* It will follow from the following lemma. We must here use precise notation, distinguishing between paths in $V^{\mathrm{reg}}$ and elements of $B$.

**18.4. Lemma.** *Let $\phi$ be an element of $N_{\mathrm{GL}(V)}(G)$ which fixes $x_0$. If $\gamma$ is a path in $V^{\mathrm{reg}}$ from $x_0$ to $gx_0$, defining an element $\mathbf{g} \in B$ (with image $g \in G$), then*

$$\mathrm{Ad}(\mathbf{g}) \cdot \mathbf{a}(\phi) \cdot \mathrm{Ad}(\mathbf{g}^{-1}) = \mathbf{a}_\gamma(g\phi g^{-1}) \, .$$

*Proof of 18.4.* Notice that $g\phi g^{-1}$ is an element $\phi_1$ of $N_{\mathrm{GL}(V)}(G)$ which fixes the regular vector $x_1 := gx_0$.

Let $\mathbf{h} \in B$, defined by a path $\eta$ in $V^{\mathrm{reg}}$ from $x_0$ to $hx_0$. Hence $\mathbf{h}$ has image $h$ in $G$. We must prove

$$\mathbf{g}\mathbf{a}(\phi)(\mathbf{g}^{-1}\mathbf{h}\mathbf{g})\mathbf{g}^{-1} = \mathbf{a}_\gamma(g\phi g^{-1})(\mathbf{h}) \, ,$$

*i.e.*, in other words

$$\begin{aligned}\gamma \cdot g\phi(g^{-1}\gamma^{-1} \cdot g^{-1}\eta \cdot g^{-1}h\gamma) \cdot (\phi_1 h\phi_1^{-1}\gamma^{-1}) \\ =& (\gamma \cdot \phi_1\gamma^{-1}) \cdot \phi_1\eta \cdot ({}^{\phi_1}h)(\phi_1\gamma \cdot \gamma^{-1}) \, ,\end{aligned}$$

an equality between two paths in $V^{\mathrm{reg}}$ from $x_0$ to ${}^{\phi_1}hx_0$ which we leave to the reader to check. $\square$

$\square$

Assume that both $g\phi$ and $\phi$ are $\zeta$–regular.

- There exists $h \in G$ such that $hg\phi h^{-1} = \phi$, *i.e.*, $g = h^{-1}\,{}^{\phi}h$.
- If $\phi x_0 = \zeta x_0$, then for $x_1 := h^{-1}x_0$, we have $g\phi x_1 = \zeta x_1$.

Let us set $\phi_1 := g\phi$. We have (see lemma 18.2)

$$\pi_{\zeta,x_1} \cdot \phi_1(\pi_{\zeta,x_1}) \cdots \phi_1^{d-1}(\pi_{\zeta,x_1}) = \pi^m \, .$$

*The case of an inner automorphism : roots of powers of $\pi$.*

Consider the particular case where $\phi$ belongs to $G$ (hence induces an inner automorphism of $G$).

**18.5. Proposition.** *Let $\zeta = \exp(2\pi i m/d)$ with order $d$. Assume that $g$ is a $\zeta$–regular element of $G$, such that $gx_0 = \zeta x_0$.*

(1) *The path $\pi_{\zeta,x_0}$ defines an element $\widetilde{g} \in B = \Pi^1(V^{\mathrm{reg}}/G, x_0)$ with image $g$ through the natural surjection $B \twoheadrightarrow G$.*
(2) *We have $\mathbf{a}(g) = \mathrm{Ad}(\widetilde{g})$ .*
(3) *The element $\widetilde{g}$ is a $d$–th root of $\pi^m$, i.e., $\widetilde{g}^d = \pi^m$ .*

**Lifting Springer theory.**

Let us now turn again to the general case where $\phi \in N_{\mathrm{GL}(V)}(G)$ has regular $\zeta$–eigenvector $x_0$.

We recall that $G(\phi,\zeta) := N_G(V(\phi,\zeta)) = C_G(\phi)$ , and that $V(\phi,\zeta)^{\mathrm{reg}} \subseteq V^{\mathrm{reg}}$ (see 16.6).

The composition

$$\iota(\phi,\zeta) : V(\phi,\zeta)^{\mathrm{reg}}/G(\phi,\zeta) \hookrightarrow V^{\mathrm{reg}}/G(\phi,\zeta) \twoheadrightarrow V^{\mathrm{reg}}/G$$

induces a morphism

$$\Pi^1(V(\phi,\zeta)^{\mathrm{reg}}/G(\phi,\zeta), x_0) \to \Pi^1(V^{\mathrm{reg}}/G, x_0)$$

*i.e.*, a morphism

$$\Pi^1\iota(\phi,\zeta) : B(\phi,\zeta) \to B\,.$$

**18.6. Proposition.**

(1) *The image of the map $\iota(\phi,\zeta)$ is the subvariety $(V^{\mathrm{reg}}/G)^{\langle\zeta^{-1}\phi\rangle}$ of fixed points under the action of the group $\langle\zeta^{-1}\phi\rangle$ generated by $\zeta^{-1}\phi$, and $\iota(\phi,\zeta)$ induces an homeomorphism*

$$V(\phi,\zeta)^{\mathrm{reg}}/G(\phi,\zeta) \xrightarrow{\sim} (V^{\mathrm{reg}}/G)^{\langle\zeta^{-1}\phi\rangle}\,.$$

(2) *The image of the map $\Pi^1\iota(\phi,\zeta)$ is contained in the group $C_B(\mathbf{a}(\phi))$ of fixed points of $\mathbf{a}(\phi)$ in $B$.*

*Proof of 18.7.*

(1) Let $x \in V(\phi,\zeta)^{\mathrm{reg}}$. Thus we have $\phi x = \zeta x$ , proving that $x$ is fixed under $\zeta^{-1}\phi$, hence its image in $V^{\mathrm{reg}}/G$ is also fixed by $\zeta^{-1}\phi$.

Let us prove that $\iota(\phi,\zeta)$ is surjective on $(V^{\mathrm{reg}}/G)^{\langle\zeta^{-1}\phi\rangle}$. Let $y \in V^{\mathrm{reg}}$ be such that its image modulo $G$ belongs to $(V^{\mathrm{reg}}/G)^{\langle\zeta^{-1}\phi\rangle}$. We want to prove that there is $h \in G$ such that $hy$ is fixed by $\zeta^{-1}\phi$. By assumption there is $g \in G$ such that $\zeta^{-1}g\phi y = y$. Hence $g\phi$ is $\zeta$–regular, and it results from 16.6 that $g\phi$ and $\phi$ are $G$–conjugate : there exists $h \in G$ such that $hg\phi h^{-1} = \phi$. It follows that $\zeta^{-1}\phi hy = hy$, hence $hy$ is fixed under $\zeta^{-1}\phi$.

Let us prove that $\iota(\phi,\zeta)$ is injective. Let $x, x' \in V(\phi,\zeta)^{\mathrm{reg}}$, so that $\phi x = \zeta x$ and $\phi x' = \zeta x'$. Assume that $x$ and $x'$ have the same image in $V^{\mathrm{reg}}/G$, *i.e.*, there is $g \in G$ such that $x' = gx$. It follows that $\phi^{-1}g^{-1}\phi g x = x$ , and since $x$ is regular we have $g = \phi^{-1}g\phi$ , proving that $g \in C_G(\phi) = G(\phi,\zeta)$.

(2) Let $z \in G(\phi,\zeta)$, and let $\widetilde{z}$ be a path from $x_0$ to $zx_0$ in $V(\phi,\zeta)^{\mathrm{reg}}$. It is clear that $\zeta^{-1}\phi$ fixes $\widetilde{z}$, proving that the image of $\Pi^1\iota(\phi,\zeta)$ is contained in $C_B(\mathbf{a}(\phi))$. $\square$

Springer theory (see 16.6) shows that the group $C_G(\phi)$ is a reflection group. The following conjecture may be viewed as "Braid Springer theory".

**18.8. Conjecture.** *The map* $\Pi^1\iota(\phi,\zeta)$ *is injective and its image is* $C_B(\mathbf{a}(\phi))$.

The preceding conjecture is now proved for $\phi \in G$ (see [Bes3]). The case where $\phi \notin G$ is still open. Let us state Bessis' result for completeness.

**18.9. Theorem.** *Let* $\zeta_d := e^{2i\pi/d}$.

(1) *The* $\zeta_d$*–regular elements in* $G$ *are the images of the* $d$*-th roots of* $\pi$.
(2) *All* $d$*-th roots of* $\pi$ *are conjugate in* $B$.
(3) *Let* $\mathbf{g}$ *be a* $d$*-th root of* $\pi$, *with image* $g$ *in* $G$. *Then* $C_B(\mathbf{g})$ *is the braid group of* $C_G(g)$.

## APPENDIX
## COXETER AND ARTIN LIKE PRESENTATIONS

Here are some definitions, notation, conventions, which will allow the reader to understand the diagrams.

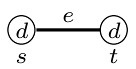
The groups have presentations given by diagrams $\mathcal{D}$ such that

- the nodes correspond to pseudo-reflections in $G$, the order of which is given inside the circle representing the node,
- two distinct nodes which do not commute are related by "homogeneous" relations with the same "support" (of cardinality 2 or 3), which are represented by links beween two or three nodes, or circles between three nodes, weighted with a number representing the degree of the relation (as in Coxeter diagrams, 3 is omitted, 4 is represented by a double line, 6 is represented by a triple line). These homogeneous relations are called the *braid relations* of $\mathcal{D}$.

More details are provided below.

**Meaning of the diagrams.**

This paragraph provides a list of examples which illustrate the way in which diagrams provide presentations for the attached groups.

- The diagram 

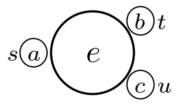
corresponds to the presentation

$$s^d = t^d = 1 \text{ and } \underbrace{ststs\cdots}_{e \text{ factors}} = \underbrace{tstst\cdots}_{e \text{ factors}}$$

- The diagram 

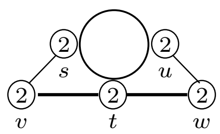
corresponds to the presentation

$$s^5 = t^3 = 1 \text{ and } stst = tsts\,.$$

- The diagram 

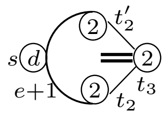
corresponds to the presentation

$$s^a = t^b = u^c = 1 \text{ and } \underbrace{stustu\cdots}_{e \text{ factors}} = \underbrace{tustus\cdots}_{e \text{ factors}} = \underbrace{ustust\cdots}_{e \text{ factors}}\,.$$

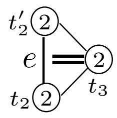
- The diagram 

corresponds to the presentation

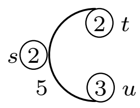
$$\begin{aligned} &s^2 = t^2 = u^2 = v^2 = w^2 = 1\,,\\ &uv = vu\,,\ sw = ws\,,\ vw = wv\,,\\ &sut = uts = tsu\,,\\ &svs = vsv\,,\ tvt = vtv\,,\ twt = wtw\,, wuw = uwu\,. \end{aligned}$$

- The diagram  corresponds to the presentation

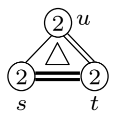
$$
\begin{aligned}
&s^d = {t'_2}^2 = t_2^2 = t_3^2 = 1\,,\ st_3 = t_3 s\,,\\
&st'_2t_2 = t'_2t_2s\,,\\
&t'_2t_3t'_2 = t_3t'_2t_3\,,\ t_2t_3t_2 = t_3t_2t_3\,,\ t_3t'_2t_2t_3t'_2t_2 = t'_2t_2t_3t'_2t_2t_3\,,\\
&\underbrace{t_2st'_2t_2t'_2t_2t'_2\cdots}_{e+1 \text{ factors}} = \underbrace{st'_2t_2t'_2t_2t'_2t_2\cdots}_{e+1 \text{ factors}}\,.
\end{aligned}
$$

- The diagram  corresponds to the presentation

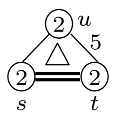
$$
\begin{aligned}
&{t'_2}^2 = t_2^2 = t_3^2 = 1\,,\\
&t'_2t_3t'_2 = t_3t'_2t_3\,,\ t_2t_3t_2 = t_3t_2t_3\,,\ t_3t'_2t_2t_3t'_2t_2 = t'_2t_2t_3t'_2t_2t_3\,,\\
&\underbrace{t_2t'_2t_2t'_2t_2t'_2\cdots}_{e \text{ factors}} = \underbrace{t'_2t_2t'_2t_2t'_2t_2\cdots}_{e \text{ factors}}\,.
\end{aligned}
$$

- The diagram  corresponds to the presentation

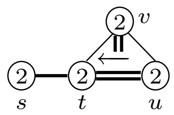
$$s^2 = t^2 = u^3 = 1\,,\ stu = tus\,,\ ustut = stutu\,.$$

- The diagram  corresponds to the presentation

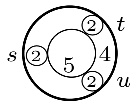
$$s^2 = t^2 = u^2 = 1\,,\ stst = tsts\,,\ tutu = utut\,,\ sus = usu\,,\ u(stu)^2 = (stu)^2t\,.$$

- The diagram  corresponds to the presentation

$$s^2 = t^2 = u^2 = 1\,,\ stst = tsts\,,\ tutut = ututu\,,\ sus = usu\,,\ (uts)^2t = s(uts)^2\,.$$

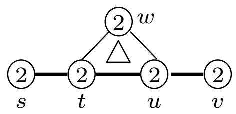
- The diagram  corresponds to the presentation

$$
\begin{aligned}
&s^2 = t^2 = u^2 = v^2 = 1\,,\ sv = vs\,,\ su = us\,,\\
&sts = tst\,,\ vtv = tvt\,,\ uvu = vuv\,,\ tutu = utut\,,\ vtuvtu = tuvtuv\,.
\end{aligned}
$$

- The diagram  corresponds to the presentation



$$s^2 = t^2 = u^2 = 1\,,\ ustus = stust\,,\ tust = ustu\,.$$

- The diagram  corresponds to the presentation



$$\begin{aligned}&s^2 = t^2 = u^2 = v^2 = w^2 = 1\,,\ vt = tv\,,\ uv = vu\,,\ tu = ut\,,\ wu = uw\,,\\ &sts = tst\,,\ tut = utu\,,\ uvu = vuv\,,\ twt = wtw\,,\ uwu = wuw\,,\\ &twvstw = wvstwv\,.\end{aligned}$$

In the following tables, we denote by $H \rtimes K$ a group which is a non-trivial split extension of $K$ by $H$. We denote by $H \cdot K$ a group which is a non-split extension of $K$ by $H$. We denote by $p^n$ an elementary abelian group of order $p^n$.

A diagram where the orders of the nodes are "forgotten" and where only the braid relations are kept is called a *braid diagram* for the corresponding group.

The groups have been ordered by their diagrams, by collecting groups with the same braid diagram. Thus, for example,

- $G_{15}$ has the same braid diagram as the groups $G(4d, 4, 2)$ for all $d \geq 2$,
- $G_4$, $G_8$, $G_{16}$, $G_{25}$, $G_{32}$ all have the same braid diagrams as groups $\mathfrak{S}_3$, $\mathfrak{S}_4$ and $\mathfrak{S}_5$,
- $G_5$, $G_{10}$, $G_{18}$ have the same braid diagram as the groups $G(d, 1, 2)$ for all $d \geq 2$,
- $G_7$, $G_{11}$, $G_{19}$ have the same braid diagram as the groups $G(2d, 2, 2)$ for all $d \geq 2$,
- $G_{26}$ has the same braid diagram as $G(d, 1, 3)$ for $d \geq 2$.

The element $\beta$ (generator of $Z(G)$) is given in the last column of our tables. Notice that the knowledge of degrees and codegrees allows then to find the order of $Z(G)$, which is not explicitely provided in the tables.

The tables provide diagrams and data for all irreducible reflection groups.

- Tables 1 and 2 collect groups corresponding to infinite families of braid diagrams,
- Table 3 collects groups corresponding to exceptional braid diagrams (notice that the fact that the diagram for $G_{31}$ provides a braid diagram is only conjectural), but $G_{24}$, $G_{27}$, $G_{29}$, $G_{33}$, $G_{34}$,
- The last table (table 4) provides diagrams for the remaining cases ($G_{24}$, $G_{27}$, $G_{29}$, $G_{33}$, $G_{34}$). It is not known nor conjectural whether these diagrams provide braid diagrams for the corresponding braid groups.

*Degrees and codegrees of a braid diagram.*

The following property may be noticed on the tables. It generalizes a property already noticed by Orlik and Solomon for the case of Coxeter–Shephard groups (see [OrSo3], (3.7)).

**18.10. Theorem.** *Let $\mathcal{D}$ be a braid diagram of rank $r$. There exist two families*

$$(\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_r) \quad \textit{and} \quad (\mathbf{d}_1^\vee, \mathbf{d}_2^\vee, \ldots, \mathbf{d}_r^\vee)$$

*of $r$ integers, depending only on $\mathcal{D}$, and called respectively the degrees and the codegrees of $\mathcal{D}$, with the following property: whenever $G$ is a complex reflection group with $\mathcal{D}$ as a braid diagram, its degrees and codegrees are given by the formulae*

$$d_j = |Z(G)|\mathbf{d}_j \quad \textit{and} \quad d_j^\vee = |Z(G)|\mathbf{d}_j^\vee \quad (j = 1, 2, \ldots, r)\,.$$

*The zeta function of a braid diagram.*

In [DeLo], Denef and Loeser compute the *zeta function of local monodromy of the discriminant* of a complex reflection group $G$, which is the element of $\mathbb{Q}[q]$ defined by the formula

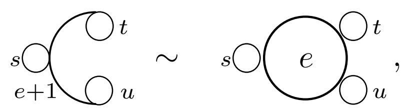
$$Z(q, G) := \prod_j \det(1 - q\mu, H^j(F_0, \mathbb{C}))^{(-1)^{j+1}}\,,$$

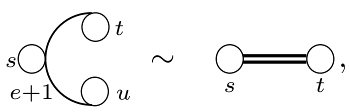
where $F_0$ denotes the Milnor fiber of the discriminant at 0 and $\mu$ denotes the monodromy automorphism (see [DeLo]).

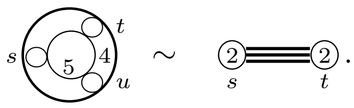
Putting together the tables of [DeLo] and our braid diagrams, one may notice the following fact.

**18.11. Theorem.** *The zeta function of local monodromy of the discriminant of a complex reflection group $G$ depends only on the braid diagram of $G$.*

*Remark.* Two different braid diagrams may be associated to isomorphic braid groups. For example, this is the case for the following rank 2 diagrams (where the sign "$\sim$" means that the corresponding groups are isomorphic) :

For $e$ even, 


for $e$ odd, 


and 


It should be noticed, however, that the above pairs of diagrams do not have the same degrees and codegrees, nor do they have the same zeta function. Thus, degrees, codegrees and zeta functions are indeed attached to the braid diagrams, not to the braid groups.

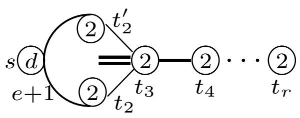
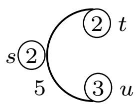
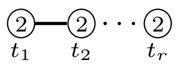
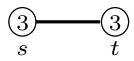
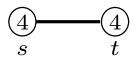
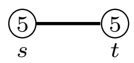
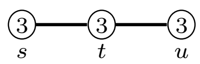
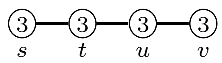
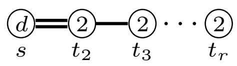
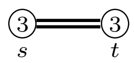
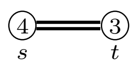
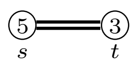
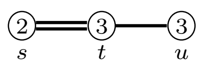
| **name** | **diagram** | **degrees** | **codegrees** | $\beta$ | **field** | $G/Z(G)$ |
|---|---|---|---|---|---|---|
| $G(de,e,r)$ $e>2,d,r\geq 2$ |  | $(ed,2ed,\ldots,(r-1)ed,rd)$ | $(0,ed,\ldots,(r-1)ed)$ | $s^{\frac{r}{(e\wedge r)}}(t_2' t_2 t_3\cdots t_r)^{\frac{e(r-1)}{(e\wedge r)}}$ | $\mathbb{Q}(\zeta_{de})$ | |
| $G_{15}$ |  | $12,24$ | $0,24$ | $ustut=s(tu)^2$ | $\mathbb{Q}(\zeta_{24})$ | $\mathfrak{S}_4$ |
| $\mathfrak{S}_{r+1}$ |  | $(2,3,\ldots,\ldots,r+1)$ | $(0,1,\ldots,\ldots,r-1)$ | $(t_1\cdots t_r)^{r+1}$ | $\mathbb{Q}$ | |
| $G_4$ |  | $4,6$ | $0,2$ | $(st)^3$ | $\mathbb{Q}(\zeta_3)$ | $\mathfrak{A}_4$ |
| $G_8$ |  | $8,12$ | $0,4$ | $(st)^3$ | $\mathbb{Q}(i)$ | $\mathfrak{S}_4$ |
| $G_{16}$ |  | $20,30$ | $0,10$ | $(st)^3$ | $\mathbb{Q}(\zeta_5)$ | $\mathfrak{A}_5$ |
| $G_{25}$ |  | $6,9,12$ | $0,3,6$ | $(stu)^4$ | $\mathbb{Q}(\zeta_3)$ | $3^2\rtimes SL_2(3)$ |
| $G_{32}$ |  | $12,18,24,30$ | $0,6,12,18$ | $(stuv)^5$ | $\mathbb{Q}(\zeta_3)$ | $PSp_4(3)$ |
| $G(d,1,r)$ $d\geq 2$ |  | $(d,2d,\ldots,\ldots,rd)$ | $(0,d,\ldots,\ldots,(r-1)d)$ | $(st_2t_3\cdots t_r)^r$ | $\mathbb{Q}(\zeta_d)$ | |
| $G_5$ |  | $6,12$ | $0,6$ | $(st)^2$ | $\mathbb{Q}(\zeta_3)$ | $\mathfrak{A}_4$ |
| $G_{10}$ |  | $12,24$ | $0,12$ | $(st)^2$ | $\mathbb{Q}(\zeta_{12})$ | $\mathfrak{S}_4$ |
| $G_{18}$ |  | $30,60$ | $0,30$ | $(st)^2$ | $\mathbb{Q}(\zeta_{15})$ | $\mathfrak{A}_5$ |
| $G_{26}$ |  | $6,12,18$ | $0,6,12$ | $(stu)^3$ | $\mathbb{Q}(\zeta_3)$ | $3^2\rtimes SL_2(3)$ |

TABLE 1

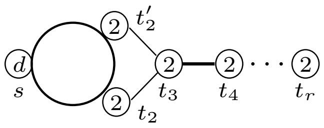
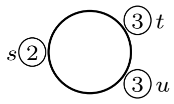
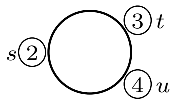
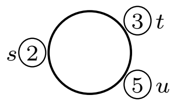
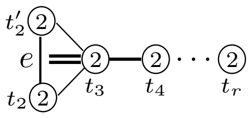
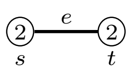
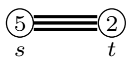
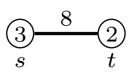
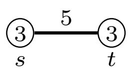
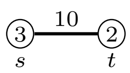
| **name** | **diagram** | **degrees** | **codegrees** | $\beta$ | **field** | $G/Z(G)$ |
|---|---|---|---|---|---|---|
| $G(2d,2,r)$ $_{d,r\geq 2}$ |  | $(2d,4d,\ldots$ $2(r-1)d,rd)$ | $(0,2d,\ldots$ $2(r-1)d)$ | $s^{\frac{r}{(2\wedge r)}}(t_2' t_2 t_3 \cdots t_r)^{\frac{2(r-1)}{(2\wedge r)}}$ | $\mathbb{Q}(\zeta_{2d})$ | |
| $G_7$ |  | $12, 12$ | $0, 12$ | $stu$ | $\mathbb{Q}(\zeta_{12})$ | $\mathfrak{A}_4$ |
| $G_{11}$ |  | $24, 24$ | $0, 24$ | $stu$ | $\mathbb{Q}(\zeta_{24})$ | $\mathfrak{S}_4$ |
| $G_{19}$ |  | $60, 60$ | $0, 60$ | $stu$ | $\mathbb{Q}(\zeta_{60})$ | $\mathfrak{A}_5$ |
| $G(e,e,r)$ $_{e\geq 2, r>2}$ |  | $(e,2e,\ldots,$ $(r-1)e,r)$ | $(0,e,\ldots,(r-2)e,$ $(r-1)e-r)$ | $(t_2' t_2 t_3 \cdots t_r)^{\frac{e(r-1)}{(e\wedge r)}}$ | $\mathbb{Q}(\zeta_e)$ | |
| $G(e,e,2)$ $_{e\geq 3}$ |  | $2, e$ | $0, e-2$ | $(st)^{e/(e\wedge 2)}$ | $\mathbb{Q}(\zeta_e+\zeta_e^{-1})$ | |
| $G_6$ |  | $4, 12$ | $0, 8$ | $(st)^3$ | $\mathbb{Q}(\zeta_{12})$ | $\mathfrak{A}_4$ |
| $G_9$ |  | $8, 24$ | $0, 16$ | $(st)^3$ | $\mathbb{Q}(\zeta_8)$ | $\mathfrak{S}_4$ |
| $G_{17}$ |  | $20, 60$ | $0, 40$ | $(st)^3$ | $\mathbb{Q}(\zeta_{20})$ | $\mathfrak{A}_5$ |
| $G_{14}$ |  | $6, 24$ | $0, 18$ | $(st)^4$ | $\mathbb{Q}(\zeta_3,\sqrt{-2})$ | $\mathfrak{S}_4$ |
| $G_{20}$ |  | $12, 30$ | $0, 18$ | $(st)^5$ | $\mathbb{Q}(\zeta_3,\sqrt{5})$ | $\mathfrak{A}_5$ |
| $G_{21}$ |  | $12, 60$ | $0, 48$ | $(st)^5$ | $\mathbb{Q}(\zeta_{12},\sqrt{5})$ | $\mathfrak{A}_5$ |

Table 2

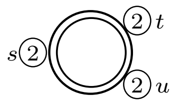
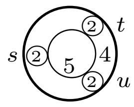
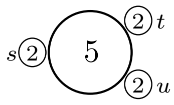
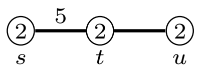
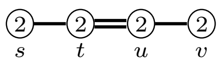
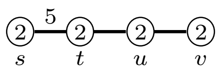
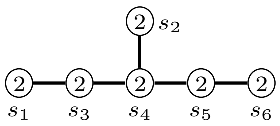
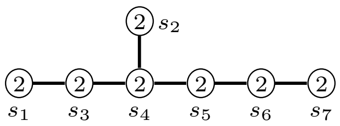
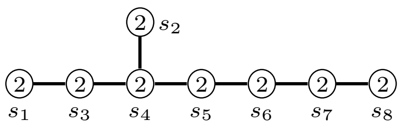
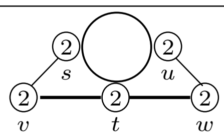
| name | diagram | degrees | codegrees | $\beta$ | field | $G/Z(G)$ |
|---|---|---|---|---|---|---|
| $G_{12}$ |  | 6,8 | 0,10 | $(stu)^4$ | $\mathbb{Q}(\sqrt{-2})$ | $\mathfrak{S}_4$ |
| $G_{13}$ |  | 8,12 | 0,16 | $(stu)^3$ | $\mathbb{Q}(\zeta_8)$ | $\mathfrak{S}_4$ |
| $G_{22}$ |  | 12,20 | 0,28 | $(stu)^5$ | $\mathbb{Q}(i,\sqrt{5})$ | $\mathfrak{A}_5$ |
| $G_{23}$ |  | 2,6,10 | 0,4,8 | $(stu)^5$ | $\mathbb{Q}(\sqrt{5})$ | $\mathfrak{A}_5$ |
| $G_{28}$ |  | 2,6, 8,12 | 0,4, 6,10 | $(stuv)^6$ | $\mathbb{Q}$ | $2^4 \rtimes (\mathfrak{S}_3 \times \mathfrak{S}_3)$ † |
| $G_{30}$ |  | 2,12, 20,30 | 0,10, 18,28 | $(stuv)^{15}$ | $\mathbb{Q}(\sqrt{5})$ | $(\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes 2$ ‡ |
| $G_{35}$ |  | 2,5,6,8, 9,12 | 0,3,4,6, 7,10 | $(s_1 \cdots s_6)^{12}$ | $\mathbb{Q}$ | $SO_6^-(2)'$ |
| $G_{36}$ |  | 2,6,8, 10,12, 14,18 | 0,4,6, 8,10, 12,16 | $(s_1 \cdots s_7)^9$ | $\mathbb{Q}$ | $SO_7(2)$ |
| $G_{37}$ |  | 2,8,12, 14,18,20, 24,30 | 0,6,10, 12,16,18, 22,28 | $(s_1 \cdots s_8)^{15}$ | $\mathbb{Q}$ | $SO_8^+(2)$ |
| $G_{31}$ |  | 8,12, 20,24 | 0,12, 16,28 | $(stuvw)^6$ | $\mathbb{Q}(i)$ | $2^4 \rtimes \mathfrak{S}_6$ ⋆ |

Table 3

† The action of $\mathfrak{S}_3 \times \mathfrak{S}_3$ on $2^4$ is irreducible.

‡ The automorphism of order 2 of $\mathfrak{A}_5 \times \mathfrak{A}_5$ permutes the two factors.

⋆ The group $G_{31}/Z(G_{31})$ is not isomorphic to the quotient of the Weyl group $D_6$ by its center.

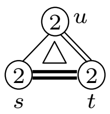
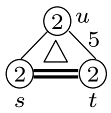
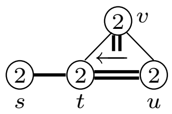
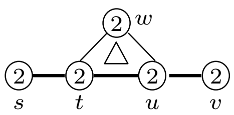
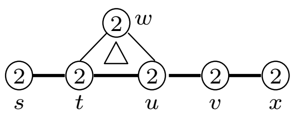
| name | diagram | degrees | codegrees | $\beta$ | field | $G/Z(G)$ |
|---|---|---|---|---|---|---|
| $G_{24}$ |  $(*)$ | 4,6,14 | 0,8,10 | $(stu)^7$ | $\mathbb{Q}(\sqrt{-7})$ | $GL_3(2)$ |
| $G_{27}$ |  $(**)$ | 6,12,30 | 0,18,24 | $(stu)^5$ | $\mathbb{Q}(\zeta_3, \sqrt{5})$ | $\mathfrak{A}_6$ |
| $G_{29}$ |  | 4,8,12,20 | 0,8,12,16 | $(stuv)^5$ | $\mathbb{Q}(i)$ | $2^4 \rtimes \mathfrak{S}_5$ † |
| $G_{33}$ |  | 4,6,10, 12,18 | 0,6,8, 12,14 | $(ustvw)^9$ | $\mathbb{Q}(\zeta_3)$ | $SO_5(3)'$ |
| $G_{34}$ |  | 6,12,18,24, 30,42 | 0,12,18,24, 30,36 | $(stuvwx)^7$ | $\mathbb{Q}(\zeta_3)$ | $PSO_6^-(3)'{\cdot}2$ |

Table 4

(*) $\triangle : u(stu)^2 = (stu)^2 t$

(**) $\triangle : (uts)^2 t = s(uts)^2$

† The group $G_{29}/Z(G_{29})$ is not isomorphic to the Weyl group $D_5$.

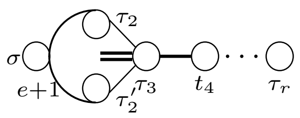
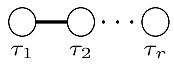
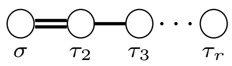
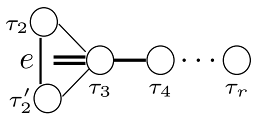
| **name** | **diagram** | **degrees** | **codegrees** | $\beta$ |
|---|---|---|---|---|
| $B(de,e,r)$ $_{e\geq 2, r\geq 2, d>1}$ |  | $e, 2e, \ldots, (r-1)e, r$ | $0, e, \ldots, (r-1)e$ | $\sigma^{\frac{r}{(e\wedge r)}}(\tau_2\tau_2'\tau_3\cdots\tau_r)^{\frac{e(r-1)}{(e\wedge r)}}$ |
| $B(1,1,r)$ |  | $2, 3, \ldots, r+1$ | $0, 1, \ldots, r-1$ | $(\tau_1\cdots\tau_r)^{r+1}$ |
| $B(d,1,r)$ $_{d>1}$ |  | $1, 2, \ldots, r$ | $0, , \ldots, (r-1)$ | $(\sigma\tau_2\tau_3\cdots\tau_r)^r$ |
| $B(e,e,r)$ $_{e\geq 2, r\geq 2}$ |  | $e, 2e, \ldots, (r-1)e, r$ | $0, e, \ldots, (r-2)e, (r-1)e-r$ | $(\tau_2\tau_2'\tau_3\cdots\tau_r)^{\frac{e(r-1)}{(e\wedge r)}}$ |

Table 5 : braid diagrams

This table provides a complete list of the infinite families of braid diagrams and corresponding data. Note that the braid diagram $B(de,e,r)$ for $e=2, d>1$ can also be described by a diagram as the one used for $G(2d,2,r)$ in Table 2. Similarly, the diagram for $B(e,e,r)$, $e=2$, can also be described by the Coxeter diagram of type $D_r$. The list of exceptional diagrams is given by with tables 3 and 4.

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