Pseudo reductive groups over $\mathbb{F}_x$?

Michel Broué

Institut Henri–Poincaré

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Joint work between Michel Broué, Gunter Malle, and Jean Michel,
Joint work between Michel Broué, Gunter Malle, and Jean Michel, initiated in the Greek island named SPETSES in 1993.
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## Unipotent characters for $G_2$

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<td>$S_3.(g_2, 1)$</td>
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<td>$S_3.(g_3, 1)$</td>
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<td>$\frac{1}{3}q\Phi_3\Phi_6$</td>
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<td>$S_3.(1, \varepsilon)$</td>
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Unipotent characters for $G_4$

$G_4 = 2 \times S_3$

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<td>$-\sqrt{-3}$</td>
<td>$\Phi_3'$</td>
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Unipotent characters for $G_4$

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<tr>
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<td>0</td>
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$\Phi_3', \Phi_3''$ (resp. $\Phi_6', \Phi_6''$) are factors of $\Phi_3$ (resp $\Phi_6$) in $\mathbb{Q}(\zeta_3)$
Finite reductive groups

- $G$ a reductive group over $\overline{F}_q$, 

\[ \begin{align*}
\text{UnCh}(G) := & \{ \text{unipotent characters} \} = \{ \text{irreducible constituents of } R^G_T \}\ \\
\text{UnSh}(G) := & \{ \text{unipotent character sheaves} \} = \text{another } \mathbb{C} \text{-basis for the space of class functions on } G.
\end{align*} \]
Finite reductive groups

- $G$ a reductive group over $\overline{F}_q$, $F$ an isogeny such that $G$ is finite.
Finite reductive groups

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- \{ \text{\( G \)-conjugacy classes of \( F \)-stable maximal tori of \( G \)} \}
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- \{ G-conjugacy classes of $F$–stable maximal tori of $G$ \} \leftrightarrow \{ conjugacy classes of the Weyl group $W$ \}
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Finite reductive groups

- **G** a reductive group over \(\overline{F}_q\), \(F\) an isogeny such that \(G\) is finite. For simplicity we assume \((G, F)\) **split**. Set \(G := G^F\).

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  \(\quad := \{\text{Irreducible constituents of } R^G_{T_w}(1)\}\)

- \(\text{UnSh}(G) := \{\text{Unipotent character sheaves}\}\)
  \(\quad = \text{Another } \mathbb{C}\text{–basis for the space of class functions on } G\text{ generated by UnCh}(G)\).

- **Lusztig’s Fourier matrix** \(S\)
Finite reductive groups

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  $= \text{Another } \mathbb{C}\text{-basis for the space of class functions on } G \text{ generated by } \text{UnCh}(G)$.

- **Lusztig’s Fourier matrix** $S$
  $:= (\text{square}) \text{ matrix between } \text{UnCh}(G) \text{ and } \text{UnSh}(G)$.
Finite reductive groups

- \( G \) a reductive group over \( \overline{F}_q \), \( F \) an isogeny such that \( G \) is finite. For simplicity we assume \((G, F)\) split. Set \( G := G^F \).

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- \text{Lusztig’s Fourier matrix} \( S \) := \( \text{(square) matrix between} \ \text{UnCh}(G) \) \text{ and UnSh}(G).

- The blocks of \( S \) correspond to \text{Lusztig’s families of unipotent characters}. 

Michel Broué  Pseudo reductive groups over \( F_x \) ?
Lusztig’s Fourier matrix for $G_2$

<table>
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<tr>
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<th>$(1, 1)$</th>
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<th>$(g_3, 1)$</th>
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<th>$(1, \varepsilon)$</th>
<th>$(g_2, \varepsilon)$</th>
<th>$(g_3, \zeta_3)$</th>
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### Lusztig’s Fourier matrix for $G_2$

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The Fourier matrix for $G_4$

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Unipotent characters for $G_4$

<table>
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<th>Character</th>
<th>Degree</th>
<th>FakeDegree</th>
<th>Eigenvalue</th>
<th>Family</th>
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<td>$\phi_{1,0}$</td>
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<td>1</td>
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<tr>
<td>$\phi_{2,1}$</td>
<td>$\frac{3-\sqrt{-3}}{6} q\Phi_3' \Phi_4 \Phi_6''$</td>
<td>$q\Phi_4$</td>
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It gives rise to $\xrightarrow{N G} V \xleftarrow{L F W G(\mathcal{L})}$ and to $\Rightarrow$ Harish–Chandra induction $\Rightarrow R_{G \mathcal{L}} := \text{Ind}_{G P} \cdot \text{Res}_{P} \rightarrow L$:

$\xrightarrow{C \mathcal{L} \text{mod}} \xrightarrow{C G \text{mod}}$ and restriction $\ast R_{G \mathcal{L}} : C G \text{mod} \rightarrow C \mathcal{L} \text{mod}$ which do not depend on the choice of $P$. 

Michel Broué | Pseudo reductive groups over $\mathbb{F}_x$?

Michel Broué
A Levi subgroup of an $F$–stable parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ is called a $\mathbf{1}$–Levi subgroup of $\mathbf{G}$.

It gives rise to

Harish-Chandra induction

$\text{Ind}^\mathbf{G}_{\mathbf{P}} \cdot \text{Res}^\mathbf{P}_\mathbf{L} \colon \mathbb{C}^\mathbf{L} \mod \to \mathbb{C}^\mathbf{G} \mod$

and restriction

$\ast \text{Res}^\mathbf{P}_\mathbf{L} \colon \mathbb{C}^\mathbf{G} \mod \to \mathbb{C}^\mathbf{L} \mod$

which do not depend on the choice of $\mathbf{P}$. 

Michel Broué

It gives rise to

\[
\begin{array}{ccc}
P^F & \rightarrow & N_{G^F}(L) \\
V^F & \downarrow & \downarrow W_{G(L)} \\
1 & \downarrow & \downarrow L^F \\
& & \end{array}
\]

It gives rise to

and to

Harish-Chandra induction $R_{G/L} := \text{Ind}_{G/P} \cdot \text{Res}_{P \to L}$, which do not depend on the choice of $P$. 

Michel Broué

It gives rise to

$$\begin{array}{ccc}
P^F & \overset{\text{P}}{\longrightarrow} & N_{G^F}(L) \\
V^F & \overset{\text{W}}{\longrightarrow} & L^F \\
1 & \overset{\text{G}}{\longrightarrow} & G \end{array}$$

and to

- Harish–Chandra induction $R_L^G := \text{Ind}_P^G \cdot \text{Res}_{P \to L}$

It gives rise to

$$
\begin{array}{c}
P^F \\
\downarrow \\
V^F \\
\downarrow \\
1
\end{array} \quad \begin{array}{c}
N_{G^F}(L) \\
\downarrow \\
L^F \\
\downarrow \\
W_G(L)
\end{array}
$$

and to

- Harish–Chandra induction $R_L^G := \text{Ind}_P^G \cdot \text{Res}_{P \to L} : C_{L \text{mod}} \to C_{G \text{mod}}$

It gives rise to

$$
\begin{array}{ccc}
F & N_{GF}(L) \\
P & V^F & L^F \\
1 & W_G(L) & \\
\end{array}
$$

and to

- Harish–Chandra induction $R_L^G := \text{Ind}_P^G \cdot \text{Res}_{P \to L} : \mathcal{C}_L \text{mod} \to \mathcal{C}_G \text{mod}$
- and restriction $^*R_L^G : \mathcal{C}_G \text{mod} \to \mathcal{C}_L \text{mod}$

It gives rise to

$$
\begin{array}{c}
P^F \\
\downarrow \\
\downarrow \\
V^F \\
\downarrow \\
\downarrow \\
L^F \\
\downarrow \\
\downarrow \\
1
\end{array}
$$

and to

- Harish–Chandra induction $R^G_L := \text{Ind}_P^G \cdot \text{Res}_{P \to L} : \mathcal{C}_L \text{mod} \to \mathcal{C}_G \text{mod}$
- and restriction $^*R^G_L : \mathcal{C}_G \text{mod} \to \mathcal{C}_L \text{mod}$
- which do not depend on the choice of $P$. 

Michel Broué Pseudo reductive groups over $\mathbb{F}_x$
Definition: cuspidal character

An irreducible character $\gamma$ of $G$ is said to be 1–cuspidal if, whenever $L$ is a proper 1–Levi subgroup, we have $\ast_R G L(\gamma) = 0$.

A pair $(L, \lambda)$ is called 1–cuspidal if $L$ is a 1–Levi subgroup of $G$ and $\lambda$ a 1–cuspidal irreducible character of $L$.

Main theorem

1. Partition: $\text{Irr}_K(G) = \bigcup (L, \lambda)$ cuspidal $\text{Irr}_R G L(\lambda)$

2. For $(L, \lambda)$ 1–cuspidal, the relative Weyl group $W_{G(L, \lambda)} := N_G(L, \lambda)/L$ is a finite Coxeter group.

3. The commuting algebra $\text{End}_K G R G L(\lambda)$ is a Hecke algebra for $W_{G(L, \lambda)}$.
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#### Main theorem

1. **Partition:**
   
   \[
   \text{Irr}_K(G) = \bigcup (L, \lambda) \text{ cuspidal} / \text{Irr}_R G L (\lambda)
   \]

2. For $(L, \lambda)$ 1-cuspidal, the relative Weyl group $W_G (L, \lambda) := N_G (L, \lambda) / L$ is a finite Coxeter group.

3. The commuting algebra $\text{End}_K R G L (\lambda)$ is a Hecke algebra for $W_G (L, \lambda)$.
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Main theorem

1. Partition: $\text{Irr}_K(G) = \bigcup (L, \lambda) \text{ cuspidal/}G \text{ Irr}_R^G(L, \lambda)$

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Main theorem

1. Partition:
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Michel Broué
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Definition: cuspidal character

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Degrees and Eigenvalues

The elements of $\text{UnCh}(G)$ and $\text{UnSh}(G)$ are parametrized by finite sets which are independent of $q$ (depend only on the "type" of $G$). The degrees of elements of $\text{UnCh}(G)$ and $\text{UnSh}(G)$ are polynomials evaluated at $q$.

Among the unipotent character sheaves are the functions $R \chi = 1 |W| \sum_{w \in W} \chi(w) R_G T_w$ and $\text{Deg} R \chi$ is the graded multiplicity of $\chi$ in the coinvariant algebra of $W$, evaluated at $q$ (the "fake degree" of $\chi$).

The degrees of the irreducible constituents of $R_G L(\lambda)$ are given by the "generic degrees" for the Hecke algebra of $W_G(L, \lambda)$.

The Fourier matrix is independent of $q$.

Michel Broué
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The Fourier matrix is independent of $q$.Michel Broué
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The elements of UnCh(\(G\)) and UnSh(\(G\)) are parametrized by finite sets which are independent of \(q\) (depend only on the “type” of \(G\)).

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Among the unipotent character sheaves are the functions

$$R_\chi = \frac{1}{|W|} \sum_{w \in W} \chi(w) R_{T_w}^G$$
Degrees and Eigenvalues

Degrees

- The elements of UnCh(G) and UnSh(G) are parametrized by finite sets which are independent of q (depend only on the “type” of G).
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$$R_\chi = \frac{1}{|W|} \sum_{w \in W} \chi(w) R^G_{T_w}$$

and $\text{Deg}R_\chi$ is the graded multiplicity of $\chi$ in the coinvariant algebra of $W$, evaluated at $q$ (the fake degree of $\chi$).

- The degrees of the irreducible constituents of $R^G_L(\lambda)$ are given by the “generic degrees” for the Hecke algebra of $W_G(L, \lambda)$.

- The Fourier matrix is independent of $q$. 

Unipotent characters for $G_2$

In red, the principal series $= \text{degrees prime to } \Phi_1 \ (i.e., \ \text{Deg}_\chi(q)_{q=1} \neq 0)$

<table>
<thead>
<tr>
<th>Character</th>
<th>Degree</th>
<th>Fake degree</th>
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<tr>
<td>$\phi_{1,0}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$C_1$</td>
</tr>
<tr>
<td>$\phi_{1,6}$</td>
<td>$q^6$</td>
<td>$q^6$</td>
<td>1</td>
<td>$C_1$</td>
</tr>
<tr>
<td>$\phi_{2,1}$</td>
<td>$\frac{1}{6} q\Phi_2^2 \Phi_3$</td>
<td>$q\Phi_8$</td>
<td>1</td>
<td>$S_3.(1,1)$</td>
</tr>
<tr>
<td>$\phi_{2,2}$</td>
<td>$\frac{1}{2} q\Phi_2^2 \Phi_6$</td>
<td>$q^2 \Phi_4$</td>
<td>1</td>
<td>$S_3.(g_2, 1)$</td>
</tr>
<tr>
<td>$\phi_{1,3}'$</td>
<td>$\frac{1}{3} q\Phi_3 \Phi_6$</td>
<td>$q^3$</td>
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<td>$S_3.(g_3, 1)$</td>
</tr>
<tr>
<td>$\phi_{1,3}''$</td>
<td>$\frac{1}{3} q\Phi_3 \Phi_6$</td>
<td>$q^3$</td>
<td>1</td>
<td>$S_3.(1, \rho)$</td>
</tr>
<tr>
<td>$G_2[1]$</td>
<td>$\frac{1}{6} q\Phi_2^2 \Phi_6$</td>
<td>0</td>
<td>1</td>
<td>$S_3.(1, \varepsilon)$</td>
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<tr>
<td>$G_2[-1]$</td>
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<td>$-1$</td>
<td>$S_3.(g_2, \varepsilon)$</td>
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<tr>
<td>$G_2[\zeta_3]$</td>
<td>$\frac{1}{3} q\Phi_2^2 \Phi_2$</td>
<td>0</td>
<td>$\zeta_3$</td>
<td>$S_3.(g_3, \zeta_3)$</td>
</tr>
<tr>
<td>$G_2[\zeta_3^2]$</td>
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<td>0</td>
<td>$\zeta_3^2$</td>
<td>$S_3.(g_3, \zeta_3^2)$</td>
</tr>
</tbody>
</table>
Unipotent characters for $G_4$

Red = the principal series
Blue = series $(L,\lambda)$
Purple = Cuspidal

<table>
<thead>
<tr>
<th>Character</th>
<th>Degree</th>
<th>FakeDegree</th>
<th>Eigenvalue</th>
<th>Family</th>
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</thead>
<tbody>
<tr>
<td>$\phi_{1,0}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$C_1$</td>
</tr>
<tr>
<td>$\phi_{2,1}$</td>
<td>$\frac{3-\sqrt{-3}}{6} q\Phi_3' \Phi_4 \Phi_6''$</td>
<td>$q\Phi_4$</td>
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<td>$X_3.01$</td>
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<tr>
<td>$\phi_{2,3}$</td>
<td>$\frac{3+\sqrt{-3}}{6} q\Phi_3' \Phi_4 \Phi_6'$</td>
<td>$q^3 \Phi_4$</td>
<td>1</td>
<td>$X_3.02$</td>
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<tr>
<td>$Z_3 : 2$</td>
<td>$\frac{\sqrt{-3}}{3} q\Phi_1 \Phi_2 \Phi_4$</td>
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<td>$\zeta_3^2$</td>
<td>$X_3.12$</td>
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<tr>
<td>$\phi_{3,2}$</td>
<td>$\frac{1}{2} q^4 \Phi_3^2 \Phi_6^2$</td>
<td>$q^2 \Phi_3 \Phi_6$</td>
<td>1</td>
<td>$C_1$</td>
</tr>
<tr>
<td>$\phi_{1,4}$</td>
<td>$\frac{-\sqrt{-3}}{6} q^4 \Phi_3'' \Phi_4 \Phi''_6$</td>
<td>$q^4$</td>
<td>1</td>
<td>$X_5.1$</td>
</tr>
<tr>
<td>$\phi_{1,8}$</td>
<td>$\frac{\sqrt{-3}}{6} q^4 \Phi_3' \Phi_4 \Phi_6'$</td>
<td>$q^8$</td>
<td>1</td>
<td>$X_5.2$</td>
</tr>
<tr>
<td>$\phi_{2,5}$</td>
<td>$\frac{1}{2} q^4 \Phi_2^2 \Phi_6$</td>
<td>$q^5 \Phi_4$</td>
<td>1</td>
<td>$X_5.3$</td>
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<tr>
<td>$Z_3 : 11$</td>
<td>$\frac{\sqrt{-3}}{3} q^4 \Phi_1 \Phi_2 \Phi_4$</td>
<td>0</td>
<td>$\zeta_3^2$</td>
<td>$X_5.4$</td>
</tr>
<tr>
<td>$G_4$</td>
<td>$\frac{1}{2} q^4 \Phi_1^2 \Phi_3$</td>
<td>0</td>
<td>$-1$</td>
<td>$X_5.5$</td>
</tr>
</tbody>
</table>
Eigenvalues of Frobenius

$\text{Eigenvectors of Frobenius}$

Each $H_i c(X_w, Q_\ell)$ is a $G \times \mathbb{F}_\ell$–module.

For any unipotent character $\rho \in \text{UnCh}(G)$, the eigenvalues of $\mathbb{F}$ on the $\rho$-isotypic part of $H_i c(X_w, Q_\ell)$ are $\lambda_\rho q^n/2$ where $\lambda_\rho$ is a root of unity independent of $i$ and of $w$.

Definition $\lambda_\rho$ is the eigenvalue of Frobenius attached to $\rho$. 

Michel Broué

Pseudo reductive groups over $\mathbb{F}_x$?
Eigenvalues of Frobenius

\[ R^G_{T_w}(g) = \sum_i (-1)^i \text{Trace}(g \mid H^i_c(X_w, \mathbb{Q}_\ell)) \]
Eigenvalues of Frobenius

- $R_T^G(g) = \sum_i (-1)^i \text{Trace}(g \mid H_c^i(X_w, \mathbb{Q}_\ell))$
- Each $H_c^i(X_w, \mathbb{Q}_\ell)$ is a $G \times <F>$–module.

\[ \lambda_{\rho}q^{n/2} \]

$n \in \mathbb{N}$

$\lambda_{\rho}$ is a root of unity independent of $i$ and of $w$.

Definition $\lambda_{\rho}$ is the eigenvalue of Frobenius attached to $\rho$. 

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Eigenvalues of Frobenius

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- Each $H_c^i(X_w, \mathbb{Q}_\ell)$ is a $G \times <F>$–module.

- For any unipotent character $\rho \in \text{UnCh}(G)$, the eigenvalues of $F$ on the $\rho$-isotypic part of $H_c^i(X_w, \mathbb{Q}_\ell)$ are $\lambda_\rho q^n/2$ where $\lambda_\rho$ is a root of unity independent of $i$ and of $w$. 

Definition $\lambda_\rho$ is the eigenvalue of Frobenius attached to $\rho$. 

Michel Broué  Pseudo reductive groups over $\mathbb{F}_x$?
Eigenvalues of Frobenius

- $R^G_{T_w}(g) = \sum_i (-1)^i \text{Trace}(g | H^i_c(X_w, \mathbb{Q}_\ell))$

- Each $H^i_c(X_w, \mathbb{Q}_\ell)$ is a $G \times <F>$–module.

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where
Eigenvalues of Frobenius

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- Each \( H^i_c(X_w, \mathbb{Q}_\ell) \) is a \( G \times <F> \)–module.
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  \[
  \lambda_\rho q^{n/2}
  \]

where
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Eigenvalues of Frobenius

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$$\lambda_\rho q^{n/2}$$

where

- $n \in \mathbb{N}$
- $\lambda_\rho$ is a root of unity independent of $i$ and of $w$. 

Michel Broué
Pseudo reductive groups over $\mathbb{F}_x$?
**Eigenvalues of Frobenius**

- \[ R^G_T(w(g)) = \sum_i (-1)^i \text{Trace}(g \mid H^i_c(X_w, \mathbb{Q}_\ell)) \]
- Each \( H^i_c(X_w, \mathbb{Q}_\ell) \) is a \( G \times <F> \)-module.
- For any unipotent character \( \rho \in \text{UnCh}(G) \), the eigenvalues of \( F \) on the \( \rho \)-isotypic part of \( H^i_c(X_w, \mathbb{Q}_\ell) \) are \( \lambda_\rho q^n/2 \)

where
- \( n \in \mathbb{N} \)
- \( \lambda_\rho \) is a root of unity independent of \( i \) and of \( w \).

**Definition**

\( \lambda_\rho \) is the **eigenvalue of Frobenius** attached to \( \rho \).
Reflection data and the Spetses game

By the isogeny theorem, $(G, F)$ is determined by its reflection datum $(X(T), Y(T), \Phi, \Phi^\vee, q)$. All the above data (unipotent characters, degrees, Fourier matrix, eigenvalues of Frobenius) can be obtained as a combinatorial game starting from $W \subset GL(V)$, where $V = X(T) \otimes \mathbb{C}$.

The Hecke algebra $H(W) = \langle s \mid (s - q)(s + 1) = 0 \rangle$.

Spetses island: there we started to play the same game, replacing the Weyl group $W$ by the complex reflection group of order 3.

Michel Broué

Pseudo reductive groups over $F_x$?
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By the isogeny theorem, \((G, F)\) is determined by its **reflection datum**

\[(X(T), Y(T), \Phi, \Phi^\vee, q)\]
Reflection data and the Spetses game

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- All the above data (unipotent characters, degrees, Fourier matrix, eigenvalues of Frobenius) can be obtained as a combinatorial game starting from

\[ W \subset \text{GL}(V), \quad V = X(T) \otimes \mathbb{C} \]
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Reflection data and the Spetses game

- By the isogeny theorem, \((G, F)\) is determined by its reflection datum
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Hecke algebras of complex reflection groups

Every complex reflection group $W$ has a nice presentation \("a la Coxeter\):

$G_2$:

\[ S T S T S T = T S T S T S \]

$G_4$:

\[ S^3 T S T S T S T S T = T S T S T S T S T S T \]

and a field of realisation $\mathbb{Q} W$:

$\mathbb{Q} G_2 = \mathbb{Q}$ and $\mathbb{Q} G_4 = \mathbb{Q}(\zeta_3)$.

The associated generic Hecke algebra is defined from such a presentation:

$H(G_2) := \langle S, T; S T S T S T = T S T S T S T S T S T S T \rangle$

$H(G_4) := \langle S, T; S^3 T S T S T S T S T S T S T = T S T S T S T S T S T S T S T \rangle$
Every complex reflection group $W$ has a nice presentation “à la Coxeter”:

$G_2 : \begin{array}{c}
\circ \rightarrow \circ \\
\circ \rightarrow \circ
\end{array}$, \quad $G_4 : \begin{array}{c}
3 \rightarrow 3
\end{array}$
Every complex reflection group $W$ has a nice presentation “à la Coxeter”:

$$G_2 : \begin{array}{c}
\circ \quad \circ \\
\hline \\
\circ \quad \circ
\end{array} \quad , \quad G_4 : \begin{array}{c}
\bullet \quad \bullet \\
\hline \\
\bullet \quad \bullet
\end{array}$$

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$$\mathbb{Q}_{G_2} = \mathbb{Q} \quad \text{and} \quad \mathbb{Q}_{G_4} = \mathbb{Q}(\zeta_3).$$
Hecke algebras of complex reflection groups

- Every complex reflection group $W$ has a nice presentation “à la Coxeter”:
  
  $G_2 : \begin{array}{c} \circ \equiv \circ \end{array}$, \hspace{1cm} $G_4 : \begin{array}{c} 3 \equiv 3 \end{array}$

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- The associated generic Hecke algebra is defined from such a presentation:

  $\mathcal{H}(G_2) := \langle S, T \rangle \begin{cases} 
  STSTST = TSTSTS \\
  (S - u_0)(S - u_1) = 0 \\
  (T - v_0)(T - v_1) = 0
  \end{cases}$

  $\mathcal{H}(G_4) := \langle S, T \rangle \begin{cases} 
  STS = TST \\
  (S - u_0)(S - u_1)(S - u_2) = 0
  \end{cases}$
The generic Hecke algebra $\mathcal{H}(W)$ is free of rank $|W|$ over the corresponding Laurent polynomial ring $\mathbb{Z}[(u_i^{\pm 1}), (v_j^{\pm 1}), \ldots]$. It becomes a split semisimple algebra over a field obtained by extracting suitable roots of the indeterminates. With suitable choice we get a bijection $\text{Irr}(W) \leftrightarrow \text{Irr}(\mathcal{H}(G))$, $\chi \mapsto \chi_{\mathcal{H}(W)}$. The generic Hecke algebra $\mathcal{H}(W)$ is endowed with a canonical symmetrizing form $t: \mathcal{H}(W) \to \mathbb{Z}[(u_i^{\pm 1}), (v_j^{\pm 1}), \ldots]$ which specialises to the canonical form of the group algebra $Q_W$, and satisfies some other condition. The Schur elements of the irreducible characters of $W$ are the elements $s_\chi \in \mathbb{Z}_W[(x^{\pm 1}), (y^{\pm 1}), \ldots]$ defined by $t = \sum_{\chi \in \text{Irr}(W)} 1 s_\chi \chi_{\mathcal{H}(W)}$. Michel Broué

Pseudo reductive groups over $\mathbb{F}_x$ ?
The generic Hecke algebra $\mathcal{H}(W)$ is free of rank $|W|$ over the corresponding Laurent polynomial ring $\mathbb{Z}[(u_i^{\pm 1}), (v_j^{\pm 1}), \ldots]$. It becomes a split semisimple algebra over a field obtained by extracting suitable roots of the indeterminates.
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2. It becomes a split semisimple algebra over a field obtained by extracting suitable roots of the indeterminates.  

3. With suitable choice we get a bijection

$$\text{Irr}(W) \sim \text{Irr}(\mathcal{H}(G)), \quad \chi \mapsto \chi_{\mathcal{H}(W)}.$$
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   ▶ which specialises to the canonical form of the group algebra $\mathbb{Q}_W W$, 
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$$t = \sum_{\chi \in \text{Irr}(W)} \frac{1}{s_\chi} \chi_{\mathcal{H}}(W).$$
The spetsial Hecke algebra is the specialisation of $H(W)$ defined as follows: if $G = s d m t \cdots$, then the relation $(S - u_0)(S - u_1) \cdots (S - u_{d-1}) = 0$ specializes to $(S - q)(S_d - 1 + \cdots + S + 1) = 0$. Thus the spetsial algebra becomes the group algebra of $W$ at $q = 1$. In our cases:

▶ For $G_2$: $(S - q)(S + 1) = 0$
▶ For $G_4$: $(S - q)(S^2 + S + 1) = 0$
The **spetsial Hecke algebra** is the specialisation of $\mathcal{H}(W)$ defined as follows:

If $G = \bigcirc s \bigcirc t \bigcirc \cdots$, then the relation

$$(S - q_0)(S - q_1) \cdots (S - q_{d-1}) = 0$$

specializes to

$$(S - q)(S_d + 1 + \cdots + S + 1) = 0$$

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The spetsial Hecke algebra is the specialisation of $\mathcal{H}(W)$ defined as follows:

$$\text{If } G = \begin{tikzpicture}[baseline=(current bounding box.center)]
  \node (s) at (0,0) {$s$};
  \node (d) at (1,0) {$d$};
  \node (e) at (2,0) {$e$};
  \node (t) at (3,0) {$t$};
  \draw (s) -- (d);
  \draw (d) -- (e);
  \draw (e) -- (t);
\end{tikzpicture},$$

then the relation

$$(S - u_0)(S - u_1) \cdots (S - u_{d-1}) = 0$$

specializes to

$$(S - q)(S + 1)(S + 2) \cdots = 0$$

Thus the spetsial algebra becomes the group algebra of $W$ at $q = 1$.

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▶ For $G_2$: $(S - q)(S + 1) = 0$
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The **spetsial Hecke algebra** is the specialisation of $\mathcal{H}(W)$ defined as follows:

If $G = \circlearrowleft \, s^m e_t \, \cdots$, then the relation

$$(S - u_0)(S - u_1) \cdots (S - u_{d-1}) = 0$$

specializes to

$$(S - q)(S - (d+1)) \cdots = 0$$

Thus the spetsial algebra becomes the group algebra of $W$ at $q = 1$.

In our cases:

- For $G_2$: $(S - q)(S + 1) = 0$
- For $G_4$: $(S - q)(S^2 + S + 1) = 0$
The **spetsial Hecke algebra** is the specialisation of $\mathcal{H}(W)$ defined as follows:

If $G = \dfrac{d}{s} m \circ \dfrac{e}{t} \cdots$, then the relation

$$(S - u_0)(S - u_1) \cdots (S - u_{d-1}) = 0$$

specializes to

$$(S - q)(S^{d-1} + \cdots + S + 1) = 0$$
The spetsial Hecke algebra is the specialisation of $\mathcal{H}(W)$ defined as follows:

If \[ G = \frac{d}{s} \overbrace{e \cdots e}^{m} \frac{t}{s} \]

then the relation

\[(S - u_0)(S - u_1) \cdots (S - u_{d-1}) = 0\]

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Thus the spetsial algebra becomes the group algebra of $W$ at $q = 1$. 
The spetsial Hecke algebra is the specialisation of $\mathcal{H}(W)$ defined as follows:

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then the relation

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specializes to

$$(S - q)(S^{d-1} + \cdots + S + 1) = 0$$

Thus the spetsial algebra becomes the group algebra of $W$ at $q = 1$.

In our cases:

$\begin{array}{l}
\hline
\begin{array}{l}
\text{For } G_2: (S - q)(S + 1) = 0 \\
\text{For } G_4: (S - q)(S^2 + S + 1) = 0 \\
\end{array}
\hline
\end{array}$
The spetsial Hecke algebra is the specialisation of $\mathcal{H}(W)$ defined as follows:

$$G = \underbrace{d \circ m \circ e \cdots}_{s \circ t},$$

then the relation

$$(S - u_0)(S - u_1) \cdots (S - u_{d-1}) = 0$$

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In our cases:

- For $G_2$: $(S - q)(S + 1) = 0$
The spetsial Hecke algebra is the specialisation of $\mathcal{H}(W)$ defined as follows:

If \( G = \genfrac(){0pt}{0}{d}{s} \cdot \genfrac(){0pt}{0}{m}{e} \cdot \genfrac(){0pt}{0}{t}{\cdots} \),

then the relation

\[(S - u_0)(S - u_1)\cdots(S - u_{d-1}) = 0\]

specializes to

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Thus, the spetsial algebra becomes the group algebra of $W$ at $q = 1$.

In our cases:

- For $G_2$: \((S - q)(S + 1) = 0\)
- For $G_4$: \((S - q)(S^2 + S + 1) = 0\)
Spetsial groups

The group $W$ is called spetsial if

$$\text{Deg} \chi(q) := q^{N(q-1)}rW(q)/S\chi \in QW[q]$$

Here is the list of the spetsial groups:

- Among the imprimitive groups: $G(e, 1, r), G(e, e, r)$.
- Among the exceptional groups:

<table>
<thead>
<tr>
<th>Group $G_n$</th>
<th>Rank</th>
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<td>2–10</td>
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<td>36–40</td>
<td>5, 6</td>
</tr>
<tr>
<td>41–50</td>
<td>7, 8</td>
</tr>
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</table>

Groups $H_3$, $F_4$, $H_4$, $E_6$, $E_7$, $E_8$.

Michel Broué

Pseudo reductive groups over $F_x$?
The group $W$ is called \textit{spetsial} if

$$\text{Deg} \chi(q) := q^N(q-1)r_{\mathcal{W}(q)} / S_{\chi} \in \mathbb{Q}_{\mathcal{W}[q]}$$
The group $W$ is called \textit{spetsial} if

$$\text{Deg}_\chi(q) := q^N(q - 1)^r P_W(q)/S_\chi$$
The group $W$ is called spetsial if

$$\text{Deg}_\chi(q) := q^N(q - 1)^r P_W(q)/S_\chi \in \mathbb{Q}_W[q]$$
The group $W$ is called spetsial if

$$\text{Deg}_\chi(q) := q^N(q - 1)^r P_W(q)/S_\chi \in \mathbb{Q}_W[q]$$

Here is the list of the spetsial groups:

- Among the imprimitive groups:
  - $G(e_1, 1, r)$,
  - $G(e_1, e_1, r)$.

- Among the exceptional groups:
  - Group
    | $G_n$  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
    | Rank  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  | 2  |
  - Group
    | $G_n$  | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 |
    | Rank  | 2  | 2  | 2  | 2  | 2  | 3  | 3  | 3  | 3  | 3  | 3  |
  - Remark
    | $H_3$  |
  - Group
    | $G_n$  | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 |
    | Rank  | 4  | 4  | 4  | 4  | 4  | 5  | 6  | 6  | 7  | 8  |
  - Remark
    | $F_4$  |
    | $H_4$  |
    | $E_6$  |
    | $E_7$  |
    | $E_8$  |
The group $W$ is called \textit{spetsial} if
\[
\text{Deg}_\chi(q) := q^N(q - 1)^r P_W(q) / S_\chi \in \mathbb{Q}_W[q]
\]

Here is the list of the spetsial groups :

- Among the imprimitive groups : $G(e, 1, r), G(e, e, r)$.
The group $W$ is called spetsial if

$$\text{Deg}_\chi(q) := q^N(q - 1)^r P_W(q)/S_\chi \in \mathbb{Q}_W[q]$$

Here is the list of the spetsial groups:

- Among the imprimitive groups: $G(e, 1, r), G(e, e, r)$.
- Among the exceptional groups:
Spetsial groups

- The group $W$ is called **spetsial** if

\[
\text{Deg}_\chi(q) := q^N(q-1)^r \frac{P_W(q)}{S_\chi} \in \mathbb{Q}_W[q]
\]

- Here is the list of the spetsial groups :

  - Among the imprimitive groups : $G(e, 1, r), G(e, e, r)$.
  - Among the exceptional groups :

<table>
<thead>
<tr>
<th>Group $G_n$</th>
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<th>$5$</th>
<th>$6$</th>
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<table>
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<tbody>
<tr>
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<tr>
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<table>
<thead>
<tr>
<th>Group $G_n$</th>
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<th>$31$</th>
<th>$32$</th>
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<th>$36$</th>
<th>$37$</th>
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<tbody>
<tr>
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<td>$4$</td>
<td>$4$</td>
<td>$4$</td>
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<td>$6$</td>
<td>$6$</td>
<td>$7$</td>
<td>$8$</td>
</tr>
<tr>
<td>Remark</td>
<td>$F_4$</td>
<td>$H_4$</td>
<td>$E_6$</td>
<td>$E_7$</td>
<td>$E_8$</td>
<td></td>
<td></td>
<td></td>
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</table>
Back to the context of a finite reductive group with Weyl group $W$. 

A unipotent character $\rho$ is in $R_{G_{T_w}(1)}$ iff $\text{Deg} \, \rho(\zeta) \neq 0$.

Conjecturally, there is a good choice of a Deligne–Lusztig variety attached to $w$ such that $H_{W_w} = \text{End}_{\mathbb{Q}_\ell} G(H^\bullet_c(\mathcal{X}_w, \mathbb{Q}_\ell))$. The conjecture also predicts the eigenvalues of Frobenius attached to constituents of $R_{G_{T_w}(1)}$.
Back to the context of a finite reductive group with Weyl group $W$. Let $w \in W$ be a $\zeta$–regular element,
Back to the context of a finite reductive group with Weyl group $W$. Let $w \in W$ be a $\zeta$–regular element, i.e., $V(w, \zeta) := \ker(w - \zeta \text{Id}_V)$ is maximal.
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- The centralizer $W(w)$ of $w$ is a complex reflection group on $V(w, \zeta)$. 

A unipotent character $\rho$ is in \( R_{G,w}(1) \) iff \( \text{Deg} \rho(\zeta) \neq 0 \).

Conjecturally, there is a good choice of a Deligne–Lusztig variety attached to $w$ such that $H_{W(w)} = \text{End}_{Q_\ell} G(H^\bullet c(X_w, Q_\ell))$.

The conjecture also predicts the eigenvalues of Frobenius attached to constituents of $R_{G,w}(1)$.

This conjecture was prompted by the abelian defect groups conjecture.

Michel Broué
\(\zeta\)-Harish-Chandra theory (simplified version)

Back to the context of a finite reductive group with Weyl group \(W\). Let \(w \in W\) be a \(\zeta\)-regular element, i.e., \(V(w, \zeta) := \ker(w - \zeta \text{Id}_V)\) is maximal.

- The centralizer \(W(w)\) of \(w\) is a complex reflection group on \(V(w, \zeta)\).
- \(N_G(T_w, \text{Id})/T_w \simeq W(w)\).
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- The centralizer $W(w)$ of $w$ is a complex reflection group on $V(w, \zeta)$.
- $N_G(T_w, \text{Id})/T_w \simeq W(w)$.
- There is a $\zeta$–cyclotomic Hecke algebra $\mathcal{H}_W(w)$ for $W(w)$ which controls $R^G_{T_w}(1)$.
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- $N_G(T_w, \text{Id})/T_w \simeq W(w)$.
- There is a $\zeta$–cyclo-Hecke algebra $\mathcal{H}_W(w)$ for $W(w)$ which controls $R^G_{T_w}(1)$.
  - A unipotent character $\rho$ is in $R^G_{T_w}(1)$ iff $\text{Deg}_\rho(\zeta) \neq 0$. 

Conjecturally, there is a good choice of a Deligne–Lusztig variety attached to $w$ such that $\mathcal{H}_W(w) = \text{End}_{\mathbb{Q}_\ell} G(H^\text{c} \left( X_w, \mathbb{Q}_\ell \right))$.

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Michel Broué
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\[\zeta\text{-Harish-Chandra theory (simplified version)}\]
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- $N_G(T_w, \mathrm{Id})/T_w \cong W(w)$.
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This conjecture was prompted by the abelian defect groups conjecture.
Cyclotomic Hecke algebras

For $G_2$ regular $\zeta$ $W(\zeta)$ $H_W(\zeta)$

$G_2(s - q)(s + 1) - 1$ $G_2(s - q)(s - 1)\zeta$ $C_6(s - q^2)(s - 1)(s - q^3)(s^3 + q^3)$

$G_4(s - q)(s + 1) - 1$ $G_4(s - q)(s - 1)\zeta$ $C_6(s - q^2)(s - 1)(s + 1)(s + \zeta^3 q)(s - \zeta^3)(s + q)$ $\zeta^4 C_4(s - q^3)(s - 1)(s - q)(s + 1)$ $\zeta^6 C_6(s - q^2)(s - q)(s - 1)(s - \zeta^2)(s^3 + \zeta^2 q^3)$

Michel Broué

Pseudo reductive groups over $\mathbb{F}_x$ ?
For $G_2$
Cyclotomic Hecke algebras

For $G_2$

<table>
<thead>
<tr>
<th>Regular $\zeta$</th>
<th>$W(\zeta)$</th>
<th>$H_W(\zeta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$G_2$</td>
<td>$(s - q)(s + 1)$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$G_2$</td>
<td>$(s - q)(s - 1)$</td>
</tr>
<tr>
<td>$\zeta_3$</td>
<td>$C_6$</td>
<td>$(s - q^2)(s - q)(s - 1)(s^3 + q^3)$</td>
</tr>
<tr>
<td>$\zeta_6$</td>
<td>$C_6$</td>
<td>$(s - q^2)(s + q)(s - 1)(s^3 - q^3)$</td>
</tr>
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</table>
Cyclotomic Hecke algebras

For $G_2$

<table>
<thead>
<tr>
<th>Regular $\zeta$</th>
<th>$W(\zeta)$</th>
<th>$\mathcal{H}_W(\zeta)$</th>
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<tr>
<td>1</td>
<td>$G_2$</td>
<td>$(s - q)(s + 1)$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$G_2$</td>
<td>$(s - q)(s - 1)$</td>
</tr>
<tr>
<td>$\zeta_3$</td>
<td>$C_6$</td>
<td>$(s - q^2)(s - q)(s - 1)(s^3 + q^3)$</td>
</tr>
<tr>
<td>$\zeta_6$</td>
<td>$C_6$</td>
<td>$(s - q^2)(s + q)(s - 1)(s^3 - q^3)$</td>
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</table>

For $G_4$

Michel Broué  Pseudo reductive groups over $\mathbb{F}_x$ ?
Cyclotomic Hecke algebras

For $G_2$

<table>
<thead>
<tr>
<th>Regular $\zeta$</th>
<th>$W(\zeta)$</th>
<th>$\mathcal{H}_W(\zeta)$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>$G_2$</td>
<td>$(s - q)(s + 1)$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$G_2$</td>
<td>$(s - q)(s - 1)$</td>
</tr>
<tr>
<td>$\zeta_3$</td>
<td>$C_6$</td>
<td>$(s - q^2)(s - q)(s - 1)(s^3 + q^3)$</td>
</tr>
<tr>
<td>$\zeta_6$</td>
<td>$C_6$</td>
<td>$(s - q^2)(s + q)(s - 1)(s^3 - q^3)$</td>
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</table>

For $G_4$

<table>
<thead>
<tr>
<th>Regular $\zeta$</th>
<th>$W(\zeta)$</th>
<th>$\mathcal{H}_W(\zeta)$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>$G_4$</td>
<td>$(s - q)(s + 1)$</td>
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<td>$G_4$</td>
<td>$(s - q)(s - 1)$</td>
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<td>$\zeta_3$</td>
<td>$C_6$</td>
<td>$(s - q^2)(s - 1)(s + 1)(s + \zeta_3 q)(s - \zeta_3)(s + q)$</td>
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<td>$(s - q^3)(s - 1)(s - q)(s + 1)$</td>
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<td>$\zeta_6$</td>
<td>$C_6$</td>
<td>$(s - q^2)(s - q)(s - 1)(s - \zeta_3^2 q)(s - \zeta_3^2)(s + 1)$</td>
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## Unipotent characters for $G_4$

In red = the $\Phi'_6$–series.

• = the $\Phi_4$–series.

<table>
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<tr>
<th>Character</th>
<th>Degree</th>
<th>FakeDegree</th>
<th>Eigenvalue</th>
<th>Family</th>
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<td>$1$</td>
<td>$1$</td>
<td>$C_1$</td>
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<tr>
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<td>$q\Phi_4$</td>
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<td>$X_3.01$</td>
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<td>$q^3\Phi_4$</td>
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<td>$\zeta_3^2$</td>
<td>$X_3.12$</td>
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<td>$q^2\Phi_3 \Phi_6$</td>
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<td>$C_1$</td>
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<tr>
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<td>$q^4$</td>
<td>$1$</td>
<td>$X_5.1$</td>
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<tr>
<td>$\phi_{1,8}$</td>
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<td>$q^8$</td>
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<td>$X_5.2$</td>
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<tr>
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### Fourier matrices: $G_4$

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<td>3+√−3 6</td>
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<td>√−3 6</td>
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<td>−√−3 6</td>
<td>1 2</td>
<td>−√−3 3</td>
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<td>.</td>
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<td>1 2</td>
<td>1 2</td>
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<td>−√−3 3</td>
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<td>.</td>
<td>1 2</td>
<td>1 2</td>
<td>−1 2</td>
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</table>
Fourier matrices: Properties

Let $S$ be the Fourier matrix.

1. $S$ is symmetric.
2. $S - 1 = S$.

Let $\Omega$ be the diagonal matrix of Frobenius eigenvalues.

4. $S^2 \Omega = \Omega S^2$.
5. $(\Omega S^2)^3 = 1$.

Thus $S$ and $\Omega$ define a representation of $\text{SL}_2(\mathbb{Z})$.

6. If $i_0$ is a row of $S$ corresponding to a special character of a Rouquier block, then for all $i, j, k$, the sums $\sum_l S_{il} S_{jl} S_{kl} S_{l i_0}$ are integral.

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Michel Broué | Pseudo reductive groups over $\mathbb{F}_x$
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(Lusztig families, blocks of Fourier matrix)

(and for all \(d\))

\[ \bigcup_{(L, \lambda)} d - \text{cuspidal} / G \text{Irr}_R(G)(\lambda) = \text{Irr}_H(G)(L, \lambda) \]

So what are the sets \(F \cap \text{Irr}_H(G)(L, \lambda)\)?

Lusztig has described the intersections with the principal series \(\text{Irr}_R(G)^{(1)}\) using the Kazhdan-Lusztig basis, thus defining families of characters of \(W\).

In general, the partition \(\text{Irr}_H(G)(L, \lambda) = \bigcup_{F \in \text{Fam}(G)} F \cap \text{Irr}_H(G)(L, \lambda)\) is the partition into Rouquier blocks of the cyclotomic Hecke algebra \(H_G(L, \lambda)\).
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- In general, the partition

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Michel Broué  
Pseudo reductive groups over $\mathbb{F}_x$ ?
Rouquier blocks

The Rouquier blocks of a cyclotomic Hecke algebra of a group $W$ are the ordinary blocks of that algebra over the ring $\mathbb{Z}_W[q, q^{-1}, (1-q^n)^n]_{n \geq 1}$.

They are, roughly speaking, the bad primes blocks of the Hecke algebra, where the bad primes are those prime ideals of $\mathbb{Z}_W$ which divide the Schur elements (in other words, the primes in the denominators of the generic degrees).

All Rouquier blocks of all cyclotomic Hecke algebras of all complex reflection groups have been determined (Malle–Rouquier, Broué–Kim, Kim, Chlouveraki).
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Spetses

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▶ Fourier matrices satisfying the above properties... and many other.

Lusztig knew already a solution for Coxeter groups which are not $W$eyl groups (except the Fourier matrix for $H_4$ which was determined by Malle in 1994).

Malle gave a solution for imprimitive spetsial complex reflection groups in 1995, and also proposed (unpublished) data for many primitive spetsial groups.

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