A generic Atlas for Spetses ?

Michel Broué

Université Paris-Diderot Paris VII

FINITE SIMPLE GROUPS: THIRTY YEARS OF THE ATLAS AND BEYOND

PRINCETON, NOVEMBER 2015

In honor of John Conway

An example of a known Unknown : Spets

An example of a known Unknown : Spets

Long lasting joint work between Gunter Malle, Jean Michel, and myself,

An example of a known Unknown : Spets

Long lasting joint work between Gunter Malle, Jean Michel, and myself,

initiated in the Greek island named SPETSES in 1993

An example of a known Unknown : Spets

Long lasting joint work between Gunter Malle, Jean Michel, and myself,

initiated in the Greek island named SPETSES in 1993

(there, we computed unipotent degrees, Frobenius eigenvalues, families, Fourier matrix, for the "generic fake finite reductive group"

An example of a known Unknown : Spets

Long lasting joint work between Gunter Malle, Jean Michel, and myself,

initiated in the Greek island named SPETSES in 1993

(there, we computed unipotent degrees, Frobenius eigenvalues, families, Fourier matrix, for the "generic fake finite reductive group" ("Spets?")

Long lasting joint work between Gunter Malle, Jean Michel, and myself,

initiated in the Greek island named SPETSES in 1993

(there, we computed unipotent degrees, Frobenius eigenvalues, families, Fourier matrix, for the "generic fake finite reductive group" ("Spets?") whose Weyl group is the cyclic group $\boldsymbol{\mu}_3$ of order 3...),

An example of a known Unknown : Spets

Long lasting joint work between Gunter Malle, Jean Michel, and myself,

initiated in the Greek island named SPETSES in 1993

(there, we computed unipotent degrees, Frobenius eigenvalues, families, Fourier matrix, for the "generic fake finite reductive group" ("Spets?") whose Weyl group is the cyclic group $\boldsymbol{\mu}_3$ of order 3...),

going on with the collaboration of Olivier Dudas, and more and more of Cédric Bonnafé.

I. Generic point of view on finite reductive groups

I. Generic point of view on finite reductive groups

Let $\mathbf{G}$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, with

Let $\mathbf{G}$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, with Weyl group W,

I. Generic point of view on finite reductive groups

Let **G** be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, with Weyl group W, group of co-characters Y,

I. Generic point of view on finite reductive groups

Let $\mathbf{G}$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, with Weyl group W, group of co-characters Y, endowed with a rational structure over $\mathbb{F}_q$ *via* a Frobenius endomorphism F.

I. Generic point of view on finite reductive groups

Let $\mathbf{G}$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, with Weyl group W, group of co-characters Y, endowed with a rational structure over $\mathbb{F}_q$ *via* a Frobenius endomorphism F.

The group $G := \mathbf{G}^F$ is called a finite reductive group.

I. Generic point of view on finite reductive groups

Let $\mathbf{G}$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, with Weyl group W, group of co-characters Y, endowed with a rational structure over $\mathbb{F}_q$ *via* a Frobenius endomorphism F.

The group $G := \mathbf{G}^F$ is called a finite reductive group.

To simplify the lecture

I. Generic point of view on finite reductive groups

Let $\mathbf{G}$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, with Weyl group W, group of co-characters Y, endowed with a rational structure over $\mathbb{F}_q$ *via* a Frobenius endomorphism F.

The group $G := \mathbf{G}^F$ is called a finite reductive group.

To simplify the lecture (but (!) this is bad)

Let **G** be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, with Weyl group W, group of co-characters Y, endowed with a rational structure over $\mathbb{F}_q$ *via* a Frobenius endomorphism F.

The group $G := \mathbf{G}^F$ is called a finite reductive group.

To simplify the lecture (but (!) this is bad) we assume that G is split, *i.e.*, F acts on Y by multiplication by q.

I. Generic point of view on finite reductive groups

Let **G** be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, with Weyl group W, group of co-characters Y, endowed with a rational structure over $\mathbb{F}_q$ *via* a Frobenius endomorphism F.

The group $G := \mathbf{G}^F$ is called a finite reductive group.

To simplify the lecture (but (!) this is bad) we assume that G is split, *i.e.*, F acts on Y by multiplication by q.

The type of G is $\mathbb{G} := (V, W)$ where $V := \mathbb{C} \otimes_{\mathbb{Z}} Y$.

I. Generic point of view on finite reductive groups

Let **G** be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, with Weyl group W, group of co-characters Y, endowed with a rational structure over $\mathbb{F}_q$ *via* a Frobenius endomorphism F.

The group $G := \mathbf{G}^F$ is called a finite reductive group.

To simplify the lecture (but (!) this is bad) we assume that G is split, *i.e.*, F acts on Y by multiplication by q.

The type of G is $\mathbb{G} := (V, W)$ where $V := \mathbb{C} \otimes_{\mathbb{Z}} Y$.

Lots of numerical data associated with G come from evaluation at $x = q$ of polynomials in x which depends only on the type $\mathbb{G}$.

Let $\mathbf{G}$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, with Weyl group W, group of co-characters Y, endowed with a rational structure over $\mathbb{F}_q$ *via* a Frobenius endomorphism F.

The group $G := \mathbf{G}^F$ is called a finite reductive group.

To simplify the lecture (but (!) this is bad) we assume that G is split, *i.e.*, F acts on Y by multiplication by q.

The type of G is $\mathbb{G} := (V, W)$ where $V := \mathbb{C} \otimes_{\mathbb{Z}} Y$.

Lots of numerical data associated with G come from evaluation at $x = q$ of polynomials in x which depends only on the type $\mathbb{G}$.

Let us give some examples.

1. The polynomial order

1. The polynomial order

The element of $\mathbb{Z}[x]$ defined by

1. The polynomial order

The element of $\mathbb{Z}[x]$ defined by

$$|\mathbb{G}|(x) = x^N \frac{1}{\frac{1}{|W|}\sum_{w\in W} \frac{1}{\det_V(1-wx)}}$$

1. The polynomial order

The element of $\mathbb{Z}[x]$ defined by

$$|\mathbb{G}|(x) = x^{N} \frac{1}{\frac{1}{|W|}\sum_{w\in W} \frac{1}{\det_V(1-wx)}}$$

is the polynomial order of $\mathbb{G}$,

The element of $\mathbb{Z}[x]$ defined by

$$|\mathbb{G}|(x) = x^N \frac{1}{\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det_V(1-wx)}}$$

is the polynomial order of $\mathbb{G}$, that is, $|\mathbb{G}|(q) = |G|$.

1. The polynomial order

The element of $\mathbb{Z}[x]$ defined by

$$|\mathbb{G}|(x) = x^{N} \frac{1}{\frac{1}{|W|} \sum_{w \in W} \frac{1}{\det_V(1 - wx)}}$$

is the polynomial order of $\mathbb{G}$, that is, $|\mathbb{G}|(q) = |G|$.

- Then

$$|\mathbb{G}|(x) = x^{N} \prod_{\zeta \bmod \mathrm{Gal}} \Phi_\zeta(x)^{a(\zeta)} ,$$

The element of $\mathbb{Z}[x]$ defined by

$$|\mathbb{G}|(x) = x^{N} \frac{1}{\frac{1}{|W|}\sum_{w\in W}\frac{1}{\det_V(1-wx)}}$$

is the polynomial order of $\mathbb{G}$, that is, $|\mathbb{G}|(q) = |G|$.

- Then
$$|\mathbb{G}|(x) = x^{N} \prod_{\zeta \bmod \mathrm{Gal}} \Phi_\zeta(x)^{a(\zeta)} ,$$
(for **L** an F-stable Levi subgroup of **G**, its polynomial order $|\mathbb{L}|(x)$ divides $|\mathbb{G}|(x)$).

1. The polynomial order

The element of $\mathbb{Z}[x]$ defined by

$$|\mathbb{G}|(x) = x^N \frac{1}{\frac{1}{|W|}\sum_{w\in W} \frac{1}{\det_V(1-wx)}}$$

is the polynomial order of $\mathbb{G}$, that is, $|\mathbb{G}|(q) = |G|$.

- Then
$$|\mathbb{G}|(x) = x^N \prod_{\zeta \bmod \mathrm{Gal}} \Phi_\zeta(x)^{a(\zeta)}\,,$$
(for $\mathbf{L}$ an F-stable Levi subgroup of $\mathbf{G}$, its polynomial order $|\mathbb{L}|(x)$ divides $|\mathbb{G}|(x)$).
- I insist : the polynomial order depends only on the type (V, W) (not on a root datum).

1. The polynomial order

The element of $\mathbb{Z}[x]$ defined by

$$|\mathbb{G}|(x) = x^{N} \frac{1}{\frac{1}{|W|}\sum_{w\in W} \frac{1}{\det_V(1-wx)}}$$

is the polynomial order of $\mathbb{G}$, that is, $|\mathbb{G}|(q) = |G|$.

- Then
$$|\mathbb{G}|(x) = x^{N} \prod_{\zeta \bmod \mathrm{Gal}} \Phi_\zeta(x)^{a(\zeta)} ,$$
(for **L** an F-stable Levi subgroup of **G**, its polynomial order $|\mathbb{L}|(x)$ divides $|\mathbb{G}|(x)$).
- I insist : the polynomial order depends only on the type (V, W) (not on a root datum). Thus
$$|\mathrm{SO}_{2n+1}(q)| = |\,\mathrm{Sp}_{2n}(q)|\,.$$

2. Unipotent characters, generic degrees (Lusztig)

2. Unipotent characters, generic degrees (Lusztig)

1. The set $\mathrm{Un}(G)$ of unipotent characters of G is parametrized by the set of unipotent generic characters $\mathrm{Un}(\mathbb{G})$ (depending only on the type). Let us denote that parametrization by

$$\mathrm{Un}(\mathbb{G}) \to \mathrm{Un}(G)\ ,\ \rho \mapsto \rho_q\,.$$

2. Unipotent characters, generic degrees (Lusztig)

1. The set $\mathrm{Un}(G)$ of unipotent characters of G is parametrized by the set of unipotent generic characters $\mathrm{Un}(\mathbb{G})$ (depending only on the type). Let us denote that parametrization by

$$\mathrm{Un}(\mathbb{G}) \to \mathrm{Un}(G)\ ,\ \rho \mapsto \rho_q\,.$$

2. Generic degree : for all $\rho \in \mathrm{Un}(\mathbb{G})$, there exists $\mathrm{Deg}_\rho(x) \in \mathbb{Q}[x]$ such that

$$\mathrm{Deg}_\rho(x)\,|_{x=q} = \mathrm{Deg}(\rho_q)\,.$$

2. Unipotent characters, generic degrees (Lusztig)

1. The set $\mathrm{Un}(G)$ of unipotent characters of G is parametrized by the set of unipotent generic characters $\mathrm{Un}(\mathbb{G})$ (depending only on the type). Let us denote that parametrization by

$$\mathrm{Un}(\mathbb{G}) \to \mathrm{Un}(G)\ ,\ \rho \mapsto \rho_q\,.$$

2. Generic degree : for all $\rho \in \mathrm{Un}(\mathbb{G})$, there exists $\mathrm{Deg}_\rho(x) \in \mathbb{Q}[x]$ such that

$$\mathrm{Deg}_\rho(x)\mid_{x=q} = \mathrm{Deg}(\rho_q)\,.$$

Of course, the (generic) degrees divide the (polynomial) order of $\mathbb{G}$.

2. Unipotent characters, generic degrees (Lusztig)

1. The set $\mathrm{Un}(G)$ of unipotent characters of G is parametrized by the set of unipotent generic characters $\mathrm{Un}(\mathbb{G})$ (depending only on the type). Let us denote that parametrization by

$$\mathrm{Un}(\mathbb{G}) \to \mathrm{Un}(G)\ ,\ \rho \mapsto \rho_q\,.$$

2. Generic degree : for all $\rho \in \mathrm{Un}(\mathbb{G})$, there exists $\mathrm{Deg}_\rho(x) \in \mathbb{Q}[x]$ such that

$$\mathrm{Deg}_\rho(x)\,|_{x=q} = \mathrm{Deg}(\rho_q)\,.$$

   Of course, the (generic) degrees divide the (polynomial) order of $\mathbb{G}$.

3. Every $\rho \in \mathrm{Un}(\mathbb{G})$ comes equipped with a Frobenius eigenvalue Fr_ρ, a root of unity

2. Unipotent characters, generic degrees (Lusztig)

1. The set $\mathrm{Un}(G)$ of unipotent characters of G is parametrized by the set of unipotent generic characters $\mathrm{Un}(\mathbb{G})$ (depending only on the type). Let us denote that parametrization by

$$\mathrm{Un}(\mathbb{G}) \to \mathrm{Un}(G)\ ,\ \rho \mapsto \rho_q\ .$$

2. Generic degree : for all $\rho \in \mathrm{Un}(\mathbb{G})$, there exists $\mathrm{Deg}_\rho(x) \in \mathbb{Q}[x]$ such that

$$\mathrm{Deg}_\rho(x)\ |_{x=q} = \mathrm{Deg}(\rho_q)\ .$$

   Of course, the (generic) degrees divide the (polynomial) order of $\mathbb{G}$.

3. Every $\rho \in \mathrm{Un}(\mathbb{G})$ comes equipped with a Frobenius eigenvalue Fr_ρ, a root of unity
   ...which has something to do with the Deligne–Lusztig varieties $\mathbf{X}_w$...

2. Unipotent characters, generic degrees (Lusztig)

1. The set $\mathrm{Un}(G)$ of unipotent characters of G is parametrized by the set of unipotent generic characters $\mathrm{Un}(\mathbb{G})$ (depending only on the type). Let us denote that parametrization by

$$\mathrm{Un}(\mathbb{G}) \to \mathrm{Un}(G)\ ,\ \rho \mapsto \rho_q\,.$$

2. Generic degree : for all $\rho \in \mathrm{Un}(\mathbb{G})$, there exists $\mathrm{Deg}_\rho(x) \in \mathbb{Q}[x]$ such that

$$\mathrm{Deg}_\rho(x)\,|_{x=q} = \mathrm{Deg}(\rho_q)\,.$$

   Of course, the (generic) degrees divide the (polynomial) order of $\mathbb{G}$.

3. Every $\rho \in \mathrm{Un}(\mathbb{G})$ comes equipped with a Frobenius eigenvalue Fr_ρ, a root of unity
   ...which has something to do with the Deligne–Lusztig varieties $\mathbf{X}_w$... but can also be defined by

$$\mathrm{sh}_{F/F}(\rho_\chi) = \sum_{\rho\in\mathrm{Un}(G)} \mathrm{Fr}_\rho \langle R_\chi\,;\rho\rangle\rho \quad \text{for } \chi \in \mathrm{Irr}(W)\,.$$

Generic unipotent characters, continued

Generic unipotent characters, continued

... lots of other properties, like the partition of $\mathrm{Un}(\mathbb{G})$ into Harish-Chandra series, or more generally

Generic unipotent characters, continued

... lots of other properties, like the partition of $\mathrm{Un}(\mathbb{G})$ into Harish-Chandra series, or more generally

5. For all $\zeta \in \boldsymbol{\mu}$, partition of $\mathrm{Un}(\mathbb{G})$ into ζ-Harish-Chandra series.

Generic unipotent characters, continued

... lots of other properties, like the partition of $\mathrm{Un}(\mathbb{G})$ into Harish-Chandra series, or more generally

5. For all $\zeta \in \boldsymbol{\mu}$, partition of $\mathrm{Un}(\mathbb{G})$ into ζ-Harish-Chandra series.
6. Description of the principal ζ-Harish-Chandra series with a ζ-cyclotomic Hecke algebra.

Generic unipotent characters, continued

... lots of other properties, like the partition of $\mathrm{Un}(\mathbb{G})$ into Harish-Chandra series, or more generally

5. For all $\zeta \in \boldsymbol{\mu}$, partition of $\mathrm{Un}(\mathbb{G})$ into ζ-Harish-Chandra series.
6. Description of the principal ζ-Harish-Chandra series with a ζ-cyclotomic Hecke algebra.

The principal 1-Harish-Chandra series is the usual principal Harish-Chandra series.

Generic unipotent characters, continued

... lots of other properties, like the partition of $\mathrm{Un}(\mathbb{G})$ into Harish-Chandra series, or more generally

5. For all $\zeta \in \boldsymbol{\mu}$, partition of $\mathrm{Un}(\mathbb{G})$ into ζ-Harish-Chandra series.

6. Description of the principal ζ-Harish-Chandra series with a ζ-cyclotomic Hecke algebra.

The principal 1-Harish-Chandra series is the usual principal Harish-Chandra series.

Let us devote some time to the notion of ζ-cyclotomic Hecke algebras.

Generic unipotent characters, continued

... lots of other properties, like the partition of $\mathrm{Un}(\mathbb{G})$ into Harish-Chandra series, or more generally

5. For all $\zeta \in \boldsymbol{\mu}$, partition of $\mathrm{Un}(\mathbb{G})$ into ζ-Harish-Chandra series.
6. Description of the principal ζ-Harish-Chandra series with a ζ-cyclotomic Hecke algebra.

The principal 1-Harish-Chandra series is the usual principal Harish-Chandra series.

Let us devote some time to the notion of ζ-cyclotomic Hecke algebras.
Then we shall come back to the generic properties of $\mathrm{Un}(\mathbb{G})$.

3. ζ-regular elements and ζ-cyclotomic Hecke algebras

3. ζ-regular elements and ζ-cyclotomic Hecke algebras

What follows holds more generally for any pair $\mathbb{G} = (V, W)$ where

3. ζ-regular elements and ζ-cyclotomic Hecke algebras

What follows holds more generally for any pair $\mathbb{G} = (V, W)$ where

- V is a finite dimensional complex vector space,

3. ζ-regular elements and ζ-cyclotomic Hecke algebras

What follows holds more generally for any pair $\mathbb{G} = (V, W)$ where

- V is a finite dimensional complex vector space,
- W is a finite subgroup of $\mathrm{GL}(V)$ generated by (pseudo)-reflections.

3. ζ-regular elements and ζ-cyclotomic Hecke algebras

What follows holds more generally for any pair $\mathbb{G} = (V, W)$ where

- V is a finite dimensional complex vector space,
- W is a finite subgroup of $\mathrm{GL}(V)$ generated by (pseudo)-reflections.

Let $\mathcal{A}$ be the set of reflecting hyperplanes of W. A root of unity ζ is called regular if there exist $w \in W$ and $x \in V^{\mathrm{reg}} := V \setminus \bigcup_{H \in \mathcal{A}} H$ such that $w(x) = \zeta x$. We then say that w is ζ-regular.

3. ζ-regular elements and ζ-cyclotomic Hecke algebras

What follows holds more generally for any pair $\mathbb{G} = (V, W)$ where

- V is a finite dimensional complex vector space,
- W is a finite subgroup of $\mathrm{GL}(V)$ generated by (pseudo)-reflections.

Let $\mathcal{A}$ be the set of reflecting hyperplanes of W. A root of unity ζ is called regular if there exist $w \in W$ and $x \in V^{\mathrm{reg}} := V \setminus \bigcup_{H \in \mathcal{A}} H$ such that $w(x) = \zeta x$. We then say that w is ζ-regular.

From now on we assume that ζ is regular. Then

3. ζ-regular elements and ζ-cyclotomic Hecke algebras

What follows holds more generally for any pair $\mathbb{G} = (V, W)$ where

- V is a finite dimensional complex vector space,
- W is a finite subgroup of $\mathrm{GL}(V)$ generated by (pseudo)-reflections.

Let $\mathcal{A}$ be the set of reflecting hyperplanes of W. A root of unity ζ is called regular if there exist $w \in W$ and $x \in V^{\mathrm{reg}} := V \setminus \bigcup_{H \in \mathcal{A}} H$ such that $w(x) = \zeta x$. We then say that w is ζ-regular.

From now on we assume that ζ is regular. Then

- [Springer] The group $W_\zeta := C_W(w)$ acts faithfully as a reflection group on the vector space $V_\zeta := \ker(w - \zeta \mathrm{Id}_V)$.

3. ζ-regular elements and ζ-cyclotomic Hecke algebras

What follows holds more generally for any pair $\mathbb{G} = (V, W)$ where

- V is a finite dimensional complex vector space,
- W is a finite subgroup of $\mathrm{GL}(V)$ generated by (pseudo)-reflections.

Let $\mathcal{A}$ be the set of reflecting hyperplanes of W. A root of unity ζ is called regular if there exist $w \in W$ and $x \in V^{\mathrm{reg}} := V \setminus \bigcup_{H \in \mathcal{A}} H$ such that $w(x) = \zeta x$. We then say that w is ζ-regular.

From now on we assume that ζ is regular. Then

- [Springer] The group $W_\zeta := C_W(w)$ acts faithfully as a reflection group on the vector space $V_\zeta := \ker(w - \zeta \mathrm{Id}_V)$.
- The group W_ζ is called the ζ-cyclotomic Weyl group.

3. ζ-regular elements and ζ-cyclotomic Hecke algebras

What follows holds more generally for any pair $\mathbb{G} = (V, W)$ where

- V is a finite dimensional complex vector space,
- W is a finite subgroup of $\mathrm{GL}(V)$ generated by (pseudo)-reflections.

Let $\mathcal{A}$ be the set of reflecting hyperplanes of W. A root of unity ζ is called regular if there exist $w \in W$ and $x \in V^{\mathrm{reg}} := V \setminus \bigcup_{H \in \mathcal{A}} H$ such that $w(x) = \zeta x$. We then say that w is ζ-regular.

From now on we assume that ζ is regular. Then

- [Springer] The group $W_\zeta := C_W(w)$ acts faithfully as a reflection group on the vector space $V_\zeta := \ker(w - \zeta \mathrm{Id}_V)$.
- The group W_ζ is called the ζ-cyclotomic Weyl group.

[Note that $W_1 = W$].

3.1. Hecke algebras everywhere

3.1. Hecke algebras everywhere

One knows that the (ordinary) principal series $\mathrm{Un}(\mathbb{G}, 1)$

3.1. Hecke algebras everywhere

One knows that the (ordinary) principal series $\mathrm{Un}(\mathbb{G}, 1)$

- corresponds to unipotent characters of G occurring in $\overline{\mathbf{Q}}_\ell(G/B)$,

3.1. Hecke algebras everywhere

One knows that the (ordinary) principal series $\mathrm{Un}(\mathbb{G}, 1)$

- corresponds to unipotent characters of G occurring in $\overline{\mathbf{Q}}_\ell(G/B)$,
- and the commutant of that module is the (ordinary) Hecke algebra of W evaluated at q.

3.1. Hecke algebras everywhere

One knows that the (ordinary) principal series $\mathrm{Un}(\mathbb{G}, 1)$

- corresponds to unipotent characters of G occurring in $\overline{\mathbf{Q}}_\ell(G/B)$,
- and the commutant of that module is the (ordinary) Hecke algebra of W evaluated at q.

One conjectures that the ζ–principal series $\mathrm{Un}(\mathbb{G}, \zeta)$

3.1. Hecke algebras everywhere

One knows that the (ordinary) principal series $\mathrm{Un}(\mathbb{G}, 1)$

- corresponds to unipotent characters of G occurring in $\overline{\mathbf{Q}}_\ell(G/B)$,
- and the commutant of that module is the (ordinary) Hecke algebra of W evaluated at q.

One conjectures that the ζ–principal series $\mathrm{Un}(\mathbb{G}, \zeta)$

(!) for a good choice of w

3.1. Hecke algebras everywhere

One knows that the (ordinary) principal series $\mathrm{Un}(\mathbb{G}, 1)$

- corresponds to unipotent characters of G occurring in $\overline{\mathbf{Q}}_\ell(G/B)$,
- and the commutant of that module is the (ordinary) Hecke algebra of W evaluated at q.

One conjectures that the ζ–principal series $\mathrm{Un}(\mathbb{G}, \zeta)$

(!) for a good choice of w

- corresponds to unipotent characters of G occurring in $\bigoplus_n H^n_c(\mathbf{X}_w, \overline{\mathbb{Q}}_\ell)$,

3.1. Hecke algebras everywhere

One knows that the (ordinary) principal series $\mathrm{Un}(\mathbb{G}, 1)$

- corresponds to unipotent characters of G occurring in $\overline{\mathbf{Q}}_\ell(G/B)$,
- and the commutant of that module is the (ordinary) Hecke algebra of W evaluated at q.

One conjectures that the ζ–principal series $\mathrm{Un}(\mathbb{G}, \zeta)$

(!) for a good choice of w

- corresponds to unipotent characters of G occurring in $\bigoplus_n H^n_c(\mathbf{X}_w, \overline{\mathbb{Q}}_\ell)$,
- and the commutant of that module is a ζ-cyclotomic Hecke algebra of W_ζ evaluated at q.

3.1. Hecke algebras everywhere

One knows that the (ordinary) principal series $\mathrm{Un}(\mathbb{G}, 1)$

- corresponds to unipotent characters of G occurring in $\overline{\mathbf{Q}}_\ell(G/B)$,
- and the commutant of that module is the (ordinary) Hecke algebra of W evaluated at q.

One conjectures that the ζ–principal series $\mathrm{Un}(\mathbb{G}, \zeta)$

(!) for a good choice of w

- corresponds to unipotent characters of G occurring in $\bigoplus_n H^n_c(\mathbf{X}_w, \overline{\mathbb{Q}}_\ell)$,
- and the commutant of that module is a ζ-cyclotomic Hecke algebra of W_ζ evaluated at q.

We shall now introduce the notion of ζ-cyclotomic Hecke algebra.

3.2. ζ-cyclotomic Hecke algebras

3.2. ζ-cyclotomic Hecke algebras

- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of W_ζ is in particular

3.2. ζ-cyclotomic Hecke algebras

- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of W_ζ is in particular
  - a $\mathbb{C}[x, x^{-1}]$-algebra,

3.2. ζ-cyclotomic Hecke algebras

- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of W_ζ is in particular
  - a $\mathbb{C}[x, x^{-1}]$-algebra,
  - an image of the group algebra of the braid group B_{W_ζ} attached to W_ζ,

3.2. ζ-cyclotomic Hecke algebras

- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of W_ζ is in particular
  - a $\mathbb{C}[x, x^{-1}]$-algebra,
  - an image of the group algebra of the braid group B_{W_ζ} attached to W_ζ,
  - a deformation (*via* x) of the group algebra of W_ζ,

3.2. ζ-cyclotomic Hecke algebras

- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of W_ζ is in particular
  - a $\mathbb{C}[x, x^{-1}]$-algebra,
  - an image of the group algebra of the braid group B_{W_ζ} attached to W_ζ,
  - a deformation (*via* x) of the group algebra of W_ζ,
  - which specializes to that algebra for $x = \zeta$.

3.2. ζ-cyclotomic Hecke algebras

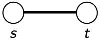
- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of W_ζ is in particular
  - a $\mathbb{C}[x, x^{-1}]$-algebra,
  - an image of the group algebra of the braid group B_{W_ζ} attached to W_ζ,
  - a deformation (*via* x) of the group algebra of W_ζ,
  - which specializes to that algebra for $x = \zeta$.
- Examples:

3.2. ζ-cyclotomic Hecke algebras

- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of W_ζ is in particular
  - a $\mathbb{C}[x, x^{-1}]$-algebra,
  - an image of the group algebra of the braid group B_{W_ζ} attached to W_ζ,
  - a deformation (*via* x) of the group algebra of W_ζ,
  - which specializes to that algebra for $x = \zeta$.
- Examples:
  - Case where $G = \mathrm{GL}_3$, $\zeta = 1$: $W_\zeta = W = \mathfrak{S}_3$

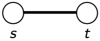
3.2. ζ-cyclotomic Hecke algebras

- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of W_ζ is in particular
  - a $\mathbb{C}[x, x^{-1}]$-algebra,
  - an image of the group algebra of the braid group B_{W_ζ} attached to W_ζ,
  - a deformation (*via* x) of the group algebra of W_ζ,
  - which specializes to that algebra for $x = \zeta$.

- Examples:
  - Case where $G = \mathrm{GL}_3$, $\zeta = 1$: $W_\zeta = W = \mathfrak{S}_3 \quad \longleftrightarrow$ 

3.2. ζ-cyclotomic Hecke algebras

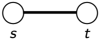
- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of W_ζ is in particular
  - a $\mathbb{C}[x, x^{-1}]$-algebra,
  - an image of the group algebra of the braid group B_{W_ζ} attached to W_ζ,
  - a deformation (*via* x) of the group algebra of W_ζ,
  - which specializes to that algebra for $x = \zeta$.

- Examples:
  - Case where $G = \mathrm{GL}_3$, $\zeta = 1$: $W_\zeta = W = \mathfrak{S}_3 \quad \longleftrightarrow$ 

$$\mathcal{H}(W) = \left\langle S, T \,;\, STS = TST \,,\, (S - x)(S + 1) = 0 \right\rangle \quad \text{is 1-cyclotomic.}$$

3.2. ζ-cyclotomic Hecke algebras

- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of W_ζ is in particular
  - a $\mathbb{C}[x, x^{-1}]$-algebra,
  - an image of the group algebra of the braid group B_{W_ζ} attached to W_ζ,
  - a deformation (*via* x) of the group algebra of W_ζ,
  - which specializes to that algebra for $x = \zeta$.

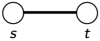
- Examples:
  - Case where $G = \mathrm{GL}_3$, $\zeta = 1$: $W_\zeta = W = \mathfrak{S}_3 \quad \longleftrightarrow$ 

$$\mathcal{H}(W) = \left\langle S, T \,;\, STS = TST \,,\, (S - x)(S + 1) = 0 \right\rangle \quad \text{is 1-cyclotomic.}$$

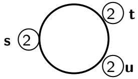
  - For $G = \mathrm{O}_8(q)$, $W = D_4$, $\zeta = i$, $W_i = G(4, 2, 2)$

3.2. ζ-cyclotomic Hecke algebras

- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of W_ζ is in particular
  - a $\mathbb{C}[x, x^{-1}]$-algebra,
  - an image of the group algebra of the braid group B_{W_ζ} attached to W_ζ,
  - a deformation (*via* x) of the group algebra of W_ζ,
  - which specializes to that algebra for $x = \zeta$.

- Examples:
  - Case where $G = \mathrm{GL}_3$, $\zeta = 1$: $W_\zeta = W = \mathfrak{S}_3$ $\longleftrightarrow$ 

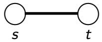
$$\mathcal{H}(W) = \left\langle S, T \,;\, STS = TST \,,\, (S - x)(S + 1) = 0 \right\rangle$$ is 1-cyclotomic.

  - For $G = \mathrm{O}_8(q)$, $W = D_4$, $\zeta = i$, $W_i = G(4, 2, 2)$ $\longleftrightarrow$ 

3.2. ζ-cyclotomic Hecke algebras

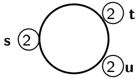
- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of W_ζ is in particular
  - a $\mathbb{C}[x, x^{-1}]$-algebra,
  - an image of the group algebra of the braid group B_{W_ζ} attached to W_ζ,
  - a deformation (*via* x) of the group algebra of W_ζ,
  - which specializes to that algebra for $x = \zeta$.

- Examples:
  - Case where $G = \mathrm{GL}_3$, $\zeta = 1$: $W_\zeta = W = \mathfrak{S}_3 \quad \longleftrightarrow$



$$\mathcal{H}(W) = \left\langle S, T \, ; STS = TST \, , (S - x)(S + 1) = 0 \right\rangle \quad \text{is 1-cyclotomic.}$$

  - For $G = \mathrm{O}_8(q)$, $W = D_4$, $\zeta = i$, $W_i = G(4, 2, 2) \quad \longleftrightarrow$



$$\mathcal{H}(W_i) = \left\langle S, T, U \, ; \begin{cases} STU = TUS = UST \\ (S - x^2)(S - 1) = 0 \end{cases} \right\rangle$$

Fundamental properties

Case by case checking...

Fundamental properties

Case by case checking...

"There is a proof, but so far I've not seen an explanation" [JHC]

Fundamental properties

Case by case checking...

"There is a proof, but so far I've not seen an explanation" [JHC]

- Such an algebra has a canonical symmetrizing form τ.

Fundamental properties Case by case checking...

"There is a proof, but so far I've not seen an explanation" [JHC]

- Such an algebra has a canonical symmetrizing form τ.
- It becomes split semisimple over $\mathbb{C}(x^{1/|ZW_\zeta|})$.

Fundamental properties Case by case checking...

"There is a proof, but so far I've not seen an explanation" [JHC]

- Such an algebra has a canonical symmetrizing form τ.
- It becomes split semisimple over $\mathbb{C}(x^{1/|ZW_\zeta|})$.

$\Rightarrow$ Hence each absolute irreducible character χ of $\mathcal{H}(W_\zeta)$ is equipped with a Schur element

Fundamental properties Case by case checking...

"There is a proof, but so far I've not seen an explanation" [JHC]

- Such an algebra has a canonical symmetrizing form τ.
- It becomes split semisimple over $\mathbb{C}(x^{1/|ZW_\zeta|})$.

$\Rightarrow$ Hence each absolute irreducible character χ of $\mathcal{H}(W_\zeta)$ is equipped with a Schur element

$$S_\chi \in \mathbb{C}[x^{1/|ZW_\zeta|}, x^{-1/|ZW_\zeta|}] \quad \text{defined by} \quad \tau = \sum_{\chi \in \mathrm{Irr}\mathcal{H}(W_\zeta)} \frac{\chi}{S_\chi} .$$

3.3. Spetsial ζ-cyclotomic Hecke algebras

Definition

A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of W_ζ is spetsial for $\mathbb{G}$ if

3.3. Spetsial ζ-cyclotomic Hecke algebras

Definition

A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of W_ζ is spetsial for $\mathbb{G}$ if

1. it satisfies various technical conditions...

Definition

A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of W_ζ is spetsial for $\mathbb{G}$ if

1. it satisfies various technical conditions...
2. for each absolute irreducible character χ of $\mathcal{H}(W_\zeta)$,

$$S_\chi \in \mathbb{C}[x, x^{-1}]$$

3.3. Spetsial ζ-cyclotomic Hecke algebras

Definition

A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of W_ζ is spetsial for $\mathbb{G}$ if

1. it satisfies various technical conditions...
2. for each absolute irreducible character χ of $\mathcal{H}(W_\zeta)$,

$$S_\chi \in \mathbb{C}[x, x^{-1}]$$

(and not only $\mathbb{C}[x^{1/|ZW_\zeta|}, x^{-1/|ZW_\zeta|}]$).

On $\mathrm{Un}(\mathbb{G})$ again

On Un($\mathbb{G}$) again

When we started speaking about ζ-regular elements, ζ-cyclotomic Weyl groups, spetsial ζ-cyclotomic Hecke algebras, we were stating "generic properties" of unipotent characters :

On $\mathrm{Un}(\mathbb{G})$ again

When we started speaking about ζ-regular elements, ζ-cyclotomic Weyl groups, spetsial ζ-cyclotomic Hecke algebras, we were stating "generic properties" of unipotent characters :

... lots of other properties, like the partition of $\mathrm{Un}(\mathbb{G})$ into ζ-Harish-Chandra series:

On Un($\mathbb{G}$) again

When we started speaking about ζ-regular elements, ζ-cyclotomic Weyl groups, spetsial ζ-cyclotomic Hecke algebras, we were stating "generic properties" of unipotent characters :

... lots of other properties, like the partition of Un($\mathbb{G}$) into ζ-Harish-Chandra series:

6. Description of the principal ζ-Harish-Chandra series with a spetsial ζ-cyclotomic Hecke algebra.

Let us come back to that long list.

On Un($\mathbb{G}$) again

When we started speaking about ζ-regular elements, ζ-cyclotomic Weyl groups, spetsial ζ-cyclotomic Hecke algebras, we were stating "generic properties" of unipotent characters :

... lots of other properties, like the partition of Un($\mathbb{G}$) into ζ-Harish-Chandra series:

6. Description of the principal ζ-Harish-Chandra series with a spetsial ζ-cyclotomic Hecke algebra.

Let us come back to that long list.

7. Partition of Un($\mathbb{G}$) into families and their Intersections with ζ-Harish-Chandra series (Rouquier blocks).

On Un($\mathbb{G}$) again

When we started speaking about ζ-regular elements, ζ-cyclotomic Weyl groups, spetsial ζ-cyclotomic Hecke algebras, we were stating "generic properties" of unipotent characters :

... lots of other properties, like the partition of Un($\mathbb{G}$) into ζ-Harish-Chandra series:

6. Description of the principal ζ-Harish-Chandra series with a spetsial ζ-cyclotomic Hecke algebra.

Let us come back to that long list.

7. Partition of Un($\mathbb{G}$) into families and their Intersections with ζ-Harish-Chandra series (Rouquier blocks).
8. Ennola permutation on Un($\mathbb{G}$).

On Un($\mathbb{G}$) again

When we started speaking about ζ-regular elements, ζ-cyclotomic Weyl groups, spetsial ζ-cyclotomic Hecke algebras, we were stating "generic properties" of unipotent characters :

... lots of other properties, like the partition of Un($\mathbb{G}$) into ζ-Harish-Chandra series:

6. Description of the principal ζ-Harish-Chandra series with a spetsial ζ-cyclotomic Hecke algebra.

Let us come back to that long list.

7. Partition of Un($\mathbb{G}$) into families and their Intersections with ζ-Harish-Chandra series (Rouquier blocks).
8. Ennola permutation on Un($\mathbb{G}$).
   [This is an abstract formulation of the fact that $\mathrm{U}_n(q) = \pm \mathrm{GL}_n(-q)$]

On Un($\mathbb{G}$) again

When we started speaking about ζ-regular elements, ζ-cyclotomic Weyl groups, spetsial ζ-cyclotomic Hecke algebras, we were stating "generic properties" of unipotent characters :

... lots of other properties, like the partition of Un($\mathbb{G}$) into ζ-Harish-Chandra series:

6. Description of the principal ζ-Harish-Chandra series with a spetsial ζ-cyclotomic Hecke algebra.

Let us come back to that long list.

7. Partition of Un($\mathbb{G}$) into families and their Intersections with ζ-Harish-Chandra series (Rouquier blocks).
8. Ennola permutation on Un($\mathbb{G}$).
   [This is an abstract formulation of the fact that $\mathrm{U}_n(q) = \pm\mathrm{GL}_n(-q)$]
9. Fourier matrices and $\mathrm{SL}_2(\mathbb{Z})$-representation.

On Un($\mathbb{G}$) again

When we started speaking about ζ-regular elements, ζ-cyclotomic Weyl groups, spetsial ζ-cyclotomic Hecke algebras, we were stating "generic properties" of unipotent characters :

... lots of other properties, like the partition of Un($\mathbb{G}$) into ζ-Harish-Chandra series:

6. Description of the principal ζ-Harish-Chandra series with a spetsial ζ-cyclotomic Hecke algebra.

Let us come back to that long list.

7. Partition of Un($\mathbb{G}$) into families and their Intersections with ζ-Harish-Chandra series (Rouquier blocks).
8. Ennola permutation on Un($\mathbb{G}$).
   [This is an abstract formulation of the fact that $\mathrm{U}_n(q) = \pm\mathrm{GL}_n(-q)$]
9. Fourier matrices and $\mathrm{SL}_2(\mathbb{Z})$-representation.

We shall review this now in the more general context of "Spetses".

II. Towards Spetses

II. Towards Spetses

- Try to treat a complex reflection group as a Weyl group: try to build a thing $\mathbb{G}(x)$ (x an indeterminate) associated with a type $\mathbb{G} = (V, W)$ where W is a (pseudo)-reflection group.

II. Towards Spetses

- Try to treat a complex reflection group as a Weyl group: try to build a thing $\mathbb{G}(x)$ (x an indeterminate) associated with a type $\mathbb{G} = (V, W)$ where W is a (pseudo)-reflection group.
- Try at least to build "unipotent characters" of $\mathbb{G}$, or at least to build their degrees (polynomials in x), Frobenius eigenvalues (roots of unity), Fourier matrices.

II. Towards Spetses

- Try to treat a complex reflection group as a Weyl group: try to build a thing $\mathbb{G}(x)$ (x an indeterminate) associated with a type $\mathbb{G} = (V, W)$ where W is a (pseudo)-reflection group.
- Try at least to build "unipotent characters" of $\mathbb{G}$, or at least to build their degrees (polynomials in x), Frobenius eigenvalues (roots of unity), Fourier matrices.
  - Lusztig knew already a solution for Coxeter groups which are not Weyl groups (except the Fourier matrix for H_4 which was determined by Malle in 1994).

II. Towards Spetses

- Try to treat a complex reflection group as a Weyl group: try to build a thing $\mathbb{G}(x)$ (x an indeterminate) associated with a type $\mathbb{G} = (V, W)$ where W is a (pseudo)-reflection group.
- Try at least to build "unipotent characters" of $\mathbb{G}$, or at least to build their degrees (polynomials in x), Frobenius eigenvalues (roots of unity), Fourier matrices.
  - Lusztig knew already a solution for Coxeter groups which are not Weyl groups (except the Fourier matrix for H_4 which was determined by Malle in 1994).
  - Malle gave a solution for imprimitive spetsial complex reflection groups in 1995.

II. Towards Spetses

- Try to treat a complex reflection group as a Weyl group: try to build a thing $\mathbb{G}(x)$ (x an indeterminate) associated with a type $\mathbb{G} = (V, W)$ where W is a (pseudo)-reflection group.
- Try at least to build "unipotent characters" of $\mathbb{G}$, or at least to build their degrees (polynomials in x), Frobenius eigenvalues (roots of unity), Fourier matrices.
  - Lusztig knew already a solution for Coxeter groups which are not Weyl groups (except the Fourier matrix for H_4 which was determined by Malle in 1994).
  - Malle gave a solution for imprimitive spetsial complex reflection groups in 1995.
  - Stating a long series of precise axioms — many of technical nature — we can now show that there is a unique solution for all primitive spetsial complex reflection groups.

II. Towards Spetses

- Try to treat a complex reflection group as a Weyl group: try to build a thing $\mathbb{G}(x)$ (x an indeterminate) associated with a type $\mathbb{G} = (V, W)$ where W is a (pseudo)-reflection group.
- Try at least to build "unipotent characters" of $\mathbb{G}$, or at least to build their degrees (polynomials in x), Frobenius eigenvalues (roots of unity), Fourier matrices.
  - Lusztig knew already a solution for Coxeter groups which are not Weyl groups (except the Fourier matrix for H_4 which was determined by Malle in 1994).
  - Malle gave a solution for imprimitive spetsial complex reflection groups in 1995.
  - Stating a long series of precise axioms — many of technical nature — we can now show that there is a unique solution for all primitive spetsial complex reflection groups.

M. Broué, G. Malle, J. Michel, Split spetses for primitive reflection groups, *Astérisque* **359** (2014)

A double object : double N

A double object : double N

$\mathbb{G} = (V, W)$, where

A double object : double N

$\mathbb{G} = (V, W)$, where

- V is a complex vector space of dimension r,

A double object : double N

$\mathbb{G} = (V, W)$, where

- V is a complex vector space of dimension r,
- W is finite (pseudo)-reflection subgroup of $\mathrm{GL}(V)$,

A double object : double N

$\mathbb{G} = (V, W)$, where

- V is a complex vector space of dimension r,
- W is finite (pseudo)-reflection subgroup of $\mathrm{GL}(V)$,

$\mathcal{A}(W) :=$ the hyperplanes arrangement of W.

A double object : double N

$\mathbb{G} = (V, W)$, where

- V is a complex vector space of dimension r,
- W is finite (pseudo)-reflection subgroup of $\mathrm{GL}(V)$,

$\mathcal{A}(W) :=$ the hyperplanes arrangement of W.

- $N_W^{\mathrm{hyp}} :=$ number of reflecting hyperplanes,

A double object : double N

$\mathbb{G} = (V, W)$, where

- V is a complex vector space of dimension r,
- W is finite (pseudo)-reflection subgroup of $\mathrm{GL}(V)$,

$\mathcal{A}(W) :=$ the hyperplanes arrangement of W.

- $N_W^{\mathrm{hyp}} :=$ number of reflecting hyperplanes,
- $N_W^{\mathrm{ref}} :=$ number of reflections.

A double object : double N

$\mathbb{G} = (V, W)$, where

- V is a complex vector space of dimension r,
- W is finite (pseudo)-reflection subgroup of $\mathrm{GL}(V)$,

$\mathcal{A}(W) :=$ the hyperplanes arrangement of W.

- $N_W^{\mathrm{hyp}} :=$ number of reflecting hyperplanes,
- $N_W^{\mathrm{ref}} :=$ number of reflections.

$N_W^{\mathrm{hyp}} = N_W^{\mathrm{ref}}$ if W is generated by true reflections.

Double polynomial order

Double polynomial order

$$|\mathbb{G}_{\mathrm{c}}|(x) := (-1)^r x^{N_W^{\mathrm{hyp}}} \frac{1}{\frac{1}{|W|}\sum_{w\in W} \frac{1}{\det_V(1-wx)^*}}$$

$$|\mathbb{G}_{\mathrm{nc}}|(x) := (-1)^r x^{N_W^{\mathrm{ref}}} \frac{1}{\frac{1}{|W|}\sum_{w\in W} \frac{1}{\det_V(1-wx)^*}}$$

Double polynomial order

$$|\mathbb{G}_{\mathrm{c}}|(x) := (-1)^r x^{N_W^{\mathrm{hyp}}} \frac{1}{\frac{1}{|W|}\sum_{w\in W} \frac{1}{\det_V(1-wx)^*}}$$

$$|\mathbb{G}_{\mathrm{nc}}|(x) := (-1)^r x^{N_W^{\mathrm{ref}}} \frac{1}{\frac{1}{|W|}\sum_{w\in W} \frac{1}{\det_V(1-wx)^*}}$$

The compact and the noncompact order coincide if W is generated by true reflections.

2. Spetsial ζ-cyclotomic Hecke algebras for $\mathbb{G}$

2. Spetsial ζ-cyclotomic Hecke algebras for $\mathbb{G}$

As above,

2. Spetsial ζ-cyclotomic Hecke algebras for $\mathbb{G}$

As above,

- $W_\zeta = C_W(w)$ is the centralizer of a ζ-regular element $w \in W$,

2. Spetsial ζ-cyclotomic Hecke algebras for $\mathbb{G}$

As above,

- $W_\zeta = C_W(w)$ is the centralizer of a ζ-regular element $w \in W$,
- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ is spetsial for $\mathbb{G}$ if

2. Spetsial ζ-cyclotomic Hecke algebras for $\mathbb{G}$

As above,

- $W_\zeta = C_W(w)$ is the centralizer of a ζ-regular element $w \in W$,
- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ is spetsial for $\mathbb{G}$ if

1. for each absolute irreducible character χ of $\mathcal{H}(W_\zeta)$,

$$S_\chi \in \mathbb{C}[x, x^{-1}] \, ,$$

2. Spetsial ζ-cyclotomic Hecke algebras for $\mathbb{G}$

As above,

- $W_\zeta = C_W(w)$ is the centralizer of a ζ-regular element $w \in W$,
- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ is spetsial for $\mathbb{G}$ if

1. for each absolute irreducible character χ of $\mathcal{H}(W_\zeta)$,

$$S_\chi \in \mathbb{C}[x, x^{-1}]\,,$$

2. and $\mathcal{H}(W_\zeta)$ satisfies various technical conditions, which split into

2. Spetsial ζ-cyclotomic Hecke algebras for $\mathbb{G}$

As above,

- $W_\zeta = C_W(w)$ is the centralizer of a ζ-regular element $w \in W$,
- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ is spetsial for $\mathbb{G}$ if

1. for each absolute irreducible character χ of $\mathcal{H}(W_\zeta)$,

$$S_\chi \in \mathbb{C}[x, x^{-1}] \,,$$

2. and $\mathcal{H}(W_\zeta)$ satisfies various technical conditions, which split into
   - ⋆ compact type conditions,

2. Spetsial ζ-cyclotomic Hecke algebras for $\mathbb{G}$

As above,

- $W_\zeta = C_W(w)$ is the centralizer of a ζ-regular element $w \in W$,
- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ is spetsial for $\mathbb{G}$ if

1. for each absolute irreducible character χ of $\mathcal{H}(W_\zeta)$,

$$S_\chi \in \mathbb{C}[x, x^{-1}]\,,$$

2. and $\mathcal{H}(W_\zeta)$ satisfies various technical conditions, which split into
   - ⋆ compact type conditions,
   - ⋆ noncompact type conditions.

2. Spetsial ζ-cyclotomic Hecke algebras for $\mathbb{G}$

As above,

- $W_\zeta = C_W(w)$ is the centralizer of a ζ-regular element $w \in W$,
- A ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ is spetsial for $\mathbb{G}$ if

  1. for each absolute irreducible character χ of $\mathcal{H}(W_\zeta)$,

     $$S_\chi \in \mathbb{C}[x, x^{-1}] \,,$$

  2. and $\mathcal{H}(W_\zeta)$ satisfies various technical conditions, which split into
     - ⋆ compact type conditions,
     - ⋆ noncompact type conditions.

These conditions coincide if W is generated by true reflections.

2.1. Spetsial 1-cyclotomic Hecke algebras, special groups

Theorem

2.1. Spetsial 1-cyclotomic Hecke algebras, special groups

Theorem

1. A 1-cyclotomic Hecke algebra can be *spetsial of compact type* for $\mathbb{G}$ only if it is the algebra $\mathcal{H}^{c}(W)$ defined by

$$\begin{cases} \mathcal{H}_{c}(W) = \langle \mathbf{s}_H \rangle_{H \in \mathcal{A}} \quad \text{with relations:} \\ (\mathbf{s}_H - x)(1 + \mathbf{s}_H + \cdots + \mathbf{s}_H^{e_H - 1}) = 0 \quad (\text{if } \mathbf{s}_H \text{ has order } e_H) \end{cases}$$

Theorem

1. A 1-cyclotomic Hecke algebra can be *spetsial of compact type* for $\mathbb{G}$ only if it is the algebra $\mathcal{H}^{\mathrm{c}}(W)$ defined by

$$\begin{cases} \mathcal{H}_{\mathrm{c}}(W) = \langle \mathbf{s}_H \rangle_{H\in\mathcal{A}} \quad \text{with relations:} \\ (\mathbf{s}_H - x)(1 + \mathbf{s}_H + \cdots + \mathbf{s}_H^{e_H-1}) = 0 \quad (\text{if } \mathbf{s}_H \text{ has order } e_H) \end{cases}$$

2. A 1-cyclotomic Hecke algebras can be *spetsial of noncompact type* for $\mathbb{G}$ only if it is the algebra $\mathcal{H}^{\mathrm{nc}}(W)$ defined by

$$\begin{cases} \mathcal{H}^{\mathrm{nc}}(W) = \langle \mathbf{s}_H \rangle_{H\in\mathcal{A}} \quad \text{with relations:} \\ (\mathbf{s}_H - x)(x^{e_H-1} + x^{e_H-2}\mathbf{s}_H + \cdots + \mathbf{s}_H^{e_H-1}) = 0\,. \end{cases}$$

2.2. Spetsial groups

2.2. Spetsial groups

Let $\mathcal{H}(W)$ denote either $\mathcal{H}^{\mathrm{c}}(W)$ or $\mathcal{H}^{\mathrm{nc}}(W)$.

Let $\mathcal{H}(W)$ denote either $\mathcal{H}^{\mathrm{c}}(W)$ or $\mathcal{H}^{\mathrm{nc}}(W)$.

Theorem (G. Malle)–Definition

Let $\mathcal{H}(W)$ denote either $\mathcal{H}^{\mathrm{c}}(W)$ or $\mathcal{H}^{\mathrm{nc}}(W)$.

Theorem (G. Malle)–Definition

Assume W acts irreducibly on V. The following assertions are equivalent.

Let $\mathcal{H}(W)$ denote either $\mathcal{H}^{c}(W)$ or $\mathcal{H}^{nc}(W)$.

Theorem (G. Malle)–Definition

Assume W acts irreducibly on V. The following assertions are equivalent.

(i) $\mathcal{H}(W)$ is spetsial.

2.2. Spetsial groups

Let $\mathcal{H}(W)$ denote either $\mathcal{H}^{\mathrm{c}}(W)$ or $\mathcal{H}^{\mathrm{nc}}(W)$.

Theorem (G. Malle)–Definition

Assume W acts irreducibly on V. The following assertions are equivalent.

(i) $\mathcal{H}(W)$ is spetsial.

(ii) For each absolutely irreducible character χ of $\mathcal{H}(W)$, $S_\chi \in \mathbb{C}[x, x^{-1}]$.

Let $\mathcal{H}(W)$ denote either $\mathcal{H}^{\mathrm{c}}(W)$ or $\mathcal{H}^{\mathrm{nc}}(W)$.

Theorem (G. Malle)–Definition

Assume W acts irreducibly on V. The following assertions are equivalent.

(i) $\mathcal{H}(W)$ is spetsial.

(ii) For each absolutely irreducible character χ of $\mathcal{H}(W)$, $S_\chi \in \mathbb{C}[x, x^{-1}]$.

(iii) W is one of the following groups (Shephard–Todd's notation), called the spetsial groups:

Let $\mathcal{H}(W)$ denote either $\mathcal{H}^{\mathrm{c}}(W)$ or $\mathcal{H}^{\mathrm{nc}}(W)$.

Theorem (G. Malle)–Definition

Assume W acts irreducibly on V. The following assertions are equivalent.

(i) $\mathcal{H}(W)$ is spetsial.

(ii) For each absolutely irreducible character χ of $\mathcal{H}(W)$, $S_\chi \in \mathbb{C}[x, x^{-1}]$.

(iii) W is one of the following groups (Shephard–Todd's notation), called the spetsial groups:

- $G(d,1,n)_{(d,n\geq 1)}$, $G(e,e,n)_{(e,n\geq 2)}$,

2.2. Spetsial groups

Let $\mathcal{H}(W)$ denote either $\mathcal{H}^{\mathrm{c}}(W)$ or $\mathcal{H}^{\mathrm{nc}}(W)$.

Theorem (G. Malle)–Definition

Assume W acts irreducibly on V. The following assertions are equivalent.

(i) $\mathcal{H}(W)$ is spetsial.

(ii) For each absolutely irreducible character χ of $\mathcal{H}(W)$, $S_\chi \in \mathbb{C}[x, x^{-1}]$.

(iii) W is one of the following groups (Shephard–Todd's notation), called the spetsial groups:

- $G(d, 1, n)_{(d,n\geq 1)}$, $G(e, e, n)_{(e,n\geq 2)}$,
- all groups G_i $(4 \leq i \leq 37)$ well generated by true reflections,

Let $\mathcal{H}(W)$ denote either $\mathcal{H}^{\mathrm{c}}(W)$ or $\mathcal{H}^{\mathrm{nc}}(W)$.

Theorem (G. Malle)–Definition

Assume W acts irreducibly on V. The following assertions are equivalent.

(i) $\mathcal{H}(W)$ is spetsial.

(ii) For each absolutely irreducible character χ of $\mathcal{H}(W)$, $S_\chi \in \mathbb{C}[x, x^{-1}]$.

(iii) W is one of the following groups (Shephard–Todd's notation), called the spetsial groups:

- $G(d, 1, n)_{(d,n\geq 1)}$, $G(e, e, n)_{(e,n\geq 2)}$,
- all groups G_i $(4 \leq i \leq 37)$ well generated by true reflections,
- G_4, G_6, G_8, G_{25}, G_{26}, G_{32} .

3. Some data associated with spetsial groups

3. Some data associated with spetsial groups

Given $\mathbb{G} = (V, W)$ where W is special, there are

3. Some data associated with spetsial groups

Given $\mathbb{G} = (V, W)$ where W is special, there are

- the set $\mathrm{Un}(\mathbb{G}_c)$ of unipotent characters (compact type),

3. Some data associated with spetsial groups

Given $\mathbb{G} = (V, W)$ where W is special, there are

- the set $\mathrm{Un}(\mathbb{G}_{\mathrm{c}})$ of unipotent characters (compact type),
- the set $\mathrm{Un}(\mathbb{G}_{\mathrm{nc}})$ of unipotent characters (noncompact type),

3. Some data associated with spetsial groups

Given $\mathbb{G} = (V, W)$ where W is special, there are

- the set $\mathrm{Un}(\mathbb{G}_{\mathsf{c}})$ of unipotent characters (compact type),
- the set $\mathrm{Un}(\mathbb{G}_{\mathsf{nc}})$ of unipotent characters (noncompact type),

which coincide if W is generated by true reflections

3. Some data associated with spetsial groups

Given $\mathbb{G} = (V, W)$ where W is special, there are

- the set $\mathrm{Un}(\mathbb{G}_{\mathrm{c}})$ of unipotent characters (compact type),
- the set $\mathrm{Un}(\mathbb{G}_{\mathrm{nc}})$ of unipotent characters (noncompact type),

which coincide if W is generated by true reflections

each of them (denoted $\mathrm{Un}(\mathbb{G})$ below), endowed with two maps

3. Some data associated with spetsial groups

Given $\mathbb{G} = (V, W)$ where W is special, there are

- the set $\mathrm{Un}(\mathbb{G}_{\mathrm{c}})$ of unipotent characters (compact type),
- the set $\mathrm{Un}(\mathbb{G}_{\mathrm{nc}})$ of unipotent characters (noncompact type),

which coincide if W is generated by true reflections

each of them (denoted $\mathrm{Un}(\mathbb{G})$ below), endowed with two maps

- the map degree

  $$\mathrm{Deg} : \mathrm{Un}(\mathbb{G}) \to \mathbb{C}[x] \ , \ \rho \mapsto \mathrm{Deg}_\rho(x) \, ,$$

  defined up to sign,

3. Some data associated with spetsial groups

Given $\mathbb{G} = (V, W)$ where W is special, there are

- the set $\mathrm{Un}(\mathbb{G}_{\mathrm{c}})$ of unipotent characters (compact type),
- the set $\mathrm{Un}(\mathbb{G}_{\mathrm{nc}})$ of unipotent characters (noncompact type),

which coincide if W is generated by true reflections

each of them (denoted $\mathrm{Un}(\mathbb{G})$ below), endowed with two maps

- the map degree
$$\mathrm{Deg} : \mathrm{Un}(\mathbb{G}) \to \mathbb{C}[x] \,,\ \rho \mapsto \mathrm{Deg}_\rho(x)\,,$$
defined up to sign,
- the map Frobenius eigenvalue $\rho \mapsto \mathrm{Fr}_\rho$, where Fr_ρ is a root of unity,

3. Some data associated with spetsial groups

Given $\mathbb{G} = (V, W)$ where W is special, there are

- the set $\mathrm{Un}(\mathbb{G}_{\mathrm{c}})$ of unipotent characters (compact type),
- the set $\mathrm{Un}(\mathbb{G}_{\mathrm{nc}})$ of unipotent characters (noncompact type),

which coincide if W is generated by true reflections

each of them (denoted $\mathrm{Un}(\mathbb{G})$ below), endowed with two maps

- the map degree
$$\mathrm{Deg} : \mathrm{Un}(\mathbb{G}) \to \mathbb{C}[x] \ , \ \rho \mapsto \mathrm{Deg}_\rho(x) \, ,$$
defined up to sign,
- the map Frobenius eigenvalue $\rho \mapsto \mathrm{Fr}_\rho$, where Fr_ρ is a root of unity,
- a bijection (Alvis–Curtis duality)
$$\mathrm{Un}(\mathbb{G}_{\mathrm{c}}) \to \mathrm{Un}(\mathbb{G}_{\mathrm{nc}}) \ , \ \rho \mapsto \rho^{\mathrm{nc}} \, ,$$

3. Some data associated with spetsial groups

Given $\mathbb{G} = (V, W)$ where W is special, there are

- the set $\mathrm{Un}(\mathbb{G}_{\mathrm{c}})$ of unipotent characters (compact type),
- the set $\mathrm{Un}(\mathbb{G}_{\mathrm{nc}})$ of unipotent characters (noncompact type),

which coincide if W is generated by true reflections

each of them (denoted $\mathrm{Un}(\mathbb{G})$ below), endowed with two maps

- the map degree
$$\mathrm{Deg} : \mathrm{Un}(\mathbb{G}) \to \mathbb{C}[x] \,,\; \rho \mapsto \mathrm{Deg}_\rho(x)\,,$$
defined up to sign,
- the map Frobenius eigenvalue $\rho \mapsto \mathrm{Fr}_\rho$, where Fr_ρ is a root of unity,
- a bijection (Alvis–Curtis duality)
$$\mathrm{Un}(\mathbb{G}_{\mathrm{c}}) \to \mathrm{Un}(\mathbb{G}_{\mathrm{nc}}) \,,\; \rho \mapsto \rho^{\mathrm{nc}}\,,$$

with lots of properties (axioms) described below.

3.1. First axioms

3.1. First axioms

Connection compact / noncompact

Connection compact / noncompact

1. $\mathrm{Deg}_{\rho^{\mathrm{nc}}}(x) = x^{N_W^{\mathrm{ref}}} \mathrm{Deg}_\rho(1/x)^*$,

Connection compact / noncompact

1. $\mathrm{Deg}_{\rho^{\mathrm{nc}}}(x) = x^{N_W^{\mathrm{ref}}} \mathrm{Deg}_{\rho}(1/x)^*$, (!) up to sign!

Connection compact / noncompact

1. $\mathrm{Deg}_{\rho^{\mathrm{nc}}}(x) = x^{N_W^{\mathrm{ref}}} \mathrm{Deg}_\rho(1/x)^*$, (!) up to sign!
2. $\mathrm{Fr}_\rho \mathrm{Fr}_{\rho^{\mathrm{nc}}} = 1$.

3.1. First axioms

Connection compact / noncompact

1. $\mathrm{Deg}_{\rho^{\mathrm{nc}}}(x) = x^{N_W^{\mathrm{ref}}}\,\mathrm{Deg}_\rho(1/x)^*$, ① up to sign!
2. $\mathrm{Fr}_\rho \mathrm{Fr}_{\rho^{\mathrm{nc}}} = 1$.

From now on we only describe the compact type case.

Connection compact / noncompact

1. $\mathrm{Deg}_{\rho^{\mathrm{nc}}}(x) = x^{N_W^{\mathrm{ref}}} \mathrm{Deg}_\rho(1/x)^*$, (!) up to sign!
2. $\mathrm{Fr}_\rho \mathrm{Fr}_{\rho^{\mathrm{nc}}} = 1$.

From now on we only describe the compact type case.

Definition

Let $\zeta \in \boldsymbol{\mu}$.

Connection compact / noncompact

1. $\mathrm{Deg}_{\rho^{\mathrm{nc}}}(x) = x^{N_W^{\mathrm{ref}}}\,\mathrm{Deg}_\rho(1/x)^*$, (!) up to sign!
2. $\mathrm{Fr}_\rho \mathrm{Fr}_{\rho^{\mathrm{nc}}} = 1$.

From now on we only describe the compact type case.

Definition

Let $\zeta \in \boldsymbol{\mu}$.
The ζ-principal series is

$$\mathrm{Un}(\mathbb{G},\zeta) := \{\rho \in \mathrm{Un}(\mathbb{G}) \mid \mathrm{Deg}_\rho(\zeta) \neq 0\}\,.$$

ζ-Axioms (compact type)

3.2. ζ-Axioms

ζ-Axioms (compact type)

3.2. ζ-Axioms

For $w \in W$ a ζ-regular element, there are

3.2. ζ-Axioms

For $w \in W$ a ζ-regular element, there are

- a spetsial ζ-cyclotomic Hecke algebra of compact type $\mathcal{H}(W_\zeta)$ associated with w ,

ζ-Axioms (compact type)

3.2. ζ-Axioms

For $w \in W$ a ζ-regular element, there are

- a spetsial ζ-cyclotomic Hecke algebra of compact type $\mathcal{H}(W_\zeta)$ associated with w ,
- and a bijection

$$\mathrm{Irr}\,\mathcal{H}(W_\zeta) \xrightarrow{\sim} \mathrm{Un}(\mathbb{G}, \zeta)\ ,\ \ \chi \mapsto \rho_\chi$$

ζ-Axioms (compact type)

3.2. ζ-Axioms

For $w \in W$ a ζ-regular element, there are

- a spetsial ζ-cyclotomic Hecke algebra of compact type $\mathcal{H}(W_\zeta)$ associated with w,
- and a bijection

$$\mathrm{Irr}\,\mathcal{H}(W_\zeta) \xrightarrow{\sim} \mathrm{Un}(\mathbb{G}, \zeta)\ ,\ \chi \mapsto \rho_\chi$$

such that

ζ-Axioms (compact type)

3.2. ζ-Axioms

For $w \in W$ a ζ-regular element, there are

- a spetsial ζ-cyclotomic Hecke algebra of compact type $\mathcal{H}(W_\zeta)$ associated with w,
- and a bijection

$$\mathrm{Irr}\,\mathcal{H}(W_\zeta) \xrightarrow{\sim} \mathrm{Un}(\mathbb{G},\zeta)\ ,\ \chi \mapsto \rho_\chi$$

such that

1. $\mathrm{Deg}_{\rho_\chi}(x) = \pm \dfrac{[|\mathbb{G}|(x) : |\mathbb{T}_w|(x)]_{x'}}{S_\chi(x)}$,

ζ-Axioms (compact type)

3.2. ζ-Axioms

For $w \in W$ a ζ-regular element, there are

- a spetsial ζ-cyclotomic Hecke algebra of compact type $\mathcal{H}(W_\zeta)$ associated with w,
- and a bijection

$$\operatorname{Irr}\mathcal{H}(W_\zeta) \xrightarrow{\sim} \operatorname{Un}(\mathbb{G},\zeta)\ ,\ \ \chi \mapsto \rho_\chi$$

such that

1. $\operatorname{Deg}_{\rho_\chi}(x) = \pm \dfrac{[|\mathbb{G}|(x) : |\mathbb{T}_w|(x)]_{x'}}{S_\chi(x)}$,
2. $\operatorname{Fr}_{\rho_\chi}$ = explicit formula depending only on $\mathcal{H}(W_\zeta)$ and χ .

3.3. Rouquier blocks

3.3. Rouquier blocks

- If the representation of W_ζ on V_ζ is rational over some cyclotomic field K, the ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ may be defined over $\mathbb{Z}_K[x, x^{-1}]$.

Definition

The Rouquier blocks of a ζ-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ are the blocks of the algebra

$$\mathbb{Z}_K[x, x^{-1}, \left((x^n - 1)^{-1}\right)_{n \geq 1}] \otimes_{\mathbb{Z}[x,x^{-1}]} \mathcal{H}(W_\zeta)\,.$$

- The Rouquier blocks of ζ-cyclotomic Hecke algebras have been classified in all cases (Malle–Rouquier, B.–Kim, Chlouveraki).
- For $\zeta = 1$ and W Coxeter group, Rouquier blocks are nothing but the characters associated with two sided cells (Kazhdan–Lusztig theory).

3.3. Rouquier blocks

OMITTED IN ORDER TO LEAVE TIME FOR ALAN BAKER STORY.

3.4. Families and Rouquier blocks

Families

There is a partition

$$\mathrm{Un}(\mathbb{G}) = \bigsqcup_{\mathcal{F}\in\mathrm{Fam}(\mathbb{G})} \mathcal{F}$$

(where the $\mathcal{F}$'s are the families of unipotent characters), hence for all regular ζ,

$$\mathrm{Un}(\mathbb{G},\zeta) = \bigsqcup_{\mathcal{F}\in\mathrm{Fam}(\mathbb{G})} \left(\mathcal{F}\cap\mathrm{Un}(\mathbb{G},\zeta)\right),$$

with the following properties.

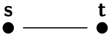
1. Through the bijection $\mathrm{Un}(\mathbb{G},\zeta) \xrightarrow{\sim} \mathrm{Irr}\,\mathcal{H}(W_\zeta)$, the nonempty intersections $\mathcal{F}\cap\mathrm{Un}(\mathbb{G},\zeta)$ are the Rouquier blocks of $\mathrm{Irr}\,\mathcal{H}(W_\zeta)$.
2. The integers a_ρ (valuation of Deg_ρ) and A_ρ (degree of Deg_ρ) are constant for ρ in a family $\mathcal{F}$.

3.4. Families and Rouquier blocks

OMITTED IN ORDER TO LEAVE TIME FOR ALAN BAKER STORY.

The Fourier matrices

Let us denote by $\mathbf{B}_2$ the braid group on three brands, generated by two elements $\mathbf{s}$ and $\mathbf{t}$ satisfying the relation

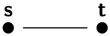


$$\mathbf{sts} = \mathbf{tst}\,.$$

Let us set $\mathbf{w}_0 := \mathbf{sts}$. It is known that

The Fourier matrices

Let us denote by $\mathbf{B}_2$ the braid group on three brands, generated by two elements $\mathbf{s}$ and $\mathbf{t}$ satisfying the relation



$$\mathbf{sts} = \mathbf{tst}\,.$$

Let us set $\mathbf{w}_0 := \mathbf{sts}$. It is known that

- the center of $\mathbf{B}_2$ is infinite cyclic and generated by $\mathbf{w}_0^2 = (\mathbf{sts})^2 = (\mathbf{st})^3$,

The Fourier matrices

Let us denote by $\mathbf{B}_2$ the braid group on three brands, generated by two elements $\mathbf{s}$ and $\mathbf{t}$ satisfying the relation

$$\overset{\mathbf{s}}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{\mathbf{t}}{\bullet} \qquad \mathbf{sts} = \mathbf{tst}\,.$$

Let us set $\mathbf{w}_0 := \mathbf{sts}$. It is known that

- the center of $\mathbf{B}_2$ is infinite cyclic and generated by $\mathbf{w}_0^2 = (\mathbf{sts})^2 = (\mathbf{st})^3$,
- the map

$$\mathbf{s} \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \ , \ \mathbf{t} \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

  induces an isomorphism $\mathbf{B}_2/\langle \mathbf{w}_0^4 \rangle \xrightarrow{\sim} \mathrm{SL}_2(\mathbb{Z})$.

Let $\mathcal{F}$ be a family in $\mathrm{Un}(\mathbb{G})$.

Let $\mathcal{F}$ be a family in $\mathrm{Un}(\mathbb{G})$.

The S-matric (Fourier matrix)

Let $\mathcal{F}$ be a family in $\mathrm{Un}(\mathbb{G})$.

The S-matric (Fourier matrix)

There is a complex matrix S with entries indexed by $\mathcal{F} \times \mathcal{F}$, such that for all $\chi_0 \in \mathrm{Irr}(W)$,

$$\sum_{\chi \in \mathrm{Irr}(W)} S_{\rho_\chi, \rho_{\chi_0}} \mathrm{Feg}_\chi = \mathrm{Deg}_{\rho_{\chi_0}} ,$$

Let $\mathcal{F}$ be a family in $\mathrm{Un}(\mathbb{G})$.

The S-matric (Fourier matrix)

There is a complex matrix S with entries indexed by $\mathcal{F} \times \mathcal{F}$, such that for all $\chi_0 \in \mathrm{Irr}(W)$,

$$\sum_{\chi \in \mathrm{Irr}(W)} S_{\rho_\chi, \rho_{\chi_0}} \mathrm{Feg}_\chi = \mathrm{Deg}_{\rho_{\chi_0}} ,$$

and with the following properties.

Let $\mathcal{F}$ be a family in $\mathrm{Un}(\mathbb{G})$.

The S-matric (Fourier matrix)

There is a complex matrix S with entries indexed by $\mathcal{F} \times \mathcal{F}$, such that for all $\chi_0 \in \mathrm{Irr}(W)$,

$$\sum_{\chi \in \mathrm{Irr}(W)} S_{\rho_\chi, \rho_{\chi_0}} \mathrm{Feg}_\chi = \mathrm{Deg}_{\rho_{\chi_0}} \, ,$$

and with the following properties.

1. S is unitary and symmetric,

Let $\mathcal{F}$ be a family in $\mathrm{Un}(\mathbb{G})$.

The S-matric (Fourier matrix)

There is a complex matrix S with entries indexed by $\mathcal{F} \times \mathcal{F}$, such that for all $\chi_0 \in \mathrm{Irr}(W)$,

$$\sum_{\chi \in \mathrm{Irr}(W)} S_{\rho_\chi, \rho_{\chi_0}} \mathrm{Feg}_\chi = \mathrm{Deg}_{\rho_{\chi_0}} ,$$

and with the following properties.

1. S is unitary and symmetric,
2. S^2 is an order 2 monomial matrix with entries in $\{\pm 1\}$,

Let $\mathcal{F}$ be a family in $\mathrm{Un}(\mathbb{G})$.

The S-matric (Fourier matrix)

There is a complex matrix S with entries indexed by $\mathcal{F} \times \mathcal{F}$, such that for all $\chi_0 \in \mathrm{Irr}(W)$,

$$\sum_{\chi \in \mathrm{Irr}(W)} S_{\rho_\chi, \rho_{\chi_0}} \mathrm{Feg}_\chi = \mathrm{Deg}_{\rho_{\chi_0}} ,$$

and with the following properties.

1. S is unitary and symmetric,
2. S^2 is an order 2 monomial matrix with entries in $\{\pm 1\}$,
3. there exists a special character of W (in the Rouquier block corresponding to $\mathcal{F}$) such that

Let $\mathcal{F}$ be a family in $\mathrm{Un}(\mathbb{G})$.

The S-matric (Fourier matrix)

There is a complex matrix S with entries indexed by $\mathcal{F} \times \mathcal{F}$, such that for all $\chi_0 \in \mathrm{Irr}(W)$,

$$\sum_{\chi \in \mathrm{Irr}(W)} S_{\rho_\chi, \rho_{\chi_0}} \mathrm{Feg}_\chi = \mathrm{Deg}_{\rho_{\chi_0}} ,$$

and with the following properties.

1. S is unitary and symmetric,
2. S^2 is an order 2 monomial matrix with entries in $\{\pm 1\}$,
3. there exists a special character of W (in the Rouquier block corresponding to $\mathcal{F}$) such that
   1. the corresponding row i_0 of S has no zero entry,

Let $\mathcal{F}$ be a family in $\mathrm{Un}(\mathbb{G})$.

The S-matric (Fourier matrix)

There is a complex matrix S with entries indexed by $\mathcal{F} \times \mathcal{F}$, such that for all $\chi_0 \in \mathrm{Irr}(W)$,

$$\sum_{\chi \in \mathrm{Irr}(W)} S_{\rho_\chi, \rho_{\chi_0}} \mathrm{Feg}_\chi = \mathrm{Deg}_{\rho_{\chi_0}} ,$$

and with the following properties.

1. S is unitary and symmetric,
2. S^2 is an order 2 monomial matrix with entries in $\{\pm 1\}$,
3. there exists a special character of W (in the Rouquier block corresponding to $\mathcal{F}$) such that
    1. the corresponding row i_0 of S has no zero entry,
    2. (Verlinde type formula) for all $i, j, k \in \mathcal{F}$, the sums $\sum_l S_{l,i} S_{l,j} S^*_{l,k} S^{-1}_{l,i_0}$ are integers.

Frobenius and Shintani matrices

Frobenius and Shintani matrices

- Let Ω be the diagonal matrix indexed by $\mathcal{F} \times \mathcal{F}$ whose diagonal term at $\rho \in \mathcal{F}$ is the Frobenius eigenvalue Fr_ρ.

Frobenius and Shintani matrices

- Let Ω be the diagonal matrix indexed by $\mathcal{F} \times \mathcal{F}$ whose diagonal term at $\rho \in \mathcal{F}$ is the Frobenius eigenvalue Fr_ρ.
- Define $\mathrm{Sh} := S \cdot \Omega \cdot S^{-1}$.

- Let Ω be the diagonal matrix indexed by $\mathcal{F} \times \mathcal{F}$ whose diagonal term at $\rho \in \mathcal{F}$ is the Frobenius eigenvalue Fr_ρ.
- Define $\mathrm{Sh} := S \cdot \Omega \cdot S^{-1}$.

Fact[a]

[a] *"There is a proof, but so far I've not seen an explanation"* [JHC]

Frobenius and Shintani matrices

- Let Ω be the diagonal matrix indexed by $\mathcal{F} \times \mathcal{F}$ whose diagonal term at $\rho \in \mathcal{F}$ is the Frobenius eigenvalue Fr_ρ.
- Define $\mathrm{Sh} := S \cdot \Omega \cdot S^{-1}$.

Fact[a]

[a] *"There is a proof, but so far I've not seen an explanation"* [JHC]

The map

$$\mathbf{s} \mapsto \Omega \ , \ \mathbf{t} \mapsto \mathrm{Sh}$$

induces a representation of $\mathrm{SL}_2(\mathbb{Z})$ onto the complex vector space with basis $\mathcal{F}$ such that $\mathbf{w}_0 \mapsto S$.

- Let Ω be the diagonal matrix indexed by $\mathcal{F} \times \mathcal{F}$ whose diagonal term at $\rho \in \mathcal{F}$ is the Frobenius eigenvalue Fr_ρ.
- Define $\mathrm{Sh} := S \cdot \Omega \cdot S^{-1}$.

Fact[a]

[a] *"There is a proof, but so far I've not seen an explanation"* [JHC]

The map

$$\mathbf{s} \mapsto \Omega \ , \ \mathbf{t} \mapsto \mathrm{Sh}$$

induces a representation of $\mathrm{SL}_2(\mathbb{Z})$ onto the complex vector space with basis $\mathcal{F}$ such that $\mathbf{w}_0 \mapsto S$.

All this makes us think of a kind of *modular datum*, and perhaps for the Spets of a kind of *triangulated modular tensor category (?)*.

The Fourier matrix for G_4

		01	02	12		01	34	04	25	13
	1	.	.	.	.	.	.	.	.	.
01	.	$\frac{1+\frac{1}{\sqrt{-3}}}{2}$	$\frac{1-\frac{1}{\sqrt{-3}}}{2}$	$\frac{-1}{\sqrt{-3}}$	.	.	.	.	.	.
02	.	$\frac{1-\frac{1}{\sqrt{-3}}}{2}$	$\frac{1+\frac{1}{\sqrt{-3}}}{2}$	$\frac{1}{\sqrt{-3}}$	.	.	.	.	.	.
12	.	$\frac{-1}{\sqrt{-3}}$	$\frac{1}{\sqrt{-3}}$	$\frac{-1}{\sqrt{-3}}$	.	.	.	.	.	.
	.	.	.	.	1	.	.	.	.	.
01	.	.	.	.	.	$\frac{1}{2\sqrt{-3}}$	$\frac{-1}{2\sqrt{-3}}$	$\frac{1}{2}$	$\frac{-1}{\sqrt{-3}}$	$\frac{1}{2}$
34	.	.	.	.	.	$\frac{-1}{2\sqrt{-3}}$	$\frac{1}{2\sqrt{-3}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{-3}}$	$\frac{1}{2}$
04	.	.	.	.	.	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	.	$-\frac{1}{2}$
25	.	.	.	.	.	$\frac{-1}{\sqrt{-3}}$	$\frac{1}{\sqrt{-3}}$	.	$\frac{-1}{\sqrt{-3}}$	.
13	.	.	.	.	.	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	.	$\frac{1}{2}$

Unipotent characters for G_4

Unipotent characters for G_4 ③—③

In red = the Φ'_6–series.

● = the Φ_4–series.

Character	Degree	FakeDegree	Eigenvalue	Family
● $\phi_{1,0}$	● 1	1	1	C_1
$\phi_{2,1}$	$\frac{3-\sqrt{-3}}{6}q\Phi'_3\Phi_4\Phi''_6$	$q\Phi_4$	1	$X_3.01$
$\phi_{2,3}$	$\frac{3+\sqrt{-3}}{6}q\Phi''_3\Phi_4\Phi'_6$	$q^3\Phi_4$	1	$X_3.02$
$Z_3 : 2$	$\frac{\sqrt{-3}}{3}q\Phi_1\Phi_2\Phi_4$	0	ζ_3^2	$X_3.12$
● $\phi_{3,2}$	● $q^2\Phi_3\Phi_6$	$q^2\Phi_3\Phi_6$	1	C_1
$\phi_{1,4}$	$\frac{-\sqrt{-3}}{6}q^4\Phi''_3\Phi_4\Phi''_6$	q^4	1	$X_5.1$
$\phi_{1,8}$	$\frac{\sqrt{-3}}{6}q^4\Phi'_3\Phi_4\Phi'_6$	q^8	1	$X_5.2$
● $\phi_{2,5}$	● $\frac{1}{2}q^4\Phi_2^2\Phi_6$	$q^5\Phi_4$	1	$X_5.3$
$Z_3 : 11$	$\frac{\sqrt{-3}}{3}q^4\Phi_1\Phi_2\Phi_4$	0	ζ_3^2	$X_5.4$
● G_4	● $\frac{1}{2}q^4\Phi_1^2\Phi_3$	0	-1	$X_5.5$

Φ'_3, Φ''_3 (resp. Φ'_6, Φ''_6) are factors of Φ_3 (resp Φ_6) in $\mathbb{Q}(\zeta_3)$