A generic Atlas for Spetses ?

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Finite Simple Groups: Thirty Years of the Atlas and Beyond

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In honor of John Conway
An example of a known Unknown : Spets

Long lasting joint work between Gunter Malle, Jean Michel, and myself, initiated in the Greek island named SPETSES in 1993 (there, we computed unipotent degrees, Frobenius eigenvalues, families, Fourier matrix, for the "generic fake finite reductive group" (Spets?) whose Weyl group is the cyclic group $\mu_3$ of order 3...), going on with the collaboration of Olivier Dudas, and more and more of Cédric Bonnafé.

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I. Generic point of view on finite reductive groups

Let $G$ be a connected reductive algebraic group over $\mathbb{F}_q$, with Weyl group $W$, group of co-characters $Y$, endowed with a rational structure over $\mathbb{F}_q$ via a Frobenius endomorphism $F$. The group $G := G_{\mathbb{F}_q}$ is called a finite reductive group.

To simplify the lecture (but this is bad) we assume that $G$ is split, i.e., $F$ acts on $Y$ by multiplication by $q$.

The type of $G$ is $G := (V, W)$ where $V := \mathbb{C} \otimes \mathbb{Z} Y$.

Lots of numerical data associated with $G$ come from evaluation at $x = q$ of polynomials in $x$ which depends only on the type $G$.

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Let us give some examples.
1. The polynomial order

The element of $\mathbb{Z}[x]$ defined by $|G|(x) = x^N \prod \zeta \mod \text{Gal} \Phi_\zeta(x) a(\zeta)$ for $L$ an $F$-stable Levi subgroup of $G$, its polynomial order $|L|(x)$ divides $|G|(x)$.

I insist: the polynomial order depends only on the type $(V, W)$ (not on a root datum).

Thus $|SO_{2n+1}(q)| = |Sp_{2n}(q)|$. Michel Broué
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- I insist: the polynomial order depends only on the type \( (V, W) \) (not on a root datum). Thus

\[
|SO_{2n+1}(q)| = |Sp_{2n}(q)|.
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2. Unipotent characters, generic degrees (Lusztig)

The set $\text{Un}(G)$ of unipotent characters of $G$ is parametrized by the set of unipotent generic characters $\text{Un}(G)$ (depending only on the type). Let us denote that parametrization by $\text{Un}(G) \rightarrow \text{Un}(G)$, $\rho \mapsto \rho_q$.

Generic degree: for all $\rho \in \text{Un}(G)$, there exists $\text{Deg}_\rho(x) \in \mathbb{Q}[x]$ such that $\text{Deg}_\rho(x) \mid x = q = \text{Deg}(\rho_q)$.

Of course, the (generic) degrees divide the (polynomial) order of $G$.

Every $\rho \in \text{Un}(G)$ comes equipped with a Frobenius eigenvalue $\text{Fr}_\rho$, a root of unity...which has something to do with the Deligne–Lusztig varieties $X_w$...but can also be defined by $
abla_{F/F}(\rho \chi) = \sum_{\rho \in \text{Un}(G)} \text{Fr}_\rho \langle R \chi; \rho \rangle$ for $\chi \in \text{Irr}(W)$.
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Generic unipotent characters, continued

For all $\zeta \in \mu$, partition of $\text{Un}(G)$ into $\zeta$-Harish-Chandra series.

Description of the principal $\zeta$-Harish-Chandra series with a $\zeta$-cyclotomic Hecke algebra.

The principal 1-Harish-Chandra series is the usual principal Harish-Chandra series.

Let us devote some time to the notion of $\zeta$-cyclotomic Hecke algebras.

Then we shall come back to the generic properties of $\text{Un}(G)$. 
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The principal $1$-Harish-Chandra series is the usual principal Harish-Chandra series.

Let us devote some time to the notion of $\zeta$-cyclotomic Hecke algebras.
Then we shall come back to the generic properties of $\text{Un}(G)$. 
What follows holds more generally for any pair $G = (V, W)$ where $V$ is a finite dimensional complex vector space, $W$ is a finite subgroup of $GL(V)$ generated by (pseudo)-reflections.

Let $A$ be the set of reflecting hyperplanes of $W$. A root of unity $\zeta$ is called regular if there exist $w \in W$ and $x \in V$ such that $w(x) = \zeta x$. We then say that $w$ is $\zeta$-regular. From now on we assume that $\zeta$ is regular. Then $[\text{Springer}]$ the group $W_\zeta := C_W(w)$ acts faithfully as a reflection group on the vector space $V_\zeta := \ker(w - \zeta \text{Id}_V)$. The group $W_\zeta$ is called the $\zeta$-cyclotomic Weyl group. [Note that $W_1 = W$.]
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3.1. Hecke algebras everywhere

One knows that the (ordinary) principal series $\text{Un}(G,1)$ corresponds to unipotent characters of $G$ occurring in $\mathbb{Q}_\ell(G/B)$, and the commutant of that module is the (ordinary) Hecke algebra of $W$ evaluated at $q$.

One conjectures that the $\zeta$–principal series $\text{Un}(G,\zeta)$ for a good choice of $w$ corresponds to unipotent characters of $G$ occurring in $\bigoplus H_n^c(X_w, \mathbb{Q}_\ell)$, and the commutant of that module is a $\zeta$-cyclotomic Hecke algebra of $W_\zeta$ evaluated at $q$.

We shall now introduce the notion of $\zeta$-cyclotomic Hecke algebra.
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One conjectures that the $\zeta$-principal series $\text{Un}(G, \zeta)_n$ for a good choice of $\omega$ corresponds to unipotent characters of $G$ occurring in $\bigoplus_n H_n^c(X_\omega, \mathcal{Q}_\ell)$, and the commutant of that module is a $\zeta$-cyclotomic Hecke algebra of $W_\zeta$ evaluated at $q$.

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We shall now introduce the notion of $\zeta$–cyclotomic Hecke algebra.
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We shall now introduce the notion of $\zeta$-cyclotomic Hecke algebra.
3.2. ζ-cyclotomic Hecke algebras

A ζ-cyclotomic Hecke algebra $H(W_ζ)$ of $W_ζ$ is in particular $\mathbb{C}[x, x−1]$-algebra, an image of the group algebra of the braid group $B_{W_ζ}$ attached to $W_ζ$, a deformation (via $x$) of the group algebra of $W_ζ$, which specializes to that algebra for $x = ζ$.

Examples:

- Case where $G = GL_3$, $ζ = 1$:
  
  $W_ζ = W_3 = S_3$ ←→ $H(W_3) = \langle S, T; STS = TST, (S−x)(S+1) = 0 \rangle$ is 1-cyclotomic.

- For $G = O_8(q)$, $W = D_4$, $ζ = i$, $W_i = G(4, 2, 2)$ ←→ $H(W_i) = \langle S, T, U; \{STU = TUS = UST \} \rangle$.
3.2. $\zeta$-cyclotomic Hecke algebras

- A $\zeta$-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of $W_\zeta$ is in particular a $C[x, x-1]$-algebra, an image of the group algebra of the braid group $B W_\zeta$ attached to $W_\zeta$, a deformation (via $x$) of the group algebra of $W_\zeta$, which specializes to that algebra for $x = \zeta$.

Examples:

- Case where $G = GL_3$, $\zeta = 1$: $W_\zeta = W = S_3$ is 1-cyclotomic.

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- Case where $G = \text{GL}_3$, $\zeta = 1$: $W_\zeta = W = S_3$ $\leftrightarrow$ $H(W) = \langle S, T; STS = TST, (S-x)(S+1) = 0 \rangle$ is 1-cyclotomic.
- For $G = \text{O}_8^-$ ($q$), $W = D_4$, $\zeta = i$, $W_i = G(4, 2, 2)$ $\leftrightarrow$ $H(W_i) = \langle S, T, U; \{ STU = TUS = UST \} \rangle$.
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Examples:
- Case where $G = \text{GL}_3$, $\zeta = 1$ : $W_\zeta = W = \mathfrak{S}_3$ $\iff$ $H(W) = \langle S, T ; STS = TST , (S - x)(S + 1) = 0 \rangle$ is 1-cyclotomic.

- For $G = \text{O}_8(q)$, $W = D_4$, $\zeta = i$, $W_i = G(4, 2, 2)$ $\iff$ $H(W) = \langle S, T , U ; TUS = STU, (S - x^2)(S + 1) = 0 \rangle$ is 2-cyclotomic.
3.2. $\zeta$-cyclotomic Hecke algebras

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- Examples:
  - Case where $G = \text{GL}_3$, $\zeta = 1$: $W_\zeta = W = \mathfrak{S}_3 \leftrightarrow s \circledast t$
    
    $$\mathcal{H}(W) = \left\langle S, T \ ; \ STS = TST \ , \ (S - x)(S + 1) = 0 \right\rangle \text{ is 1-cyclotomic.}$$

  - For $G = \text{O}_8(q)$, $W = D_4$, $\zeta = i$, $W_i = G(4, 2, 2) \leftrightarrow s \circledast t \circledast u$
    
    $$\mathcal{H}(W_i) = \left\langle S, T, U \ ; \begin{cases} STU = TUS = UST \\ (S - x^2)(S - 1) = 0 \end{cases} \right\rangle$$
Fundamental properties

Case by case checking...

There is a proof, but so far I've not seen an explanation.

Such an algebra has a canonical symmetrizing form $\tau$. It becomes split semisimple over $\mathbb{C}(\frac{x_1}{|ZW\zeta|}, \frac{x-1}{|ZW\zeta|})$. Hence each absolute irreducible character $\chi$ of $H(W\zeta)$ is equipped with a Schur element $S\chi \in \mathbb{C}[\frac{x_1}{|ZW\zeta|}, \frac{x-1}{|ZW\zeta|}]$ defined by $\tau = \sum_{\chi \in \text{Irr}H(W\zeta)} \chi S\chi$. 

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$$S_\chi \in \mathbb{C}[x^{1/|ZW_\zeta|}, x^{-1/|ZW_\zeta|}] \quad \text{defined by} \quad \tau = \sum_{\chi \in \text{Irr} \mathcal{H}(W_\zeta)} \frac{\chi}{S_\chi}. $$
3.3. Spetsial $\zeta$-cyclotomic Hecke algebras

**Definition**

A $\zeta$-cyclotomic Hecke algebra $\mathcal{H}(W_\zeta)$ of $W_\zeta$ is spetsial for $G$ if...
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2. for each absolute irreducible character $\chi$ of $\mathcal{H}(W_\zeta)$,

$$S_\chi \in \mathbb{C}[x, x^{-1}]$$

(and not only $\mathbb{C}[x^{1/|ZW_\zeta|}, x^{-1/|ZW_\zeta|}]$).
On $\text{Un}(G)$ again

When we started speaking about $\zeta$-regular elements, $\zeta$-cyclotomic Weyl groups, spetsial $\zeta$-cyclotomic Hecke algebras, we were stating “generic properties” of unipotent characters:

... lots of other properties, like the partition of $\text{Un}(G)$ into $\zeta$-Harish-Chandra series:

6. Description of the principal $\zeta$-Harish-Chandra series with a spetsial $\zeta$-cyclotomic Hecke algebra.

Let us come back to that long list.

7. Partition of $\text{Un}(G)$ into families and their intersections with $\zeta$-Harish-Chandra series (Rouquier blocks).

8. Ennola permutation on $\text{Un}(G)$.

[This is an abstract formulation of the fact that $\text{U}_n(\mathbb{Q}) = \pm \text{GL}_n(\mathbb{Q})$]


We shall review this now in the more general context of “Spetses.”
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[This is an abstract formulation of the fact that \(U_n(q) = \pm GL_n(−q)\)]

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9 Fourier matrices and $\text{SL}_2(\mathbb{Z})$-representation.

We shall review this now in the more general context of “Spetses”.
II. Towards Spetses

Try to treat a complex reflection group as a Weyl group: try to build a thing $G(x^a)$ associated with a type $G = (V, W)$ where $W$ is a (pseudo)-reflection group.

Try at least to build "unipotent characters" of $G$, or at least to build their degrees (polynomials in $x$), Frobenius eigenvalues (roots of unity), Fourier matrices.

Lusztig knew already a solution for Coxeter groups which are not Weyl groups (except the Fourier matrix for $H_4$ which was determined by Malle in 1994).

Malle gave a solution for imprimitive spetsial complex reflection groups in 1995.

Stating a long series of precise axioms — many of technical nature — we can now show that there is a unique solution for all primitive spetsial complex reflection groups.

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A double object: double $N$

$N = (V, W)$, where $V$ is a complex vector space of dimension $r$, $W$ is a finite (pseudo)-reflection subgroup of $\text{GL}(V)$, $\mathcal{A}(W) :=$ the hyperplanes arrangement of $W$, $N_{\text{hyp}} :=$ number of reflecting hyperplanes, $N_{\text{ref}} :=$ number of reflections. $N_{\text{hyp}} = N_{\text{ref}}$ if $W$ is generated by true reflections.
A double object: double $N$

$\mathbb{G} = (V, W)$, where

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\[ G = (V, W), \text{ where} \]

- \( V \) is a complex vector space of dimension \( r \),
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A double object: double $\mathcal{N}$

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$\mathcal{G} = (V, W)$, where

- $V$ is a complex vector space of dimension $r$,
- $W$ is finite (pseudo)-reflection subgroup of $\text{GL}(V)$,
- $A(W) :=$ the hyperplanes arrangement of $W$.

- $N^\text{hyp}_W :=$ number of reflecting hyperplanes,
- $N^\text{ref}_W :=$ number of reflections.

$N^\text{hyp}_W = N^\text{ref}_W$ if $W$ is generated by true reflections.
Double polynomial order

\[ G_c(x) := (-1)^r x^N \sum_{w \in W_1} \text{det} V(1 - wx)^* \]

The compact and the noncompact order coincide if \( W \) is generated by true reflections.

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Double polynomial order

\[ |G_c|(x) := (-1)^r x^{N^\text{hyp}_W} \frac{1}{|W| \sum_{w \in W} \det_V (1 - wx)^*} \]

\[ |G_{nc}|(x) := (-1)^r x^{N^\text{ref}_W} \frac{1}{|W| \sum_{w \in W} \det_V (1 - wx)^*} \]

The compact and the noncompact order coincide if \( W \) is generated by true reflections.
Double polynomial order

\[ |G_c| (x) := (-1)^r x^{N_{W}^{hyp}} \frac{1}{|W| \sum_{w \in W} \frac{1}{\det_V (1 - wx)^*}} \]

\[ |G_{nc}| (x) := (-1)^r x^{N_{W}^{ref}} \frac{1}{|W| \sum_{w \in W} \frac{1}{\det_V (1 - wx)^*}} \]

The compact and the noncompact order coincide if \( W \) is generated by true reflections.
2. Spetsial $\zeta$-cyclotomic Hecke algebras for $G$

As above, $W_\zeta = C_W(w)$ is the centralizer of a $\zeta$-regular element $w \in W$, $A_\zeta$-cyclotomic Hecke algebra $H(W_\zeta)$ is spetsial for $G$ if

1. For each absolute irreducible character $\chi$ of $H(W_\zeta)$, $S_\chi \in \mathbb{C}[x, x^{-1}]$.

2. $H(W_\zeta)$ satisfies various technical conditions, which split into

- compact type conditions,
- noncompact type conditions.

These conditions coincide if $W$ is generated by true reflections.
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  1. for each absolute irreducible character $\chi$ of $\mathcal{H}(W_\zeta)$,
     \[ S_\chi \in \mathbb{C}[x, x^{-1}] , \]
  2. and $\mathcal{H}(W_\zeta)$ satisfies various technical conditions, which split into
     - compact type conditions,
     - noncompact type conditions.

These conditions coincide if $W$ is generated by true reflections.
Theorem

A 1-cyclotomic Hecke algebra can be spetsial of compact type for $G$ only if it is the algebra $H^c(W)$ defined by

$$H^c(W) = \langle s_H \rangle_{H \in A}$$

with relations:

$$(s_H - x)(x^{e_H-1} + x^{e_H-2}s_H + \cdots + s_{e_H-1}) = 0$$

if $s_H$ has order $e_H$.

A 1-cyclotomic Hecke algebra can be spetsial of noncompact type for $G$ only if it is the algebra $H^{nc}(W)$ defined by

$$H^{nc}(W) = \langle s_H \rangle_{H \in A}$$

with relations:

$$(s_H - x)(x^{e_H-1} + x^{e_H-2}s_H + \cdots + s_{e_H-1}) = 0$$.
2.1. Spetsial 1-cyclotomic Hecke algebras, special groups

Theorem

A 1-cyclotomic Hecke algebra can be *spetsial of compact type* for $\mathbb{G}$ only if it is the algebra $\mathcal{H}_{c}(W)$ defined by

$$
\begin{aligned}
\mathcal{H}_{c}(W) &= \langle s_{H} \rangle_{H \in \mathcal{A}} \quad \text{with relations:} \\
(s_{H} - x)(1 + s_{H} + \cdots + s_{H}^{e_{H} - 1}) &= 0 \quad \text{(if $s_{H}$ has order $e_{H}$)}
\end{aligned}
$$

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2.1. Spetsial 1-cyclotomic Hecke algebras, special groups

**Theorem**

1. A 1-cyclotomic Hecke algebra can be *spetsial of compact type* for $\mathbb{G}$ only if it is the algebra $\mathcal{H}^c(W)$ defined by

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2. A 1-cyclotomic Hecke algebras can be *spetsial of noncompact type* for $\mathbb{G}$ only if it is the algebra $\mathcal{H}^{nc}(W)$ defined by

$$\mathcal{H}^{nc}(W) = \langle s_H \rangle_{H \in \mathcal{A}} \quad \text{with relations:}$$

$$\left(s_H - x \right) \left(x^{e_H-1} + x^{e_H-2} s_H + \cdots + s_H^{e_H-1} \right) = 0.$$
2.2. Spetsial groups

Let $H(W)$ denote either $H_c(W)$ or $H_{nc}(W)$.

Theorem (G. Malle)–Definition

Assume $W$ acts irreducibly on $V$. The following assertions are equivalent.

(i) $H(W)$ is spetsial.

(ii) For each absolutely irreducible character $\chi$ of $H(W)$, $S_\chi \in \mathbb{C}[x, x^{-1}]$.

(iii) $W$ is one of the following groups (Shephard–Todd's notation), called the spetsial groups:

- $G(d, 1, n)$ ($d, n \geq 1$),
- $G(e, e, n)$ ($e, n \geq 2$),
- all groups $G_i$ ($4 \leq i \leq 37$) well generated by true reflections, $G_4, G_6, G_8, G_{25}, G_{26}, G_{32}$. 

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**Theorem (G. Malle)–Definition**

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(i) $\mathcal{H}(W)$ is special.

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- $G_4$, $G_6$, $G_8$, $G_{25}$, $G_{26}$, $G_{32}$. 
3. Some data associated with spetsial groups

Given $G = (V, W)$ where $W$ is special, there are the set $Un(G_c)$ of unipotent characters (compact type), the set $Un(G_{nc})$ of unipotent characters (noncompact type), which coincide if $W$ is generated by true reflections each of them (denoted $Un(G)$ below), endowed with two maps the map degree $\text{Deg} : Un(G) \to \mathbb{C}[x]$, $\rho \mapsto \text{Deg} \rho(x)$, defined up to sign, the map Frobenius eigenvalue $\rho \mapsto \text{Fr} \rho$, where $\text{Fr} \rho$ is a root of unity, a bijection (Alvis–Curtis duality) $Un(G_c) \to Un(G_{nc})$, $\rho \mapsto \rho_{nc}$, with lots of properties (axioms) described below.
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- the map degree
  \[ \text{Deg} : \text{Un}(\mathcal{G}) \to \mathbb{C}[x], \quad \rho \mapsto \text{Deg}_\rho(x), \]
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defined up to sign,

- the map Frobenius eigenvalue $\rho \mapsto \text{Fr}_\rho$, where $\text{Fr}_\rho$ is a root of unity,

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  \[ \text{Un}(G_c) \rightarrow \text{Un}(G_{nc}), \quad \rho \mapsto \rho^{nc}, \]
Given $G = (V, W)$ where $W$ is special, there are

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with lots of properties (axioms) described below.
3.1. First axioms

Connection compact / noncompact

\[ \deg \rho_{nc}(x) = x_{Nref} W \deg \rho(\frac{1}{x})^\ast, \]

up to sign!

Fr \rho_{nc} = 1.

From now on we only describe the compact type case.

Definition Let \( \zeta \in \mu \).

The \( \zeta \)-principal series is

\[ \text{Un}(G, \zeta) := \{ \rho \in \text{Un}(G) | \deg \rho(\zeta) \neq 0 \}. \]
3.1. First axioms

Connection compact / noncompact
3.1. First axioms

Connection compact / noncompact

1. $\text{Deg}_{\rho}^{nc}(x) = x^{N_{W}^{\text{ref}}} \text{Deg}_{\rho}(1/x)^{*}$,
3.1. First axioms

Connection compact / noncompact

1. \( \text{Deg}_{\rho_{\text{nc}}} (x) = x^{N_W^{\text{ref}}} \text{Deg}_{\rho} (1/x)^* , \) up to sign!
3.1. First axioms

Connection compact / noncompact

1. \( \text{Deg}_{\rho_{nc}}(x) = x^{N^\text{ref}_W} \text{Deg}_{\rho}(1/x)^* \), \( \uparrow \) up to sign!

2. \( \text{Fr}_\rho \text{Fr}_{\rho_{nc}} = 1 \).
3.1. First axioms

Connection compact / noncompact

1. $\text{Deg}_{\rho_{nc}}(x) = x^{N^\text{ref}_{W}} \text{Deg}_{\rho}(1/x)^*$, \(\uparrow\) up to sign!

2. $\text{Fr}_{\rho} \text{Fr}_{\rho_{nc}} = 1$.

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3.1. First axioms

Connection compact / noncompact

1. $\text{Deg}_{\rho_{nc}}(x) = x^{N_{W}^{\text{ref}}} \text{Deg}_{\rho}(1/x)^{\ast}$,  \(\uparrow\) up to sign!

2. $\text{Fr}_{\rho} \text{Fr}_{\rho_{nc}} = 1$.

From now on we only describe the compact type case.

**Definition**

Let $\zeta \in \mu$. 
3.1. First axioms

Connection compact / noncompact

1. \( \text{Deg}_{\rho^{nc}}(x) = x^{N_{\overline{W}}} \text{Deg}_\rho(1/x)^* \), up to sign!

2. \( \text{Fr}_\rho \text{Fr}_{\rho^{nc}} = 1 \).

From now on we only describe the compact type case.

Definition

Let \( \zeta \in \mu \).
The \( \zeta \)-principal series is

\[
\text{Un}(G, \zeta) := \{ \rho \in \text{Un}(G) \mid \text{Deg}_\rho(\zeta) \neq 0 \}.
\]
3.2. $\zeta$-Axioms

For $w \in W$ a $\zeta$-regular element, there are a special $\zeta$-cyclotomic Hecke algebra of compact type $H(W)$ associated with $w$, and a bijection $\text{Irr} H(W) \xrightarrow{\sim} \text{Un}(G, \zeta)$, $\chi \mapsto \rho_\chi$ such that

1. $\text{Deg} \rho_\chi(x) = \pm |G|(x) : |T_w|(x) \cdot |S_\chi(x)|$,

2. $\text{Fr} \rho_\chi$ explicit formula depending only on $H(W)$ and $\chi$. 

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3.2. $\zeta$-Axioms

For $w \in W$ a $\zeta$-regular element, there are...
3.2. $\zeta$-Axioms

For $w \in W$ a $\zeta$-regular element, there are

- a special $\zeta$-cyclotomic Hecke algebra of compact type $\mathcal{H}(W_\zeta)$ associated with $w$, 

\[\text{Deg } \rho_\chi(x) = \pm |G(x) : |T_w(x)| x', S_\chi(x),\]

\[\text{Fr } \rho_\chi = \text{explicit formula depending only on } \mathcal{H}(W_\zeta) \text{ and } \chi.\]
3.2. $\zeta$-Axioms

For $w \in W$ a $\zeta$-regular element, there are

- a spetsial $\zeta$-cyclotomic Hecke algebra of compact type $\mathcal{H}(W_\zeta)$
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- and a bijection

$$\text{Irr} \, \mathcal{H}(W_\zeta) \xrightarrow{\sim} \text{Un}(G, \zeta) \, , \, \chi \mapsto \rho_\chi$$
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$$\text{Irr } \mathcal{H}(W_\zeta) \xrightarrow{\sim} \text{Un}(G, \zeta), \quad \chi \mapsto \rho_\chi$$

such that

$\text{Deg } \rho_\chi(x) = \pm \frac{|\mathcal{G}(x)|}{|T_w(x)|} x'$
3.2. \( \zeta \)-Axioms

For \( w \in W \) a \( \zeta \)-regular element, there are

- a spetsial \( \zeta \)-cyclotomic Hecke algebra of compact type \( \mathcal{H}(W_\zeta) \)
  associated with \( w \),

- and a bijection

\[
\text{Irr } \mathcal{H}(W_\zeta) \sim \rightarrow \text{Un}(G, \zeta), \; \chi \mapsto \rho_\chi
\]

such that

\[ \text{Deg}_{\rho_\chi}(x) = \pm \frac{[|G|(x) : |T_w|(x)]_{x'}}{S_\chi(x)}, \]

where

- \( \text{Deg}_{\rho_\chi}(x) \) is the degree of the character \( \rho_\chi 
- \( |G|(x) \) is the size of the group
- \( |T_w|(x) \) is the size of the stabilizer
- \( S_\chi(x) \) is a function of \( x \)

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3.2. \( \zeta \)-Axioms

For \( w \in W \) a \( \zeta \)-regular element, there are

- a spetsial \( \zeta \)-cyclotomic Hecke algebra of compact type \( \mathcal{H}(W_{\zeta}) \) associated with \( w \),
- and a bijection

\[
\text{Irr} \mathcal{H}(W_{\zeta}) \xrightarrow{\sim} \text{Un}(G, \zeta), \ \chi \mapsto \rho_{\chi}
\]

such that

1. \( \text{Deg}_{\rho_{\chi}}(x) = \pm \frac{[G : \text{T}_{w}(x)]_{x'}}{S_{\chi}(x)} \),
2. \( \text{Fr}_{\rho_{\chi}} = \) explicit formula depending only on \( \mathcal{H}(W_{\zeta}) \) and \( \chi \).
3.3. Rouquier blocks

If the representation of $W$ on $V$ is rational over some cyclotomic field $K$, the $\zeta$-cyclotomic Hecke algebra $H(W_\zeta)$ may be defined over $\mathbb{Z}_K[x, x^{-1}]$.

**Definition**

The Rouquier blocks of a $\zeta$-cyclotomic Hecke algebra $H(W_\zeta)$ are the blocks of the algebra $\mathbb{Z}_K[x, x^{-1}, (x^n - 1) - 1] \otimes \mathbb{Z}_K[x, x^{-1}] H(W_\zeta)$.

The Rouquier blocks of $\zeta$-cyclotomic Hecke algebras have been classified in all cases (Malle–Rouquier, B.–Kim, Chlouveraki).

For $\zeta = 1$ and $W$ Coxeter group, Rouquier blocks are nothing but the characters associated with two sided cells (Kazhdan–Lusztig theory).

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A generic Atlas for Spetses?
3.3. Rouquier blocks

- If the representation of $W_{\zeta}$ on $V_{\zeta}$ is rational over some cyclotomic field $K$, the $\zeta$-cyclotomic Hecke algebra $\mathcal{H}(W_{\zeta})$ may be defined over $\mathbb{Z}_K[x, x^{-1}]$.

**Definition**

The **Rouquier blocks** of a $\zeta$-cyclotomic Hecke algebra $\mathcal{H}(W_{\zeta})$ are the blocks of the algebra

$$\mathbb{Z}_K[x, x^{-1}, ((x^n - 1)^{-1})_{n \geq 1}] \otimes_{\mathbb{Z}[x, x^{-1}]} \mathcal{H}(W_{\zeta}).$$

- The Rouquier blocks of $\zeta$-cyclotomic Hecke algebras have been classified in all cases (Malle–Rouquier, B.–Kim, Chlouveraki).
- For $\zeta = 1$ and $W$ Coxeter group, Rouquier blocks are nothing but the characters associated with two sided cells (Kazhdan–Lusztig theory).
3.3. Rouquier blocks

OMITTED in order to leave time for Alan Baker story.
3.4. Families and Rouquier blocks

There is a partition $\text{Un}(\mathcal{G}) = \bigsqcup_{F \in \text{Fam}(\mathcal{G})} F$, hence for all regular $\zeta$, $\text{Un}(\mathcal{G}, \zeta) = \bigsqcup_{F \in \text{Fam}(\mathcal{G})} (F \cap \text{Un}(\mathcal{G}, \zeta))$, with the following properties.

1. Through the bijection $\text{Un}(\mathcal{G}, \zeta) \sim \rightarrow \text{Irr}_H(W_\zeta)$, the nonempty intersections $F \cap \text{Un}(\mathcal{G}, \zeta)$ are the Rouquier blocks of $\text{Irr}_H(W_\zeta)$.

2. The integers $a_\rho$ (valuation of $\text{Deg}_\rho$) and $A_\rho$ (degree of $\text{Deg}_\rho$) are constant for $\rho$ in a family $F$. 

Michel Broué
A generic Atlas for Spetses ?
3.4. Families and Rouquier blocks

Families

There is a partition

\[ \text{Un}(G) = \bigsqcup_{F \in \text{Fam}(G)} F \]

(where the \( F \)'s are the families of unipotent characters), hence for all regular \( \zeta \),

\[ \text{Un}(G, \zeta) = \bigsqcup_{F \in \text{Fam}(G)} (F \cap \text{Un}(G, \zeta)) , \]

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1. Through the bijection \( \text{Un}(G, \zeta) \sim \rightarrow \text{Irr} \mathcal{H}(W_\zeta) \), the nonempty intersections \( F \cap \text{Un}(G, \zeta) \) are the Rouquier blocks of \( \text{Irr} \mathcal{H}(W_\zeta) \).
2. The integers \( a_\rho \) (valuation of \( \text{Deg}_\rho \)) and \( A_\rho \) (degree of \( \text{Deg}_\rho \)) are constant for \( \rho \) in a family \( F \).
OMITTED in order to leave time for Alan Baker story.
Let us denote by $B_2$ the braid group on three brands, generated by two elements $s$ and $t$ satisfying the relation

$$s \cdot t \cdot st = tst.$$ 

Let us set $w_0 := sts$. It is known that
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$$w_0^2 = (sts)^2 = (st)^3,$$
The Fourier matrices

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- the center of $B_2$ is infinite cyclic and generated by $w_0^2 = (sts)^2 = (st)^3$,
- the map
  $$s \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$ 
  induces an isomorphism $B_2/\langle w_0^4 \rangle \sim \text{SL}_2(\mathbb{Z})$. 

Michel Broué

A generic Atlas for Spetses?
Let $\mathcal{F}$ be a family in $\text{Un}(G)$. 
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The $S$-matric (Fourier matrix)

1. $S$ is unitary and symmetric,
2. $S^2$ is an order 2 monomial matrix with entries in $\{\pm 1\}$,
3. there exists a special character of $W$ (in the Rouquier block corresponding to $\mathcal{F}$) such that
   - the corresponding row $i_0$ of $S$ has no zero entry,
   - (Verlinde type formula) for all $i, j, k \in \mathcal{F}$, the sums
     \[ \sum_l S^l, i S^l, j S^*, k S^{-1} l, i_0 \]
     are integers.
Let \( \mathcal{F} \) be a family in \( \text{Un}(G) \).

**The S-matric (Fourier matrix)**

There is a complex matrix \( S \) with entries indexed by \( \mathcal{F} \times \mathcal{F} \), such that for all \( \chi_0 \in \text{Irr}(W) \),

\[
\sum_{\chi \in \text{Irr}(W)} S_{\rho_\chi, \rho_{\chi_0}} \text{Feg}_\chi = \text{Deg}_{\rho_{\chi_0}},
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Frobenius and Shintani matrices

Let $\Omega$ be the diagonal matrix indexed by $F \times F$ whose diagonal term at $\rho \in F$ is the Frobenius eigenvalue $F_r$. Define $S := S \cdot \Omega \cdot S^{-1}$.

Fact: There is a proof, but so far I've not seen an explanation.

The map $s \mapsto \Omega, t \mapsto S$ induces a representation of $SL_2(\mathbb{Z})$ onto the complex vector space with basis $F$ such that $w_0 \mapsto S$. All this makes us think of a kind of modular datum, and perhaps for the Spets of a kind of triangulated modular tensor category (?).
Frobenius and Shintani matrices

- Let $\Omega$ be the diagonal matrix indexed by $\mathcal{F} \times \mathcal{F}$ whose diagonal term at $\rho \in \mathcal{F}$ is the Frobenius eigenvalue $\text{Fr}_\rho$. 

"There is a proof, but so far I've not seen an explanation" [JHC]

The map $s \mapsto \Omega$, $t \mapsto Sh$ induces a representation of $\text{SL}_2(\mathbb{Z})$ onto the complex vector space with basis $F_0 \mapsto S$. 

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The Fourier matrix for $G_4$

<table>
<thead>
<tr>
<th></th>
<th>01</th>
<th>02</th>
<th>12</th>
<th>01</th>
<th>34</th>
<th>04</th>
<th>25</th>
<th>13</th>
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<td>.</td>
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<tr>
<td>02</td>
<td>.</td>
<td>$1 + \frac{1}{\sqrt{-3}}$</td>
<td>$1 - \frac{1}{\sqrt{-3}}$</td>
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<td>.</td>
<td>.</td>
<td>.</td>
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</tr>
<tr>
<td>12</td>
<td>.</td>
<td>$1 - \frac{1}{\sqrt{-3}}$</td>
<td>$1 + \frac{1}{\sqrt{-3}}$</td>
<td>$1$</td>
<td>.</td>
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<tr>
<td></td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>$\frac{1}{2\sqrt{-3}}$</td>
<td>$\frac{1}{2\sqrt{-3}}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>04</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>$-\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
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</tr>
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</tr>
</tbody>
</table>
Unipotent characters for $G_4$

In red = the $\Phi'_6$–series.
• = the $\Phi_4$–series.

<table>
<thead>
<tr>
<th>Character</th>
<th>Degree</th>
<th>FakeDegree</th>
<th>Eigenvalue</th>
<th>Family</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_{1,0}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$C_1$</td>
</tr>
<tr>
<td>$\phi_{2,1}$</td>
<td>3</td>
<td>$-\sqrt{-3}/6$</td>
<td>$q\Phi'_3\Phi_4\Phi''_6$</td>
<td>$X_3.01$</td>
</tr>
<tr>
<td>$\phi_{2,3}$</td>
<td>$3+\sqrt{-3}/6$</td>
<td>$q\Phi'_3\Phi_4\Phi''_6$</td>
<td>$X_3.02$</td>
<td></td>
</tr>
<tr>
<td>$\phi_{3,2}$</td>
<td>2</td>
<td>$q^{1/2}\Phi_3\Phi_6$</td>
<td>$C_1$</td>
<td></td>
</tr>
<tr>
<td>$\phi_{1,4}$</td>
<td>4</td>
<td>$-\sqrt{-3}/6$</td>
<td>$q\Phi'_3\Phi_4\Phi''_6$</td>
<td>$X_5.1$</td>
</tr>
<tr>
<td>$\phi_{1,8}$</td>
<td>8</td>
<td>$\sqrt{-3}/6$</td>
<td>$q\Phi'_3\Phi_4\Phi''_6$</td>
<td>$X_5.2$</td>
</tr>
<tr>
<td>$\phi_{2,5}$</td>
<td>2</td>
<td>$q^{1/2}\Phi_2\Phi_6$</td>
<td>$Z_3:2$</td>
<td></td>
</tr>
<tr>
<td>$\phi_{3,4}$</td>
<td>4</td>
<td>$-\sqrt{-3}/3$</td>
<td>$q\Phi_1\Phi_2\Phi_4$</td>
<td>$X_5.3$</td>
</tr>
<tr>
<td>$\phi_{2,7}$</td>
<td>7</td>
<td>$q^{1/2}\Phi_2\Phi_6$</td>
<td>$Z_3:11$</td>
<td></td>
</tr>
<tr>
<td>$\Phi'_3$, $\Phi'_6$</td>
<td>(resp. $\Phi''_3$, $\Phi''_6$)</td>
<td>are factors of $\Phi_3$ (resp $\Phi_6$) in $\mathbb{Q}(\zeta_3)$</td>
<td></td>
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In red = the $\Phi_6'$–series.
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<tr>
<td>$\bullet \phi_{1,0}$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td>$C_1$</td>
</tr>
<tr>
<td>$\phi_{2,1}$</td>
<td>$\frac{3-\sqrt{-3}}{6} q \Phi_3 \Phi_4 \Phi_6''$</td>
<td>$q\Phi_4$</td>
<td>$1$</td>
<td>$X_3.01$</td>
</tr>
<tr>
<td>$\phi_{2,3}$</td>
<td>$\frac{3+\sqrt{-3}}{6} q \Phi_3'' \Phi_4 \Phi_6'$</td>
<td>$q^3\Phi_4$</td>
<td>$1$</td>
<td>$X_3.02$</td>
</tr>
<tr>
<td>$Z_3 : 2$</td>
<td>$\sqrt{-\frac{3}{3}} q \Phi_1 \Phi_2 \Phi_4$</td>
<td>$0$</td>
<td>$\zeta_3^2$</td>
<td>$X_3.12$</td>
</tr>
<tr>
<td>$\bullet \phi_{3,2}$</td>
<td>$q^2\Phi_3 \Phi_6$</td>
<td>$q^2\Phi_3 \Phi_6$</td>
<td>$1$</td>
<td>$C_1$</td>
</tr>
<tr>
<td>$\phi_{1,4}$</td>
<td>$-\frac{\sqrt{-3}}{6} q^4 \Phi_3'' \Phi_4 \Phi_6'$</td>
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<td>$X_5.1$</td>
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<tr>
<td>$\bullet \phi_{2,5}$</td>
<td>$\frac{1}{2} q^4 \Phi_2^2 \Phi_6$</td>
<td>$q^5\Phi_4$</td>
<td>$1$</td>
<td>$X_5.3$</td>
</tr>
<tr>
<td>$Z_3 : 11$</td>
<td>$\sqrt{-\frac{3}{3}} q^4 \Phi_1 \Phi_2 \Phi_4$</td>
<td>$0$</td>
<td>$\zeta_3^2$</td>
<td>$X_5.4$</td>
</tr>
<tr>
<td>$\bullet G_4$</td>
<td>$\frac{1}{2} q^4 \Phi_1^2 \Phi_3$</td>
<td>$0$</td>
<td>$-1$</td>
<td>$X_5.5$</td>
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$\Phi_3', \Phi_3''$ (resp. $\Phi_6', \Phi_6''$) are factors of $\Phi_3$ (resp $\Phi_6$) in $\mathbb{Q}(\zeta_3)$