$\mathrm{GL}_n(x)$ for x an indeterminate ?

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February 2012

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They can be viewed from the *algebraic groups* point of view, as follows.

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As if there were an object $\mathbb{G}(x)$ such that $\mathbb{G}(x)|_{x=q} = \mathbf{G}(q)$.

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- N is the number of reflecting hyperplanes of the Weyl group of $\mathbf{G}$. Hence $\mathbf{G}$ has a trivial Weyl group, *i.e.*, $\mathbf{G}$ is a torus

$$\mathbf{G} \cong \overline{\mathbb{F}}_q^\times \times \cdots \times \overline{\mathbb{F}}_q^\times$$

if and only if its (polynomial) order is not divisible by x.

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Note that, for $d = 1$ and $\varphi = \pm 1$, one has $W_1 = W$.

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where $\boldsymbol{\mu}_d$ denotes the cyclic group of all d-th roots of unity.

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If *"not general", then a Sylow ℓ-subgroup of G an extension of $Z^0(L_d)_\ell$ by W_ℓ.*

Unipotent characters, generic degrees

1. The set $\mathrm{Un}(G)$ of unipotent characters of G is parametrized by a "generic" (*i.e.*, independant of q) set $\mathrm{Un}(\mathbb{G})$. We denote by

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Ennola : $$\lambda_{\mathrm{U}_n(q)}(u^\mu_{\mathrm{U}_n(q)}) = \pm V_{\lambda,\mu}(x)|_{x=-q} \, .$$

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- For each element $w \in W$, one may define the Deligne–Lusztig variety $\mathbf{X}_w$, a subvariety of $\mathbf{G}/\mathbf{B}$, acted on by G.

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Complex reflection groups

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- Irreducible finite reflection groups over $\mathbb{C}$ have been classified (Shephard–Todd, 1954).

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  - Stating now a long series of precise axioms — many of technical nature — we can now show that there is a unique solution for all primitive *spetsial* complex reflection groups.

Spetsial groups

Spetsial groups in red.

$G(e,1,r)$, $G(e,e,r)$, and

Group G_n	4	5	6	7	8	9	10	11	12	13	14	15	16
Rank	2	2	2	2	2	2	2	2	2	2	2	2	2

Group G_n	17	18	19	20	21	22	23	24	25	26	27
Rank	2	2	2	2	2	2	3	3	3	3	3
Remark							H_3				

Group G_n	28	29	30	31	32	33	34	35	36	37
Rank	4	4	4	4	4	5	6	6	7	8
Remark	F_4		H_4					E_6	E_7	E_8

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ρ	$\mathrm{Deg}(\rho)$	$\mathrm{Fr}(\rho)$
χ_a	1	1
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Where is the Steinberg character ?

Unipotent characters for G_4

Unipotent characters for G_4 ③—③

In red = the Φ'_6–series.

● = the Φ_4–series.

Character	Degree	FakeDegree	Eigenvalue	Family
● $\phi_{1,0}$	● 1	1	1	C_1
$\phi_{2,1}$	$\frac{3-\sqrt{-3}}{6}q\Phi'_3\Phi_4\Phi''_6$	$q\Phi_4$	1	$X_3.01$
$\phi_{2,3}$	$\frac{3+\sqrt{-3}}{6}q\Phi''_3\Phi_4\Phi'_6$	$q^3\Phi_4$	1	$X_3.02$
$Z_3 : 2$	$\frac{\sqrt{-3}}{3}q\Phi_1\Phi_2\Phi_4$	0	ζ_3^2	$X_3.12$
● $\phi_{3,2}$	● $q^2\Phi_3\Phi_6$	$q^2\Phi_3\Phi_6$	1	C_1
$\phi_{1,4}$	$\frac{-\sqrt{-3}}{6}q^4\Phi''_3\Phi_4\Phi''_6$	q^4	1	$X_5.1$
$\phi_{1,8}$	$\frac{\sqrt{-3}}{6}q^4\Phi'_3\Phi_4\Phi'_6$	q^8	1	$X_5.2$
● $\phi_{2,5}$	● $\frac{1}{2}q^4\Phi_2^2\Phi_6$	$q^5\Phi_4$	1	$X_5.3$
$Z_3 : 11$	$\frac{\sqrt{-3}}{3}q^4\Phi_1\Phi_2\Phi_4$	0	ζ_3^2	$X_5.4$
● G_4	● $\frac{1}{2}q^4\Phi_1^2\Phi_3$	0	-1	$X_5.5$

Φ'_3, Φ''_3 (resp. Φ'_6, Φ''_6) are factors of Φ_3 (resp Φ_6) in $\mathbb{Q}(\zeta_3)$