GL$_n(x)$ for $x$ an indeterminate?

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Let $q$ be a prime power. There is (up to non unique isomorphism) a single field with $q$ elements, denoted $\mathbb{F}_q$. 

Since Chevalley (1955), one knows how to construct Lie groups over $\mathbb{F}_q$, analog of the usual complex reductive Lie groups: 

$GL_n(\mathbb{F}_q)$, $O_n(\mathbb{F}_q)$, $Sp_n(\mathbb{F}_q)$, $U_n(\mathbb{F}_q)$,...

denoted respectively $GL_n(q)$, $O_n(q)$, $Sp_n(q)$, $U_n(q)$,...

For example $U_n(q) := \{ U \in \text{Mat}_n(\mathbb{F}_q^2) \mid U \cdot t U^* = 1 \}$. 

There are also groups of exceptional types $G_2$, $F_4$, $E_6$, $E_7$, $E_8$ over $\mathbb{F}_q$.

They can be viewed from the algebraic groups point of view, as follows.
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For example $U_n(\mathbb{F}_q) := \{ U \in \text{Mat}_{n \times n}(\mathbb{F}_q^2) | U^t U = 1 \}$. There are also groups of exceptional types $G_2$, $F_4$, $E_6$, $E_7$, $E_8$ over $\mathbb{F}_q$. They can be viewed from the algebraic groups point of view, as follows.
Finite Reductive Groups

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There are also groups of exceptional types $G_2, F_4, E_6, E_7, E_8$ over $\mathbb{F}_q$.

They can be viewed from the algebraic groups point of view, as follows.
Let $G$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, endowed with a Frobenius endomorphism $F$ which defines an $\mathbb{F}_q$-rational structure. Then the group $G := G(\overline{\mathbb{F}}_q) := G^F$ is a finite reductive group over $\mathbb{F}_q$.

Example: Assume $G = GL_n(\mathbb{F}_q)$. Then $G = GL_n(\overline{\mathbb{F}}_q)$. For $F: (a_{ij}) \mapsto (a_{q^i,j})$, $G = U_n(\mathbb{F}_q)$.

Let $T \sim = \mathbb{F}_q \times \mathbb{F}_q \times \cdots \mathbb{F}_q$ be an $\mathbb{F}_q$-stable maximal torus of $G$.

The Weyl group of $G$ is $W := N_G(T) / T$.

Example: For $G = GL_n(\mathbb{F}_q)$, $T = \begin{pmatrix} \mathbb{F}_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbb{F}_q \end{pmatrix}$ and $W = S_n$. Michel Broué
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\textbf{Example:}

Assume $G = \text{GL}_n(\mathbb{F}_q)$.

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F & \cdots & 0 \\
0 & \ddots & \vdots \\
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Set $V := \mathbb{C} \otimes_{\mathbb{Z}} Y(T)$, a finite dimensional complex vector space. Then $W$ acts on $V$ as a reflection group, and the Frobenius endomorphism $F$ acts on $V$ as $q\varphi$, where $\varphi$ is a finite order element of $N_{\text{GL}(V)}(W)$. 

Example:

For $G = \text{GL}_n(q)$, its type is $G = \text{GL}_n(q) = (C_n, S_n)$.

For $G = \text{U}_n(q)$, its type is $G = \text{U}_n(q) = (C_n, -S_n)$. 

Main fact: Lots of data about $G = \text{GL}_n(q)$ are values at $x = q$ of polynomials in $x$ which depend only on the type $G$. As if there were an object $G(x)$ such that $G(x) |_{x = q} = G(q)$. 

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As if there were an object $G(x)$ such that $G(x)|_{x=q} = G(q)$. 
Let $G = (V, W\varphi)$. 

R. Steinberg (1967): There is a polynomial (element of $\mathbb{Z}[x]$) $|G|_G(x) = x \prod_{d|\varphi} \Phi_d(x)^{a(d)}$ such that $|G|_G(q) = |G(q)| = |G|$. 

Example | $|GL_n|_G(x) = x^{n^2} \prod_{d|n} \Phi_d(x)^{n/d}$ | $|U_n|_G(x) = \pm |GL_n|_{-x}$ (well, precisely $(-1)^{n^2/2} |GL_n|_{-x}$).
Polynomial order

Let $G = (V, W_\varphi)$.

R. Steinberg (1967): There is a polynomial (element of $\mathbb{Z}[x]$)

$$|G|(x) = x^N \prod_{d} \Phi_d(x)^{a(d)}$$

such that $|G|(q) = |G(q)| = |G|$. 

Example $|GL_n(x)| = x^{n^2} \prod_{d} (x^d - 1)^{\lfloor n/d \rfloor}$
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**Example**

- $|GL_n|(x) = x^{\binom{n}{2}} \prod_{d=1}^{d=n}(x^d - 1) = x^{\binom{n}{2}} \prod_{d=1}^{d=n} \Phi_d(x)^{[n/d]}$
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Example

\begin{itemize}
  \item $|GL_n|(x) = x^{\binom{n}{2}} \prod_{d=1}^{\lfloor n/d \rfloor} (x^d - 1) = x^{\binom{n}{2}} \prod_{d=1}^{\lfloor n/d \rfloor} \Phi_d(x)^{[n/d]}
  \item $|U_n|(x) = \pm |GL_n|(-x)$
\end{itemize}
Let $G = (V, W_\varphi)$.

R. Steinberg (1967) : There is a polynomial (element of $\mathbb{Z}[x]$)

$$\left| G \right|(x) = x^N \prod_{d} \Phi_d(x)^{a(d)}$$

such that

$$\left| G \right|(q) = \left| G(q) \right| = \left| G \right|.$$ 

Example

- $\left| GL_n \right|(x) = x^\binom{n}{2} \prod_{d=1}^{d=n} (x^d - 1) = x^\binom{n}{2} \prod_{d=1}^{d=n} \Phi_d(x)^{[n/d]}$

- $\left| U_n \right|(x) = \pm \left| GL_n \right|(-x)$ (well, precisely $(-1)^{\binom{n}{2}} \left| GL_n \right|(-x)$).
Remarks

The prime divisors of $|G|$ are $x$ and cyclotomic polynomials $\Phi_d(x)$.

$N$ is the number of reflecting hyperplanes of the Weyl group of $G$.

Hence $G$ has a trivial Weyl group, i.e., $G \cong F \times q \times \cdots \times F \times q$ if and only if its (polynomial) order is not divisible by $x$.

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- The prime divisors of $|G|(x)$ are $x$ and cyclotomic polynomials $\Phi_d(x)$.

- $N$ is the number of reflecting hyperplanes of the Weyl group of $G$. Hence $G$ has a trivial Weyl group, i.e., $G$ is a torus $G \cong \mathbb{F}_q^\times \times \cdots \times \mathbb{F}_q^\times$ if and only if its (polynomial) order is not divisible by $x$. 

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$GL_n(x)$ for $x$ an indeterminate?
Admissible subgroups

The tori of $G$ are the subgroups of the shape $T = T(q) = T_F \times T_F \times \cdots \times T_F \times q \times \cdots$ where $T_F$ is an $F$–stable torus of $G$.

The Levi subgroups of $G$ are the subgroups of the shape $L = L(q) = L_F$ where $L = C_G(T)$ is the centralizer of an $F$–stable torus in $G$.

Examples for $GL_n(q)$:

- The split maximal torus $T_1 = (F \times q)^n$ of order $(q^n - 1)^n$.
- The Coxeter torus $T_c = GL_1(F_q^n)$ of order $q^n - 1$.
- Levi subgroups have shape $GL_n(q_{a_1}) \times \cdots \times GL_n(q_{a_s})$.
Admissible subgroups

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The following definition is the generic version of $p$-subgroups of $G$.

$\Phi_d(x)$–groups

For $\Phi_d(x)$ a cyclotomic polynomial, a $\Phi_d(x)$–group is a finite reductive group whose (polynomial) order is a power of $\Phi_d(x)$.

A $\Phi_d(x)$–group is a torus.

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Michel Broué
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Maximal $\Phi_d(x)$-subgroups ("Sylow $\Phi_d(x)$-subgroups") $S_d$ of $G$ have as (polynomial) order the contribution of $\Phi_d(x)$ to the (polynomial) order of $G$:

$$|S_d| = |S_d(q)| = \Phi_d(q) a(d).$$

Notation: Set $L_d := C_G(S_d)$ and $N_d := N_G(S_d) = N_G(L_d)$.

Sylow $\Phi_d(x)$-subgroups are all conjugate by $G$.

The (polynomial) index $|G:N_d|$ is congruent to 1 modulo $\Phi_d(x)$.

$W_d := N_d/L_d$ is a true finite group, a complex reflection group in its action on $V_d := \mathbb{C} \otimes Y(S_d)$.

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Example for $\text{GL}_n$

Recall that

$$|\text{GL}_n(x)| = x^\left(\begin{array}{c} n \\ 2 \end{array}\right) \prod_{d=1}^{d=n} \Phi_d(x)^{[n/d]}$$

For each $d$ ($1 \leq d \leq n$), $\text{GL}_n(q^d)$ contains a subtorus of (polynomial) order $\Phi_d(x)^{[n/d]}$.

Assume $n = md + r$ with $r < d$. Then

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Generic and ordinary Sylow subgroups

Let \( \ell \) be a prime number.

If \( \ell \) divides \( |G| = |G^q| \), let \( d \) be the order of \( q \) modulo \( \ell \) (so \( \ell \) divides \( \Phi_d(q) \)).

Let \( S_d \) be a Sylow \( \Phi_d(x) \)-subgroup of \( G \), and let \( S_\ell \) be the Sylow \( \ell \)-subgroup of \( S_d \).

Let \( W_\ell \) be a Sylow \( \ell \)-subgroup of the \( d \)-cyclotomic Weyl group \( W_d \).

(M. Enguehard)

1. A Sylow \( \ell \)-subgroup of \( N_d = N_{G}(S_d) \) is a Sylow \( \ell \)-subgroup of \( G \).

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Michel Broué
The set $\text{Un}(G)$ of \textit{unipotent characters} of $G$ is parametrized by a “generic” (\textit{i.e.}, independant of $q$) set $\text{Un}(\mathbb{G})$. We denote by

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Unipotent characters, generic degrees

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2. Generic degree: For $\rho \in \text{Un}(\mathbb{G})$, there exists $\text{Deg}_\rho(x) \in \mathbb{Q}[x]$ such that

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Example for $\text{GL}_n$: For $\lambda = (\lambda_1 \leq \cdots \leq \lambda_m)$ a partition of $n$, let $\beta_i : \lambda_i + i - 1$. Then

$$\text{Deg}_\lambda(x) = \frac{(x - 1) \cdots (x^n - 1) \prod_{j>i}(x^{\beta_j} - x^{\beta_i})}{x^{(m_2-1)+(m_2-2)+\cdots} \prod_i \prod_{j=1}^{\beta_i} (x^j - 1)}.$$
The (polynomial) degree $\text{Deg}_\rho(x)$ of a unipotent character divides the (polynomial) order $|G|(x)$. 

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For $\lambda$ and $\mu$ partitions of $n$, let $\lambda_q$ be the corresponding unipotent character of $\text{GL}_n(q)$, and let $u^\mu_q$ be a unipotent element of $\text{GL}_n(q)$ of type $\mu$. There exists a polynomial $V_{\lambda,\mu}(x)$ such that $\lambda_q(u^\mu_q) = V_{\lambda,\mu}(x) |x= q$.

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In $\text{U}_n(q)$, unipotent classes and unipotent characters are parametrized by partitions of $n$ as well. For $\lambda$ and $\mu$ partitions of $n$, let $\lambda_{U_n}(q)$ be the corresponding unipotent character, and let $u^\mu_{U_n}(q)$ be a unipotent element of type $\mu$. Ennola: $\lambda_{U_n}(q)(u^\mu_{U_n}(q)) = \pm V_{\lambda,\mu}(x) |x= -q$. 
The (polynomial) degree $\text{Deg}_\rho(x)$ of a unipotent character divides the (polynomial) order $|G|(x)$.

More: character values!
The (polynomial) degree \( \text{Deg}_\rho(x) \) of a unipotent character divides the (polynomial) order \( |G|(x) \).

More: character values! In \( \text{GL}_n(q) \), unipotent classes are also parametrized by partitions of \( n \).
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More: $\text{U}_n(q) = \text{GL}_n(-q)$!

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Michel Broué

$\text{GL}_n(x)$ for $x$ an indeterminate?
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More generic evidence

Lots of other behaviors or data for $GL_n$ may be viewed as obtained from its type $GL_n$ evaluated at $x = q$.

The $\ell$–modular representation theory of $G$ (here for simplicity we only consider the type $GL_n$).

One may define a notion of $\Phi_d(x)$–blocks of characters of $GL_n$ (the so-called $\Phi_d(x)$–Harish-Chandra theory).

Now, given $\ell$ which divides $\Phi_d(q)$, in order to find the $\ell$–blocks:

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Michel Broué
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Complex reflection groups

A finite reflection group on a field $K$ is a finite subgroup of $\text{GL}_k(V)$, where $V$ is a finite-dimensional $K$–vector space, generated by reflections, i.e., linear maps represented by
\[
\begin{pmatrix}
\zeta & \cdots & 0 \\
0 & \ddots & \vdots \\
\vdots & \ddots & \zeta \\
0 & \cdots & 1
\end{pmatrix}
\]

A finite reflection group on $\mathbb{R}$ is called a Coxeter group. A finite reflection group on $\mathbb{Q}$ is called a Weyl group.

Irreducible finite reflection groups over $\mathbb{C}$ have been classified (Shephard–Todd, 1954).
A finite reflection group on a field \( K \) is a finite subgroup of \( \text{GL}_K(V) \) (\( V \) a finite dimensional \( K \)-vector space) generated by reflections, i.e., linear maps represented by

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\vdots & \vdots & \ddots & \vdots \\
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Try to treat a complex reflection group as a Weyl group: try to build a family of objects providing polynomials depending only on the type $G$ associated with a complex reflection group... “like” the finite reductive groups are associated with their Weyl group. Try at least to build unipotent characters of $G$, or at least to build their degrees (polynomials in $x$), satisfying all the machinery of Harish-Chandra series, families, Frobenius eigenvalues, Fourier matrices... Lusztig knew already a solution for Coxeter groups which are not Weyl groups (except the Fourier matrix for $H_4$ which was determined by Malle in 1994). Malle gave a solution for imprimitive spetsial complex reflection groups in 1995. Stating now a long series of precise axioms — many of technical nature — we can now show that there is a unique solution for all primitive spetsial complex reflection groups.
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Spetsial groups

Spetsial groups in red.

\(G(e, 1, r), G(e, e, r)\), and

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The case of the cyclic group of order $3 : \{1, \zeta, \zeta^2\}$

Unipotent degrees and Frobenius eigenvalues
The case of the cyclic group of order 3: \( \{1, \zeta, \zeta^2\} \)

Unipotent degrees and Frobenius eigenvalues

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<td>( \zeta^2 )</td>
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Two families: \( \{\chi_a\} \), \( \{\chi_b, \chi_c, \gamma\} \)
The case of the cyclic group of order 3: \{1, \zeta, \zeta^2\}

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<tr>
<td>χ_c</td>
<td>\frac{1}{1-\zeta}x(x - \zeta)</td>
<td>1</td>
</tr>
<tr>
<td>γ</td>
<td>\frac{\zeta}{1-\zeta^2}x(x - 1)</td>
<td>\zeta^2</td>
</tr>
</tbody>
</table>

Two families: \{χ_a\}, \{χ_b, χ_c, γ\}

Where is the Steinberg character?
Unipotent characters for $G_4$

In red = the $\Phi'_6$–series.
• = the $\Phi_4$–series.

<table>
<thead>
<tr>
<th>Character</th>
<th>Degree</th>
<th>FakeDegree</th>
<th>Eigenvalue</th>
<th>Family</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi_1,0$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$C_1$</td>
</tr>
<tr>
<td>$\Phi_2,1$</td>
<td>3</td>
<td>$-\sqrt{-3}/6$</td>
<td>$q \Phi'_3 \Phi_4 \Phi''_6$</td>
<td>$\Phi_4^1 X_3.1$</td>
</tr>
<tr>
<td>$\Phi_2,3$</td>
<td>3+</td>
<td>$\sqrt{-3}/6$</td>
<td>$q \Phi'_3 \Phi_4 \Phi''_6$</td>
<td>$\Phi_4^1 X_3.2$</td>
</tr>
<tr>
<td>$\Phi_3,2$</td>
<td>2</td>
<td>$q^2 \Phi_3 \Phi_6$</td>
<td>$q^2 \Phi_3 \Phi_6$</td>
<td>$C_1$</td>
</tr>
<tr>
<td>$\Phi_1,4$</td>
<td>4</td>
<td>$-\sqrt{-3}/4$</td>
<td>$q \Phi'_3 \Phi_4 \Phi''_6$</td>
<td>$\Phi_4^1 X_5.1$</td>
</tr>
<tr>
<td>$\Phi_1,8$</td>
<td>8</td>
<td>$\sqrt{-3}/4$</td>
<td>$q \Phi'_3 \Phi_4 \Phi''_6$</td>
<td>$\Phi_4^1 X_5.2$</td>
</tr>
<tr>
<td>$\Phi_2,5$</td>
<td>5</td>
<td>$1/2 q^4 \Phi_2 \Phi_6$</td>
<td>$1/2 q^4 \Phi_2 \Phi_6$</td>
<td>$\Phi_4^1 X_5.3$</td>
</tr>
</tbody>
</table>

$\Phi'_3, \Phi_4, \Phi''_6$ (resp. $\Phi'_6, \Phi_4, \Phi''_6$) are factors of $\Phi_3$ (resp $\Phi_6$) in $Q(\zeta_3)$. 

Michel Broué
Unipotent characters for $G_4$

In red = the $\Phi'_6$–series.
• = the $\Phi_4$–series.

<table>
<thead>
<tr>
<th>Character</th>
<th>Degree</th>
<th>FakeDegree</th>
<th>Eigenvalue</th>
<th>Family</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_{1,0}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$C_1$</td>
</tr>
<tr>
<td>$\phi_{2,1}$</td>
<td>$\frac{3-\sqrt{-3}}{6}q\Phi'_3\Phi_4\Phi''_6$</td>
<td>$q\Phi_4$</td>
<td>1</td>
<td>$X_3.01$</td>
</tr>
<tr>
<td>$\phi_{2,3}$</td>
<td>$\frac{3+\sqrt{-3}}{6}q\Phi''_3\Phi_4\Phi'_6$</td>
<td>$q^3\Phi_4$</td>
<td>1</td>
<td>$X_3.02$</td>
</tr>
<tr>
<td>$Z_3 : 2$</td>
<td>$\sqrt{-3}q\Phi_1\Phi_2\Phi_4$</td>
<td>0</td>
<td>$\zeta_3^2$</td>
<td>$X_3.12$</td>
</tr>
<tr>
<td>$\phi_{3,2}$</td>
<td>$q^2\Phi_3\Phi_6$</td>
<td>$q^2\Phi_3\Phi_6$</td>
<td>1</td>
<td>$C_1$</td>
</tr>
<tr>
<td>$\phi_{1,4}$</td>
<td>$\frac{-\sqrt{-3}}{6}q^4\Phi'_3\Phi_4\Phi''_6$</td>
<td>$q^4$</td>
<td>1</td>
<td>$X_5.1$</td>
</tr>
<tr>
<td>$\phi_{1,8}$</td>
<td>$\frac{\sqrt{-3}}{6}q^4\Phi'_3\Phi_4\Phi'_6$</td>
<td>$q^8$</td>
<td>1</td>
<td>$X_5.2$</td>
</tr>
<tr>
<td>$\phi_{2,5}$</td>
<td>$\frac{1}{2}q^4\Phi_2^2\Phi_6$</td>
<td>$q^5\Phi_4$</td>
<td>1</td>
<td>$X_5.3$</td>
</tr>
<tr>
<td>$Z_3 : 11$</td>
<td>$\sqrt{-3}q^4\Phi_1\Phi_2\Phi_4$</td>
<td>0</td>
<td>$\zeta_3^2$</td>
<td>$X_5.4$</td>
</tr>
<tr>
<td>$G_4$</td>
<td>$\frac{1}{2}q^4\Phi_1^2\Phi_3$</td>
<td>0</td>
<td>$-1$</td>
<td>$X_5.5$</td>
</tr>
</tbody>
</table>

$\Phi'_3, \Phi''_3$ (resp. $\Phi'_6, \Phi''_6$) are factors of $\Phi_3$ (resp $\Phi_6$) in $\mathbb{Q}(\zeta_3)$