# Rickard Equivalences and Block Theory

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## 1. Introduction

### Control of fusion.

Let $G$ be a finite group, and let $p$ be a prime number.

**1.1. Definition.** *We say that a subgroup $H$ of $G$ controls the fusion of $p$–subgroups of $G$ if the following two conditions are fulfilled :*

(C1) *$H$ contains a Sylow $p$–subgroup $S_p$ of $G$,*
(C2) *whenever $P$ is a subgroup of $S_p$ and $g$ is an element of $G$ such that $gPg^{-1} \subseteq S_p$, there exist $z$ in the centralizer $C_G(P)$ of $P$ in $G$, and $h$ in $H$, such that $g = hz$.*

**1.2. Basic example.** We denote by $\mathrm{O}_{p'}G$ the largest normal subgroup of $G$ with order prime to $p$. Then if $H$ is a subgroup of $G$ which "covers the quotient" $G/\mathrm{O}_{p'}G$ (*i.e.,* if $G = H\mathrm{O}_{p'}G$), then $H$ controls the fusion of $p$–subgroups of $G$.

The following two results provide fundamental examples where the converse is true. The first one is due to Frobenius and was proved in 1905. The second one was proved by Glauberman for the case $p = 2$ (see [Gl]), and for odd $p$ it is a consequence of the classification of non abelian finite simple groups (see also [Ro] for an approach not using the classification).

**1.3. Theorems.**

(Fr) *Assume that a Sylow $p$–subgroup $S_p$ of $G$ controls the fusion of $p$–subgroups of $G$. Then $G = S_p\mathrm{O}_{p'}G$.*
(Gl) *Assume that there exists a $p$–subgroup $P$ of $G$ whose centralizer $C_G(P)$ controls the fusion of $p$–subgroups of $G$. Then $G = C_G(P)\mathrm{O}_{p'}G$.*

### Groups with abelian Sylow $p$–subgroups.

A classical example where a subgroup controls the fusion of $p$–subgroups is given by an old result of Burnside :

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*Assume that the Sylow $p$–subgroups of $G$ are abelian. Let $H$ be the normalizer of one of them. Then $H$ controls the fusion of $p$–subgroups of $G$.*

If $G$ is $p$–solvable, this is once again a particular case of 1.2, since it is not difficult to prove the following result.

**1.4. Proposition.** *Let $G$ be $p$–solvable and let $H$ be the normalizer of a Sylow $p$–subgroup. If the Sylow $p$–subgroups of $G$ are abelian, then $G = H\mathrm{O}_{p'}G$.*

The situation may look quite different if $G$ is a non abelian simple group with abelian Sylow $p$–subgroups. Indeed, in this case, we have $G \neq H\mathrm{O}_{p'}G$ whenever $H$ normalizes a non-trivial $p$–subgroup of $G$.

For example, there seems to be an enormous difference between the Monster group, a non abelian simple group of order

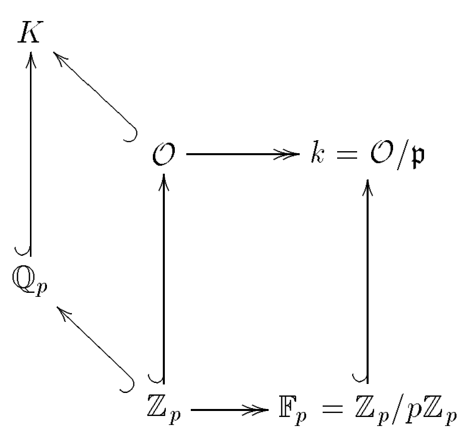
$$2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \simeq 8.10^{53} \,,$$

and the normalizer of one of its Sylow 11–subgroups, a group of order 72600, isomorphic to $(C_{11} \times C_{11}) \rtimes (C_5 \times \mathrm{SL}_2(5))$ (here we denote by $C_m$ the cyclic group of order $m$).

Nevertheless, there still is a strong connection between $G$ and the normalizer of a Sylow $p$–subgroup, which is a kind of generalization of the "factorization situation" given by theorem 1.3. In order to express this connection, we need to introduce the language of block theory.

## The principal block.

Let $K$ be a finite extension of the field of $p$–adic numbers $\mathbb{Q}_p$ which contains the $|G|$-th roots of unity. Thus the group algebra $KG$ is a split semi–simple $K$–algebra. Let $\mathcal{O}$ be the ring of integers of $K$ over $\mathbb{Z}_p$. We denote by $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$, and we set $k := \mathcal{O}/\mathfrak{p}$. If $JkG$ denotes the Jacobson radical of the group algebra $kG$, the algebra $kG/JkG$ is a split semi–simple $k$–algebra.




The decomposition of the unity element of $\mathcal{O}G$ into a sum of orthogonal primitive central idempotents $1 = \sum e$ corresponds to the decomposition of the algebra $\mathcal{O}G$ into a direct sum of indecomposable two–sided ideals $\mathcal{O}G = \bigoplus B$ ($B = \mathcal{O}Ge$), called the *blocks* of $\mathcal{O}G$. For $B$ a block of $\mathcal{O}G$, we set $KB := K \otimes_{\mathcal{O}} B$ and $kB := k \otimes_{\mathcal{O}} B$.

By reduction modulo $\mathfrak{p}$, a primitive central idempotent remains primitive central, and consequently $kG = \bigoplus kB$ is still a decomposition into a direct sum of indecomposable two–sided ideals, called the blocks of $kG$.

$$\begin{array}{ccccc} \mathcal{O}G & = & \bigoplus & B \\ \downarrow & & & \downarrow \\ kG & = & \bigoplus & kB \end{array}$$

The augmentation map $\mathcal{O}G \to \mathcal{O}$ factorizes through a unique block of $\mathcal{O}G$ called *the principal block* and denoted by $B_p(G)$.

If $B$ is a block of $\mathcal{O}G$, we denote by $\mathrm{Irr}(B)$ the set of all isomorphism classes of irreducible representations of the algebra $KB$. The set $\mathrm{Irr}(B)$ will be identified with a subset of the set $\mathrm{Irr}(G)$ of characters of irreducible representations of $G$ over $K$.

For $\chi \in \mathrm{Irr}(G)$, we denote by $\ker(\chi)$ the kernel of the corresponding representation of $G$. In other words, we have

$$\ker(\chi) = \{z \in G\,;\, (\forall g \in G)(\chi(zg) = \chi(g))\}\,.$$

The factorization $G = H\mathrm{O}_{p'}G$ can be interpreted in terms of principal blocks as follows.

**1.5. Proposition.**

*(1) We have* $\bigcap_{\chi \in \mathrm{Irr}(B_p(G))} \ker(\chi) = \mathrm{O}_{p'}G$.

*(2) If $H$ is a subgroup of $G$, the following assertions are equivalent*

(i) $G = H\mathrm{O}_{p'}G$ ,

(ii) *the map* $\mathrm{Res}_H^G$ *induces a bijection from* $\mathrm{Irr}(B_p(G))$ *onto* $\mathrm{Irr}(B_p(H))$.

What happens in the general case (where the Sylow $p$–subgroups of $G$ are abelian and $H$ is the normalizer of one of them) will first be illustrated by the example of the group $G = \mathfrak{A}_5$. The bijection $\mathrm{Res}_H^G$ of the previous proposition is replaced by a "bijection with signs" between $\mathrm{Irr}(B_p(G))$ and $\mathrm{Irr}(B_p(H))$.

**The case of $G = \mathfrak{A}_5$.**

Let $G$ be the alternating group on five letters. Then $|G| = 2^2 \cdot 3 \cdot 5$, and for all prime number $p$ which divides $|G|$, the Sylow $p$–subgroups of $G$ are abelian. Let us examine the principal $p$–blocks of $G$ and their connections with the corresponding Sylow normalizers.

1.6. Character table of $\mathfrak{A}_5$

| | (1) | (2) | (3) | (5) | (5′) |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_4$ | 4 | 0 | 1 | $-1$ | $-1$ |
| $\chi_5$ | 5 | 1 | $-1$ | 0 | 0 |
| $\chi_3$ | 3 | $-1$ | 0 | $(1+\sqrt{5})/2$ | $(1-\sqrt{5})/2$ |
| $\chi_3'$ | 3 | $-1$ | 0 | $(1-\sqrt{5})/2$ | $(1+\sqrt{5})/2$ |

We have

$$\begin{aligned} \mathrm{Irr}(B_2(G)) &= \{1\,,\,\chi_5\,,\,\chi_3\,,\,\chi_3'\}\,, \\ \mathrm{Irr}(B_3(G)) &= \{1\,,\,\chi_4\,,\,\chi_5\}\,, \\ \mathrm{Irr}(B_5(G)) &= \{1\,,\,\chi_4\,,\,\chi_3\,,\,\chi_3'\}\,,\,. \end{aligned}$$

For each $p \in \{2,3,5\}$, let us denote by $S_p$ a Sylow $p$–subgroup of $G$. We shall point out that there exists an isomorphism

$$I_p \colon \mathbb{Z}\mathrm{Irr}(B_p(N_G(S_p))) \xrightarrow{\sim} \mathbb{Z}\mathrm{Irr}(B_p(G))$$

such that :

(I1) $I_p$ is an isometry,
(I2) it preserves the character degrees modulo $p$,
(I3) it preserves the values on the $p$-elements.

The preceding properties express the fact that $I_p$ is an *isotypy* between $B_p(G)$ and $B_p(N_G(S_p))$, as we shall explain below in §2.

*The case* $p = 2$.

The normalizer $N_G(S_2)$ of a Sylow 2–subgroup $S_2$ of $G$ is isomorphic to the alternating group $\mathfrak{A}_4$. The principal block $B_2(N_G(S_2))$ is its only 2–block. Let us change the sign of certain irreducible characters in its character table.

1.7. Character table of $\mathfrak{A}_4$

| | (1) | (2) | (3) | (3′) |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| $-\alpha_3$ | $-3$ | 1 | 0 | 0 |
| $-\alpha_1$ | $-1$ | $-1$ | $(1+\sqrt{-3})/2$ | $(1-\sqrt{-3})/2$ |
| $-\alpha_1'$ | $-1$ | $-1$ | $(1-\sqrt{-3})/2$ | $(1+\sqrt{-3})/2$ |

We denote by $I_2$ the map

$$I_2 \colon \begin{pmatrix} 1 \\ -\alpha_3 \\ -\alpha_1 \\ -\alpha_1' \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ \chi_5 \\ \chi_3 \\ \chi_3' \end{pmatrix}.$$

*The case* $p = 3$.

The normalizer $N_G(S_3)$ of a Sylow 3–subgroup $S_3$ of $G$ is isomorphic to the symmetric group $\mathfrak{S}_3$. The principal block $B_3(N_G(S_3))$ is its only 3–block.

1.8. Character table of $\mathfrak{S}_3$

| | (1) | (2) | (3) |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| $\beta_1$ | 1 | $-1$ | 1 |
| $\beta_2$ | 2 | 0 | $-1$ |

We denote by $I_3$ the map

$$I_3 : \begin{pmatrix} 1 \\ \beta_1 \\ \beta_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ \chi_4 \\ \chi_5 \end{pmatrix} .$$

*The case* $p = 5$.

The normalizer $N_G(S_5)$ of a Sylow 5–subgroup $S_5$ of $G$ is isomorphic to the dihedral group $\mathrm{D}_5$. The principal block $B_5(N_G(S_5))$ is its only 5–block. Let us change the sign of certain irreducible characters in its character table.

1.9. Character table of $\mathrm{D}_5$

| | (1) | (2) | (5) | (5′) |
|---|---|---|---|---|
| $1$ | $1$ | $1$ | $1$ | $1$ |
| $-\gamma_1$ | $-1$ | $1$ | $-1$ | $-1$ |
| $-\gamma_2$ | $-2$ | $0$ | $(1+\sqrt{5})/2$ | $(1-\sqrt{5})/2$ |
| $-\gamma_2'$ | $-2$ | $0$ | $(1-\sqrt{5})/2$ | $(1+\sqrt{5})/2$ |

We denote by $I_5$ the map

$$I_5 : \begin{pmatrix} 1 \\ -\gamma_1 \\ -\gamma_2 \\ -\gamma_2' \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ \chi_4 \\ \chi_3 \\ \chi_3' \end{pmatrix} .$$

*Remark.* We can also notice that

- if one changes 5 to $-3$, then $I_2(\alpha)$ takes also on the 3–elements of $G$ the same values as $\alpha$ takes on the 5–elements of $N_G(S_2)$,
- $I_3(\beta)$ takes also on the 2–elements of $G$ the same values as $\beta$ takes on the 5–elements of $N_G(S_3)$,
- $I_5(\gamma)$ takes also on the 2–elements of $G$ the same values as $\gamma$ takes on the 3–elements of $N_G(S_5)$,

*i.e.*, in other words, if $\{p,q,r\} = \{2,3,5\}$, not only do $\zeta$ and $I_p(\zeta)$ take the same values on non trivial $p$–elements, but (with suitable trick for $p = 2$) they exchange values on non trivial $q$–elements and $r$–elements. This last property will not be explained in what follows.

## 2. Isotypies

The case of $\mathfrak{A}_5$ provides particular examples of what should replace the factorization type theorem (see above theorem 1.3 and proposition 1.5) in the case where $H$ is the normalizer of an abelian Sylow $p$–subgroup.

**Notation.**

*Various class functions.*

For $R = K$ or $\mathcal{O}$, we denote by $\mathrm{CF}(G,R)$ the $R$–module of all class functions from $G$ into $R$. For $B$ a block of $G$, we denote by $\mathrm{CF}(B,K)$ the subspace of $\mathrm{CF}(G,K)$ consisting of functions which are linear combination of characters of $KB$, and we set $\mathrm{CF}(B,\mathcal{O}) := \mathrm{CF}(G,\mathcal{O}) \cap \mathrm{CF}(B,K)$.

We denote by $\mathrm{CF}_{p'}(G,R)$ the submodule of $\mathrm{CF}(G,R)$ consisting of all class functions on $G$ which vanish outside the set $G_{p'}$ of $p'$–elements of $G$. For $B$ a block of $G$, we set $\mathrm{CF}_{p'}(B,R) := \mathrm{CF}_{p'}(G,R) \cap \mathrm{CF}(B,R)$.

*Remark.* The Brauer character of a $kG$–module is usually defined as a class function on $G_{p'}$. Extending it by 0 outside $G_{p'}$ , *we shall make here the convention that the Brauer characters are elements of* $\mathrm{CF}_{p'}(G,\mathcal{O})$ (hence elements of $\mathrm{CF}_{p'}(B,\mathcal{O})$ for the characters of $B$–modules). It results from our convention that the set $\mathrm{BrIrr}(B)$ of Brauer irreducible characters of $B$–modules is an $\mathcal{O}$–basis of $\mathrm{CF}_{p'}(B,\mathcal{O})$.

Finally, we denote by $\mathrm{CF}_{p'}^{\mathrm{pr}}(G,\mathcal{O})$ the dual submodule of $\mathrm{CF}_{p'}(G,\mathcal{O})$ in $\mathrm{CF}_{p'}(G,K)$, consisting of all elements of $\mathrm{CF}_{p'}(G,K)$ whose scalar product with the elements of $\mathrm{CF}_{p'}(G,\mathcal{O})$ belongs to $\mathcal{O}$, and we set $\mathrm{CF}_{p'}^{\mathrm{pr}}(B,\mathcal{O}) := \mathrm{CF}_{p'}^{\mathrm{pr}}(G,\mathcal{O}) \cap \mathrm{CF}(B,K)$. From our previous convention it follows that the set $\mathrm{Prim}(B)$ of characters of indecomposable projective $\mathcal{O}G$–modules is an $\mathcal{O}$–basis of $\mathrm{CF}_{p'}^{\mathrm{pr}}(G,\mathcal{O})$.

*Decomposition maps.*

For $x$ a $p$–element of $G$, we denote by

$$d_G^x \colon \mathrm{CF}(G,\mathcal{O}) \to \mathrm{CF}_{p'}(C_G(x),\mathcal{O})$$

the linear map defined by

$$d_G^x(\chi)(y) := \begin{cases} \chi(xy) & \text{if } y \text{ is a } p\text{–regular element of } C_G(x) \\ 0 & \text{if } y \text{ is a } p\text{–singular element of } C_G(x)\,. \end{cases}$$

It results from Brauer's second and third main theorems (see for example [Fe]) that the map $d_G^x$ sends $\mathrm{CF}(B_p(G),\mathcal{O})$ into $\mathrm{CF}_{p'}(B_p(C_G(x)),\mathcal{O})$, and so induces by restriction a map still denoted by

$$d_G^x \colon \mathrm{CF}(B_p(G),\mathcal{O}) \to \mathrm{CF}_{p'}(B_p(C_G(x)),\mathcal{O})\,.$$

**Isotypies.**

From now on, the following hypothesis and notation will be in force : We denote by $G$ a finite group whose Sylow $p$–subgroups are *abelian*. We denote by $S_p$ one of the Sylow $p$–subgroups, and we set $H := N_G(S_p)$.

The following definition is a slight modification of the analogous definition given in [Br1] (see Remarque 2 following definition 4.6 in *loc. cit.*).

*Remark.* A more general definition is available for non–principal blocks with *abelian defect groups*. Its statement requires the use of the "local structure" associated with a block (see [AlBr] or [Br4]). We refer the reader to [Br1] for details.

**2.1. Definition.** *An isotypy $I$ between $B_p(G)$ and $B_p(H)$ is the datum, for every $p$-subgroup $P$ of $S_p$, of a bijective isometry*

$$I(P)\colon \mathbb{Z}\mathrm{Irr}(B_p(C_H(P))) \xrightarrow{\sim} \mathbb{Z}\mathrm{Irr}(B_p(C_G(P)))$$

*such that the following conditions are fullfilled :*

(Equi) *(Equivariance) For all $h \in H$, we have $I(P)^h = I(P^h)$.*

(Com) *(Compatibility condition) For every subgroup $P$ of $S_p$ and every $x \in S_p$, we still denote by*

$$I(P)\colon \mathrm{CF}(B_p(C_H(P)),K)\colon \xrightarrow{\sim} \mathrm{CF}(B_p(C_G(P)),K)$$

*the bijective isometry defined by linear extension of $I(P)$. The following diagram is commutative :*

$$\begin{array}{ccc} \mathrm{CF}(B_p(C_H(P)),K) & \xrightarrow{I(P)} & \mathrm{CF}(B_p(C_G(P)),K) \\ {\scriptstyle d^x_{C_H(P)}}\big\downarrow & & \big\downarrow{\scriptstyle d^x_{C_G(P)}} \\ \mathrm{CF}(B_p(C_H(P\langle x\rangle)),K) & \xrightarrow{I(P\langle x\rangle)} & \mathrm{CF}(B_p(C_G(P\langle x\rangle)),K) \end{array}$$

(Triv) *$I(S_p)$ is the identity map.*

*Moreover, we say that the isotypy $I$ is normalized if $I(P)(1_{C_H(P)}) = 1_{C_G(P)}$ for all $P \subseteq S_p$.*

Let us list some of the straighforward properties of an isotypy. In what follows, for every subgroup $P$ of $S_p$ we denote by $R(P)$ the inverse map of $I(P)$. We set $I(1) = I(\{1\})$ and $R(1) = R(\{1\})$.

**2.2. Proposition.**

(Loc) *(Local isotypies) Let $P$ be any subgroup of $S_p$. Set $I_P(Q) := I(PQ)$. The collection of maps $(I_P(Q))_{Q\subseteq S_p}$ defines an isotypy between $B_p(C_H(P))$ and $B_p(C_G(P))$.*

(Int) *(Integrality property) By restriction, the map $I(1)$ induces bijective isometries*

$$\begin{cases} I(1)\colon \mathrm{CF}(B_p(H),\mathcal{O}) & \xrightarrow{\sim} \mathrm{CF}(B_p(G),\mathcal{O}) \\ I(1)\colon \mathrm{CF}_{p'}(B_p(H),\mathcal{O}) & \xrightarrow{\sim} \mathrm{CF}_{p'}(B_p(G),\mathcal{O}) \\ I(1)\colon \mathrm{CF}^{\mathrm{pr}}_{p'}(B_p(H),\mathcal{O}) & \xrightarrow{\sim} \mathrm{CF}^{\mathrm{pr}}_{p'}(B_p(G),\mathcal{O}) \end{cases}$$

(Loc) is obvious. We give a proof of (Int). By 2.1, (Com), applied with $x = 1$, we see first that $I(1)$ sends $\mathrm{CF}_{p'}(B_p(H),K)$ into $\mathrm{CF}_{p'}(B_p(G),K)$. Moreover, since, for any finite group $G$, the ordinary decomposition map $d^1_G\colon \mathbb{Z}\mathrm{Irr}(B_p(G)) \to \mathbb{Z}\mathrm{BrIrr}(B_p(G))$ is onto, and since $\mathrm{BrIrr}(B_p(G))$ is an $\mathcal{O}$–basis of $\mathrm{CF}_{p'}(B_p(G),\mathcal{O})$, we see that $I(1)$ and $R(1)$ define inverse isometries between $\mathrm{CF}_{p'}(B_p(H)),\mathcal{O})$ and $\mathrm{CF}_{p'}(B_p(G),\mathcal{O})$. It then follows by adjunction that they induce inverse isometries between $\mathrm{CF}^{\mathrm{pr}}_{p'}(B_p(H),\mathcal{O})$ and $\mathrm{CF}^{\mathrm{pr}}_{p'}(B_p(G),\mathcal{O})$.

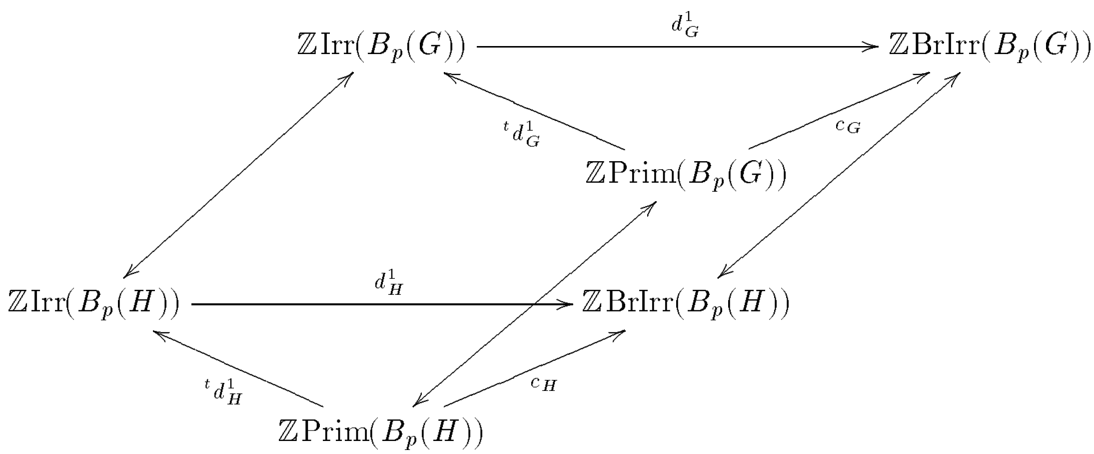
To check that $I(1)$ and $R(1)$ induce inverse isomorphisms between $\mathrm{CF}(B_p(H),\mathcal{O})$ and $\mathrm{CF}(B_p(G),\mathcal{O})$, it suffices to check that the image of $\mathrm{CF}(B_p(H),\mathcal{O})$ under $I(1)$

is contained in $\mathrm{CF}(B_p(G),\mathcal{O})$, *i.e.*,that, for $\zeta\in\mathrm{CF}(B_p(H),\mathcal{O})$ and $g\in G$, we have $I(1)(\zeta)(g)\in\mathcal{O}$. Let $x$ be the $p$–component of $g$ and let $x'$ be its $p'$–component. We have $I(1)(\zeta)(g)=(I(\langle x\rangle)(d_H^x(\zeta)))(x')$ and the result follows from the fact that $I(\langle x\rangle)$ sends $\mathrm{CF}_{p'}(C_H(x),\mathcal{O})$ into $\mathrm{CF}_{p'}(C_G(x),\mathcal{O})$, by the "local isotypies" property (Loc) and what preceds. □

*Remark.*

The integrality properties (Int) show that the map $I(1)$ is a "perfect isometry" as defined in [Br1].

It follows in particular (see [Br1] or [Br5]) that an isotypy induces an isomorphism between the associated "Cartan–decomposition" triangles between Grothendieck groups (see [Se] or [Br1] for the definitions of the triangles).




*The character of an isotypy.*

Any linear map $F\colon \mathbb{Z}\mathrm{Irr}(H)\to\mathbb{Z}\mathrm{Irr}(G)$ between the character groups of $H$ and $G$ defines a character $\mu_F$ of $H\times G$ by the formula

$$\mu_F := \sum_{\zeta\in\mathrm{Irr}(H)} \zeta\otimes F(\zeta)\,.$$

Let $I=(I(P))_{P\subseteq S_p}$ be an isotypy between $B_p(G)$ and $B_p(H)$. We set $\mu_P:=\mu_{I(P)}$.
Then the defining properties of an isotypy (cf. 2.1) translate as follows :

(Equi) *For all $h\in H$, we have $\mu_P^h=\mu_{P^h}$ .*

(Com) $$\mu_P(xx',yy')=\begin{cases} 0 & \textit{if } x \textit{ and } y \textit{ are not conjugate in } G\,,\\ \mu_{P\langle x\rangle}(x',y') & \textit{if } x=y\,.\end{cases}$$

The following statement describes the case of a $p$–group. If $G$ is an (abelian) $p$–group, then $H=G$, and $B_p(G)=\mathcal{O}G$.

**2.3. Lemma.** *Let $G$ be an abelian $p$–group. Let $I = (I(P))_{P \subseteq G}$ be a "self–isotypy" of $\mathcal{O}G$. If $p = 2$, assume moreover that $I$ is normalized. Then $I(P)$ is the identity for all $p$–subgroups $P$ of $G$.*

*Proof of 2.3.* By definition of an isotypy (property (Triv)), we know that $I(G)$ is the identity. We prove by induction on $|G : P|$ that $I(P) = \mathrm{Id}$ for any subgroup $P$ of $G$. Let $P$ be a proper subgroup of $G$. By the induction hypothesis, we may assume that $I(P') = \mathrm{Id}$ whenever $P'$ is a subgroup of $G$ which strictly contains $P$, and we must prove that $I(P) = \mathrm{Id}$. For $x \in G$, the map $d_G^x$ is identified with the map $\chi \mapsto \chi(x)$. For $\chi \in \mathrm{Irr}(G)$ we have $I(P)(\chi)(x) = d_G^x(I(P)(\chi)) = I(P\langle x\rangle)(d_G^x(\chi))$ and so $I(P)(\chi)(x) = \chi(x)$ for all $x \notin P$. Now let $y \in P$. For $x \notin P$, we have $xy \notin P$, so $I(P)(\chi)(xy) = \chi(xy)$. Set $I(P)(\chi) = \varepsilon_\chi \chi'$ where $\varepsilon_\chi = \pm 1$ and $\chi' \in \mathrm{Irr}(G)$. It follows that

$$\chi'(z) = \begin{cases} \varepsilon_\chi \chi(z) & \text{for} \quad z \notin P \\ \chi(z) & \text{for} \quad z \in P\,. \end{cases}$$

For $p$ odd, this implies that $\varepsilon_\chi = 1$, and so $I(P) = \mathrm{Id}$.

Assume $p = 2$. Since $I$ is normalized, $I(P)$ fixes the trivial character, and so for $x \in P$ we have $I(P)(\chi)(x) = I(P\langle x\rangle)(d_G^x(\chi)) = I(P)(d_G^x(\chi)) = d_G^x(\chi) = \chi(x)$ which also shows that $\varepsilon_\chi = 1$ and that $I(P) = \mathrm{Id}$. □

*Remark.* If $G = \mathbb{Z}/2\mathbb{Z}$ and if $\mathrm{Irr}(G) = \{1, \sigma\}$, then $I := \{I(1), I(G)\}$ where $I(G) = \mathrm{Id}$ and

$$I(1)\colon \begin{cases} 1 \mapsto -\sigma \\ \sigma \mapsto -1 \end{cases}$$

is a non trivial self–isotypy.

**A conjecture.**

The following conjecture is a particular case of a more general conjecture concerning blocks with abelian defect groups (see [Br1]).

**2.4. Conjecture.** *Let $G$ be a finite group whose Sylow $p$-subgroups are abelian. Let $H$ be the normalizer of a Sylow $p$-subgroup. Then there is a normalized isotypy between $B_p(G)$ and $B_p(H)$.*

The preceding conjecture is true if $G$ is $p$–solvable, by 1.4 and 1.5, (2), above.

It has also been checked in the following cases.

- $G$ is a symmetric group (Rouquier, [Rou1]), or an alternating group (Fong),
- $G$ is a sporadic non abelian simple group (Rouquier, [Rou1]),
- $G$ is a "finite reductive group" in non-describing characteristic (Broué–Malle–Michel, [BMM] and [BrMi]),
- $p = 2$ — and $G$ is any finite group with abelian Sylow 2–subgroups (Fong–Harris, [FoHa]).

## 3. Rickard equivalences

We shall explain now why the existence of an isotypy between the principal blocks of two finite groups must be the "shadow" of a much deeper connection between the two blocks, which we call a "Rickard equivalence".

## $p$–permutation modules and Rickard complexes.

*$p$–permutation modules and the Brauer functor.*

Let $R$ denote either $\mathcal{O}$ or $k$. We call $p$–permutation $RG$–modules the summands of the permutation $G$–modules over $R$. The following characterization of $p$–permutation $RG$–modules is well known (see for example [Br3]).

**3.1. Proposition.** *The $p$-permutation $RG$–modules are the modules which, once restricted to a Sylow $p$–subgroup of $G$, are permutation modules.*

Let us denote by ${}_{RG}\mathbf{perm}_p$ the category of all $p$–permutation $RG$–modules. For $P$ a $p$–subgroup of $G$, we set $\overline{N}_G(P) := N_G(P)/P$ .

For $\Omega$ a finite $G$–set, we denote by $\Omega^P$ the set of fixed points of $\Omega$ under $P$, viewed as a $\overline{N}_G(P)$–set.

**3.2. Proposition.** *There is a functor*

$$\mathrm{Br}_P \colon {}_{\mathcal{O}G}\mathbf{perm}_p \to {}_{k\overline{N}_G(P)}\mathbf{perm}_p$$

*which "induces" the "fixed points" functor, i.e., which is such that the diagram of natural transformations*

$$\begin{array}{ccc} {}_G\mathbf{set} & \xrightarrow{\ .^P\ } & {}_{\overline{N}_G(P)}\mathbf{set} \\ \downarrow & & \downarrow \\ {}_{\mathcal{O}G}\mathbf{perm}_p & \xrightarrow{\ \mathrm{Br}_P\ } & {}_{k\overline{N}_G(P)}\mathbf{perm}_p \end{array}$$

*is commutative.*

*Sketch of proof of 3.2.* For $X$ a $p$–permutation $\mathcal{O}G$–module and $Q$ a $p$–subgroup of $G$, we define

$$\mathrm{Tr}_Q^P \colon X^Q \to X^P \quad \text{by}$$
$$\mathrm{Tr}_Q^P(x) := \sum_{g \in P/Q} g(x)\,.$$

We set

$$\mathrm{Br}_P(X) := (X/\mathfrak{p}X)^P / \sum_{Q \subsetneq P} \mathrm{Tr}_Q^P((X/\mathfrak{p}X)^Q)\,.$$

□

For $V$ any $kG$–module, and $g$ a $p'$–element of $G$, we denote by $\mathrm{Brtr}(g\,;\,V)$ the value at $g$ of the Brauer character of $V$. Recall that we view the Brauer character $\mathrm{Brtr}(\cdot\,;\,V)$ as a class function on $G$ vanishing outside the set $G_{p'}$ of $p$–regular elements of $G$. The following proposition generalizes to $p$–permutation modules a result which is well known for actual permutation modules.

**3.3. Proposition.** *Let $X$ be a $p$-permutation $\mathcal{O}G$–module. If $g = g_p g_{p'}$ where $g_p$ is a $p$–element, $g_{p'}$ is $p$-regular and $g_p g_{p'} = g_{p'} g_p$, then*

$$\mathrm{tr}(g\,;\,X) = \mathrm{Brtr}(g_{p'}\,;\,\mathrm{Br}_{\langle g_p \rangle}(X))\,.$$

*Rickard complexes.*

Let us start with some notation related to complexes of modules.

Let $\Gamma := \left( \cdots \to 0 \to \Gamma^m \xrightarrow{d^m} \Gamma^{m+1} \xrightarrow{d^{m+1}} \cdots \xrightarrow{d^{m+a-1}} \Gamma^{m+a} \to 0 \to \cdots \right)$ be a complex of modules on some $\mathcal{O}$–algebra. The $\mathcal{O}$–dual of $\Gamma$ is by definition the complex $\Gamma^* := \left( \cdots \to 0 \to \mathrm{Hom}_{\mathcal{O}}(\Gamma^{m+a}, \mathcal{O}) \xrightarrow{{}^t d^{m+a-1}} \cdots \xrightarrow{{}^t d^m} \mathrm{Hom}_{\mathcal{O}}(\Gamma^m, \mathcal{O}) \to 0 \to \cdots \right)$.

From now on, the following hypothesis and notation will be in force :

*We denote by $G$ a finite group whose Sylow $p$–subgroups are abelian. We denote by $S_p$ one of the Sylow $p$–subgroups, and we set $H := N_G(S_p)$.*

**3.4. Definition.** *A Rickard complex for the principal blocks $B_p(G)$ and $B_p(H)$ is a bounded complex of $(B_p(G), B_p(H))$–bimodules*

$$\Gamma := \left( \cdots \to 0 \to \Gamma^m \xrightarrow{d^m} \Gamma^{m+1} \xrightarrow{d^{m+1}} \cdots \xrightarrow{d^{m+a-1}} \Gamma^{m+a} \to 0 \to \cdots \right)$$

*with the following properties :*

(1) *Each constituant $\Gamma^n$ of $\Gamma$, viewed as an $\mathcal{O}[G \times H]$–module, is a $p$-permutation module with vertex contained in $\Delta_{G\times H^\circ}(S_p)$ (where $\Delta_{G\times H^\circ} \colon S_p \to G \times H$ is defined by $\Delta_{G\times H^\circ}(x) := (x, x^{-1})$).*

(2) *We have homotopy equivalences :*

$$\Gamma \underset{\mathcal{O}H}{\otimes} \Gamma^* \simeq B_p(G) \quad \textit{as complexes of } (B_p(G), B_p(G))\textit{–bimodules,}$$
$$\Gamma^* \underset{\mathcal{O}G}{\otimes} \Gamma \simeq B_p(H) \quad \textit{as complexes of } (B_p(H), B_p(H))\textit{–bimodules.}$$

One of the main properties of Rickard complexes is that they automatically define Rickard complexes at the "local level" as well, as shown by the following result.

**3.5. Theorem.** *(J. Rickard)*

*Let $\Gamma$ be a Rickard complex for $B_p(G)$ and $B_p(H)$. Then, for every subgroup $P$ of $S_p$, there is a finite complex $\Gamma_P$ of $(B_p(C_G(P)), B_p(C_H(P)))$–bimodules, unique up to isomorphism, such that*

(1) *$\Gamma_P$ is a Rickard complex for $B_p(C_G(P))$ and $B_p(C_H(P))$,*

(2) *we have*

$$\mathrm{Br}_{\Delta_{G\times H^\circ}(P)}(\Gamma) = k \otimes_{\mathcal{O}} \Gamma_P \,.$$

**Rickard complexes and derived equivalences.**

For $A$ an $\mathcal{O}$–algebra (finitely generated as an $\mathcal{O}$-module), we denote by $\mathcal{D}^b(A)$ the *derived bounded category of the module category* ${}_A\mathbf{mod}$, *i.e.,* the triangulated category whose

- objects are the complexes

$$X := \left( \cdots \to X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \cdots \xrightarrow{d^{a-1}} X^a \to 0 \to \cdots \right)$$

of finitely generated projective $A$–modules, bounded on the right, and exact almost everywhere,
- morphisms are chain maps modulo homotopy.

If $\Gamma$ is a Rickard complex for $B_p(G)$ and $B_p(H)$, it is easy to see that the functor $Y \mapsto \Gamma \underset{B_p(H)}{\otimes} Y$ defines an equivalence of triangulated categories from $\mathcal{D}^b(B_p(H))$ to $\mathcal{D}^b(B_p(G))$ (see for example [Br5] for more details).

Thus the datum of a Rickard complex for $B_p(G)$ and $B_p(H)$ induces a consistent family of derived equivalences between $B_p(C_G(P))$ and $B_p(C_H(P))$, where $P$ runs over the set of subgroups of $S_p$.

**Rickard complexes and isotypies.**

- Let

$$\Gamma := \left( \cdots \to 0 \to \Gamma^m \xrightarrow{d^m} \Gamma^{m+1} \xrightarrow{d^{m+1}} \cdots \xrightarrow{d^{m+a-1}} \Gamma^{m+a} \to 0 \to \cdots \right)$$

be a Rickard complex for $B_p(G)$ and $B_p(H)$. We denote by $\mu_\Gamma$ the character of $\Gamma$ as a complex of $(\mathcal{O}G, \mathcal{O}H)$–bimodules, *i.e.*, $\mu_\Gamma := \sum_n (-1)^n \mathrm{tr}(\cdot\,;\,\Gamma^n)$.

- For every subgroup $P$ of $S_p$, let $\Gamma_P$ be the complex of $(B_p(C_G(P)), B_p(C_H(P)))$–bimodules defined as in 3.5 above. We denote by $\mu_{\Gamma_P}$ the character of $\Gamma_P$ as a complex of $(\mathcal{O}C_G(P), C_H(P))$–bimodules.

The following result is a consequence of the definition of a Rickard complex, of 3.3 and of 3.5. It shows that a Rickard complex for $B_p(G)$ and $B_p(H)$ provides a natural isotypy between $B_p(G)$ and $B_p(H)$.

**3.6. Theorem.** *There is an isotypy $I = (I(P))_{P \subseteq S_p}$ such that, for each subgroup $P$ of $S_p$, we have*

$$\mu_{\Gamma_P} = \sum_{\zeta \in \mathrm{Irr}(B_p(C_H(P)))} \zeta \otimes I(P)(\zeta)\,.$$

**A conjecture.**

The following conjecture (one of J. Rickard and the author's dreams) makes more precise earlier conjectures about derived equivalences between blocks (see [Br1] and [Br5]).

**3.7. Conjecture.** *Let $G$ be a finite group with abelian Sylow $p$–subgroups, and let $H$ be the normalizer of one of the Sylow $p$–subgroups. Then there exists a Rickard complex for $B_p(G)$ and $B_p(H)$.*

It follows from 3.6 that the preceding conjecture implies the conjecture 2.4.

Notice that conjecture 3.7 holds if $G$ is $p$–solvable by 1.4. Indeed, in this case the $(B_p(G), B_p(H))$–bimodule $\mathcal{O}(G/\mathrm{O}_{p'}G)$ is a Rickard complex for $B_p(G)$ and $B_p(H)$ (in this case, the algebras $B_p(G)$ and $B_p(H)$ are actually isomorphic).

Conjecture 3.7 is known to hold only in a few cases if $G$ is not $p$–solvable.

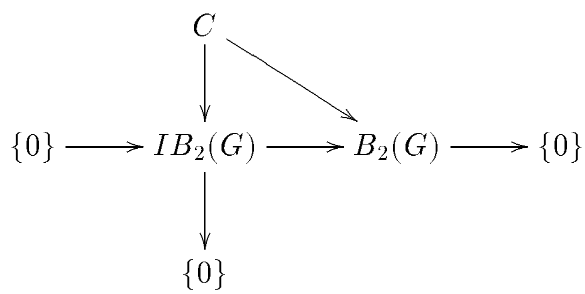
- Some examples are provided below for the case $G = \mathfrak{A}_5$, and others are provided in §4 for the case where $G$ is a finite reductive group.
- It follows from recent work of Rouquier ([Rou2]) that conjecture 3.7 holds when $S_p$ is cyclic.

**The case of $G = \mathfrak{A}_5$.**

*The case $p = 2$.*

The following explanation of the isotypy $I_2$ described in §1 is due to Rickard (cf. [Ri4]).

We set $H := N_G(S_2)$, where $S_2$ denote a Sylow 2–subgroup of $G$.
We view $B_2(G)$ as a $(B_2(G), B_2(H))$–bimodule, where $B_2(G)$ acts by left multiplication, while $B_2(H)$ acts by right multiplication.
Let us denote by $IB_2(G)$ the kernel of the augmentation map $B_2(G) \to \mathcal{O}$. Thus $IB_2(G)$ is a $(B_2(G), B_2(H))$–sub-bimodule of $B_2(G)$. Let $C$ denote a projective cover of the bimodule $IB_2(G)$.




We denote by

$$\Gamma_2 := (\{0\} \to C \to B_2(G) \to \{0\})$$

the complex of $(B_2(G), B_2(H))$–bimodules defined by the preceding diagonal arrow, where $B_2(G)$ is in degree 0 (and $C$ in degree $-1$).

We denote by $K\Gamma_2$ the complex of $(KB_2(G), KB_2(H))$–bimodules deduced by extension of scalars up to $K$. Let $H^0(K\Gamma_2)$ and $H^{-1}(K\Gamma_2)$ be the corresponding homology groups, viewed as $(KG, KH)$–bimodules.
It is clear that $H^0(K\Gamma_2)$ is the trivial $(KG, KH)$–bimodule, hence its character is $1 \otimes 1$.

**3.8. Theorem.** *(J. Rickard)*

(1) *The character of $H^{-1}(K\Gamma_2)$ (with suitables choices of $\sqrt{5}$ and $\sqrt{-3}$ in the field $K$) is*

$$(\chi_5 \otimes \alpha_3) + (\chi_3 \otimes \alpha_1) + (\chi'_3 \otimes \alpha'_1).$$

*In particular the character of $\sum_n (-1)^n H^n(K\Gamma_2)$ is*

$$(1 \otimes 1) - ((\chi_5 \otimes \alpha_3) + (\chi_3 \otimes \alpha_1) + (\chi'_3 \otimes \alpha'_1)).$$

(2) *We have homotopy equivalences :*

$$\Gamma_2 \underset{\mathcal{O}H}{\otimes} \Gamma_2^* \simeq B_2(G) \quad \text{as complexes of } (B_2(G), B_2(G))\text{–bimodules,}$$
$$\Gamma_2^* \underset{\mathcal{O}G}{\otimes} \Gamma_2 \simeq B_2(H) \quad \text{as complexes of } (B_2(H), B_2(H))\text{–bimodules.}$$

*Remark.* The above theorem 3.8 may be viewed now as a particular consequence of Rouquier's recent theorem (cf. [Rou2]) which provides Rickard complexes from certain stable equivalences.

*The case $p = 3$.*

Now we view $G$ as $\mathrm{SL}_2(4)$. We then denote by $T$ the group of diagonal matrices in $G$ (the split torus, which is also a Sylow 3–subgroup of $G$), and by $U$ the Sylow 2–subgroup of $G$, consisting of unipotent uppertriangular matrices.

We set $H := N_G(T)$, and $\sigma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We have $H = T \rtimes \langle \sigma \rangle$ .

Let $\Gamma_3$ denote the free $\mathbb{Z}_3$–module with basis $G/U$. Then $\Gamma_3$ is a $(\mathbb{Z}_3 G, \mathbb{Z}_3 T)$–bimodule, where $G$ acts by left multiplication while $T$ acts by right mutiplication.

**3.9. Theorem.**

(1) *The $(\mathbb{Z}_3 G, \mathbb{Z}_3 T)$–bimodule $\Gamma_3$ extends to a $(\mathbb{Z}_3 G, \mathbb{Z}_3 H)$–bimodule, whose character as a $(KG, KH)$–bimodule is*

$$(1 \otimes 1) + (\chi_4 \otimes \beta_1) + (\chi_5 \otimes \beta_2) .$$

(2) *We have isomorphisms :*

$$\Gamma_3 \underset{\mathcal{O}H}{\otimes} \Gamma_3^* \simeq B_3(G) \quad \text{as } (B_3(G), B_3(G))\text{–bimodules,}$$
$$\Gamma_3^* \underset{\mathcal{O}G}{\otimes} \Gamma_3 \simeq B_3(H) \quad \text{as } (B_3(H), B_3(H))\text{–bimodules.}$$

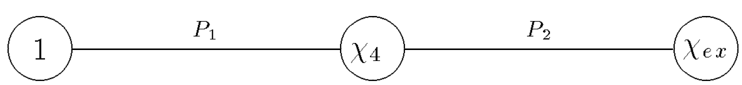
*Remark.* The previous example is a particular case of a much more general situation, as will be shown in §4.

*The case $p = 5$.*

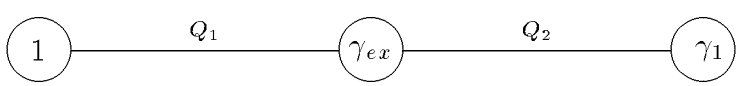
Now we set $H := N_G(S_5)$, where $S_5$ is a Sylow 5–subgroup of $G$.

Since $S_5$ is cyclic, we can apply Rouquier's constructions as in [Rou2].
The Brauer trees of the blocks $B_5(G)$ and $B_5(H)$ are respectively

(for $G$)




(for $H$)




where the exceptional vertices correspond to characters

$$\chi_{ex} := \chi_3 + \chi_3'$$
$$\gamma_{ex} := \gamma_2 + \gamma_2' .$$

In other words, we have minimal projective resolutions of the trivial representations, periodic of period 4, of the following shapes :

$$\cdots \to P_1 \to P_2 \to P_2 \to P_1 \to \mathcal{O} \to \{0\}$$
$$\cdots \to Q_1 \to Q_2 \to Q_2 \to Q_1 \to \mathcal{O} \to \{0\} .$$

The $(B_5(G), B_5(H))$–bimodule $B_5(G)$ is indecomposable and its projective cover has the shape (cf. [Rou2])

$$(P_1 \otimes \mathrm{Hom}_{\mathcal{O}}(Q_1, \mathcal{O})) \oplus (P_2 \otimes \mathrm{Hom}_{\mathcal{O}}(Q_2, \mathcal{O})) \xrightarrow{\pi} B_5(G)\,.$$

Following [Rou2], we then define

$$\Gamma_5 := \Big(\{0\} \to P_2 \otimes \mathrm{Hom}_{\mathcal{O}}(Q_2, \mathcal{O}) \xrightarrow{\pi} B_5(G) \to \{0\}\Big)\,.$$

**3.10. Theorem.** *(R. Rouquier)*

(1) *The character of* $H^{-1}(K\Gamma_5)$ *is*

$$(\chi_4 \otimes \gamma_1) + (\chi_3 \otimes \gamma_2) + (\chi_3' \otimes \gamma_2')\,.$$

*In particular the character of* $\sum_n (-1)^n H^n(K\Gamma_5)$ *is*

$$(1 \otimes 1) - ((\chi_4 \otimes \gamma_1) + (\chi_3 \otimes \gamma_2) + (\chi_3' \otimes \gamma_2'))\,.$$

(2) *We have homotopy equivalences :*

$$\Gamma_5 \underset{\mathcal{O}H}{\otimes} \Gamma_5^* \simeq B_5(G) \quad \textit{as complexes of } (B_5(G), B_5(G))\textit{–bimodules,}$$

$$\Gamma_5^* \underset{\mathcal{O}G}{\otimes} \Gamma_5 \simeq B_5(H) \quad \textit{as complexes of } (B_5(H), B_5(H))\textit{–bimodules.}$$

## 4. The case of the finite reductive groups

In the case where $G$ is a "finite reductive group", the conjecture 3.7 can be made more precise and closely linked with the underlying algebraic geometry (for more details, see [BrMa]).

In this paragraph, we change our notation to fit with the usual notation of finite reductive groups : our prime $p$ (the characteristic of our field $k := \mathcal{O}/\mathfrak{p}$) is now denoted by $\ell$, and $q$ denotes a power of another prime $p \neq \ell$.

From now on, we denote by $\mathbf{G}$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$, endowed with a Frobenius endomorphism $F$ which defines a rational structure on $\mathbb{F}_q$. The finite group $\mathbf{G}^F$ of fixed points of $\mathbf{G}$ under $F$ is called a finite reductive group.

### The Deligne–Lusztig variety and its $\ell$–adic cohomology.

*The Deligne–Lusztig variety.*

Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$, with unipotent radical $\mathbf{U}$, and with $F$–stable Levi subgroup $\mathbf{L}$.

We denote by $\mathrm{Y}(\mathbf{U})$ the associated Deligne–Lusztig variety defined (cf. [DeLu] and [Lu]) by

$$\mathrm{Y}(\mathbf{U}) := \{g(\mathbf{U} \cap F(\mathbf{U})) \in \mathbf{G}/\mathbf{U} \cap F(\mathbf{U})\,;\; g^{-1}F(g) \in F(\mathbf{U})\}\,.$$

It is clear that $\mathbf{G}^F$ acts on $\mathrm{Y}(\mathbf{U})$ by left multiplication while $\mathbf{L}^F$ acts on $\mathrm{Y}(\mathbf{U})$ by right multiplication.

It is known (cf. [Lu]) that $\mathrm{Y}(\mathbf{U})$ is an $\mathbf{L}^F$–torsor on a variety $\mathrm{X}(\mathbf{U})$, which is smooth of pure dimension equal to $\dim(\mathbf{U}/\mathbf{U}\cap F(\mathbf{U}))$, and which is affine (at least if $q$ is large enough). In particular $\mathrm{X}(\mathbf{U})$ is endowed with a left action of $\mathbf{G}^F$. If $R$ is a commutative ring, the image of the constant sheaf $R$ on $\mathrm{Y}(\mathbf{U})$ through the finite morphism $\pi\colon \mathrm{Y}(\mathbf{U}) \to \mathrm{X}(\mathbf{U})$ is a locally constant constant sheaf $\pi_*(R)$ on $\mathrm{X}(\mathbf{U})$. We denote this sheaf by $\mathcal{F}_{R\mathbf{L}^F}$.

*A consequence of yet another theorem of J. Rickard.*

From now on, we denote by $\ell$ a prime number which does not divide $q$, and which is *good for* $\mathbf{G}$.

The following theorem is a consequence of the main result of [Ri5] (for a "character theoretic approach" of this result, see [Br1], §2.A).

**4.1. Theorem.** *(J. Rickard) There exists a bounded complex*

$$\Lambda_c(\mathrm{X}(\mathbf{U}),\mathcal{F}_{\mathbb{Z}_\ell\mathbf{L}^F}) = \left(\cdots \to 0 \to \Lambda^m \xrightarrow{d^m} \Lambda^{m+1} \xrightarrow{d^{m+1}} \cdots \xrightarrow{d^{m+a-1}} \Gamma^{m+a} \to 0 \to \cdots\right)$$

*of $(\mathbb{Z}_\ell\mathbf{G}^F,\mathbb{Z}_\ell\mathbf{L}^F)$–bimodules, with the following properties :*

(1) *For each positive integer $n$, $(\mathbb{Z}_\ell/\ell^n\mathbb{Z}_\ell)\otimes\Lambda_c(\mathrm{X}(\mathbf{U}),\mathcal{F}_{\mathbb{Z}_\ell\mathbf{L}^F})$ is a representative, in the derived bounded category of $((\mathbb{Z}_\ell/\ell^n\mathbb{Z}_\ell)\mathbf{G}^F,(\mathbb{Z}_\ell/\ell^n\mathbb{Z}_\ell)\mathbf{L}^F)$–bimodules, of the "$\ell$–adic cohomology complex" $\mathrm{R}\Gamma_c(\mathrm{X}(\mathbf{U}),\mathcal{F}_{(\mathbb{Z}_\ell/\ell^n\mathbb{Z}_\ell)\mathbf{L}^F})$.*

(2) *For each integer $n$, the $\mathbb{Z}_\ell[\mathbf{G}^F\times\mathbf{L}^F]$–module $\Lambda^n$ is an $\ell$–permutation module, such that each of its indecomposable constituant has a vertex contained in $\Delta_{\mathbf{G}^F\times(\mathbf{L}^F)^o}(\mathbf{L}^F)$.*

If $\mathcal{O}$ is the ring of integers of a finite extension $K$ of $\mathbb{Q}_\ell$, we set

$$\Lambda_c(\mathrm{X}(\mathbf{U}),\mathcal{F}_{\mathcal{O}\mathbf{L}^F}) := \mathcal{O}\otimes_{\mathbb{Z}_\ell}\Lambda_c(\mathrm{X}(\mathbf{U}),\mathcal{F}_{\mathbb{Z}_\ell\mathbf{L}^F}).$$

**A conjecture.**

*Notation.*

From now on, we assume that $\ell$ is a prime number, $\ell\neq p$, which is good for $\mathbf{G}$, and such that the Sylow $\ell$–subgroups of $\mathbf{G}^F$ are *abelian*.

• Let $\mathcal{O}$ be the ring of integers of a finite unramified extension $k$ of the field of $\ell$–adic numbers $\mathbb{Q}_\ell$, with residue field $k$, such that the finite group algebra $k\mathbf{G}^F$ is split.

• Let $e$ be the principal block idempotent of $\mathcal{O}\mathbf{G}^F$, so that $\mathcal{O}\mathbf{G}^Fe$ is the principal block $B_\ell(\mathbf{G}^F)$ of $\mathcal{O}\mathbf{G}^F$.

• Let $S$ be a Sylow $\ell$–subgroup of $\mathbf{G}^F$ and let $\mathbf{L} := C_{\mathbf{G}}(S)$. The group $\mathbf{L}$ is a rational Levi subgroup of $\mathbf{G}$.

We have $N_{\mathbf{G}^F}(S) = N_{\mathbf{G}^F}(\mathbf{L})$. The group $S$ is a Sylow $\ell$–subgroup of $Z(\mathbf{L})^F$, and $\ell$ does not divide $|N_{\mathbf{G}^F}(\mathbf{L})/\mathbf{L}^F|$.

Let $f$ be the principal block idempotent of $\mathcal{O}\mathbf{L}^F$, so that $\mathcal{O}\mathbf{L}^Ff$ is the principal block $B_\ell(\mathbf{L}^F)$ of $\mathcal{O}\mathbf{L}^F$.

**4.2. Conjecture.** *There exists a parabolic subgroup of* $\mathbf{G}$ *with unipotent radical* $\mathbf{U}$ *and Levi complement* $\mathbf{L}$*, such that*

(C1) *the idempotent* $e$ *acts as the identity on the complex* $\Lambda_c(\mathrm{X}(\mathbf{U}), \mathcal{F}_{\mathcal{O}\mathbf{L}^F}).f$*,*

(C2) *the structure of complex of* $(\mathcal{O}\mathbf{G}^F e, \mathcal{O}\mathbf{L}^F f)$*–bimodules of* $\Lambda_c(\mathrm{X}(\mathbf{U}), \mathcal{F}_{\mathcal{O}\mathbf{L}^F}).f$ *extends to a structure of complex of* $(\mathcal{O}\mathbf{G}^F e, \mathcal{O}N_{\mathbf{G}^F}(\mathbf{L})f)$*–bimodules,*

(C3) *we have homotopy equivalences :*

$$\Lambda_c(\mathrm{X}(\mathbf{U}), \mathcal{F}_{\mathcal{O}\mathbf{L}^F}).f \underset{\mathcal{O}N_{\mathbf{G}^F}(\mathbf{L})f}{\otimes} f.\Lambda_c(\mathrm{X}(\mathbf{U}), \mathcal{F}_{\mathcal{O}\mathbf{L}^F})^* \simeq \mathcal{O}\mathbf{G}^F e$$

*as complexes of* $(\mathcal{O}\mathbf{G}^F e, \mathcal{O}\mathbf{G}^F e)$*–bimodules,*

$$f.\Lambda_c(\mathrm{X}(\mathbf{U}), \mathcal{F}_{\mathcal{O}\mathbf{L}^F})^* \underset{\mathcal{O}\mathbf{G}^F e}{\otimes} \Lambda_c(\mathrm{X}(\mathbf{U}), \mathcal{F}_{\mathcal{O}\mathbf{L}^F}).f \simeq \mathcal{O}N_{\mathbf{G}^F}(\mathbf{L})f$$

*as complexes of* $(\mathcal{O}N_{\mathbf{G}^F}(\mathbf{L})f, \mathcal{O}N_{\mathbf{G}^F}(\mathbf{L})f)$*–bimodules.*

By 4.1, one sees that if the above conjecture is true, the complex $\Lambda_c(\mathrm{X}(\mathbf{U}), \mathcal{F}_{\mathcal{O}\mathbf{L}^F})$ is indeed a Rickard complex for $\mathcal{O}\mathbf{G}^F e$ and $\mathcal{O}N_{\mathbf{G}^F}(\mathbf{L})f$.

Although some evidence in favour of conjecture 4.2 is indeed available (see [BrMa]), it is actually known to be true in very few cases.

The particular case where $\ell$ divides $(q-1)$ (which had been conjectured and almost proved by Hiß) may be deduced from some results of Puig ([Pu]).

**4.3. Theorem.** *Assume that* $\ell$ *divides* $q-1$ *and does not divide the order of the Weyl group of* $\mathbf{G}$*.*

(1) *We have* $\mathbf{L} = \mathbf{T}$*, a quasi–split maximal torus of* $\mathbf{G}$*. For* $\mathbf{P} = \mathbf{B}$*, a rational Borel subgroup of* $\mathbf{G}$ *containing* $\mathbf{T}$ *with unipotent radical* $\mathbf{U}$*, we have*

$$\Lambda_c(\mathrm{X}(\mathbf{U}), \mathcal{F}_{\mathcal{O}\mathbf{L}^F}) \simeq \mathcal{O}[\mathbf{G}^F/\mathbf{U}^F]\,.$$

(2) *(L. Puig) We have isomorphims :*

$$\mathcal{O}[\mathbf{G}^F/\mathbf{U}^F].f \underset{\mathcal{O}N_{\mathbf{G}^F}(\mathbf{T})f}{\otimes} f.\mathcal{O}[\mathbf{U}^F\backslash\mathbf{G}^F] \simeq \mathcal{O}\mathbf{G}^F e$$

*as* $(\mathcal{O}\mathbf{G}^F e, \mathcal{O}\mathbf{G}^F e)$*–bimodules,*

$$f.\mathcal{O}[\mathbf{U}^F\backslash\mathbf{G}^F] \underset{\mathcal{O}\mathbf{G}^F e}{\otimes} \mathcal{O}[\mathbf{G}^F/\mathbf{U}^F].f \simeq \mathcal{O}N_{\mathbf{G}^F}(\mathbf{T})f$$

*as* $(\mathcal{O}N_{\mathbf{G}^F}(\mathbf{T})f, \mathcal{O}N_{\mathbf{G}^F}(\mathbf{T})f)$*–bimodules,*

*providing a Morita equivalence between* $\mathcal{O}\mathbf{G}^F e$ *and* $\mathcal{O}N_{\mathbf{G}^F}(\mathbf{T})f$*.*

*Remark.* The case where $\mathbf{G}^F = \mathrm{SL}_2(4)$ and $\ell = 3$ (see theorem 3.9 above) is a particular case of the preceding theorem.

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