1. Arrangements de pseudodroites et de double pseudodroites
2. Algorithmique des arrangements de (double) pseudodroites
3. Arrangements de pseudohyperplans

Références

Bibliographie

Arrangements de pseudolignes (support de cours) M. Pocchiola IMJ, Janvier 2012

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1. Topic in Discrete and Computational Geometry


http://dx.doi.org/10.1007/s00454-010-9298-4


http://arxiv.org/abs/1101.1022


The real projective plane $\mathbb{R}P^2$ is a fundamental object in algebraic topology and geometry. It is a non-orientable manifold that cannot be embedded in three-dimensional Euclidean space. The real projective plane has several important properties:

- **Double Point Pairing**: Two points $p$ and $\bar{p}$ in $\mathbb{R}P^2$ are considered equivalent if they are antipodal. This equivalence relation defines a quotient space $\mathbb{R}P^2 = \mathbb{R}P^2 / \{p \sim \bar{p} \mid p \in \mathbb{R}P^2 \} = \mathbb{R}P^2$.

- **Induced Cell Complex**: The real projective plane can be constructed as a quotient of an induced cell complex, where the cells are indexed by their indices, with $0$-cells, $1$-cells, and $2$-cells representing the points, lines, and planes, respectively.

- **Arrangements of Pseudolines**: In the context of discrete and computational geometry, arrangements of pseudolines on the real projective plane $\mathbb{R}P^2$ are studied for their combinatorial and topological properties.

In particular, the study of arrangements of pseudolines in the projective plane $\mathbb{R}P^2$ involves understanding the interplay between the topological structure of the plane and the combinatorial properties of the pseudolines. This area of research has applications in various fields, including computer science, graphics, and geometric modeling.
NOTES

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1.

http://www.research.att.com/~njas/sequences/A006248
http://www.research.att.com/~njas/sequences/A063800

THEOREM 1.1.1: In the On-Line Encyclopedia of Integer Sequences, the value of $a_{11}$ is due to F. Aurenhammer, 2002.

NUMBER OF ARRANGEMENTS

CIRCULAR ARRANGEMENTS OF SIZE 3, 4, 5 AND 6

Hasse/Flag diagrams of the simple arrangements of size 3, 4, 5 and 6

1.

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13. Lemma 1.

Let $\Gamma$ be a simple arrangement of pseudolines, let $\gamma, \gamma' \in \Gamma$, and let $C$ be one of the two 2-cells of the subarrangement $\{\gamma, \gamma'\}$. Assume that there exists a vertex of $\Gamma$ lying in $C$. Then there exists a triangular 2-cell of $\Gamma$ included in $C$ with a side supported by $\gamma'$ and a vertex included in $C$.


Any arrangement of $n$ pseudolines can be extended to an arrangement of $n+1$ pseudolines such that the added pseudoline goes through two given points not on the same pseudoline of the initial arrangement.


Any arrangement of pseudolines can be extended to an infinite family of pseudolines such that any two distinct points are contained in exactly one pseudoline of the family. In other words, any arrangement of pseudolines can be extended to a family of pseudolines each containing a fixed point of the plane.
A (real two-dimensional) projective geometry is a topological point-line incidence structure \((P,L,\in)\) whose point space \(P\) is a projective plane and whose line space \(L\) is a subspace of the space of pseudolines of \(P\) such that (1) any two distinct points are contained in exactly one line which depends continuously on the two points; (2) any two distinct lines intersect in exactly one point which depends continuously on the two lines.

The line space of a projective geometry is a projective plane and the dual of a point of a projective geometry (i.e., the pencil of lines through that point) is a pseudoline of its line space.

The standard projective geometry is the projective geometry whose point space is the standard projective plane \(\mathbb{P}^2\) and whose line space is the image under the canonical projection \(S^2 \to \mathbb{P}^2\) of the space of great circles of \(S^2\).

The stretchability problem is NP-hard.
**Proof of the Goodman et al. Enlargement Lemma**

\[ F = \phi - 1 F(p) \phi \]

**TH 9 (Representation Theorem for PA)** Any arrangement of pseudolines is isomorphic to the dual family of a finite family of points of a projective geometry.

**Th 10 (Ringel, 1956 – Folkman and Lawrence, 1978)**.

The map that assigns to the isomorphism class of an indexed arrangement of oriented pseudolines its chirotope is one-to-one and its range is the set of maps \( \chi \) defined on the set of triples of a finite set \( I \) such that for every 3-, 4-, and 5-subset \( J \) of \( I \) the restriction of \( \chi \) to the set of triples of \( J \) is the chirotope of an arrangement of oriented pseudolines.

**Axiomatization of Arrangement of Pseudolines**

The map that assigns to the isomorphism class of an indexed arrangement of oriented pseudolines its chirotope is one-to-one and its range is the set of maps \( \tilde{\chi} \) defined on the set of triples of a finite set \( I \) such that for every 3-, 4-, and 5-subset \( J \) of \( I \) the restriction of \( \tilde{\chi} \) to the set of triples of \( J \) is the chirotope of an arrangement of oriented pseudolines.

**Chirotopes of Arrangements of Pseudolines**

The map that assigns to the isomorphism class of an indexed arrangement of oriented pseudolines its chirotope is one-to-one and its range is the set of maps \( \chi \) defined on the set of triples of a finite set \( I \) such that for every 3-, 4-, and 5-subset \( J \) of \( I \) the restriction of \( \chi \) to the set of triples of \( J \) is the chirotope of an arrangement of oriented pseudolines.
The map that assigns to an indexed configuration of oriented points the isomorphism class of its dual arrangement is compatible with the isomorphism relation on indexed configurations of oriented points and the induced quotient map is one-to-one and onto.
where \([\mathcal{I}]\) denotes the parallel class of \([l]\).

\[\{[l] \cap \{l \cap \mathcal{I}\} = \mathcal{I}\] \[\{[l] \cap \mathcal{V} = \mathcal{V}\]}

Hence every plane has the form \([\Omega]\), with \(\Omega\) a set of new points on \([l]\), which is the projective plane obtained from \([l]\) by adding a new line \(A\) which corresponds to the parallel class \([l]\) of \([l]\). The pencil of \([l]\) through \(p\) consists of \(l, k\) such that \(p, l\), \(p, k\) have no point in common; we call \(l, k\) a pair of parallel lines, and \(l, k\) is a unique line passing through \(p\) and \(l, k\). A line is a set of three points which are not collinear.

There exists a non-degenerate quadrangle, i.e., a set of four points no three of which are collinear. A quadrangle in \([l]\) is a set of four points \(p, l, k, m\) such that \(p, l, k, m\) are collinear and \(p, l\), \(p, k\), \(p, m\) have no point in common. A quadrangle is a set of four points \(p, l, k, m\) such that \(p, l\), \(p, k\), \(p, m\) have no point in common; we call \(p, l\), \(p, k\), \(p, m\) a pair of parallel lines, and \(p, l\), \(p, k\), \(p, m\) is a unique line passing through \(p\) and \(p, l\), \(p, k\), \(p, m\). A line is a set of three points which are not collinear.

A projective plane is an incidence structure \([\Omega, \mathcal{I}])\) be a projective plane and let \(p, l, k, m\) be an arbitrary line of \([\Omega]\). Then the pencil \([l\,A\,k]\) of sets with \(l\neq m\) and \(l\neq k\) consists of \(l, k\) such that \(l, k\) have no point in common and \(l, k\) is a unique line passing through \(p\) and \(l, k\). A line is a set of three points which are not collinear.

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A definition of arrangements of double pseudolines is given as follows:

**Definition:**
Let $\mathcal{P}$ be a projective plane. An arrangement of double pseudolines in $\mathcal{P}$ is a finite family of double pseudolines in $\mathcal{P}$ that intersect pairwise in exactly four transversal intersection points and that induce pairwise a cell structure on $\mathcal{P}$.

Examples of arrangements of double pseudolines are given by the dual families of finite families of pairwise disjoint convex bodies of projective geometries.

**Theorem 14 (Pumping Lemma):**
Let $\mathcal{C}$ be a simple arrangement of double pseudolines. Assume there is a vertex of $\mathcal{C}$ lying in the interior of the Möbius strip $\mathcal{M}(\mathcal{C})$ bounded by $\mathcal{C}$. Then there is a vertex of $\mathcal{C}$ lying in the interior of the Möbius strip $\mathcal{M}(\mathcal{C})$ bounded by $\mathcal{C}$. Let $\mathcal{C}$ be a simple arrangement of double pseudolines and let $\mathcal{C}$.

**Theorem 15 (Duality and Homotopy Theorems):**
Any arrangement of double pseudolines is isomorphic to the dual family of a finite family of pairwise disjoint convex bodies of a projective geometry. Any two arrangements of double pseudolines of the same size and living in the same projective plane are homotopic via a finite sequence of mutations followed by an isotopy. In other words, the graph of mutations on the space of arrangements of double pseudolines of given size is connected.

**Pumping Lemma:** Let $\Gamma$ be a simple arrangement of double pseudolines and let $\gamma \in \Gamma$. Assume that there is a vertex of $\Gamma$ lying in the interior of the Möbius strip $\mathcal{M}(\gamma)$ bounded by $\gamma$. Then there is a vertex of $\Gamma$ lying in the interior of the Möbius strip $\mathcal{M}(\gamma)$ bounded by $\gamma$. Let $\Gamma$ be a simple arrangement of double pseudolines and let $\Gamma$. The pumping lemma states that for any arrangement of double pseudolines, if there is a vertex in the interior of a Möbius strip bounded by the arrangement, then there is another vertex in the interior of the same Möbius strip.

**Remark:** The pumping lemma is used to prove the connectedness of the space of arrangements of double pseudolines.
CHIROTOPES AND AXIOMATIZATION

TH 16. The map that assigns to an indexed configuration of oriented convex bodies its chirotope is compatible with the isomorphism relation on indexed configurations of oriented convex bodies and the induced quotient map is one-to-one.

TH 15. The map that assigns to an indexed configuration of oriented convex bodies the isomorphism class of its dual arrangement is compatible with the isomorphism relation on indexed configurations of oriented convex bodies and the induced quotient map is onto.

TH 15. The map that assigns to an indexed configuration of oriented convex bodies its chirotope is one-to-one and its range is the set of maps defined on the set of triples of a finite set \( I \) such that for every 3-, 4-, and 5-subset \( J \) of \( I \) the restriction of \( \chi \) to the set of triples of \( J \) is the chirotope of an indexed arrangement of oriented double pseudolines.

TH 16. The map that assigns to an indexed configuration of oriented convex bodies the isomorphism class of its dual arrangement is compatible with the isomorphism relation on indexed configurations of oriented convex bodies and the induced quotient map is one-to-one and onto.
admissible triangles and their derivatives

cycles

maragons

TH 17. Two indexed arrangements of oriented double pseudolines are isomorphic if and only if they have the same indexed family of cycles.

TH 2. Two indexed arrangements of oriented double pseudolines are isomorphic if and only if

\[
\{\begin{array}{l}
ABCD = \{1,2,3,4\} \rightarrow \{a,b,c,d\} \\
\{1,2,3,4\} = \{a,b,c,d\} \\
\{1,2,3,4\} = \{a,b,c,d\} \\
\{1,2,3,4\} = \{a,b,c,d\}
\end{array}\}
\]

\[
\{1,2,3,4\} = \{a,b,c,d\} \quad \{1,2,3,4\} = \{a,b,c,d\} \quad \{1,2,3,4\} = \{a,b,c,d\} \quad \{1,2,3,4\} = \{a,b,c,d\}
\]

I = 1 : a, b, c, d

I = 0 : a, b, c, d

two-covering
Axiomatization Theorem

Let $I$ be a finite indexing set.

**DF 8.** A $k$-arrangement of double pseudolines is a finite indexed by $I$ family $\tau$ of simple closed oriented curves embedded in a compact surface $S_\tau$ with the properties that (1) $\tau$ induces a regular cell decomposition $X_\tau$ of $S_\tau$; and (2) any subfamily of size at least $2$ and at most $k$ considered as embedded in $S_\tau$ but not in $S_\nu$ for any subset $\nu$ of size at most $2$ and at most $k$, is an arrangement of double pseudolines. We denote by $A_k$ the set of $k$-arrangements and by $A_k \rightarrow C_k$ the map that assigns to a $k$-arrangement its chirotope.

**Lemma 3.** The graph of mutations on $A_k$ is connected.

**Lemma 4.** $A_k \rightarrow C_k$ is one-to-one and onto.

**TH 1.** The set of $k$-arrangements.

Let $I$ be a finite indexing set.
NOTES

A TWO-ARRANGEMENT
The $k$ edges of the visibility graph of a planar family of $n$ pairwise-disjoint convex bodies presented by its chirotope is computable in time $O(k + n \log n)$ and linear working space.

**The Problem of Piano Movers**

Visibility graph of polygonal obstacles.
Physique.

Le facteur de forme est la fraction d'énergie diffuse quittant une surface et atteignant directement une autre surface.

Géométrie.

Le facteur de forme $F_{ij}$ entre les surfaces $s_i$ et $s_j$ est la mesure de l'ensemble $X_{ij}$ des droites passant par deux points mutuellement visibles de $s_i$ et $s_j$. Il existe une unique mesure de cette ensemble de droites qui est invariante par déplacements.
1. A set of $n$ pairwise disjoint convex bodies $A$.
2. The set of $k \leq 2n - (n - 1)$ undirected free bitangents $B$.
3. A pseudotriangulation $G$ is a maximal (for $\subseteq$) planar subset of $B$.
5. $(\forall G) (\forall b \in G \setminus C) (\exists! G' \neq G) (G \setminus G' = \{b\})$.

For any flip $G \rightarrow G'$ of the pseudotriangulation $G$, we define a maximal planar set $\tilde{G}$ of $G$ such that $\tilde{G} \cap G = \emptyset$.

### Greedy Flip Property

For any filter $I$ of the poset $(B^K, \prec^K)$, we define a maximal planar set of bitangents $G(I) = \{v_1, v_2, \ldots, v_k\}$ recursively:

- $v_1$ is minimal in $I$,
- for $i > 1$, $v_i$ is minimal in the set of bitangents $v \in I \setminus \{v_1, \ldots, v_{i-1}\}$ such that $K \cup \{v_1, \ldots, v_i\}$ is planar—note that this set is empty after $3n - 3$ steps.

TH 16 (PV96, AP03).

Let $b$ be a minimal element of the filter $I$ of the pseudo-arrangement $A$.

- $G(I) = I \setminus \phi(I; \kappa)$.
- $G(I \setminus \{b\})$ is obtained from $G(I)$ by flipping $b$.

TH 17 (PV96, AP03).

A pseudotriangulation $G$ is a maximal planar subset of a pseudo-arrangement $A$.

$G$ is the intersection of the pseudotriangulations of $A$.

$\{G \subseteq \phi(I; \kappa) \mid G \neq \emptyset, G \in \forall \prec^K \forall I \in \forall I \}$.

### Flip Graph of Pseudotriangulations

For any filter $I$ of the poset $(B^K, \prec^K)$, we define a maximal planar set of bitangents $G(I) = \{v_1, v_2, \ldots, v_k\}$ recursively:

- $v_1$ is minimal in $I$,
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TH 16 (PV96, AP03).

Let $b$ be a minimal element of the filter $I$ of the pseudo-arrangement $A$.

- $G(I) = I \setminus \phi(I; \kappa)$.
- $G(I \setminus \{b\})$ is obtained from $G(I)$ by flipping $b$.
Let $I$ be a filter (for example the filter of bitangents with positive angle) of $(\mathcal{B}, \preceq)$, and let $G_0 = G(I)$ be the associated greedy pseudotriangulation. For $H = (F, G)$ a pseudotriangulation, we define
\[
\omega(H) = (F \cup \{v\}, G \setminus \{v\})
\]
and
\[
\phi(H) = (F, G \setminus \{v\} \cup \{v'\})
\]
where $v = \min < H G$ and $v'$ is the shadow of $v$ in $F \cup G$.

(Observe that $\omega$ and $\phi$ are defined only if $G \neq \emptyset$.) For $G \supseteq H$ a pseudotriangulation of the $o'$s we define a sequence $H(G) = H_1, H_2, \ldots, H_i = (F_i, G_i)$ by setting $H_0 = (\emptyset, G_0)$, and $H_{i+1} = \omega(H_i)$ if $\min < H G_i \in G_i$; otherwise $H_{i+1} = \phi(H_i)$.

It is easy to see that the sequence $H(G)$ is finite, that its last term is $H(G) = (G, \emptyset)$, and that $H(G) = H(G')$ iff. $G = G'$. This proves the strong connectivity of the flip graph of pseudotriangulations and, with some extra work, provides a linear space traversal algorithm of the flip graph whose complexity is linear amortized per node.

**Algorithm Greedy Flip Algorithm**

1. Compute the greedy pseudo-triangulation $G := G(I(0))$.
2. Repeat
3. Select a minimal bitangent $b$ in $G$ with angle less than $\pi$;
4. Flip $b$; (i.e., replace $b$ by $\phi(b)$);
5. Until there are no more bitangents of angle less than $\pi$.

**Figure 1:** The greedy flip algorithm. At each step the internal bitangent of minimal slope in the current pseudo-triangulation is flipped.

The flip operation of the greedy flip algorithm can be implemented in constant amortized time using only the chirotope of the convex bodies.

Furthermore, the only necessary data structures are linked structures to represent the incidence relations in the sweep structure $S$ and its associated pseudotriangulations $G(S)$ and $G^*(S)$, a list to store the minimal bitangents of $G(S)$, and for each minimal bitangent $b$ pointers to the arcs of $G(S)$ that $\phi_R(b)$ leaves and pointers to the arcs of $G^*(S)$ that $\phi_R(b)$ enters.

**Enumeration of Pseudotriangulations**
The number of pairs of free bitangents \((a, b)\) such that 

\[ \phi(a, b) = \phi(b, a) \]

and 

\[ \phi^*(a, b) = \phi^*(b, a) \]

is bounded by a constant times the number of free bitangents.
1. Soit $o_i$ une famille de $n$ corps convexes disjoint deux-à-deux d'un plan topologique $A$ sur $\mathbb{R}^2$ ; l'espace libre est noté $F$ et est défini comme le complémentaire dans $A$ de l'union des intérieurs des corps convexes, i.e. $F = A \setminus \bigcup_i \text{Interior}(o_i)$ ; l'espace des droites de $A$ est noté $L$ ; la courbe des tangentes de $L$ à $o_i$ est notée $\gamma_i (\approx S^1)$ ;

2. Le complexe de visibilité des $o_i$ est noté $V$ et est défini comme l'espace des droites de $F$ ; la courbe des tangentes de $V$ à $o_i$ est notée $\gamma_i + (\approx S^1)$, $\gamma_i - (\approx S^1)$.

3. $p : V \to L$ ;

4. Versions orientées de $V$ et $L$ ; les courbes (orientées) $z_i^+, z_i^-$, $\gamma_i^+, \gamma_i^-$.
cross-sections of the visibility complex of two convex bodies

Local topology of the visibility complex

Figure 2: Example of arrangement of curves $z$-$\varepsilon_0$'s for a collection of three convex bodies: each curve $z$-$\varepsilon_0$ is stretched and bent so that it stays roughly parallel to the equator of the space of directed lines. There is a one-to-one and onto correspondence between the set of vertices of the arrangement and the set of free and non-free directed bitangents. The (6) vertices marked with a square correspond to non-free bitangents. The dashed region represents the projection in the space of directed lines of the 2-cell containing the elements of the visibility complex with backward view the disk labeled 3 and forward view the disk labeled 2.
Cross-sections of the Visibility Complex

Computing the Right Boundary of a 2-cell

Incidence Operators

Incidence Relations
Typology of the 2-cells of a cross-section

1. \( F_{i+1,j+1} \to F_{j,i} \)

2. \( z_\theta = \inf_{p \in o} \langle p, 1 \exp(1 \theta) \rangle \in \mathbb{R} \)

3. \( z_{\theta} - o(\theta) = \sup_{p \in o} \langle p, 1 \exp(1 \theta) \rangle \in \mathbb{R} \)

4. \( z_{\theta+\pi} = -z_{\theta} + o(\theta) \)

5. \( \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_k < \pi \) la suite croissante des angles des bitangentes libres orientées vers le haut

6. \( \text{death}(\sigma) \) la plus petite solution supérieure à \( \theta_{i-1} \) de l'équation

7. Les \( k \) bitangentes libres d'une collection de \( n \) corps convexes disjoint deux à deux d'un plan topologique peuvent être calculées en temps \( O(k \log n) \) et espace de travail linéaire.

8. \( \text{Example} \)

9. \( 2 \in F_{22}, 9 \in F_{32}, 7, 11 \in F_{33}, 12 \in F_{31}, 1, 3 \in F_{13}, 6, 10 \in F_{21}, 8 \in F_{11}, 4 \in F_{12}, 5 = 0, 0 = 1 \).

10. \( \text{transition} \) : \( I+I+I \)
1. Proper filter of \((X_0, \prec)\): I ↔ \(S(I) = E(I) \cup F(I)\) maximal antichains of \((X_{\geq 1}, \prec)\).
2. The planar directed graph \(\Gamma(I)\) induced on \(S(I)\) by the incidence relations.
3. If \(e \in X_1\) supported by a \(z\) one has \(\text{sink}(e) = \text{sink}(e^r)\) or \(\text{sink}(e) = \text{sink}(e^f)\).
4. Let \(\sigma \in F(I)\). \(\text{sink}(\sigma)\) is minimal in \(I\) iff \(\text{sink}(\sigma +) = \text{sink}(\sigma -)\) iff \(\text{sink}(\sigma +) = \text{sink}(\sigma -)\).

**Flip Property**

\[ ((I)_L)_L = ((I)_S)_S = ((I)_C)_C \]

**Topological Cross-sections**

**Partial Order**