Topics in Discrete and Computational Geometry

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CONTENT

1. Arrangements de pseudodroites et de double pseudodroites
2. Algorithmique des arrangements de (double) pseudodroites
3. Arrangements de pseudohyperplans

BIBLIOGRAPHY

References

[1] Jacob E. Goodman. Pseudoline arrangements. In Jacob E. Goodman and Joseph O'Rourke, editors, Handbook of Discrete and Computational Geometry, chapter 5, pages 97–128. Chapman & Hall/CRC, 2004.

[2] J. E. Goodman, R. Pollack, R. Wenger, and T. Zamfirescu. Arrangements and topological planes. Amer. Math. Monthly, 101(9):866–878, November 1994. http://www.jstor.org.ezproxy.mir.math.upmc.fr/stable/pdfplus/2975135.pdf

[3] Julien Ferté, Vincent Pilaud, and Michel Pocchiola. On the number of simple arrangements of five double pseudolines. Discrete Comput. Geom. 45 (2): 279-302, 2011. http://dx.doi.org/10.1007/s00454-010-9298-4.

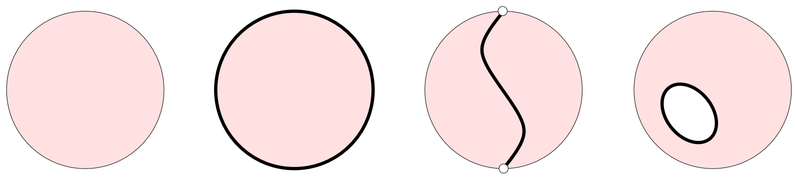
[4] L. Habert et M. Pocchiola. Arrangements of double pseudolines. Submitted to Disc. Comput. Geom. (Preliminary version on SoCG'09) http://arxiv.org/abs/1101.1022.

[5] Jürgen Bokowski. Computational Oriented Matroids. Cambridge, 2006.

[6] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler. Oriented Matroids. Cambridge University Press, 2nd edition, 1999.

NOTES

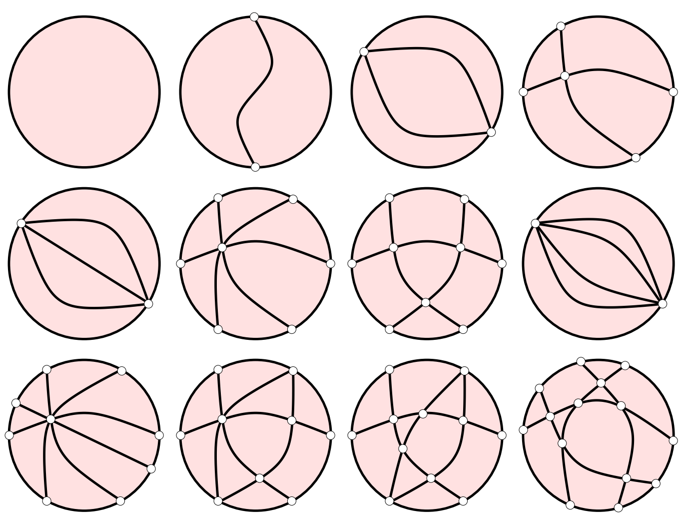
THE REAL PROJECTIVE PLANE $\mathbb{P}^2$



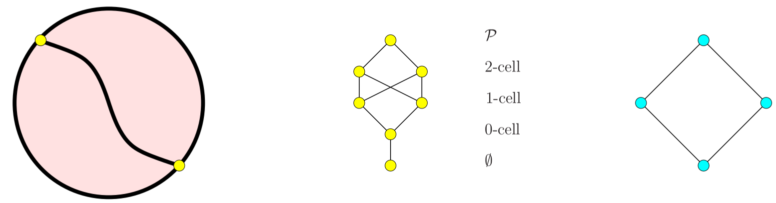
$$\mathbb{P}^2 = \{\mathbb{R}u \mid u \in \mathbb{R}^3, u \neq 0\} \approx \mathbb{S}^2/\{x \sim -x\} \approx \mathbb{D}^2/\{x \sim -x \mid x \in \partial\mathbb{D}^2\}$$

The real projective plane $\mathbb{P}^2$ - its representation by circular diagrams - classification up to homeomorphism of the simple closed curves embedded in the projective plane - pseudolines and double pseudolines

ARRANGEMENTS OF $1, 2, 3, 4$ AND 5 PSEUDOLINES



ARRANGEMENTS OF PSEUDOLINES



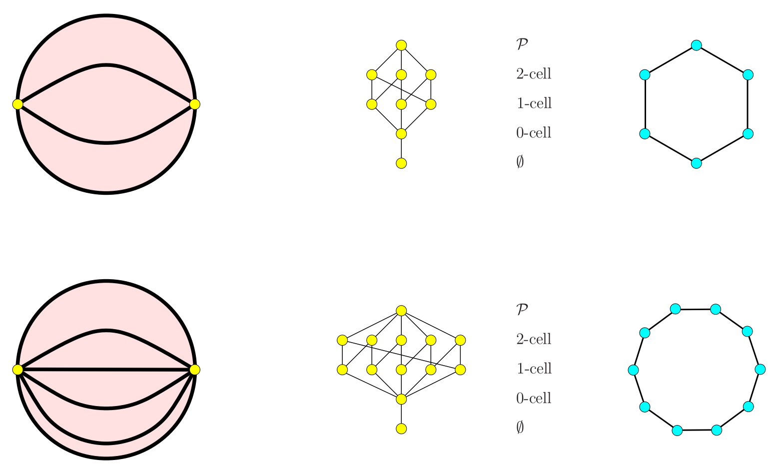
DF 1. *Let $\mathcal{P}$ be a projective plane. An* **arrangement of pseudolines in** $\mathcal{P}$ *is a finite family of pseudolines in $\mathcal{P}$ that cross pairwise in exactly one point. Two arrangements of pseudolines are called* **isomorphic** *if there is some homeomorphism of their underlying projective planes that maps the pseudolines in one arrangement to those in the other.* □

Induced cell complex, induced lattice, flag, flag diagram, etc.

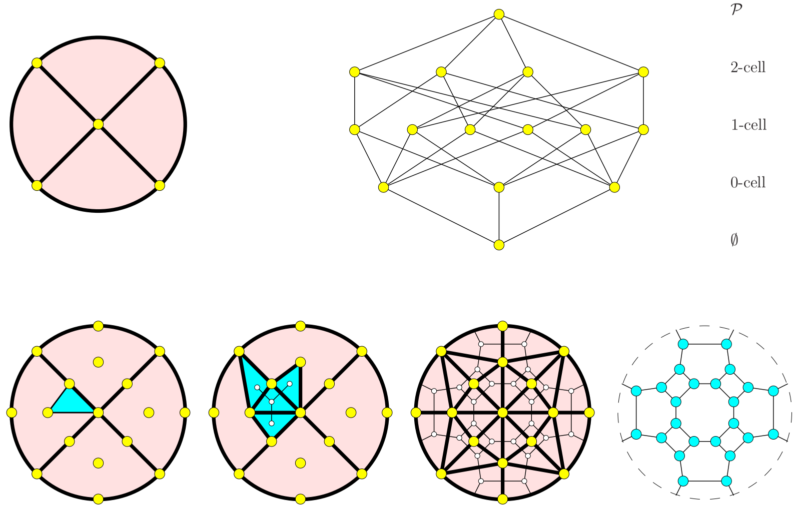
TH 1. *Two arrangements are isomorphic if and only if their induced lattices are isomorphic.* □

Extensively studied in the field of discrete and computational geometry: F. Levi (1926), G. Ringel (1956), B. Grünbaum (1972), J. E. Goodmann (1980), R. Cordovil (1982), J-P. Roudneff (1988), M.E. Mnëv (1988), P. Shor (1991), J.E. Goodmann & R. Pollack & R. Wenger and T. Zamfirescu (1994), . . . , Edelsbrunner & Guibas (1989), . . . , Agarwal & Sharir (2005), etc.

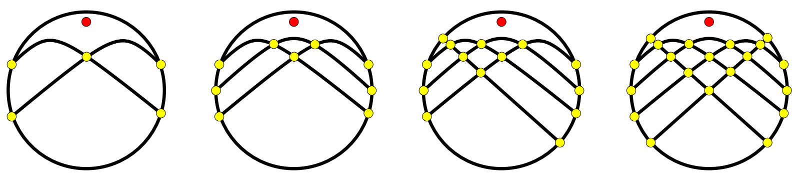
PENCILS



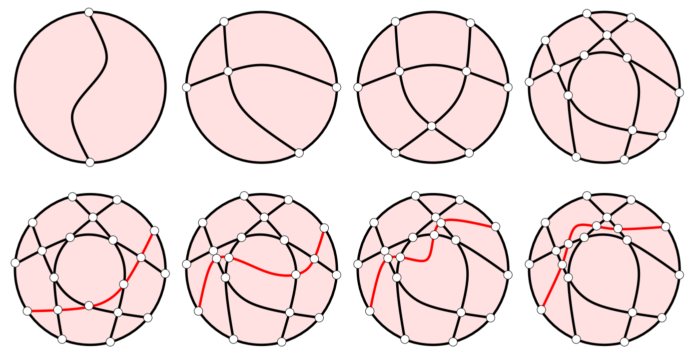
HASSE/FLAG DIAGRAM OF THE SIMPLE ARRANG. OF SIZE 3



CYCLIC ARRANGEMENTS OF SIZE 3, 4, 5 AND 6



NUMBER OF ARRANGEMENTS

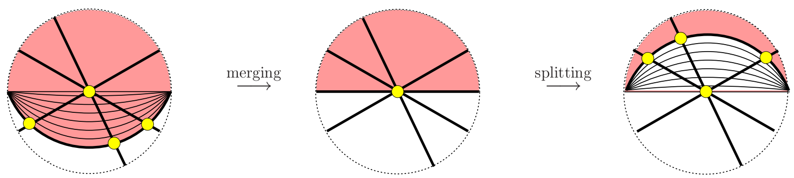


n	2	3	4	5	6	7	8	9	10	11
a_n^S	1	1	1	1	4	11	135	4 382	312 356	41 848 591
a_n	1	1	2	4	17	143	4 890	461 053	95 052 532	

according to the On-Line Encyclopedia of Integer Sequences the value of a_{11}^S is due to F. Aurenhammer, 2002 -
`http://www.research.att.com/~njas/sequences/A006248` -
`http://www.research.att.com/~njas/sequences/A063800` -

NOTES

RINGEL HOMOTOPY THEOREM

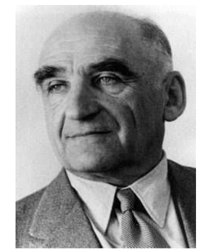


TH 2 (**Ringel Theorem, 1956**). *Any pair of simple arrangements of the same size and living in the same projective plane are homotopic via a finite sequence of* **mutations** *followed by an isotopy. In other words the graph of mutations on the space of arrangements of a given size is connected.* □

PUMPING LEMMA FOR ARRANG. OF PSEUDOLINES

Lemma 1. *Let Γ be a simple arrangement of pseudolines, let $\gamma, \gamma' \in \Gamma$, and let C be one of the two 2-cells of the subarrangement $\{\gamma, \gamma'\}$. Assume that there exists a vertex of Γ lying in C. Then there exists a triangular 2-cell of Γ included in C with a side supported by γ' and a vertex included in C.* □

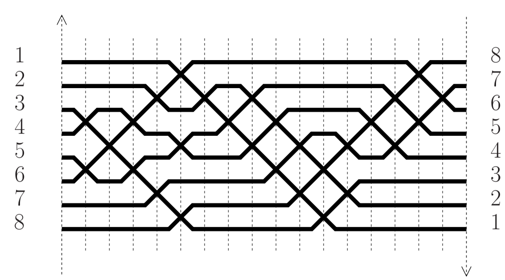
ENLARGEMENT LEMMAS



Lemma 2 (**Levi Enlargement Lemma, 1926.**). *Any arrangement of n pseudolines can be extended to an arrangement of $n+1$ pseudolines such that the added pseudoline goes through two given points not on the same pseudoline of the initial arrangement.* □

TH 3 (**J-E. Goodman, R. Pollack, R. Wenger and T. Zamfirescu., 1994**). *Any arrangement of pseudolines can be extended to an infinite family of pseudolines such that any two distinct points are contained in exactly one pseudoline of the family. In other words any arrangement of pseudolines can be extended to the line space of a* **real two-dimensional projective geometry**. □

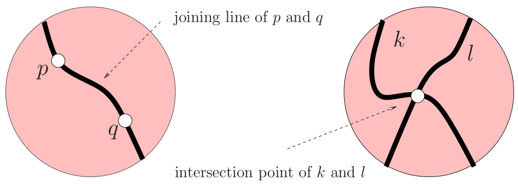
PROOF OF THE LEVI ENLARGEMENT LEMMA



$$\begin{bmatrix} 1&1&1&1&1&6&6&6&6&6&6&6&6&6&6&8&8\\ 2&2&2&2&6&1&2&2&5&5&5&5&5&5&8&6&7\\ 3&4&4&6&2&2&1&5&2&4&4&4&4&8&5&7&6\\ 4&3&6&4&4&5&5&1&4&2&7&7&8&4&7&5&5\\ 5&6&3&5&5&4&4&4&1&7&2&8&7&7&4&4&4\\ 6&5&5&3&7&7&7&7&7&1&8&2&3&3&3&3&3\\ 7&7&7&7&3&8&8&8&8&8&1&3&2&2&2&2&2\\ 8&8&8&8&8&3&3&3&3&3&3&1&1&1&1&1&1 \end{bmatrix}$$

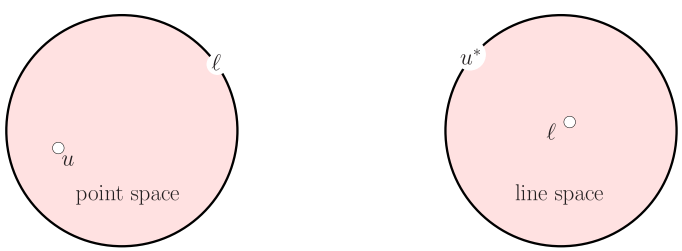
REAL TWO-DIMENSIONAL PROJECTIVE GEOMETRIES

Hilbert 1899, Kolmogoroff 1932, Köthe 1939, Skornjakov 1954, Salzmann 1955, Freudenthal 1957



DF 2. *A* **(real two-dimensional) projective geometry** *is a topological point-line incidence structure* $(\mathcal{P}, \mathcal{L}, \in)$ *whose point space* $\mathcal{P}$ *is a projective plane and whose line space* $\mathcal{L}$ *is a subspace of the space of pseudolines of* $\mathcal{P}$ *such that (1) any two distinct points are contained in exactly one line which depends continuously on the two points; (2) any two distinct lines intersect in exactly one point which depends continuously on the two lines.* □

DUALITY IN PROJECTIVE GEOMETRIES



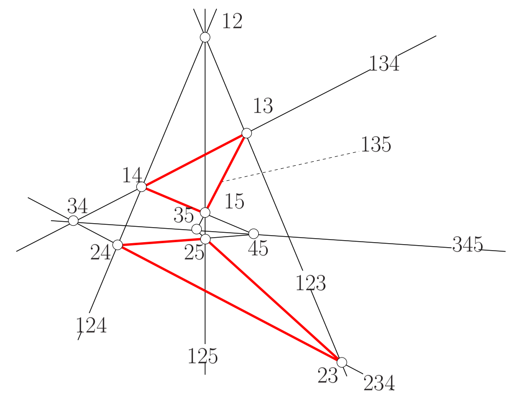
TH 6. *The line space of a projective geometry is a projective plane and the dual of a point of a projective geometry (i.e., the pencil of lines through that point) is a pseudoline of its line space.* □

$$(\mathcal{P}, \mathcal{L}) \to (\mathcal{L}, \mathcal{P}^*) \to (\mathcal{P}^*, \mathcal{L}^*) \approx (\mathcal{P}, \mathcal{L})$$

STANDARD PROJECTIVE GEOMETRY

DF 3. *The* **standard projective geometry** *is the projective geometry whose point space is the standard projective plane* $\mathbb{P}^2$ *and whose line space is the image under the canonical projection* $\mathbb{S}^2 \to \mathbb{P}^2$ *of the space of great circles of* $\mathbb{S}^2$. □

TH 7 (Hilbert, 1899). *The standard projective geometry is the unique* **desarguesian** *projective geometry.* □



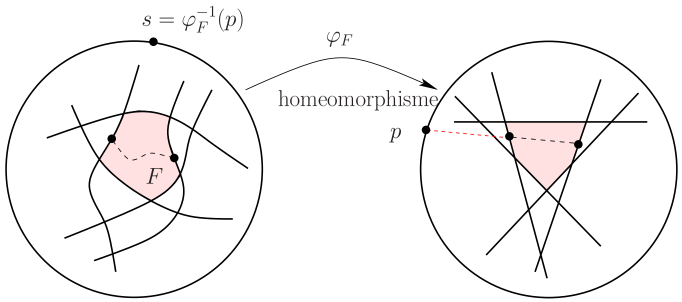
STRETCHABLE ARRANGEMENTS

n	a_n^S	stretchable	non-stretchable
2	1	1	0
3	1	1	0
4	1	1	0
5	1	1	0
6	4	4	0
7	11	11	0
8	135	135	0
9	4 382	4 381	1
10	312 356	312 114	242
11	41 848 591	41 693 377	155 214

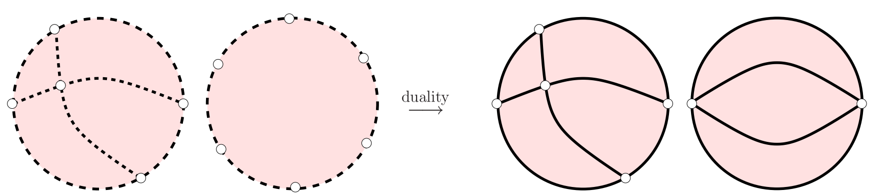
`http://www.research.att.com/~njas/sequences/A018242` -

TH 8 (Mnëv, 1988 – Shor, 1991). *The stretchability problem is NP-hard.* □

PROOF OF THE GOODMAN ET AL ENLARGEMENT LEMMA

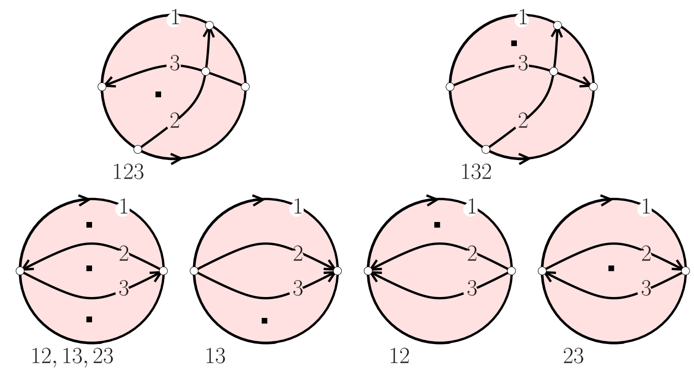


REPRESENTATION THEO. FOR PSEUDOLINE ARRANG.



TH 9 (**Representation Theorem for PA**). *Any arrangement of pseudolines is isomorphic to the dual family of a finite family of points of a projective geometry.* □

CHIROTOPES OF ARRANGEMENTS OF PSEUDOLINES



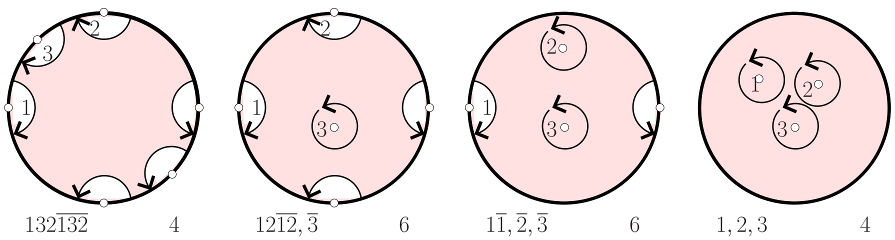
DF 4. *The* **chirotope** *of an indexed arrangement of oriented pseudolines is the map that assigns to each subset of indices of size at most three the isomorphism class of the subarrangement indexed by this subset (note that two isomorphic indexed and oriented arrangements have the same chirotope).* □

AXIOMATIZATION OF ARRANG. OF PSEUDOLINES

TH 10 (**Ringel, 1956 – Folkman and Lawrence, 1978**). *The map that assigns to the isomorphism class of an indexed arrangement of oriented pseudolines its chirotope is one-to-one and its range is the set of map χ defined on the set of triples of a finite set I such that for every 3-, 4-, and 5-subset J of I the restriction of χ to the set of triples of J is the chirotope of an arrangement of oriented pseudolines.* □

n	0	1	2	3	4	5
a_n^S	1	1	1	1	1	1
ρ_n^S	1	1	1	2	16	384
a_n	1	1	1	1	2	4
ρ_n	1	1	1	2	48	2 064

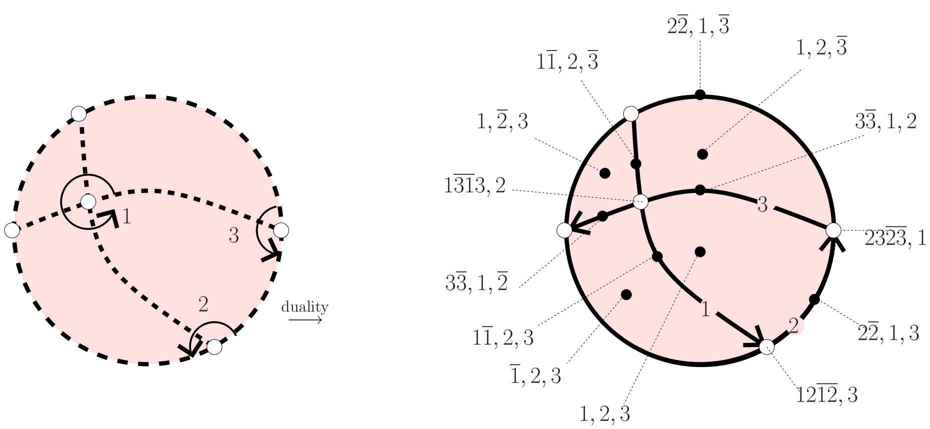
CONFIGURATIONS OF POINTS



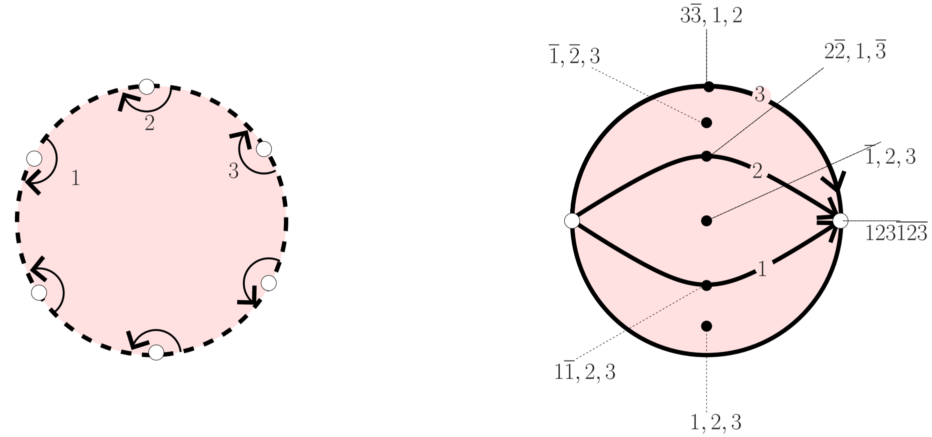
configurations of points - cocycles - cocycle-map - isomorphic configurations - chirotope

TH 11. *The map that assigns to an indexed configuration of oriented points the isomorphism class of its dual arrangement is compatible with the isomorphism relation on indexed configurations of oriented points and the induced quotient map is one-to-one and onto.* □

CONFIGURATIONS OF POINTS



CONFIGURATIONS OF POINTS



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INCIDENCE STRUCTURES/PROJECTIVE PLANES

An *incidence structure* is a triple (P, L, F) of sets with $F \subseteq P \times L$. The elements of P are termed points, the elements of L are termed lines, and the elements of F are termed flags or incidences. A set of points is termed collinear if there exists a line incident with all these points.

DF 5. *A projective plane $\mathcal{P}$ is an incidence structure (P, L, F) which satisfies the three following conditions*

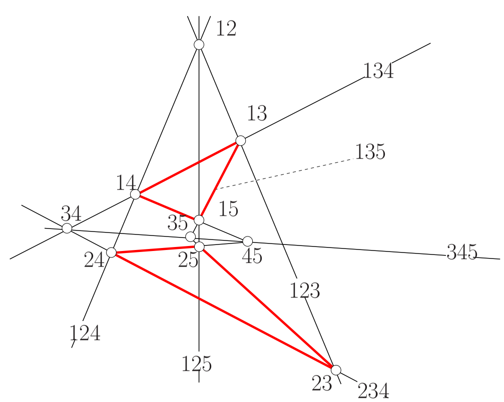
1. *any two distinct points are incident with a unique line, their joining line;*
2. *any two distinct lines are incident with a unique point, their intersection point;*
3. *there exists a non-degenerate quadrangle, i.e., a set of four points no three of which are collinear.*

Often (but not always) the lines are identified with their sets of incident points. In that case we write $p \in l$ instead of $(p, l) \in F$. The pencil L_p of a point p is the set of its incident lines.

Classical examples. Let K be a not necessarily commutative field, and consider K^3 as a right vector space over K. Denote by P the set of all one-dimensional subspaces of K^3, and by L the set of all two-dimensional subspaces. Then $\mathcal{P}_2(K) = (P, L, \subseteq)$, with inclusion $\subseteq$ as incidence, is a projective plane, *the projective plane over* K.

THE DESARGUES CONFIGURATION

The Desargues configuration is the incidence structure $\mathcal{D} = (\binom{\Omega}{2}, \binom{\Omega}{3}, \subseteq)$, where $\Omega = \{1, 2, 3, 4, 5\}$.



DESARGUESIAN PLANES

A projective plane $\mathcal{P}$ is said to be *desarguesian*, if the Desargues configuration 'always closes' in $\mathcal{P}$; this means that the following 'configurational' proposition holds: label ten distinct points and ten distinct lines of $\mathcal{P}$ by the elements of $\binom{\Omega}{2}$ and $\binom{\Omega}{3}$, respectively, and assume that 29 of the 30 incidences of $\mathcal{D}$ hold in P. Then also the remaining incidence of $\mathcal{D}$ holds in $\mathcal{P}$ (it does not matter which incidence is singled out as the conclusion, because $\mathcal{D}$ has a flag-transitive collineation group).

TH 11 (Hilbert, 1899)**.** *The desarguesian projective planes are precisely the planes $\mathcal{P}_2(K)$ where K is a (not necessarily commutative) field.* □

[1] Burkard Polster and Günter Steinke. Geometries on Surfaces. Number 84 in Encyclopedia of Mathematics and its applications. Cambridge, 2001.

[2] Helmut Salzmann, Dieter Betten, Theo Grundhöfer, Hermann Hähl, Rainer Löwen, and Markus Stroppel. Compact projective planes. Number 21 in De Gruyter expositions in mathematics. Walter de Gruyter, 1995.

AFFINE PLANES AND PROJECTIVE COMPLETION

DF 6. *An affine plane $\mathcal{A}$ is an incidence structure (A, L, F) that satisfies the three following conditions*

1. *any two distinct points are incident with a unique line;*
2. *for every pair $(p, l) \in A \times L$ there exists a unique line k of the pencil of p such that either $k = l$ or k and l have no point in common; we call k the parallel to l through p;*
3. *there exists a triangle in $\mathcal{A}$, i.e., a set of three points which are not collinear.*

Let $\mathcal{P} = (P, L, \in)$ be a projective plane and let W be an arbitrary line of $\mathcal{P}$. Then the *affine part*

$$\mathcal{P}^W = (P \setminus W, L \setminus \{W\}, \in)$$

is an affine plane. Conversely, every affine plane $\mathcal{A} = (A, L, F)$ has a *projective completion* $\overline{\mathcal{A}}$, which is the projective plane obtained from $\mathcal{A}$ by adding a new line W (called the line at infinity) and a set of new points on W (called the points at infinity) which correspond to the parallel classes of $\mathcal{A}$. (Hence every affine plane has the form $\mathcal{P}^W$.) More precisely we set

$$\begin{aligned} \overline{A} &= A \cup [L], \\ \overline{L} &= \{l \cup [l] \mid l \in L\} \cup \{[L]\}. \end{aligned}$$

where $[l]$ denotes the parallel class of $l \in L$.

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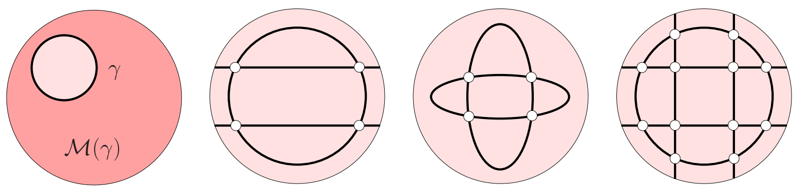
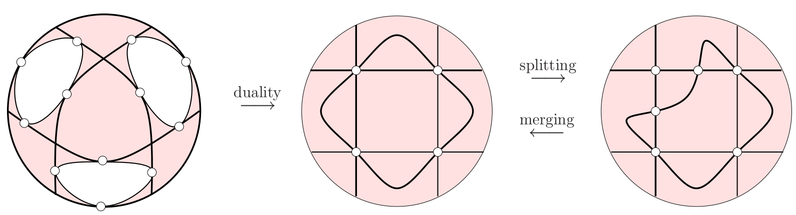
DEFINITION OF ARRANGEMENTS OF DOUBLE PSEUDOLINES



DF 7. *Let $\mathcal{P}$ be a projective plane. An* **arrangement of double pseudolines** *in $\mathcal{P}$ is a finite family of double pseudolines in $\mathcal{P}$ that intersect pairwise in exactly four transversal intersection points and that induce pairwise a cell structure on $\mathcal{P}$.* □

Examples of arrangements of double pseudolines are given by the dual families of finite families of pairwise disjoint convex bodies of projective geometries.

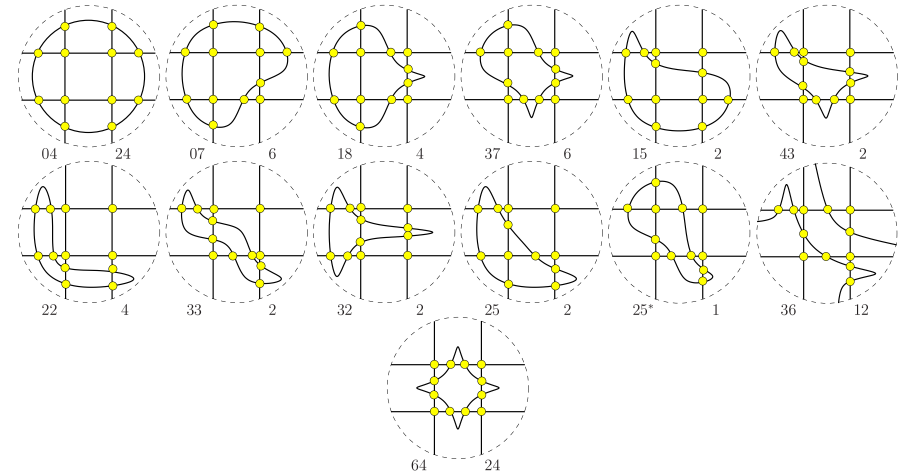
DUALITY AND HOMOTOPY THEOREMS



TH 12. *Any arrangement of double pseudolines is isomorphic to the dual family of a finite family of pairwise disjoint convex bodies of a projective geometry.* □

TH 13. *Any two arrangements of double pseudolines of the same size and living in the same projective plane are homotopic via a finite sequence of mutations followed by an isotopy. In other words the graph of mutations on the space of arrangements of double pseudolines of given size is connected.* □

ISO. CLASSES SIMPLE ARRANG. THREE DOUBLE PSEUDOLINES



n	2	3	4	5
a_n^S/a_n	1/1	13/46	6570/153 528	181 403 533/???

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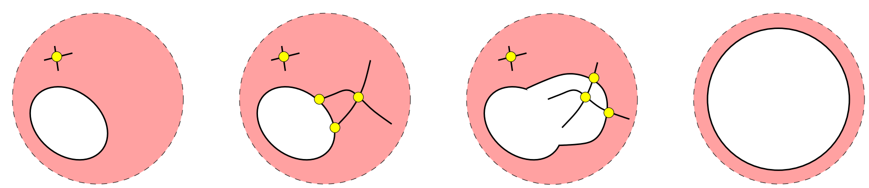
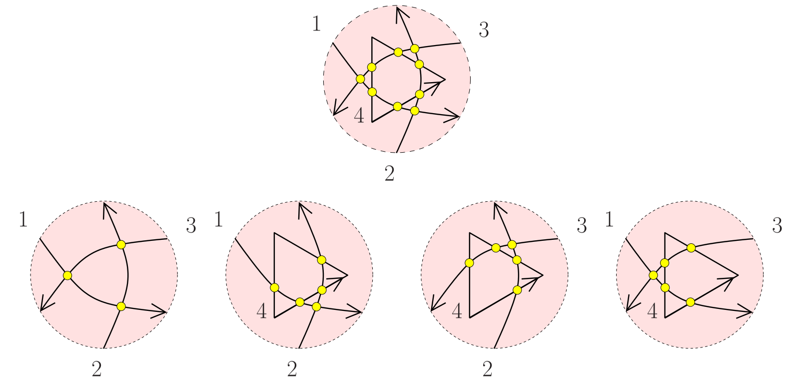
THE PUMPING LEMMA



TH 14 (**Pumping Lemma).** *Let Γ be a simple arrang. of double pseudolines and let $\gamma \in \Gamma$. Assume that there is a vertex of Γ lying in the interior of the Möbius strip $\mathcal{M}(\gamma)$ bounded by γ. Then there is a triangular face of Γ included in $\mathcal{M}(\gamma)$ with a side supported by γ.* □

CHIROTOPES AND AXIOMATIZATION



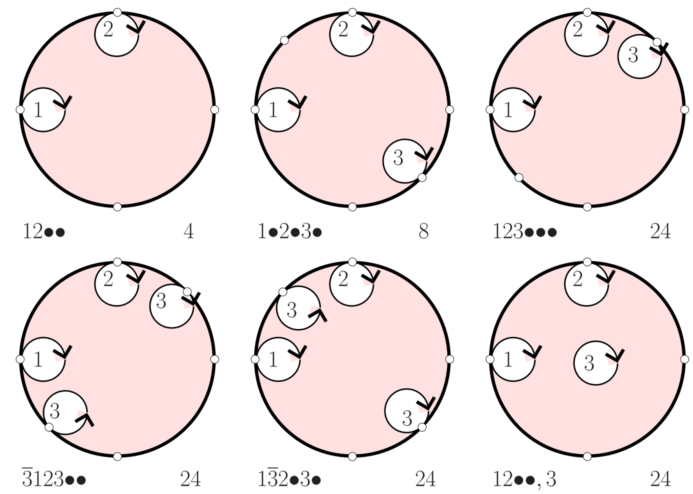
DF 8. *The* **chirotope** *of an indexed arrangement of oriented double pseudolines is the map that assigns to each subset of indices of size at most three the isomorphism class of the subarrangement indexed by this subset (note that two isomorphic arrangements have the same chirotope).* □

AXIOMATIZATION

TH 15. *The map that assigns to an isomorphism class of indexed arrangements of oriented double pseudolines its chirotope is one-to-one and its range is the set of map χ defined on the set of triples of a finite set I such that for every 3-, 4-, and 5-subset J of I the restriction of χ to the set of triples of J is the chirotope of an indexed arrangement of oriented double pseudolines.* □

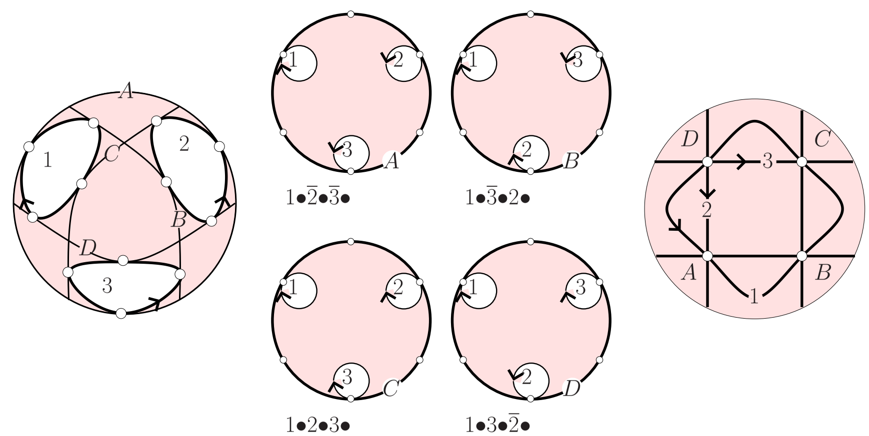
n	0	1	2	3	4	5
a_n^S	1	1	1	13	6 570	180 403 533
ρ_n^S	1	1	1	214	2 415 112	nc
$2^n n! a_n^S$				624	2 822 580	692 749 566 720

CONFIGURATIONS OF CONVEX BODIES



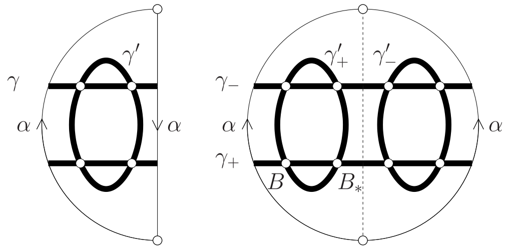
configurations of convex bodies - cocycles - isomorphism relations - chirotopes

DUALITY

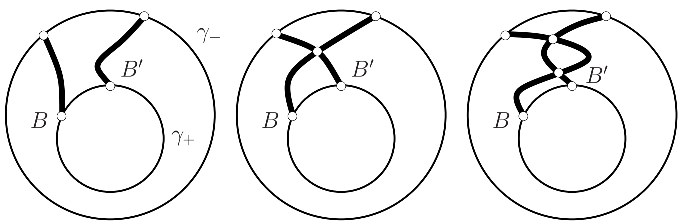


TH 16. *The map that assigns to an indexed configuration of oriented convex bodies the isomorphism class of its dual arrangement is compatible with the isomorphism relation on indexed configurations of oriented convex bodies and the induced quotient map is one-to-one and onto.* □

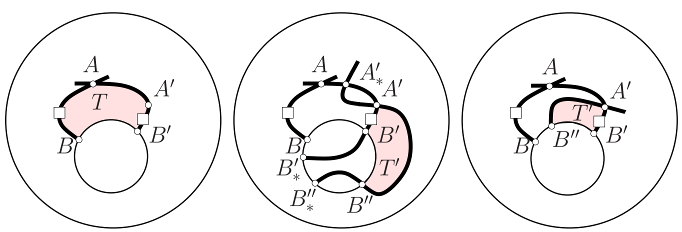
TWO-COVERING



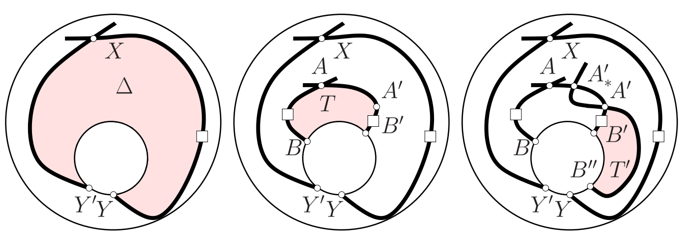
$\mathcal{P} : \alpha\alpha = 1$ $\tilde{\mathcal{P}} : \alpha\alpha^{-1} = 1$



ADMISSIBLE TRIANGLES AND THEIR DERIVATIVES

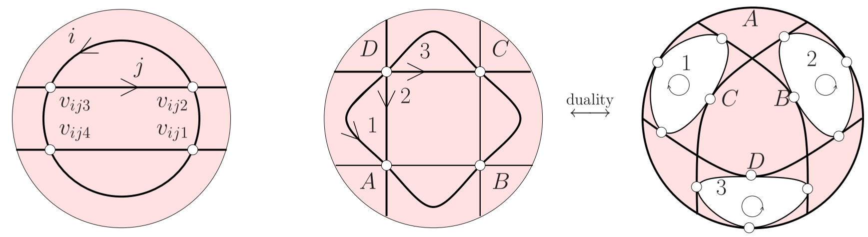


$B'' \in \overline{\mathrm{bs}}(T)$ $B'' \in \mathrm{bs}(T)$



$B'' \in \overline{\mathrm{bs}}(T)$

CYCLES

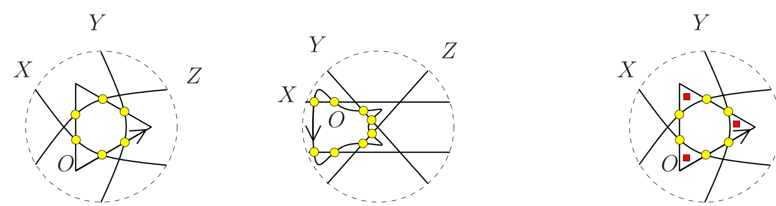


$I, \mathcal{N}(I) = \{ijk \mid i, j \in I, i \neq j, k \in \mathbb{Z}_4\}/\{ij2 = ji3, ij1 = ji1, ij4 = ji4\}$

$$\mathcal{V}(\Upsilon) = \{A, B, C, D\}, \quad \begin{cases} A = \{121, 134, 232\} \\ B = \{122, 131, 234\} \\ C = \{123, 132, 233\} \\ D = \{124, 133, 231\} \end{cases}, \quad \begin{cases} \mathcal{C}_1(\Upsilon) = ABCD \\ \mathcal{C}_2(\Upsilon) = ACBD \\ \mathcal{C}_3(\Upsilon) = ABDC. \end{cases}$$

TH 17. *Two indexed arrangements of oriented double pseudolines are isomorphic if and only if they have the same indexed family of cycles.* □

MARTAGONS



$\Gamma^1(X, Y, Z)$ $\Gamma^2(X, Y, Z)$ $\Gamma^1(X, Y, Z)$

AXIOMATIZATION THEOREM

Let I be a finite indexing set.

DF 9. *A k-chirotope (of double pseudoline arrangements) is a map χ defined on the set of triples of I such that for any subset J of I of size at most k the restriction of χ to the set of triples of J is the chirotope of a double pseudoline arrangement indexed by J. We denote by $\mathcal{C}_k$ the set of k-chirotopes.* □

TH 13. $\mathcal{C}_3 \supsetneq \mathcal{C}_4 \supsetneq \mathcal{C}_5 = \mathcal{C}_6 = \mathcal{C}_7 = \cdots$. □

AXIOMATIZATION THEOREM

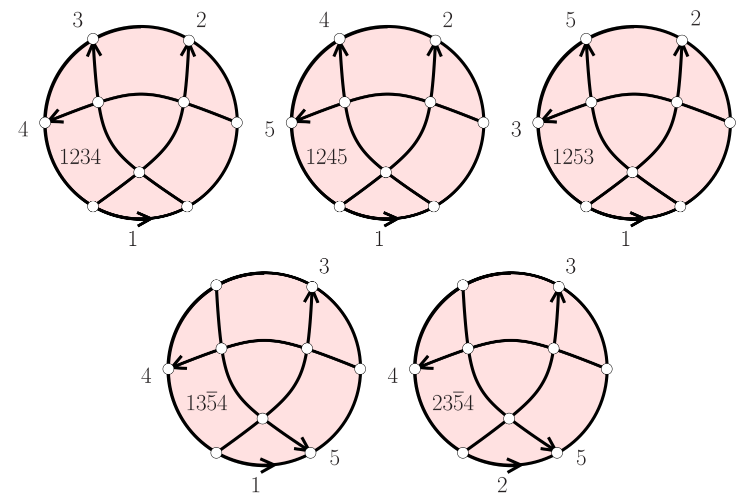
Let I be a finite indexing set.

DF 7. *A k-chirotope (of double pseudoline arrangements) is a map χ defined on the set of triples of I such that for any subset J of I of size at most k the restriction of χ to the set of triples of J is the chirotope of a double pseudoline arrangement indexed by J. We denote by $\mathcal{C}_k$ the set of k-chirotopes.* □

TH 13. $\mathcal{C}_3 \supsetneq \mathcal{C}_4 \supsetneq \mathcal{C}_5 = \mathcal{C}_6 = \mathcal{C}_7 = \cdots$. □

n	0	1	2	3	4	5
a_n^S	1	1	1	13	6 570	180 403 533
ρ_n^S	1	1	1	214	2 415 112	nc
$2^n n! a_n^S$				624	2 822 580	692 749 566 720
b_n^S	1	1	1	16	11502	238 834 187
τ_n^S	1	1	1	118	541 820	nc
$2n! b_n^S$				192	552 096	57 320 204 880

A 4-CHIROTOPE



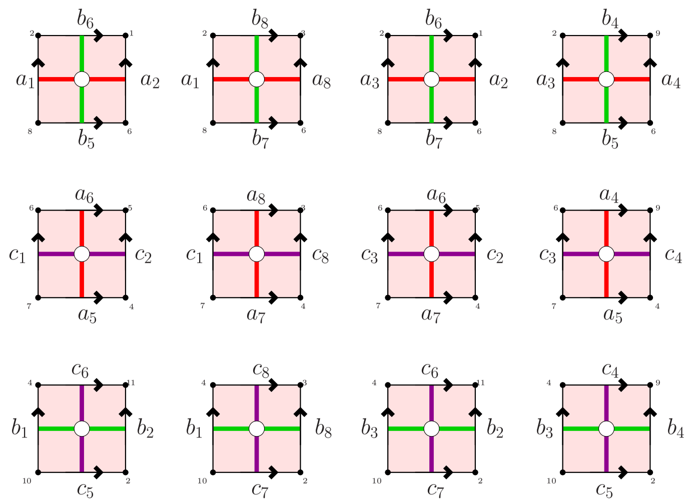
k-CHIROTOPES – k-ARRANGEMENTS

DF 8. *A k-arrangement of double pseudolines is a finite indexed by I family τ of simple closed oriented curves embedded in a compact surface S_τ with the properties that (1) τ induces a regular cell decomposition X_τ of S_τ; and (2) any subfamily ν of τ of size at least 2 and at most k, considered as embedded not in S_τ but in the compact surface S_ν implicitly defined by by a tubular neighborhood of $\bigcup \nu$ in S_τ, is an arrangement of double pseudolines. We denote by $\mathcal{A}_k$ the set of k-arrangements and by $\mathcal{A}_k \to \mathcal{C}_k$ the map that assigns to a k-arrangement its chirotope.* □

Lemma 3. *The graph of mutations on $\mathcal{A}_5$ is connected.* □

Lemma 4. *$\mathcal{A}_5 \to \mathcal{C}_5$ is one-to-one and onto.* □

A TWO-ARRANGEMENT



A TWO-ARRANGEMENT

$$a_1b_6 = b_5a_2 \quad a_1b_8 = b_7a_8 \quad a_3b_6 = b_7a_2 \quad a_3b_4 = b_5a_4$$
$$c_1a_6 = a_5c_2 \quad c_1a_8 = a_7c_8 \quad c_3a_6 = a_7c_2 \quad c_3a_4 = a_5c_4$$
$$b_1c_6 = c_5b_2 \quad b_1c_8 = c_7b_8 \quad b_3c_6 = c_7b_2 \quad b_3c_4 = c_5b_4$$

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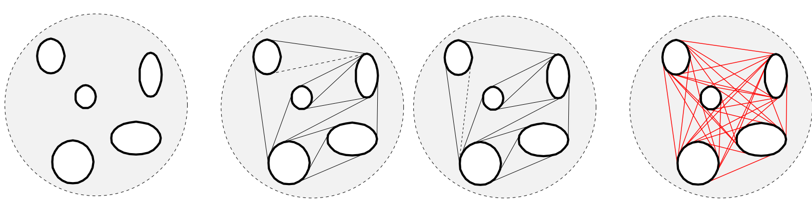
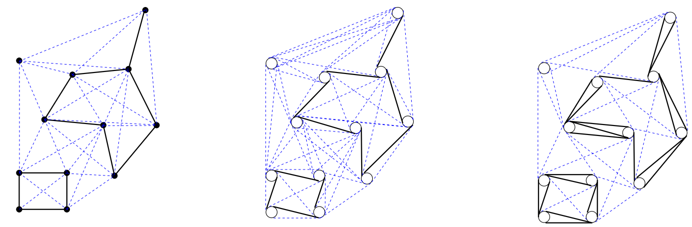
CHIROTOPES AND VIS. GRAPH ALGORITHMS

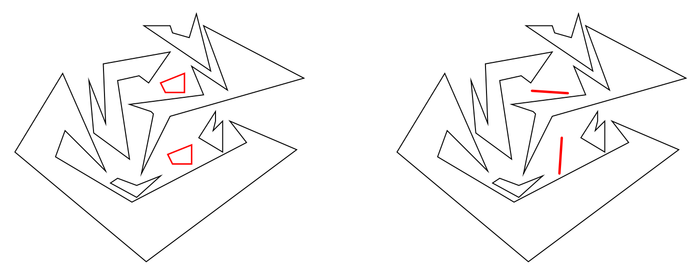


TH 14. *The k edges of the visibility graph of a planar family of n pairwise disjoint convex bodies presented by its chirotope is computable in time $O(k + n \log n)$ and linear working space.* □

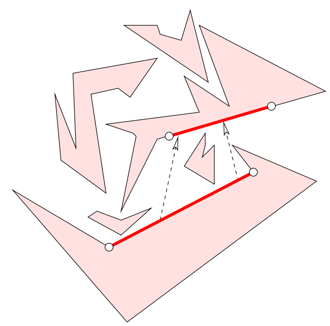
VISIBILITY GRAPH OF POLYGONAL OBSTACLES



LE PROBLÈME DU DÉMÉNAGEUR DE PIANO



FACTEUR DE FORME



Physique. Le facteur de forme est la fraction d'energie diffuse quittant une surface et atteignant directement une autre surface.

Géométrie. Le facteur de forme F_{ij} entre les surfaces s_i et s_j est la mesure de l'ensemble X_{ij} des droites passant par deux points mutuellement visibles de s_i et s_j. Il existe une unique mesure sur l'espace des droites invariante par déplacements.

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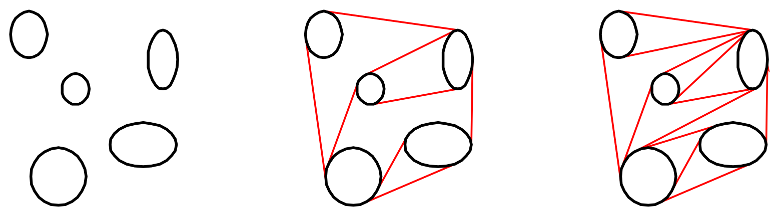
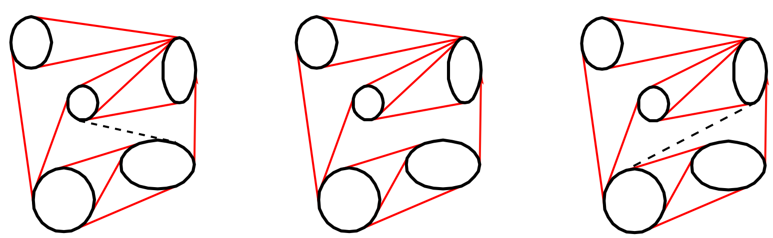
PSEUDOTRIANGULATIONS



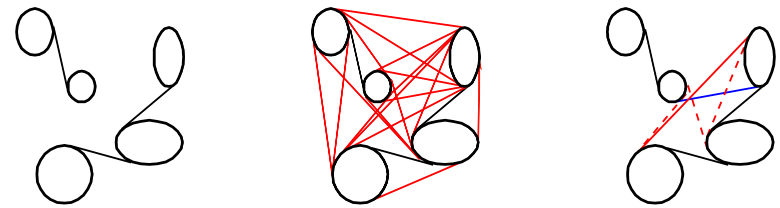
1. O a set of n pairwise disjoint convex bodies
2. B the set of $k \leq 2n(n-1)$ undirected free bitangents
3. A pseudotriangulation G is a maximal (for $\subseteq$) planar subset of B
4. $C \subset G$, $|G| = 3n-3$
5. $(\forall G)\,(\forall b \in G \setminus C)\,(\exists!\, G' \neq G)\ \ (G \setminus G' = \{b\})$.

FLIP GRAPH OF PSEUDOTRIANGULATIONS



1. G a pseudotriangulation, $C \subset G$, $|G| = 3n-3$
2. **[Flips]** $(\forall G)\,(\forall b \in G \setminus C)\,(\exists!\, G' \neq G)\ \ (G \setminus G' = \{b\})$ $\qquad G \xrightarrow{b} G'$.
3. **[Connectivity]** Le graphe d'adjacence des pseudotriangulations est 'fortement' connexe.
$$\begin{cases} G = G_0 \xrightarrow{b_1} G_1 \cdots \xrightarrow{b_k} G_k = G' \\ G \cap G' \subseteq G_i \qquad (i = 0, \ldots, k) \end{cases}$$
4. $\{H \mid C \subseteq H, H$ planaire$\}$ ordonné par inclusion est un polytope abstrait de dimension $3n - 3 - |C|$.

GREEDY PSEUDOTRIANGULATIONS



1. $\mathcal{K} \subset \mathcal{B}$ planar
2. $\mathcal{B}_\mathcal{K} = \{v \in \mathcal{B} \mid \mathcal{K} \cup \{v\}$ is planar$\}$, $\Theta(v)$, $\mathbb{B}_\mathcal{K} = \mathcal{B}_\mathcal{K} \times \mathbb{Z}$, $\Theta(v,k) = \Theta(v) + k\pi$
3. We define a partial order $\prec_\mathcal{K}$ on $\mathbb{B}_\mathcal{K}$ as follows : $v \prec_\mathcal{K} v'$ if there exists a sequence of bitangents $v = v_1, v_2, \ldots, v_k = v'$ such that v_i and v_{i+1} touch the same directed disk and $\Theta(v_i) < \Theta(v_{i+1})$

TH 15 (PV96,AP03)**.** *Two crossing bitangents of $\mathbb{B}_\mathcal{K}$ are comparable with respect to $\prec_\mathcal{K}$ and the operator $\varphi(.;\mathcal{K}) : \mathbb{B}_\mathcal{K} \to \mathbb{B}_\mathcal{K}$ that associates with $v \in \mathbb{B}_\mathcal{K}$ the minimum element of the set of $u \in \mathbb{B}_\mathcal{K}$ such that u crosses v and $v \prec_\mathcal{K} u$ is well-defined, one-to-one, and onto.* □

GREEDY FLIP PROPERTY

For any filter I of the poset $(\mathbb{B}_\mathcal{K}, \prec_\mathcal{K})$, we define a maximal planar set of bitangents $\mathrm{G}(I) = \{v_1, v_2, \ldots, v_k\}$ recursively: v_1 is minimal in I, and, for $i > 1$, v_{i+1} is minimal in the set of bitangents $v \in I \setminus \{v_1, \ldots, v_i\}$ such that $\mathcal{K} \cup \{v_1, \ldots, v_i, v\}$ is planar—note that this set is empty after $3n-3$ steps.

TH 16 (PV96,AP03)**.** *Let b be a minimal element of the filter I of the poset $(\mathbb{B}_H, \prec_H)$. Then*

1. $\mathrm{G}(I) = I \setminus \varphi(I;\kappa)$;

2. $\mathrm{G}(I \setminus \{b\})$ *is obtained from* $\mathrm{G}(I)$ *by flipping* b. □

ENUMERATION OF PSEUDOTRIANGULATIONS

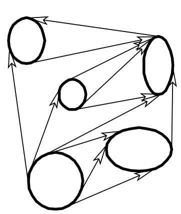
Let I be a filter (for example the filter of bitangents with positive angle) of $(\mathbb{B}_H, \prec_H)$, and let $G_0 = G(I)$ be the associated greedy pseudotriangulation. For $H = (F, G)$, $F \cap G = \emptyset$, $H \subset F \cup G \subset I$ a pseudotriangulation, we define $\omega(H)$ to be the pair $(F \cup \{v\}, G \setminus \{v\})$ and $\phi(H)$ to be the pair $(F, G \setminus \{v\} \cup \{v'\})$ where $v = \min_{<_H} G$ and v' is the shadow of v in $F \cup G$. (Observe that ω and ϕ are defined only if $G \neq \emptyset$.) For $G \supseteq H$ a pseudotriangulation of the o_i's we define a sequence $H(G) = H_1, H_2, \ldots, H_i = (F_i, G_i)$ by setting $H_0 = (\emptyset, G_0)$, and

$$H_{i+1} = \begin{cases} \omega(H_i), & \text{if } \min_{<_H} G_i \in G; \\ \phi(H_i), & \text{otherwise.} \end{cases} \quad (1)$$

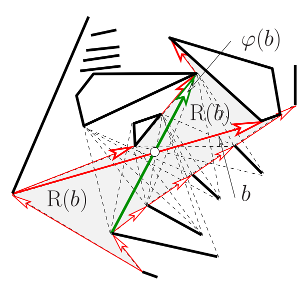
Il est aisé de voir that the sequence $H(G)$ is finite, that its last term is $\overline{H}(G) = (G, \emptyset)$, and that $H(G) = H(G')$ iff. $G = G'$. This proves the strong connectivity of the flip graph of pseudotriangulations and, with some extra work, provides a linear space traversal algorithm of the flip graph whose complexity is linear amortized per node

VISIBILITY GRAPH ALGORITHM

Algorithm GREEDY FLIP ALGORITHM
1 compute the greedy pseudo-triangulation $G := \mathrm{G}(I(0))$;
2 **repeat**
3 select a minimal bitangent b in G with angle less than π;
4 flip b; (i.e., replace b by $\varphi(b)$)
5 **until** there are no more bitangents of angle less than π.



G(S)



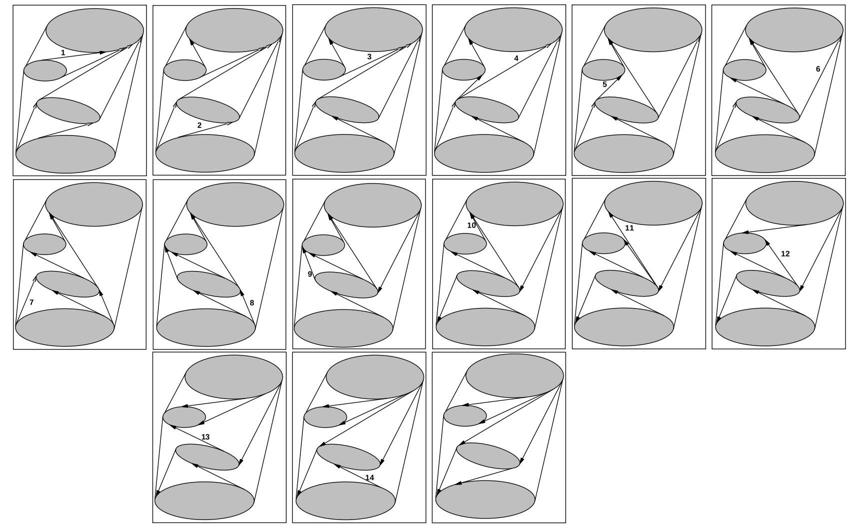
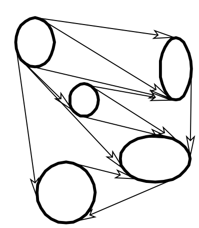


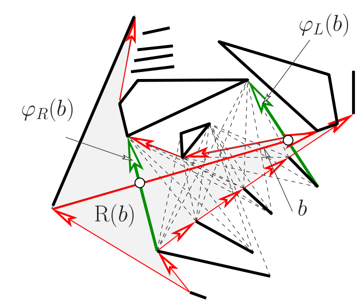
Figure 1: The greedy flip algorithm. At each step the internal bitangent of minimal slope in the current pseudo-triangulation is flipped

THE FLIP OPERATION (VERSION AP03)

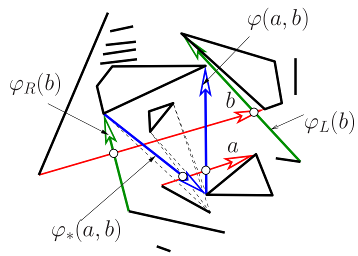
TH 17 (AP03)**.** *The flip operation of the greedy flip algorithm can be implemented in constant amortized time* <u>*using only the chirotope of the convex bodies.*</u> *Furthermore the only necessary data structures are linked structures to represent the incidence relations in the sweep structure S and its associated pseudotriangulations* G(S) *and* G$_*(S)$, *a list to store the minimal bitangents of* G(S), *and for each minimal bitangent b pointers to the arcs of* G(S) *that $\varphi_R(b)$ and $\varphi_L(b)$ leave and pointers to the arcs of* G$_*(S)$ *that $\varphi_R(b)$ and $\varphi_L(b)$ enter.* □

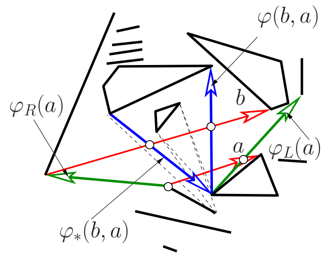


G$_*(S)$



"SUM OF SQUARES" THEOREM

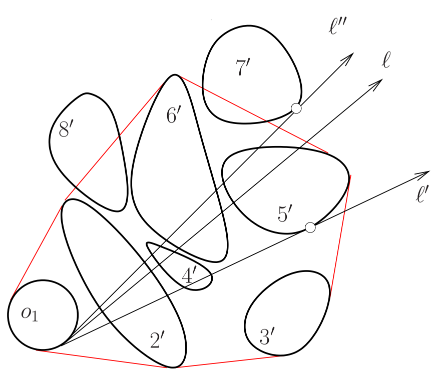




$\varphi(a,b) = \varphi(a; \{\varphi_R(b), \varphi_L(b)\})$.

TH 18 (AP03)**.** *The number of pairs of free bitangents (a,b) such that $\varphi(a,b) = \varphi(b,a)$ and $\varphi_*(a,b) = \varphi_*(b,a)$ is bounded by a constant times the number of free bitangents.* □

INITIALIZATION OF THE GFA



TH 19 (HP03)**.** *There is a* G(I) *computable in $O(n \log n)$ time using only the chirotope of the convex bodies.* □

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[5] M. Pocchiola and G. Vegter. Topologically sweeping visibility complexes via pseudo-triangulations. Discrete Comput. Geom., 16(4):419–453, December 1996.

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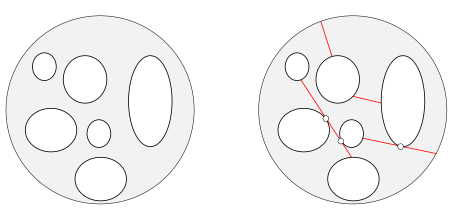
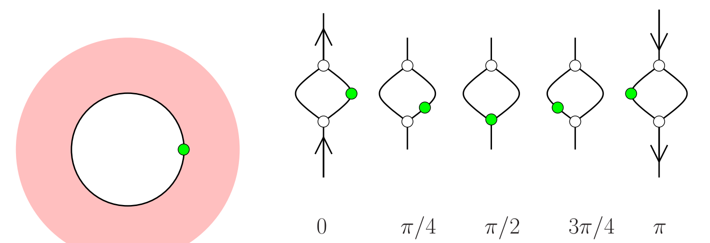
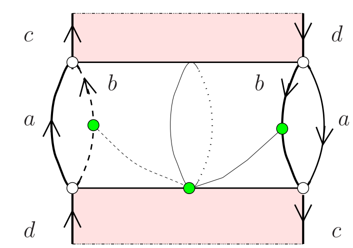
COMPLEXES DE VISIBILITÉ



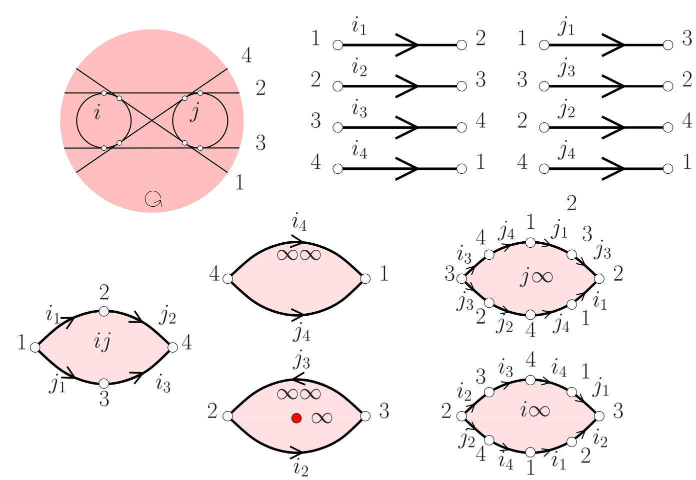
1. Soit o_i une famille de n corps convexes disjoint deux-à-deux d'un plan topologique $\mathbb{A}$ sur $\mathbb{R}^2$; l'espace libre est noté $\mathbb{F}$ et est défini comme le complémentaire dans $\mathbb{A}$ de l'union des intérieurs des corps convexes, i.e. $\mathbb{F} = \mathbb{A} \setminus \bigcup_i \text{Interior}(o_i)$;
   l'espace des droites de $\mathbb{A}$ est noté $\mathbb{L}$; la courbe des tangentes de $\mathbb{L}$ à o_i est notée $z_i (\approx \mathbb{S}^1)$;
2. Le complexe de visibilité des o_i est noté $\mathbb{V}$ et est défini comme l'espace des droites de $\mathbb{F}$; la courbe des tangentes de $\mathbb{V}$ à o_i est notée $\gamma_i (\approx \mathbb{S}^1)$;
3. $p : \mathbb{V} \to \mathbb{L}$;
4. Versions orientées de $\mathbb{V}$ et $\mathbb{L}$; les courbes (orientées) $z_{+i}, z_{-i}, \gamma_{+i}, \gamma_{-i}$.

VISIBILITY COMPLEX OF A PLANAR CONVEX BODY

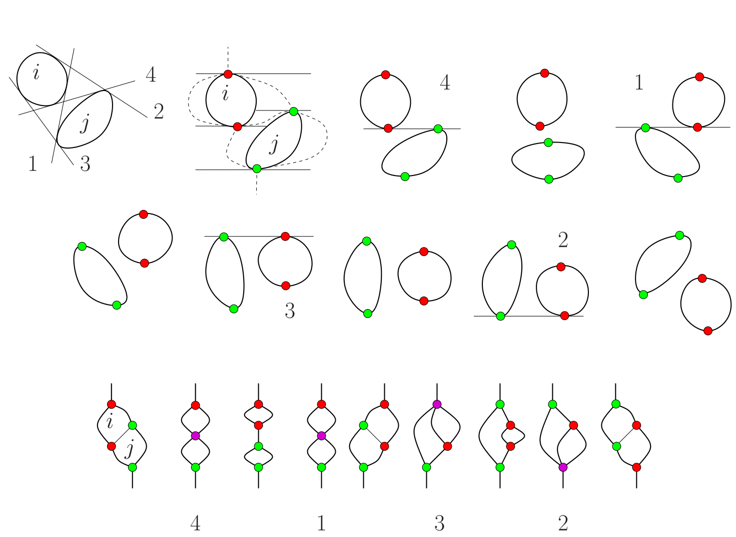




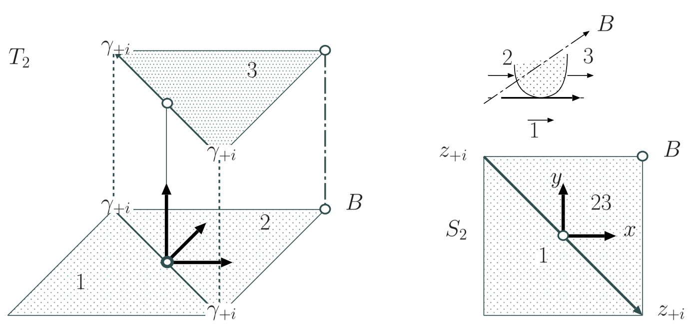
VISIBILITY COMPLEX OF TWO CONVEX BODIES



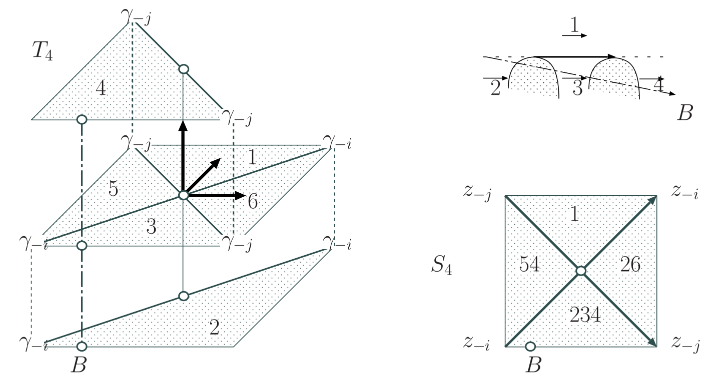
CROSS-SECTIONS OF THE VISIBILITY COMPLEX OF TWO CONVEX BODIES



LOCAL TOPOLOGY OF THE VISIBILITY COMPLEX



LOCAL TOPOLOGY OF THE VISIBILITY COMPLEX



BORDS GAUCHE ET DROIT D'UNE 2-CELLULE

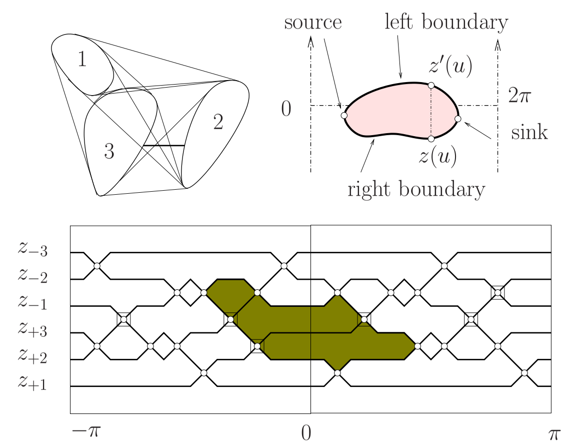
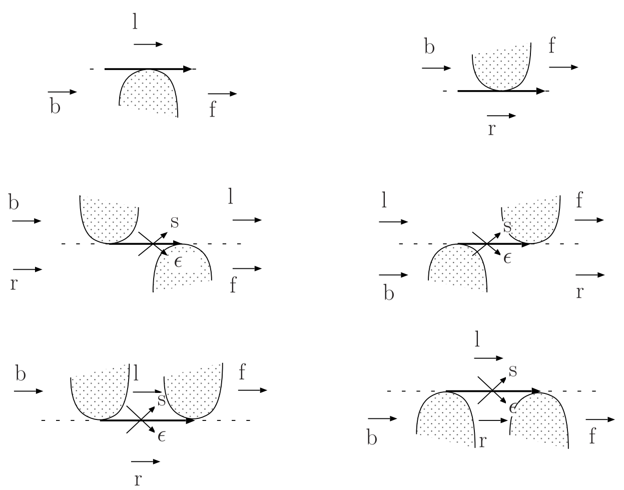


Figure 2: Example of arrangement of curves z_o's for a collection of three convex bodies : each curve z_o is stretched and bended so that it stays roughly parallel to the equator of the space of directed lines. There is a one-to-one and onto correspondence between the set of vertices of the arrangement and the set of free and non-free directed bitangents. The (6) vertices marked with a square correspond to non-free bitangents. The dashed region represents the projection in the space of directed lines of the 2-cell containing the elements of the visibility complex with backward view the disk labeled 3 and forward view the disk labeled 2.

INCIDENCE RELATIONS



INCIDENCE OPERATORS

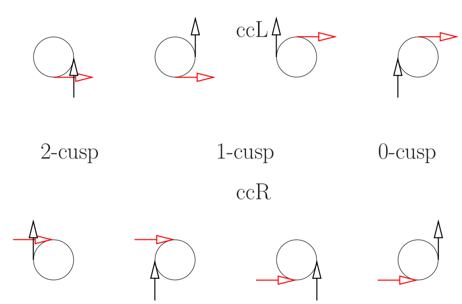


Figure 3: Regular, cusp and double cusp arcs.

1. Sink and source of a 1- or 2-cell;
2. ccR and ccL operators;
3. For $\sigma \in X_0 \cup X_1$ and $\alpha \in \{\mathrm{l}, \mathrm{b}, \mathrm{r}, \mathrm{f}, \mathrm{s}\}$ we set (if defined)

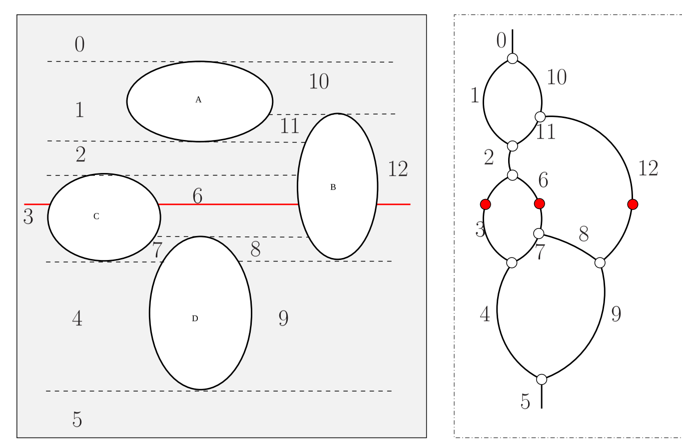
$$\varphi_\alpha(\sigma) = \mathrm{sink}(\sigma_\alpha), \qquad \alpha \in \{\mathrm{l}, \mathrm{b}, \mathrm{r}, \mathrm{f}, \mathrm{s}\}. \tag{2}$$

COMPUTING THE RIGHT BOUNDARY OF A 2-CELL

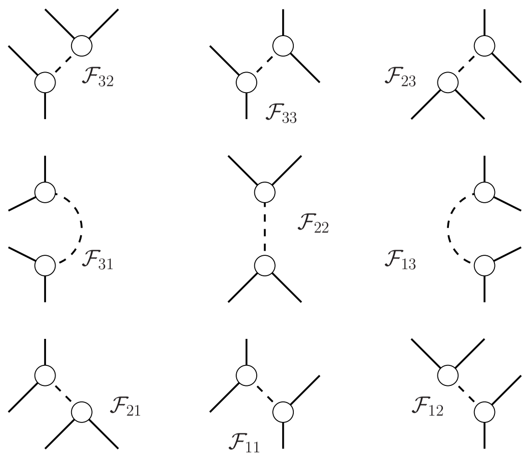
Here is an algorithm to compute the right boundary of the 2-cell σ whose source is b.

1 push(b); $b_2 \leftarrow$ ccr(b);
2 **if** (b, b_2) 2-cusp **then goto 6**
3 **while**(b, b_2) 0-cusp {
push(b_2); /* b_2 IS 'FAT' */
$b_2 \leftarrow$ ccl(b_2);
}
4 **do** {
push (b_2); /*b_2 IS 'REGULAR' */
$(b_1, b_2) \leftarrow (b_2, \mathrm{ccl}(b_2))$;
} **while** (b_1, b_2) 0-cusp
5 **while** (b_1, b_2) 2-cusp {
push (b_2); /* b_2 IS 'FAT'*/
$b_2 \leftarrow$ ccr(b_2);
}
6 push(b_2); /*b_2 IS THE SINK OF σ, I.E., $b_2 = \varphi(b)$ */
exit;

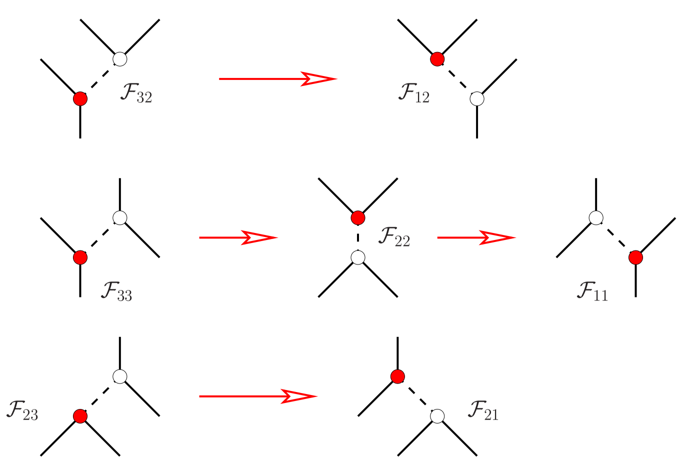
CROSS-SECTIONS OF THE VISIBILITY COMPLEX



TYPOLOGY OF THE 2-CELLS OF A CROSS-SECTION



TRANSITIONS : $\mathcal{F}_{i+1,j+1} \longrightarrow \mathcal{F}_{j,i}$



BALAYAGE DROIT (CAS D'UN PLAN AFFINE)

1. Fonctions supports des $\{o_i\}$ (cas du plan euclidien)

$$\theta \in \mathbb{R} \mapsto z_{+o}(\theta) = \inf_{p \in o} \langle p, \imath \exp(\imath\theta) \rangle \in \mathbb{R}$$
$$\theta \in \mathbb{R} \mapsto z_{-o}(\theta) = \sup_{p \in o} \langle p, \imath \exp(\imath\theta) \rangle \in \mathbb{R}$$

2. $z_{-o}(\theta + \pi) = -z_{+o}(\theta)$

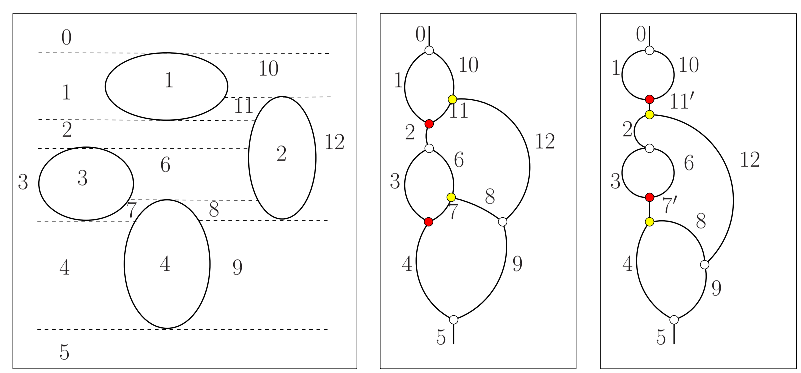
TH 20. *Soit $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_k < \pi$ la suite croissante des angles des bitangentes libres orientées vers le haut et pour toute arête σ de la section $S_i \approx S(\theta)$ pour $\theta \in (\theta_{i-1}, \theta_i)$ du complexe de visibilite soit $\mathrm{death}(\sigma)$ la plus petite solution supérieure à θ_{i-1} de l'équation $z_{\sigma^+}(\theta) = z_{\sigma^-}(\theta)$. Alors θ_i est le plus petit des $\mathrm{death}(\sigma)$ pour σ arête de S_i.* □

TH 21. *Les k bitangentes libres d'une collection de n corps convexes disjoint deux à deux d'un plan topologique affine sont calculables en temps $O(k \log n)$ et espace de travail linéaire.* □

EXAMPLE

$2 \in \mathcal{F}_{22}$, $9 \in \mathcal{F}_{32}$, $7, 11 \in \mathcal{F}_{33}$, $12 \in \mathcal{F}_{31}$, $1, 3 \in \mathcal{F}_{13}$, $6, 10 \in \mathcal{F}_{21}$, $8 \in \mathcal{F}_{11}$, $4 \in \mathcal{F}_{12}$, $5 = \hat{0}$, $0 = \hat{1}$. $\mathrm{death}(2) = v_{-3,1}$, $\mathrm{death}(11) = v_{1,-2}$, $\mathrm{death}(9) = v_{4,2}$, $\mathrm{death}(7) = v_{3,-4}$.

$$\mathrm{death}(7) = \mathrm{death}(11) < \mathrm{death}(2) < \mathrm{death}(9)$$



PARTIAL ORDER – TOPOLOGICAL CROSS-SECTIONS

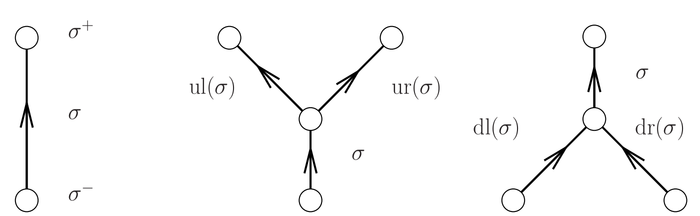
$$\begin{array}{ccc} \mathbb{W} & \xrightarrow{p'} & \hat{\mathbb{L}} \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{V} & \xrightarrow{p} & \mathbb{L} \end{array}$$

1. $\mathbb{W} = \{(x,l) \in \mathbb{V} \times \hat{L} \mid p(x) = q(l)\}$ où $q : \hat{\mathbb{L}} \to \mathbb{L}$ est un revêtement universel de $\mathbb{L}$;
2. X_0, X_1, X_2 denote the sets of 0-, 1-, and 2-cells of the visibility complex $\mathbb{W}$;
3. Partial order $\prec$ defined on $X_{\leq 2}$ via the covering relations :

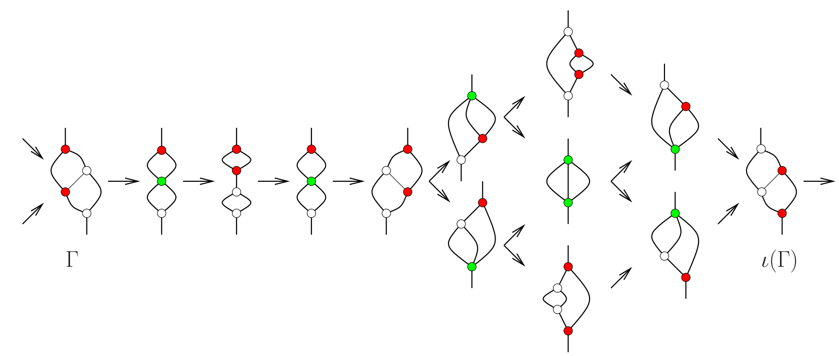
$$\mathrm{sour}(\sigma) \prec \sigma \prec \mathrm{sink}(\sigma).$$

TOPOLOGICAL CROSS-SECTIONS

1. proper filter of $(X_0, \prec)$ $\qquad I \leftrightarrow \mathrm{S}(I) = \mathcal{E}(I) \cup \mathcal{F}(I)$ $\qquad$ maximal antichains of $(X_{\geq 1}, \prec)$
2. The planar directed graph $\Gamma(I)$ induced on $\mathrm{S}(I)$ by the incidence relations.
3. For $e \in X_1$ supported by a z_{+i} one has $\mathrm{sink}(e) = \mathrm{sink}(e_\mathrm{r})$ or $\mathrm{sink}(e_\mathrm{f})$;
4. Let $\sigma \in \mathcal{F}(I)$. $\mathrm{sink}(\sigma)$ is minimal in I iff. $\mathrm{sink}(\sigma) = \mathrm{sink}(\sigma^+) = \mathrm{sink}(\sigma^-)$ iff. $\mathrm{sink}(\sigma^+) = \mathrm{sink}(\sigma^-)$



TOPOLOGICAL CROSS-SECTIONS OF THE VISIBILITY COMPLEX OF TWO CONVEX BODIES



FLIP PROPERTY

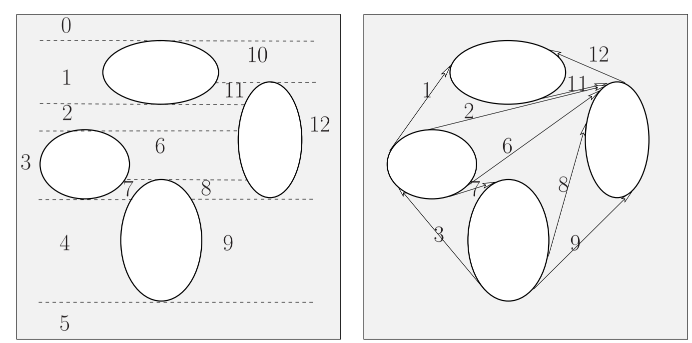


Figure 4: (Left) The horizontal map of the set of disks is a representation of the cross-section associated with the filter of vertices with positive angles : each strip-shaped region corresponds to the subset of elements with angles zero of a 2-cell of the cross-section. (Right) The horizontal greedy pseudotriangulation : the bitangent labeled i 'is' the sink of the 2-cell labeled i.

TH 22 (PV96,AP03)**.** $\mathrm{G}(I) = \mathrm{sink}(\mathrm{S}^\circ(I)) = \mathrm{sink}(\mathcal{F}^\circ(I))$. □

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