Multi-pseudotriangulations

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Abstract

We introduce a natural generalization of both pseudotriangulations and multitriangulations, that we call *multi-pseudotriangulations*. We propose an enumeration algorithm for multi-pseudotriangulations, based on certain greedy multi-pseudotriangulations that are closely related with sorting networks.

The proofs of the results of this extended abstract are skipped for space reasons.

1 Definition

Let $\mathcal{M}$ denote the *Möbius strip* (without boundary), that is, the quotient set of $\mathbb{R}^2$ by the relation $(x, y) \sim (x + \pi, -y)$. Let π denote the canonical projection.

A *pseudoline* is a non-separating simple closed curve in $\mathcal{M}$. Throughout this paper, a pseudoline is always assumed to be *monotone*, and thus can be seen as the image under the projection π of the graph $\{(x, f(x)) \,|\, x \in \mathbb{R}\}$ of a continuous and π-antiperiodic function $f : \mathbb{R} \to \mathbb{R}$.

A *pseudoline arrangement* (see Fig. 1) is a finite set of pseudolines such that any two of them have exactly one crossing point and possibly some contact points. We are only interested in *simple* arrangements, that is, where no three pseudolines meet in a common point. The *support* of a pseudoline arrangement is the union of its pseudolines. The *first level* of a pseudoline arrangement is its external hull, that is, the boundary of the cell at infinity defined by the support of the arrangement. Similarly, we define recursively the *pth level* of an arrangement to be the external hull of (the closure of) its support minus its first $p - 1$ levels.

The main objects with which this paper will be dealing are the following (see Fig. 1):

Definition 1 *Let L be an arrangement of n pseudolines and k be an integer. A k-*pseudotriangulation *of L is a set U of vertices of L containing all contact points of L and obtained as the union of*

1. *the set of vertices of the first k levels of L, and*
2. *the set of contact points of an arrangement U' of $n - 2k$ pseudolines whose support covers the support of L minus its first k levels.*

In this definition, observe that U' is completely determined by U (and *vice versa*). We denote it $\Lambda(U)$.

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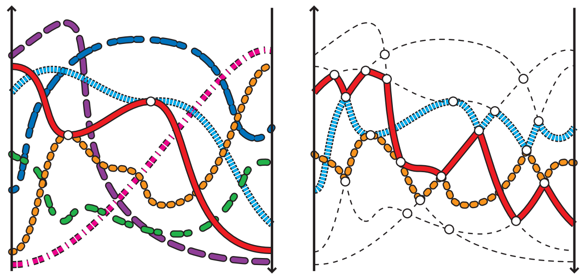

Figure 1: An arrangement of seven pseudolines and a 2-pseudotriangulation of it.

2 Pseudotriangulations and multitriangulations

Duality Remember that the Möbius strip $\mathcal{M}$ is the line space of the plane : we parametrize a line by its angle with the horizontal axis and its algebraic distance to the origin. Via this parametrization, the set of lines passing through a point p of the plane forms a pseudoline p^* of $\mathcal{M}$. If P is a point set in general position, then the set $P^* = \{p^* \,|\, p \in P\}$ is a simple pseudoline arrangement (without contact points).

In this section, we use the following notation: if p and q are two points of P, e is the edge pq and u is the crossing point of p^* and q^* (*i.e.*, the line pq), then we denote $e^{\succ} = u$ and $u^{\prec} = e$. We also denote $E^{\succ} = \{e^{\succ} \,|\, e \in E\}$ and $U^{\prec} = \{u^{\prec} \,|\, u \in U\}$.

Pseudotriangulations Let P be a point set in general position. A set E of edges of $\binom{P}{2}$ is *pointed* if for any vertex $p \in P$, there exists a line which passes through p and defines a half-plane containing all the edges of E incident to p. A *pseudotriangulation* of P is a maximal crossing-free pointed set of edges of $\binom{P}{2}$ [8, 9, 10, 11]. The following theorem is one of our main motivations, and justifies the title of this paper:

Theorem 1
1. *If T is a pseudotriangulation of P, then $T^{\succ}$ is a 1-pseudotriangulation of P^*.*
2. *If U is a 1-pseudotriangulation of P^*, then $U^{\prec}$ is a pseudotriangulation of P.*

To make clear the previous theorem, let us interpret the pseudolines of $\Lambda(T^{\succ})$. Remember that the pseudotriangulation T decomposes the convex hull of P into *pseudotriangles* (plane polygons with only three convex vertices, joined by three concave polygonal chains; see Fig. 2). A pseudoline of $\Lambda(T^{\succ})$ is the set of all inner tangents to a pseudotriangle of T.

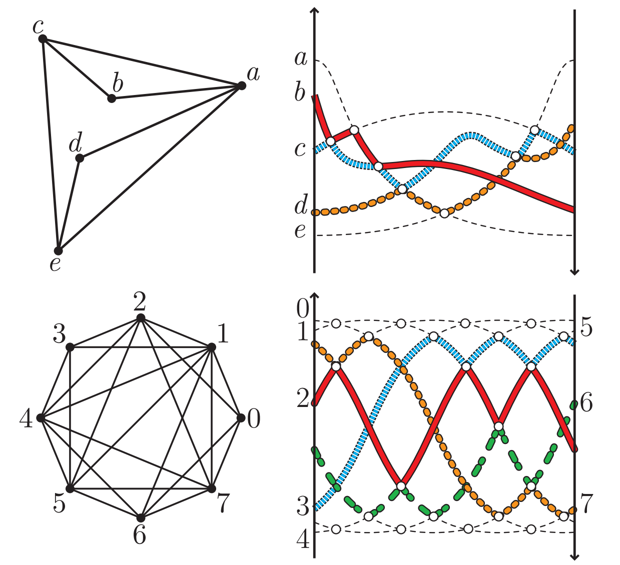

Figure 2: A pseudotriangulation and its dual arrangement. A 2-triangulation and its dual arrangement.

Multitriangulations Let P be a point set in convex position. For $\ell \in \mathbb{N}$, an *ℓ-crossing* is a set of ℓ mutually crossing edges. A *k-triangulation* of P is a maximal $(k+1)$-crossing-free subset of edges of $\binom{P}{2}$ [6, 3, 7]. Again, there is a duality between multi-pseudotriangulations of P and multitriangulations:

Theorem 2 *1. If T is a k-triangulation of P, then $T^{\succ}$ is a k-pseudotriangulation of P^*.*

2. If U is a k-pseudotriangulation of P^*, then $U^{\prec}$ is a k-triangulation of P.

The pseudolines of $\Lambda(T^{\succ})$ are easy to interpret with the following natural generalization of triangles introduced in [7]: a k-star is a set of edges of the form $\{s_j s_{j+k} \mid j \in \mathbb{Z}_{2k+1}\}$, where $s_0, \dots, s_{2k}$ are cyclically ordered. A pseudoline of $\Lambda(T^{\succ})$ is the set of all bisectors of a k-star of T.

3 Flips and greedy multi-pseudotriangulations

Flips Let L be a pseudoline arrangement without contact points. Let U be a k-pseudotriangulation of L. Let $u \in U$ be the contact point of two pseudolines of $\Lambda(U)$, and v denote their crossing point. Then that $U \triangle \{u, v\}$ is a new k-pseudotriangulation of L (we use the symbol $\triangle$ for the symmetric difference: $U \triangle \{u, v\} = U \cup \{v\} \smallsetminus \{u\}$). We say that we obtain $U \triangle \{u, v\}$ from U by flipping u. The primal notions of flips (for pseudotriangulations and for multitriangulations) obviously correspond to this dual version.

Let $G^k(L)$ be the graph whose vertices are the k-pseudotriangulations of L and whose edges are the pairs of k-pseudotriangulations linked by a flip.

Theorem 3 *The graph $G^k(L)$ is connected.*

Cuts A *cut* of L is a pseudoline λ of $\bar{\mathcal{M}} = \mathcal{M} \cup \{\infty\}$ passing through the point at infinity and such that $L \cup \{\lambda\}$ is a simple pseudoline arrangement of $\bar{\mathcal{M}}$. We orient all the pseudolines of L in the same arbitrary direction (*e.g.*, the abscisse increasing direction). Then, a cut λ naturally defines a partial order $\preccurlyeq_\lambda$ on the set of vertices of L: $u \preccurlyeq_\lambda v$ if there exists an oriented path on the support of L from u to v and avoiding λ.

Let U be a k-pseudotriangulation of L, u be a flippable contact point of U, and v denote the corresponding crossing point. It is easy to see that u and v are comparable for $\preccurlyeq_\lambda$. We say that the flip of u is *λ-increasing* if $u \preccurlyeq_\lambda v$ and *λ-decreasing* otherwise. We denote $G^k_\lambda(L)$ the directed graph of λ-increasing flips on k-pseudotriangulations of L. This directed graph is acyclic. In the following paragraph, we characterize its unique sink in terms of sorting networks.

Sorting networks We cut the Möbius strip along the cut λ, and denote λ_- and λ_+ the two copies of λ such that an oriented path on the support of L goes from λ_- to λ_+. We orient λ_- and λ_+ in the same arbitrary direction.

A *sweep* of L is a pencil $\lambda_0, \dots, \lambda_N$ of cuts of L such that for all $i \geq 1$, λ_i is obtained from λ_{i-1} by sweeping a crossing point u_i of L, minimal for the order $\preccurlyeq_{\lambda_{i-1}}$. We consider a sweep $\lambda_0, \dots, \lambda_N$ of L with $\lambda_0 = \lambda_-$ and $\lambda_N = \lambda_+$ (thus, $N = \binom{|L|}{2}$). We orient all pseudolines $\lambda_1, \dots, \lambda_{N-1}$ in the same direction as λ_- and λ_+. We denote p_i the integer such that the pseudolines of L that cross at u_i are the p_ith and (p_i+1)th pseudolines of L on λ_i.

Let U be a k-pseudotriangulation of L. Since we have cut $\mathcal{M}$ along λ, each of the first k levels of L is divided into two pieces. Thus, the support of L is decomposed into $n = |L|$ curves: the $2k$ pieces of the first k levels of L and the $n-2k$ pseudolines of $\Lambda(U)$. We denote these curves $\ell_1, \dots, \ell_n$ in the order they reach λ_+. With this convention, the order in which the curves leave λ_- is given by the permutation τ_k of $\{1, \dots, n\}$ defined by $\tau_k(i) = n-i$ if $i \in \{k+1, \dots, n-k\}$, and $\tau_k(i) = i$ otherwise (see Fig. 3).

To any oriented cut μ of L, we associate the permutation $\pi_{\mu,U}$ of $\{1, \dots, n\}$ given by the order of $\ell_1, \dots, \ell_n$ on μ. For example, $\pi_{\lambda_+,U}$ is the identity permutation, while $\pi_{\lambda_-,U}$ is the permutation τ_k. We consider the sequence of permutations $(\pi_{\lambda_i,U})_{0 \leq i \leq N}$. By definition, this sequence starts with τ_k, and ends with the identity permutation. Furthermore, for all i,

1. if u_i is in U, then $\pi_{\lambda_i,U} = \pi_{\lambda_{i-1},U}$
2. otherwise, $\pi_{\lambda_i,U}$ is obtained from $\pi_{\lambda_{i-1},U}$ by inverting its p_ith and (p_i+1)th entries.

Theorem 4 *The directed graph of flips $G^k_\lambda(L)$ has a unique sink U characterized by the property that for all i, the permutation $\pi_{\lambda_i,U}$ is obtained from $\pi_{\lambda_{i-1},U}$ by sorting its p_ith and (p_i+1)th entries.*

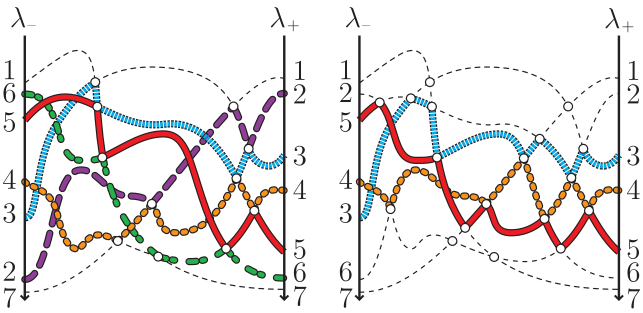

Figure 3: Greedy multi-pseudotriangulations of the pseudoline arrangement of Fig. 1.

Let us reformulate Theorem 4 in the context of sorting networks [4, Section 5.3.4]. Let $i < j$ be two integers. A *comparator* $[i : j]$ transforms a sequence of numbers $(x_1, \dots, x_n)$ by sorting (x_i, x_j), *i.e.*, replacing x_i by $\min(x_i, x_j)$ and x_j by $\max(x_i, x_j)$. A comparator $[i : j]$ is *primitive* if $j = i + 1$. A *sorting network* is a sequence of comparators that sorts any sequence $(x_1, \dots, x_n)$.

A pseudoline arrangement L together with a sweep $\lambda_0, \dots, \lambda_N$ define a sequence of primitive comparators $S_L = [p_1 : p_1 + 1], \dots, [p_N : p_N + 1]$ [5, Section 8]. In this setting, Theorem 4 affirms that,

1. given a pseudoline arrangement L, a cut λ of L and any sweep starting at λ_- and ending at λ_+, sorting the permutation τ_k according to S_L provides a k-pseudotriangulation of L.
2. this k-pseudotriangulation only depends upon L, k, and λ (not on the total order given by the sweep).

We call this k-pseudotriangulation the *λ-greedy k-pseudotriangulation of L* and denote it $\Gamma^k_\lambda(L)$.

When $k = 1$, the greedy 1-pseudotriangulation is in fact the dual of the *greedy pseudotriangulation* of [1] (originally [8] for convex bodies). This pseudotriangulation is obtained by a recursive choice of edges: we start from the empty set and add at each step a maximal (for $\preccurlyeq_\lambda$) remaining edge e_i such that the set of edges $\{e_1, \dots, e_i\}$ remains pointed and non-crossing, until we obtain a pseudotriangulation.

Constrained greedy multi-pseudotriangulation Let V be a subset of crossing points of L. We denote by $G^k_\lambda(L \,|\, V)$ the subgraph of $G^k_\lambda(L)$ induced by the k-pseudotriangulations of L that contain V.

Theorem 5 *The directed graph of flips $G^k_\lambda(L \,|\, V)$ either is empty or has a unique sink U characterized by the property that for all i,*

1. *if $u_i \in V$, then $\pi_{\lambda_i, U} = \pi_{\lambda_{i-1}, U}$*
2. *if $u_i \notin V$, then $\pi_{\lambda_i, U}$ is obtained from $\pi_{\lambda_{i-1}, U}$ by sorting its p_ith and $(p_i + 1)$th entries.*

We denote $\Gamma^k_\lambda(L \,|\, V)$ this unique sink.

4 Greedy flip property and enumeration

We are now ready to state the *greedy flip property*, that says how to update the greedy k-pseudotriangulation of L when the cut λ varies:

Theorem 6 (Greedy Flip Property) *Let L be a pseudoline arrangement. Let λ be a cut of L, and μ be a cut of L obtained from λ by sweeping a maximal (for $\preccurlyeq_\lambda$) vertex v of L. Let V be a set of crossing points of L contained in at least one k-pseudotriangulation of L. Then,*

1. *if v is in $\Gamma^k_\lambda(L \,|\, V)$, but neither in V, nor in the first k levels of L, then $\Gamma^k_\mu(L \,|\, V)$ is obtained from $\Gamma^k_\lambda(L \,|\, V)$ by flipping v. Otherwise, $\Gamma^k_\mu(L \,|\, V) = \Gamma^k_\lambda(L \,|\, V)$;*
2. *if v is in $\Gamma^k_\lambda(L \,|\, V)$, then $\Gamma^k_\mu(L \,|\, V \cup \{v\}) = \Gamma^k_\lambda(L \,|\, V)$. Otherwise, $V \cup \{v\}$ is not contained in a k-pseudotriangulation of L.*

From this greedy flip property, we derive an algorithm to enumerate multi-pseudotriangulations, similar to the enumeration algorithm of [1].

Let L be a pseudoline arrangement and λ be a cut of L. We say that a k-pseudotriangulation of L is *colored* if its contact points are colored either in blue or in red. A red contact point is fixed for the end of the algorithm, while a blue one can be flipped. In the algorithm, we construct a binary tree $\mathcal{T}$, whose nodes are colored k-pseudotriangulations of L, as follows:

1. the root of the tree is the λ-greedy k-pseudotriangulation of L, entirely colored in blue;
2. for any colored k-pseudotriangulation U of L, if there is no λ-decreasing flippable point (*i.e.*, if there only remains red contact points, or if the flip of any blue contact point of U is λ-increasing), then U is a leaf of $\mathcal{T}$;
3. otherwise, we choose a maximal blue point u of U whose flip is λ-decreasing. The right child of U is obtained by flipping u and its left child is obtained by changing the color of u into red.

This algorithm depends upon what maximal blue point we choose at each step. However, no matter what this choice is, we obtain all k-pseudotriangulations of L:

Theorem 7 *The set of k-pseudotriangulations of L is exactly the set of red-colored leafs of $\mathcal{T}$.*

Let us briefly discuss the complexity of this algorithm. We assume that the input of the algorithm is a pseudoline arrangement and we consider a flip as an elementary operation. Then, this algorithm requires a polynomial running time per k-pseudotriangulation. As for many enumeration algorithms, the crucial point of this algorithm is that its working space is also polynomial (while the number of k-pseudotriangulations of L is exponential for fixed k).

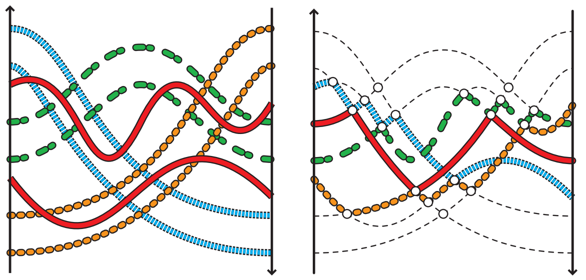

Figure 4: A double pseudoline arrangement and a 2-pseudotriangulation of it.

5 Further topics and open questions

Double pseudoline arrangements A double pseudoline is a non-contractible and non-separating simple closed curve in $\mathcal{M}$ (see Fig. 4). A *double pseudoline arrangement* [2] is a finite set of double pseudolines such that any two of them

1. have exactly four intersection points and cross transversally at these points; and
2. induce a cell decomposition of the Möbius strip.

The set C^* of tangents to a convex body C of the plane is a double pseudoline. If Q is a set of pairwise disjoint convex bodies, then $Q^* = \{C^* \,|\, C \in Q\}$ is a double pseudoline arrangement.

Defining similarly the *support* and the *levels* of a double pseudoline arrangement, we can define multi-pseudotriangulations of double pseudoline arrangements (see Fig. 4): a *k-pseudotriangulation* of an arrangement L of n double pseudolines is the set U of contact points (including the vertices of the first k levels of L) of an arrangement $\Lambda(U)$ of $2n - 2k$ pseudolines such that the support of $\Lambda(U)$ covers the support of L minus its first k levels (see Fig. 4d).

Multi-pseudotriangulations of double pseudoline arrangements have similar structural properties and can also be enumerated by the greedy flip algorithm.

The graph of flips Let L be a pseudoline arrangement and k be an integer. Let $\Theta^k(L)$ denote the partially ordered set of sets of contact points of L that contain the vertices of the first k levels and the contact points of L and that are contained in a k-pseudotriangulation of L, ordered by reverse inclusion. Theorems 3 and 5 ensure that $\Theta^k(L)$ is an abstract polytope whose 1-skeleton is the graph of flips $G^k(L)$ (see the discussion in [1, Subsection 2.2]). When $k = 1$ and L is the dual pseudoline arrangement of a point set of the Euclidean plane, it turns out that this abstract polytope can be realized effectively as a polytope of $\mathbb{R}^d$ (where d is the number of flippable edges), known as the *polytope of pointed pseudotriangulations* [9]. We naturally would like to know whether $\Theta^k(L)$ is a polytope in general.

Acknowledgments

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