Rigidity of graphs

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Preliminaries

Bar-and-joint frameworks

Structure $\mathcal{S}$: rigid rods (bars) connected at their ends (joints).

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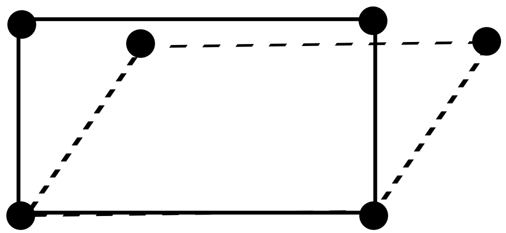
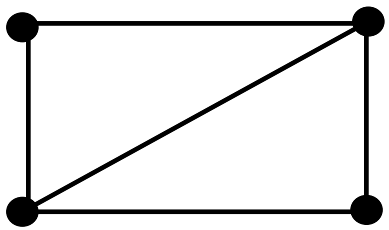

Non-rigid


Rigid

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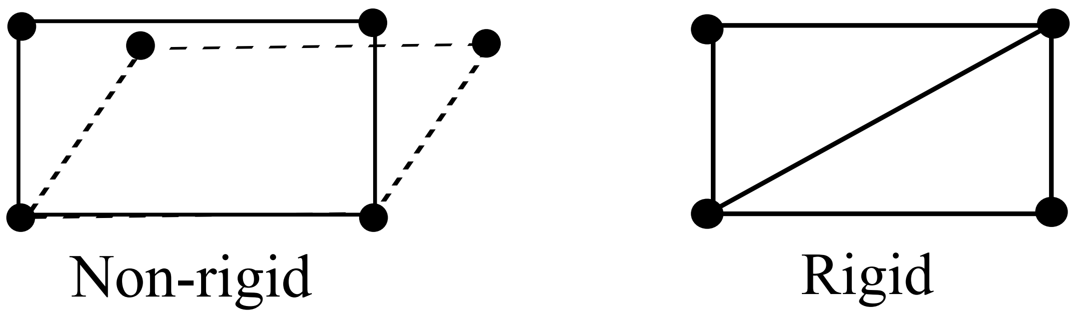

- $\mathcal{S} \sim (G, \mathbf{p})$.
  - V : set of joints of $\mathcal{S}$; $E(G)$: set of bars of $\mathcal{S}$.
  - $\mathbf{p} : V \to \mathbb{R}^d$, embedding.

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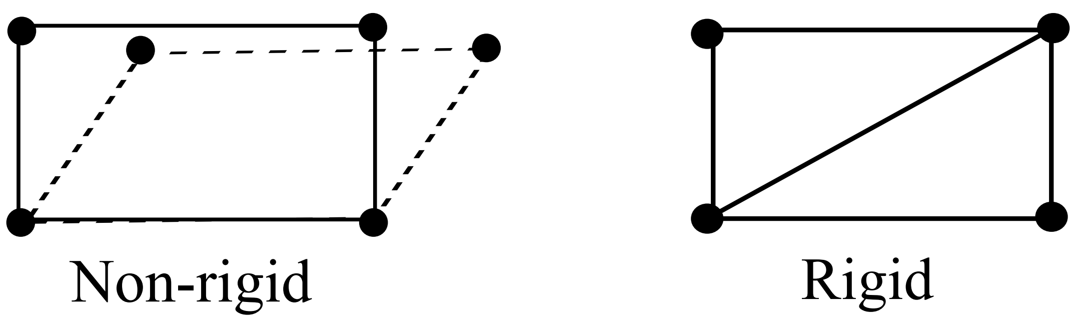

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  - V : set of joints of $\mathcal{S}$; $E(G)$: set of bars of $\mathcal{S}$.
  - $\mathbf{p} : V \to \mathbb{R}^d$, embedding.
- $(G, \mathbf{p})$: a d-dim. bar-and-joint framework.

- $(G, \mathbf{p})$ is congruent to $(G, \mathbf{q})$
  $\Leftrightarrow ||p(u) - p(v)|| = ||q(u) - q(v)||, \quad \forall u, v \in V.$

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  $\Leftrightarrow ||p(u) - p(v)|| = ||q(u) - q(v)||, \quad \forall u, v \in V.$
- $(G, \mathbf{p})$ is rigid
  $\Leftrightarrow$ every continuous motion of $(G, \mathbf{p})$ preserving the length of edges results in a framework congruent to $(G, \mathbf{p})$.

- An infinitesimal motion of $(G, \mathbf{p})$ is a $\mu : V \to \mathbb{R}^d$ s.t.

$$(\mathbf{p}(u) - \mathbf{p}(v))(\mu(u) - \mu(v)) = 0, \quad uv \in E(G).$$

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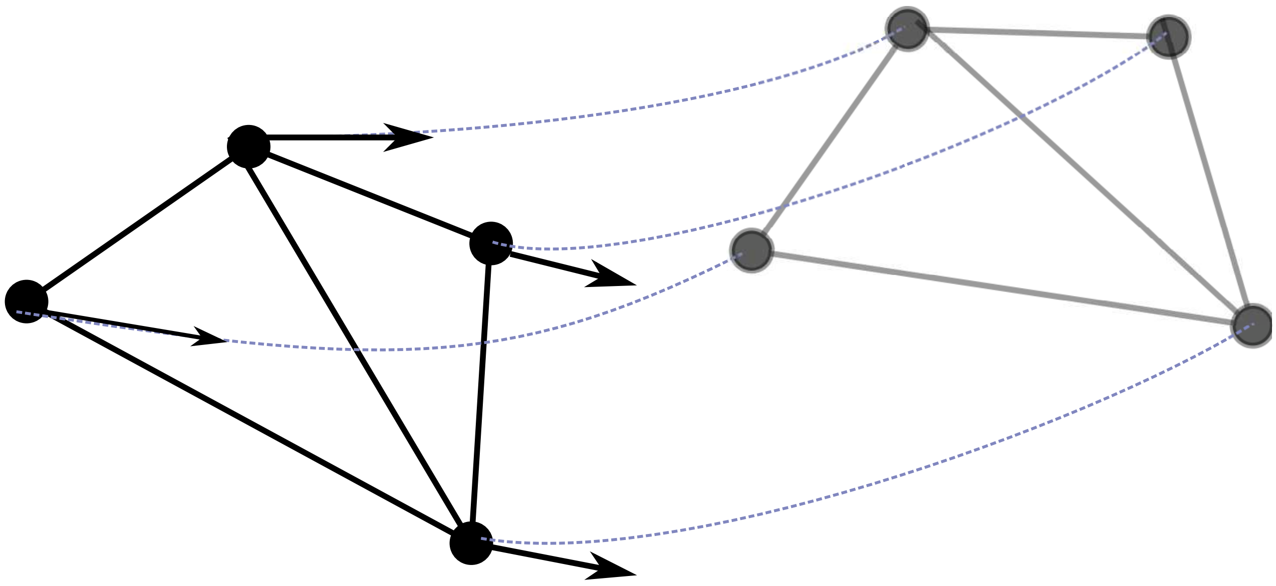

The *instantaneous velocity* of $(G, \mathbf{p})$ is an infinitesimal motion.

Rigidity matrix

- Rigidity matrix $R(G, \mathbf{p})$: $|E| \times d|V|$ matrix.

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} & & u & & v & & \end{array} \\
uv & \begin{pmatrix}
\vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
0 & \cdots & \mathbf{p}(u) - \mathbf{p}(v) & \cdots & \mathbf{p}(v) - \mathbf{p}(u) & \cdots & 0 \\
\vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots
\end{pmatrix}
\end{array}.
$$

Rigidity matrix

Preliminaries
Known results
Open problems

- Rigidity matrix $R(G,\mathbf{p}) : |E| \times d|V|$ matrix.

$$\begin{array}{cc} & \begin{array}{ccccccc} & & u & & v & & \end{array} \\ uv & \begin{pmatrix} \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{p}(u)-\mathbf{p}(v) & \cdots & \mathbf{p}(v)-\mathbf{p}(u) & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \end{pmatrix} \end{array}.$$

- μ is an infinitesimal motion of $(G,\mathbf{p})$ if and only if $R(G,\mathbf{p})\mu = 0$ (i.e $\mu \in \ker R(G,\mathbf{p})$).
- The space of infinitesimal motions induced by translations and rotations is of dimension $d(d+1)/2$.
$\Rightarrow \operatorname{rank} R(G,\mathbf{p}) \leq d|V| - d(d+1)/2$.

Determining rigidity

Determine rigidity by calculating $\operatorname{rank} R(G, \mathbf{p})$?

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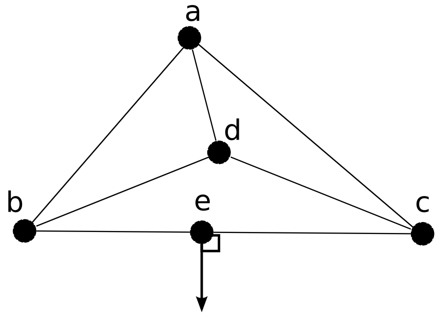
Degenerate

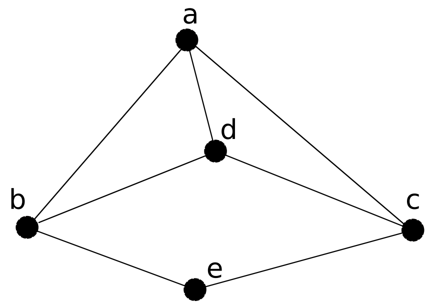


Rigid but

$\operatorname{rank} R(G, \mathbf{p}) < d|V| - d(d+1)/2$

(d=2).

Generic



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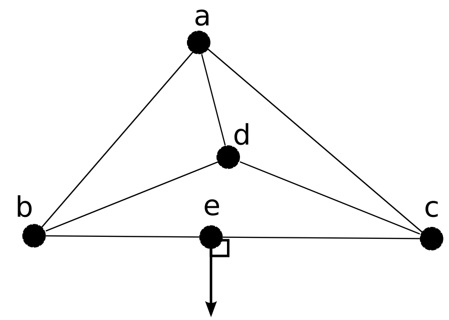

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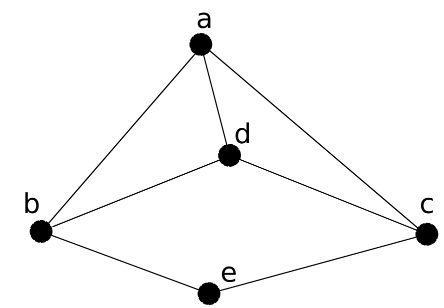

OK!

Theorem (Asimow, Roth 1979). *For generic* $\mathbf{p}$, $(G, \mathbf{p})$ *is rigid* $\Leftrightarrow \mathrm{rank} R(G, \mathbf{p}) = d|V| - d(d+1)/2$.

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Determine rigidity by calculating $\mathrm{rank}\,R(G, \mathbf{p})$?

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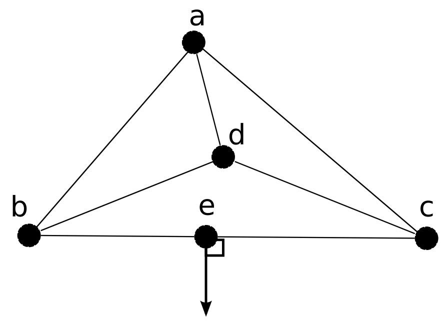

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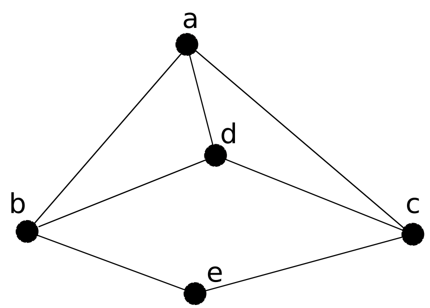

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$(G, \mathbf{p})$ *is rigid* $\Leftrightarrow \mathrm{rank} R(G, \mathbf{p}) = d|V| - d(d+1)/2$*.*

Almost all embeddings are generic and for all generic $\mathbf{p}$, $\mathrm{rank}\,R(G, \mathbf{p})$ depends uniquely on G.

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Solution:
Using combinatorial structure of the rigidity matrix
$\Leftrightarrow$ a matroid on $E(G)$ (linear matroid defined on the rows of $R(G, \mathbf{p})$).

- G is rigid (in dimension d) if $(G, \mathbf{p})$ is rigid for some generic $\mathbf{p}$ (and hence for all generic $\mathbf{p}$).
- G (or E) is independent (in dimension d) if the rows of $R(G, \mathbf{p})$ is independent for some generic $\mathbf{p}$ (and hence for all generic $\mathbf{p}$).

- Minimally rigid graphs = Maximally independent graphs.

Known results

Characterizations in dimension 2

Theorem (Laman 1970). *Graph (V, E) is rigid in dimension 2 if and only if there exists $E' \subseteq E$ s.t.*

- $|E'| = 2|V| - 3,$
- $|F| \leq 2|V(F)| - 3, \qquad \emptyset \neq F \subseteq E'.$

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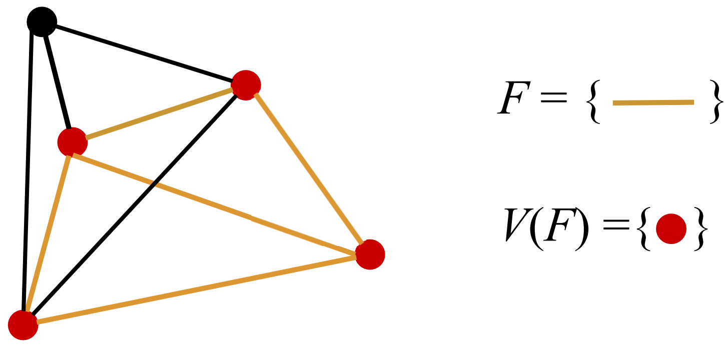

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Equivalently,

A graph G is independent in dim. 2 if and only if it satisfies

$|F| \leq 2|V(F)| - 3$ for all $\emptyset \neq F \subseteq E$.

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Theorem (Lovász&Yemini 1982). *G is rigid in dim. 2 if and only if*

$$\sum_{i=1}^{t}(2|V(G_i)|-3) \geq 2|V|-3,$$

for every $G_1, \ldots, G_t \quad (E(G_i) \neq \emptyset)$ s.t. $G_1 \cup G_2 \cup \cdots \cup G_t = G$.

Connectivity and rigidity

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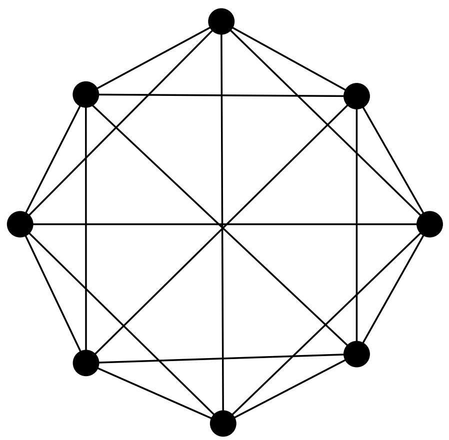

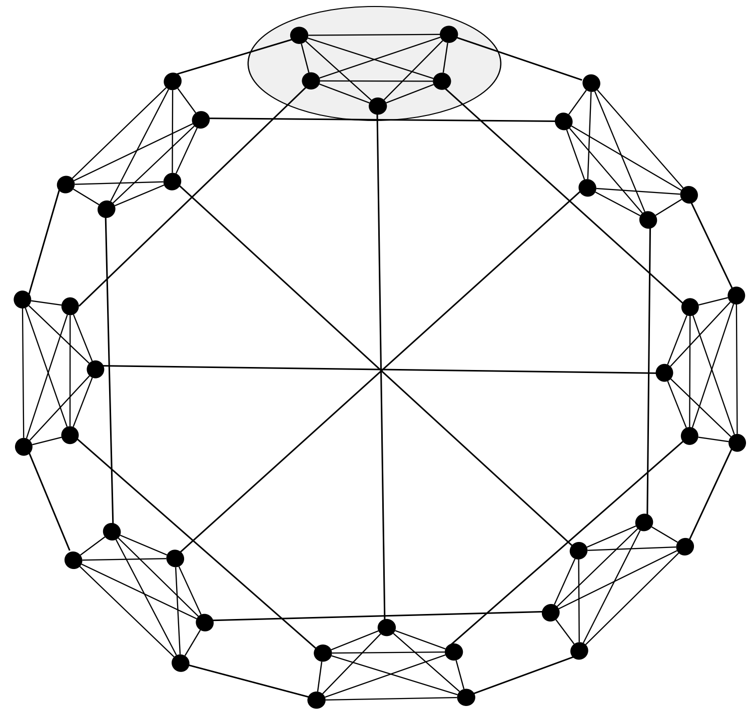

$$\sum(2|V(G_i)|-3)=7n+\frac{5n}{2}=\frac{19n}{2}<10n-3=2|V|-3.$$

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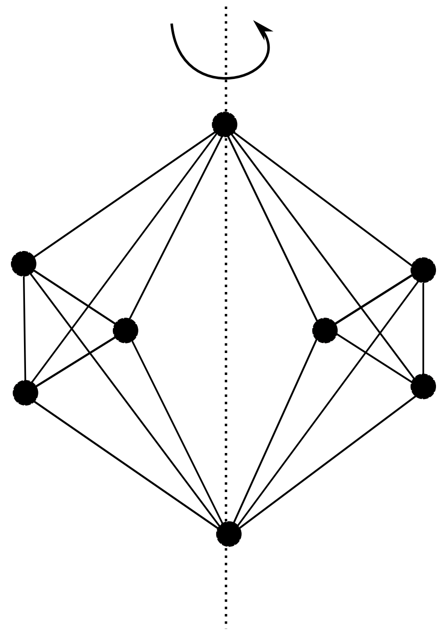

The “double banana” satisfies

$$|F| \leq 3|V(F)| - 6 \quad \text{for all} \quad F \subseteq E, |F| \geq 2,$$

and

$$E = 3|V| - 6$$

but is not rigid.

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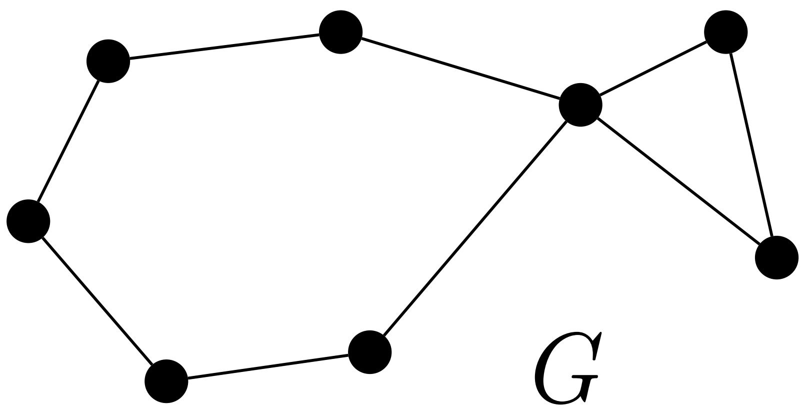

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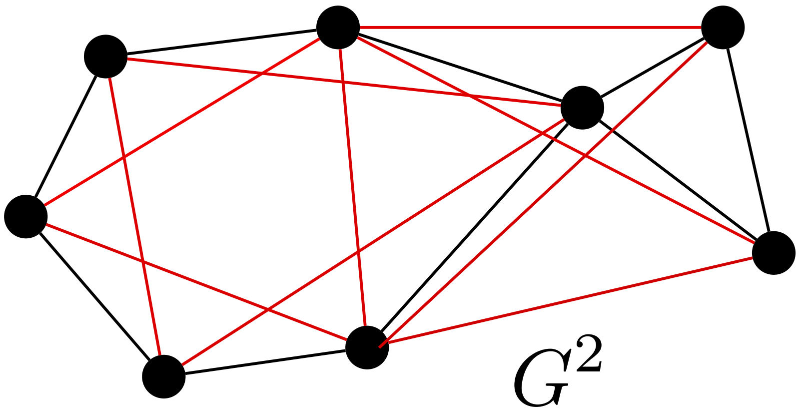

G^2

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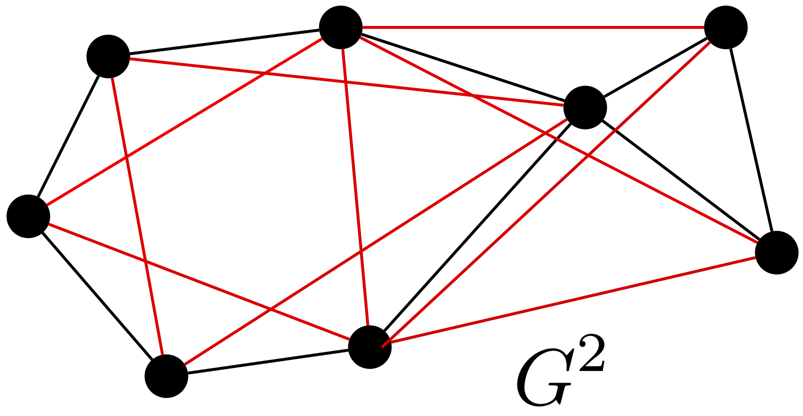

Theorem (Kato&Tanigawa 2009)**.** *Let G be a graph of minimum degree at least 2. Then the graph G^2 is rigid in dim. 3 if and only if $5G$ contains 6 disjoint spanning trees.*

Open problems

Characterization

Open problem

Characterize rigid/independent graphs in dimension $d \geq 3$.

Connectivity and rigidity

Are highly connected graphs rigid?

Conjecture (Lovász &Yemini 1982) :
There is a constant k_d such that every k_d-connected graph is rigid in dimension d. ($k_d = d(d+1)$?)

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Theorem (Jordán 2010)**.** *Every 7-connected square graph is rigid in dim. 3.*