On polygonal covers[1]

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Abstract

A polygonal cover of a finite collection of pairwise disjoint convex compact sets in the plane is a finite collection of non-overlapping bounded convex polygons such that each polygon covers exactly one convex set. We show that computing a polygonal cover with worst case minimal number of sides reduces to computing a pseudo-triangulation of the collection of convex sets. We obtain a similar reduction for two related problems concerning convex compact sets in the plane : computing a lighting set and computing a family of separating lines. Our main tool is the notion of *screen graph* (introduced in this paper) associated with a pseudo-triangulation.

Keywords: Pseudotriangle, pseudo-triangulation, screen graph, polygonal cover, translation query, Art gallery theorem, packing, covering.

1 Introduction

A *polygonal cover* of a finite collection of pairwise disjoint convex compact sets in the plane (we refer to a convex compact set as a convex disk) is a finite collection of non-overlapping bounded convex polygons such that each polygon covers exactly one convex disk; see Figure 1. In 1950, L. Fejes Tóth [8] has shown the existence of polygonal covers with a total of no

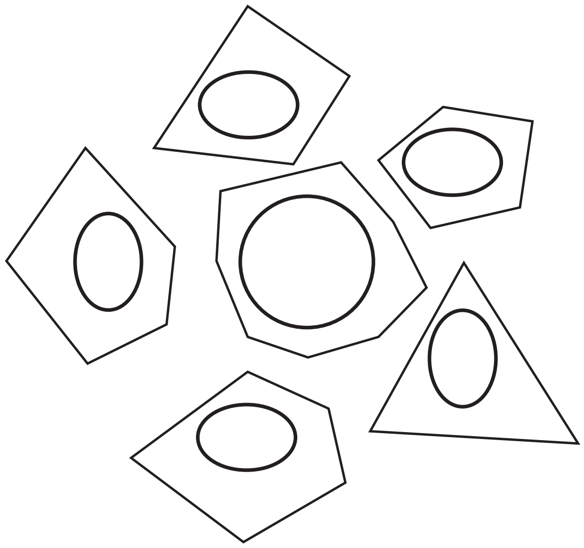

Figure 1: A polygonal cover of a collection of six disks.

more than $6n-9$ sides for $n \geq 3$ disks, this bound being tight in the worst case [7]. He used this result in order to show that the upper density of a packing with congruent copies of a convex disk in the plane is bounded by the ratio of the area of the disk to the area of a hexagon of minimum area circumscribed around the disk (see [14] and the reference cited therein). More recently, polygonal covers have been used to show that, in any collection of n pairwise disjoint convex disks, there is a disk separated by a line from at least $n/12$ other disks (see [12] and [1, 4, 5] for more separation problems) and to design efficient algorithms to answer translation queries for sets of polygons (see [6]).

The proof of L. Fejes Tóth proceeds by growing the convex disks unboundedly in all directions but the growth (for each direction) is limited by the condition that the convex disks remain pairwise interior disjoint. In this way the convex disks expand into convex polygons that fill the plane except for a finite number of gaps that are also convex polygons. A simple application of the Euler's relation for planar graph leads to the announced upper bound. The purpose of this note is to give a constructive proof of the result of L. Fejes Tóth. Our main result is the following.

Theorem 1 *A polygonal cover with worst case minimal number of sides for a collection of n pairwise disjoint convex disks is computable in $O(n \log n)$ time under the assumption that each disk has constant computational complexity.*

Our proof is based on properties of the so-called pseudo-triangulations introduced by the authors [15, 16] to design efficient algorithms to compute visibility graphs. In Section 2 we review the terminology on pseudo-triangulations. In Section 3 we give the proof of Theorem 1 : the main ingredient of the proof is the notion of *screen graph* (introduced in Section 3) associated with a pseudo-triangulation. Finally, in Section 4 we elaborate on two related problems concerning pairwise disjoint convex disks in the plane, namely computing a lighting set and computing a family of separating lines.

2 Background material

Consider a finite collection of pairwise disjoint convex disks in the plane $\mathbb{R}^2$. A *bitangent*

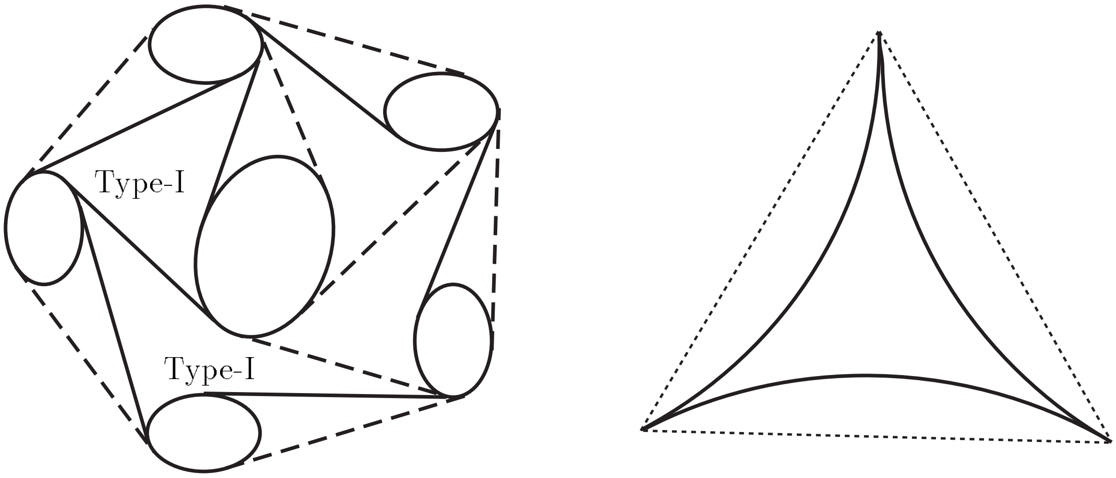

Figure 2: A pseudo-triangulation (left)—exterior bitangents are shown dashed and Type-I pseudotriangles are labeled Type-I—and a pseudotriangle (right).

is a closed line segment whose supporting line is tangent to two disks at its endpoints. A bitangent is said to be *interior* if its supporting line separates the two convex disks to which it is tangent, otherwise the bitangent is said to be *exterior*. It is called *free* if it lies in free space, i.e., the complement of the union of the interior of the disks.

For ease of the exposition we assume that the boundary of each disk is a smooth closed curve such that for each ϕ, $0 \leq \phi \leq 2\pi$, the curve has exactly one point where its (directed) tangent makes an angle of ϕ with the positive x-axis; furthermore we assume that no line is tangent to three disks and that the arrangement of the supporting lines of the bitangents is simple. Note that according to our assumptions two bitangents are disjoint or they cross at an interior point (but their supporting lines are crossing).

A *pseudo-triangulation* ([15, 16]) is a maximal subset of non-crossing free bitangents. The boundaries of the convex disks and the bitangents in a pseudo-triangulation induce a regular cell decomposition of the plane, still called a pseudo-triangulation; see the left part of

Figure 2. According to [15, 16], a bounded free face of a pseudo-triangulation is a *pseudo-triangle*, that is a simply connected subset of the plane whose boundary consists of three smooth convex curves that are tangent at their endpoints, and which is contained in the triangle formed by the three endpoints of these convex curves; the convex curves are called the *sides* of the pseudotriangle, and their endpoints are called the *cusp points* of the pseudo-triangle; see the right part of Figure 2. The number of pseudotriangles is $2n-2$ for n disks, and the number of bitangents is $3n-3$. A pseudo-triangulation of a collection of n disks is computable in $O(n \log n)$ time under the assumption that each disk has constant computational complexity in the sense that the tangent to a disk with a given direction, the common tangents to two disks or the common tangents to a disk and a point are computable in constant time. See [15, 16] for more details. (If we assume that the convex disks are composed of a total of m arcs, each of constant computational complexity, then a pseudo-triangulation is computable in time $O(m + n \log m)$ time.)

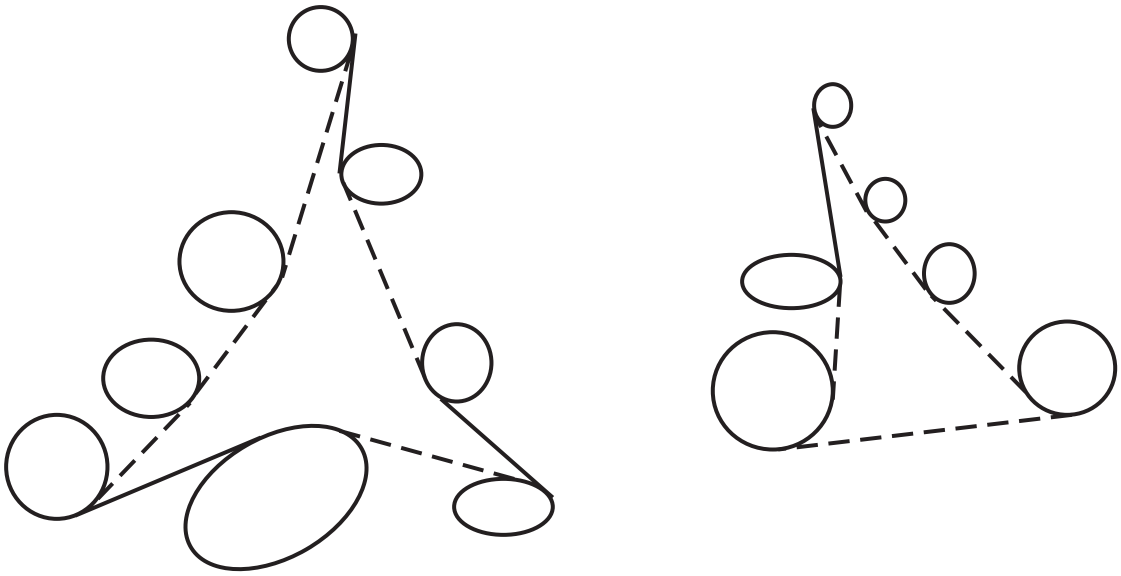

Figure 3: A Type-I (left) and a Type-II (right) pseudotriangle.

The sides of the pseudotriangles (of a pseudo-triangulation) are sequences of alternate bitangents and segments of boundaries of the convex disks (*arcs* for short), possibly reduced to single bitangents or to single arcs. A pseudotriangle is said to be of type I if none of its sides is a bitangent, otherwise the pseudotriangle is said to be of type II; see Figure 3. In this latter case one of the sides of the pseudotriangle is a (necessarily exterior) bitangent. Conversely any exterior bitangent is the bitangent side of a unique Type-II pseudotriangle: hence we have a one-to-one correspondence between the set of exterior bitangents and the set of Type-II pseudotriangles. (It follows that the number of exterior bitangents in a pseudo-triangulation of a collection of n disks is at most $2n-2$, and consequently the number of interior bitangents is at least $n-1$.) The boundary of a Type-I pseudotriangle contains exactly three interior bitangents, namely the bitangents at its cusp points. The boundary of a Type-II pseudotriangle contains exactly one interior bitangent, namely the bitangent at the opposite cusp point of the bitangent side. The following lemma gives a lower bound on the number of exterior bitangents in a pseudo-triangulation.

Lemma 1 *Consider a pseudo-triangulation of a collection of at least three convex disks. We claim that*

1. *The number of exterior bitangents is at least 3 (even 4 if the number of exterior bitangents lying on the convex hull of the convex disks is 2).*
2. *The number of exterior bitangents is at least 4 under the assumption that the supporting line of one of the interior bitangent separates a disk from the other disks and is disjoint from the other interior bitangents. (If the number of disks is at least 4 then, under the same assumption, the number of exterior bitangents is at least 5.)*

Proof. Consider the graph whose nodes are the convex disks and whose edges are the pairs of convex disks connected by an interior bitangent of some pseudo-triangulation. Clearly this graph is a plane graph without multiple edges; thus its number of edges is at most $3n-6$, where n is the number of disks. Since the total number of bitangents in a pseudo-triangulation is $3n-3$ the first part of claim (1) follows. A similar argument applies to the other claims. □

Collections of n interior-disjoint circles whose contact graphs are triangular planar graphs on n points are examples of configurations of n disks with exactly three free exterior bitangents (according to the 'Koebe-Andreev-Thurston Circle Packing Theorem' [18, page 117] or [14, pages 95–96] any triangular planar graph on n points is realizable as the contact graph of a collection of n interior-disjoint circles).

3 Polygonal covers via screen graphs

The idea is to generate a convex tessellation of the plane that covers and separates the convex disks. To this end we start from the geometric graph G consisting of the union of interior bitangents of a pseudo-triangulation (see Figure 4 part (a)), and we extend its half-edges along their supporting lines until their endpoints hit the current extension $G(t)$ of the geometric graph G. We envision the extension as a continuous process $t \in [0, +\infty) \mapsto G(t) \subset \mathbb{R}^2$ with $G(0) = G$ and $G(t) \subseteq G(t')$ for $t \leq t'$, during which the half-edges are extended continuously and concurrently.

If the extension process is not subject to further constraints there is, in general, no reason for the tessellation corresponding to $G(+\infty)$ to cover and separate the convex disks, which is our goal. (For example if we first extend the topmost interior bitangent in the left part of Figure 3, its extension is going to split the lowest disk.)

In order to generate a tessellation that covers and separates the convex disks we are going to proceed in two stages: $G \to SG \to ESG$. The first step produces the *screen graph* SG (see Figure 4 part (b)), named because it "screens" the disks from one another (see Lemma 2).

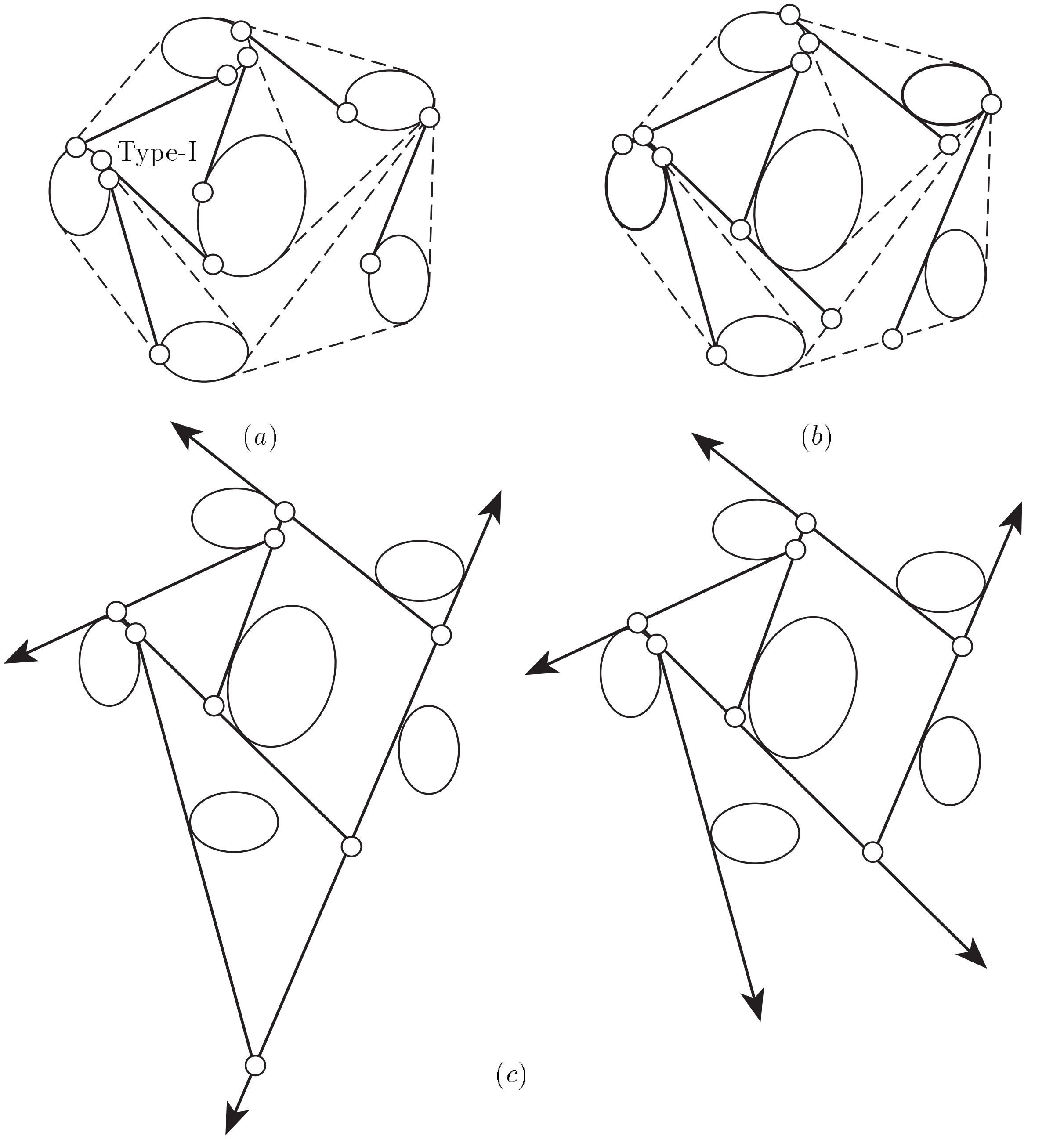

Figure 4: (a) The interior bitangents of a pseudo-triangulation (with a sole Type-I pseudotriangle), (b) its screen graph, and (c) two extensions of the screen graph.

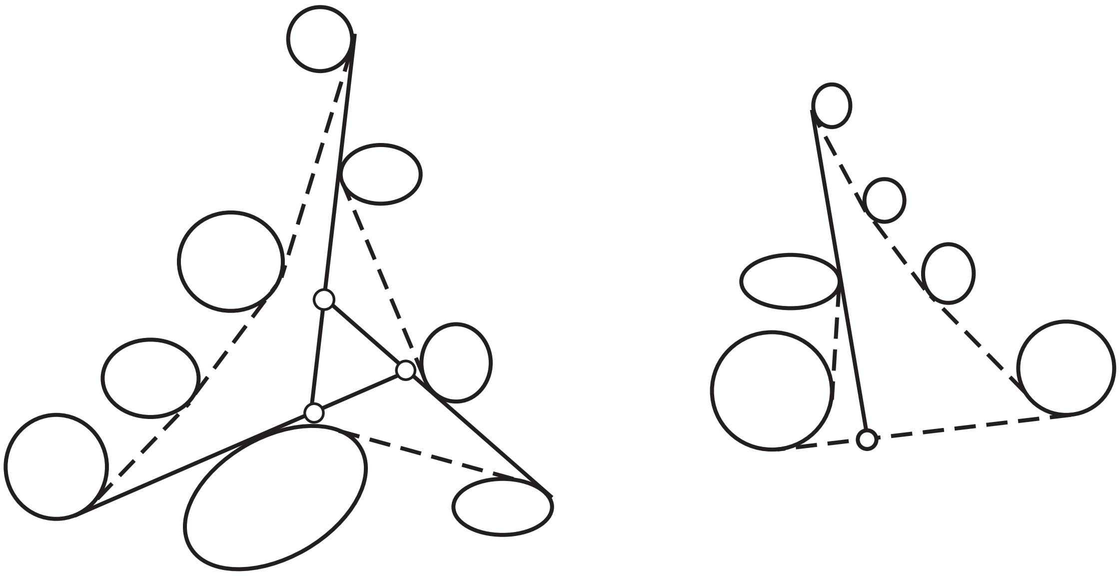

Figure 5: The edges of the screen graph inside a Type-I (left) or a Type-II (right) pseudotriangle.

This step is subject to further constraints, that turn the extension phase into a deterministic process producing a unique result. The second step does follow freely the continuous plan, and produces what we'll call an *extension* ESG of the screen graph, which is not unique (see Figure 4 part (c) for an illustration of this point) but which does cover and separate the disks. Note that according to our assumptions there is at most one region of the resulting tessellation which is a half-plane. The screen graph is defined as follows.

1. The three interior bitangents of each Type-I pseudotriangle are extended up to and including the intersection points of their supporting lines. Note that the extensions of the bitangents lie inside the pseudotriangle because tangent lines to a pseudotriangle cross inside the pseudotriangle. A Type-I pseudotriangle is therefore split into three curved triangles, one per side of the pseudotriangle, plus the straight triangle whose vertices are the intersection points of the supporting lines of the three bitangents; see the left part of Figure 5.

2. The interior bitangent of each Type-II pseudotriangle is extended up to include the intersection point of its supporting line with the bitangent side of the pseudotriangle. A Type-II pseudotriangle is therefore split into two curved triangles, one per non-bitangent side of the pseudotriangle; see the right part of Figure 5.

By construction, any closed line segment joining two non-bitangent sides of a pseudotriangle, other than the interior bitangents of the pseudo-triangulation that defines the screen graph, intersects the screen graph transversally. (The intersection point may be an endpoint of the line segment.) In particular this is the case for the exterior bitangents of the pseudo-triangulation that defines the screen graph. This simple observation has the following key extension.

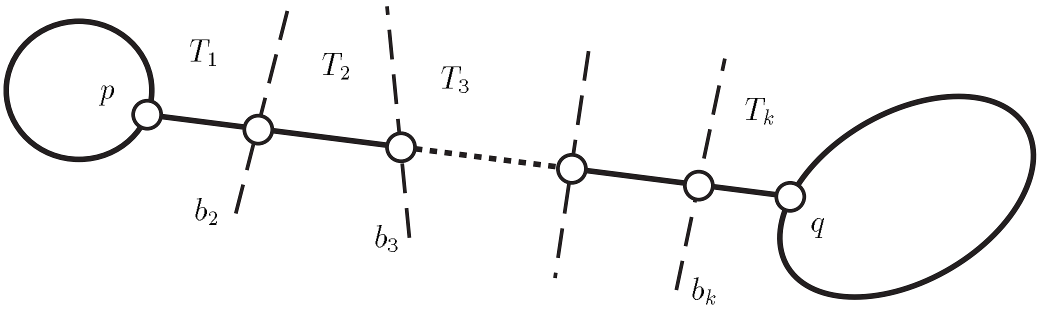

Figure 6: Illustratin of the proof of Lemma 2. If b_2 and b_3, for example, are bitangent sides of T_1 and T_3, respectively, then T_2 is crossed by joining two of its non-bitangent sides. If b_2 is not a bitangent side of T_1, then T_1 is crossed by joining two of its non-bitangent sides because p does not lie on a bitangent side of T_1.

Lemma 2 *Any closed line segment joining two distinct convex disks, other than the interior bitangents of the pseudo-triangulation that defines the screen graph, intersects the screen graph transversally. (The intersection point may be an endpoint of the line segment.)*

Proof. Consider a closed line segment, with endpoints p and q, joining two distinct convex disks and not reduced to an interior bitangent of the pseudo-triangulation (that defines the screen graph). It is sufficient to prove the lemma in the case where the interior of the line segment is disjoint from the obstacles. As noted previously, our claim is true if the line segment reduces to an exterior bitangent of the pseudo-triangulation. So we assume now that the line segment contains neither interior nor exterior bitangent of the pseudo-triangulation.

Let $T_1, b_2, T_2, b_3, \ldots, T_k$, with $k \geq 1$, be the sequence of alternate pseudotriangles and bitangents crossed by the interior of the line segment. See Figure 6 for an illustration. Because a bitangent is a bitangent side of at most one pseudotriangle, because we have k pseudotriangles T_i, but only $k-1$ bitangents b_i, and because the endpoints of the line segment lie on non-bitangent sides of T_1 and T_k, respectively, at most $k-1$ of the $2k$ crossed sides of the crossed pseudotriangles are bitangent sides. This leaves at least one pseudotriangle crossed by joining two of its non-bitangent sides. As noted previously this implies that the line segment intersects the screen graph transversally. This proves our lemma. $\square$

The screen graph is clearly computable in $O(n)$ time (assuming that each pseudotriangle is represented by its three cusp points). Once the screen graph has been constructed one of its possible extensions can be computed in $O(n \log n)$ time using a straightforward adaptation of the Bentley-Ottmann sweep algorithm to compute the intersections of a collection of line segments. From the above lemma we easily deduce the following theorem.

Theorem 2 *Any extension of the screen graph induces a convex tessellation of the plane that covers and separates the convex disks.* $\square$

Proof of Theorem 1. In view of Theorem 2 (and the above discussion on the computational complexity issues) it suffices to show that the polygonal cover induced by an extension of the screen graph has a worst case minimal number of sides, i.e., at most $6n-9$ sides for $n \geq 3$ disks. To count the number of sides we count the number of corners of the polygonal cover. Assume first that no region of the tessellation is a half-plane. It is convenient to put one corner on each infinite edge of the tessellation so that these additional corners span a convex polygon surrounding all the disks. Note that the vertices of the screen graph are of degree one or three. The number of vertices of degree three is three times the number of Type-I pseudotriangles while the number of vertices of degree one is the number of Type-II pseudotriangles. Each vertex of degree three of the screen graph accounts for one corner and each vertex of degree one of the screen graph generates a vertex of its extension that accounts for 2 corners. Therefore the number of sides (or corners) is $2a + 3(2n-2-a) = 6n-6-a$ where a is the number of Type-II pseudotriangles or, equivalently, the number of exterior bitangents. Since, according to Lemma 1, a is at least three the result follows. Assume now that a region (which is necessarily unique) of the tessellation is a half-plane : we put one corner on each infinite edge of the tessellation plus one corner in the half-plane region so that these additional corners span a convex polygon surrounding all the disks. A similar counting argument shows that the number of sides of the polygonal cover is $6n-6-a+1$. Since, according to Lemma 1, claim (2), a is at least four, the result follows. $\square$

4 Two related problems

4.1 Separating lines

A family of *separating lines* for a collection of convex disks is a collection of lines such that each pair of disks is separated by at least one of the lines of the family; see Figure 7. It is

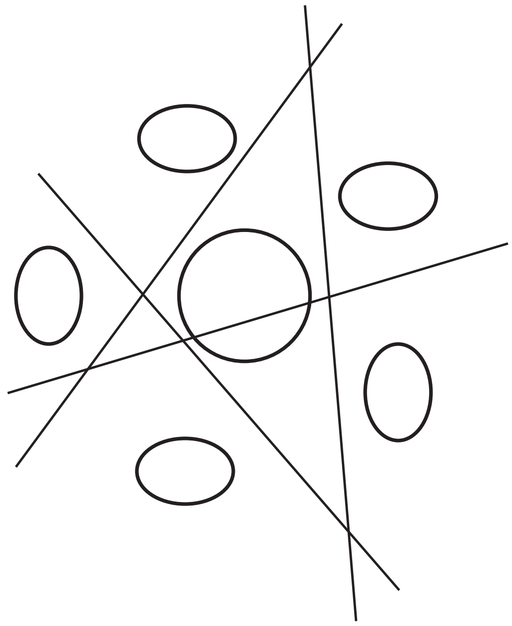

Figure 7: A family of separating lines.

well known (and easy to prove) that the family of supporting lines of the sides of a polygonal cover of the collection of convex disks is a family of separating lines. From the previous discussion we deduce the following theorem.

Theorem 3 *The collection of supporting lines of the interior bitangents of a pseudo-triangulation of a collection of n pairwise disjoint convex disks is a family of separating lines for the collection of disks. In particular, such a family can be computed in $O(n \log n)$ time under the assumption that each disk has constant computational complexity.* □

4.2 L. Fejes Tóth illumination problem revisited

Partitioning theorems and algorithms are key tools to solve Art gallery problems [2, 11, 13, 17]. For example Fisk's proof [10] of 'Chvátal's Art Gallery Theorem' [3], which asserts that $\lfloor n/3 \rfloor$ lighting points are occasionally necessary and sometimes sufficient to illuminate a polygon with n vertices, can be summarized as follow: computing a lighting set of at most $\lfloor n/3 \rfloor$ lighting points for a given polygon with n vertices reduces in $O(n)$ time to computing a

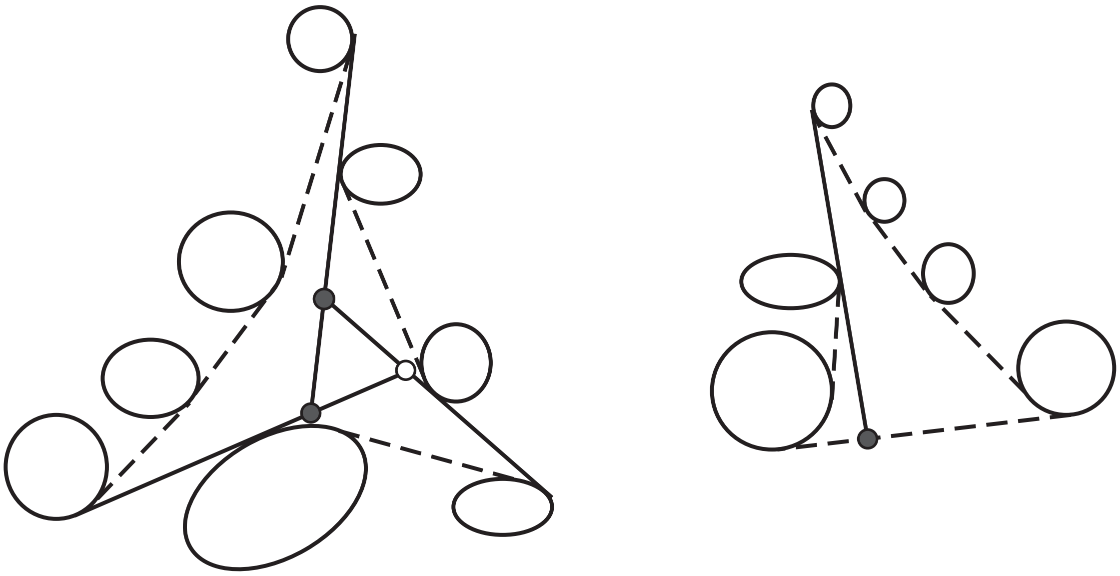

Figure 8: The boundary of a Type-I pseudotriangle can be illuminated by two of the three vertices of its screen graph (left). The boundary of a Type-II pseudotriangle can be illuminated by the vertex of its screen graph (right).

triangulation of the polygon. We give now a similar reduction for L. Fejes Tóth's art gallery theorem [9], which asserts that the boundary of a set of $n \geq 4$ pairwise disjoint convex disks can be illuminated by $4n-7$ points. The proof of L. Fejes Tóth consists of locating the lighting points at suitable vertices of the gaps of a polygonal cover obtained by the growing process of the convex disks described in the introduction (see [9] for the details). Since an extension of a screen graph can be viewed as the result of a growing process applied to the convex disks we can use (and determine efficiently), as Fejes Tóth did, at most $4n-7$ of its vertices to illuminate the boundary of the convex disks. However one can obtain a simpler solution using only a subset of the vertices of a screen graph. Our result is the following.

Theorem 4 *Computing a lighting set with at most $4n-7$ lighting points for a collection of $n \geq 4$ pairwise disjoint convex disks reduces in linear time to computing (the screen graph of) a pseudo-triangulation of the collection of convex disks. In particular, such a lighting set can be computed in $O(n \log n)$ time under the assumption that each disk has constant computational complexity.*

Proof. Since the boundary points of the convex disks are the boundary points of the pseudotriangles (forget for a moment the points lying on the convex hull of the convex disks) it is sufficient to find a lighting set for the pseudotriangles of a pseudo-triangulation. How many lighting points are necessary for a pseudotriangle? Two in general (computable in $O(1)$ time) but only one if a side of the pseudotriangle reduces to a line segment, as illustrated in Figure 8. Let a be the number of pseudotriangles with a line segment side or, equivalently, the number of exterior bitangents in the pseudo-triangulation. From the above discussion it follows that a lighting set with $4n-4-a$ lighting points exists and is computable in

$O(n)$ time from the pseudo-triangulation. According to Lemma 1, the number a is at least three, from which the result follows. (We take care to send the lighting points on the convex hull far enough to infinity in suitable directions to guarantee that they also illuminate the boundary of the convex hull; in case a region of the tessellation is a half-plane we have to use the extra lighting point given by the second claim of Lemma 1 to illuminate the whole boundary of the convex hull). □

L. Fejes Tóth also provides collections of $n \geq 4$ convex disks which cannot be illuminated by less than $4n - 7$ points but leaves open a practical characterization of all cases in which this number of lighting points is necessary. From our discussion we see that a necessary condition is that the number of free exterior bitangents is three in which case they all lie on the boundary of the convex hull of the disks—here we use Lemma 1. (Slight perturbations of) collections of n interior-disjoint circles whose contact graphs are triangular planar graphs on n points are examples of configurations of n disks with exactly three free exterior bitangents (according to the Koebe-Andreev-Thurston Circle Packing Theorem' [18, page 117] or [14, pages 95–96] any triangular planar graphs on n points is realizable as the contact graph of a collection of n interior-disjoint circles). However this condition is not sufficient since configurations of circles require only $2n - 2$ lighting points (see [9]).

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