Master of Science in Mathematics *Michel Waldschmidt* Master Training Program
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Real Analysis

Final Exam, October 20, 2010 (*3 hours*)
You are allowed to use the textbook by W. Trench, nothing else.
Please switch off your cell phone – thanks

Exercise 1. Is the function $f(x) = x^3 - x$ bounded on $(-\infty, +\infty)$? Is there a local maximum? Is there a local minimum?

Exercise 2. Consider the function

$$f(x) = \frac{(x^2 + 5x - 3)\sin(x^2 - 1) + e^{x\cos x} + \sqrt{x^2 + 1}}{x^2 + 1}$$

on the closed bounded interval $[0, 1]$.
a) Is f continuous on $[0, 1]$?
b) Is f uniformly continuous on $[0, 1]$?
c) Is f differentiable on $[0, 1]$?
d) Is f integrable on $[0, 1]$?

Exercise 3.
a) Give the values of

$$\lim_{x\to 0+} \left(\frac{|x|}{x} + \sin x + \cos x \right) \quad \text{and} \quad \lim_{x\to 0-} \left(\frac{|x|}{x} + \sin x + \cos x \right).$$

b) Give the values of

$$\limsup_{x\to+\infty} \frac{1}{2 + \sin x}, \qquad \liminf_{x\to+\infty} \frac{1}{2 + \sin x},$$

and of

$$\limsup_{x\to+\infty} \left(1 + \frac{1}{x}\right) \sin x, \qquad \liminf_{x\to+\infty} \left(1 + \frac{1}{x}\right) \sin x.$$

Exercise 4. Let t be a positive real number. Consider the function $f(x)$ defined on $(-\infty, +\infty)$ by

$$f(x) = \frac{x}{|x| + t}.$$

a) Show that this function is monotonous.
b) Show that this function is bounded and compute $\sup_{x\in\mathbf{R}} f(x)$ and $\inf_{x\in\mathbf{R}} f(x)$.
c) Is f continuous? For which values of k is f differentiable k times?

Exercise 5. Compute the value of the proper integral

$$\int_0^1 x^2 e^{-x} dx.$$

Exercise 6. Let t be a positive real number. Compute

$$\int_0^t x \cos x dx.$$

Exercise 7.
Let n be a relative integer. Is the improper integral

$$\int_1^{+\infty} t^n e^{-t} dt$$

convergent?

Exercise 8. Is the improper integral

$$\int_{-\infty}^{+\infty} \frac{\sin x}{1+x^2} dx$$

convergent?

Exercise 9. For $n \geq 1$ integer, define

$$u_n = \int_0^1 \frac{dt}{(1+t)^n}.$$

a) Compute u_n for $n \geq 1$.
b) Is the series

$$\sum_{n \geq 1} u_n$$

convergent?

Exercise 10.
a) Let t be a real number. Compute

$$\lim_{n \to +\infty} \left(1 + \frac{t}{n}\right)^n.$$

b) Compute

$$\lim_{n \to +\infty} \int_0^1 \left(1 + \frac{t}{n}\right)^n dt.$$

Exercise 11. Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Check that for all $k \geq 0$ the function f is k times differentiable. What is the Taylor series of f? Is f the sum of a power series?

Real Analysis

Final Exam, October 20, 2010— solutions

Solution exercise 1
The function $x^3 - x$ is not bounded above, not bounded below. Since the derivatives is $3x^2 - 1$, there is a local maximum at $x_1 = -1/\sqrt{3} = -\sqrt{3}/3$ with $f(x_1) = -(2/3)x_1 = 2\sqrt{3}/9$ and a local minimum at $x_2 = 1/\sqrt{3} = \sqrt{3}/3$ with $f(x_2) = -(2/3)x_2 = -2\sqrt{3}/9$.

Solution exercise 2
The answers are all yes: the sums, products, composites of continuous functions are continuous, also the quotient when the denominator does not vanish. A function which is continuous on a closed bounded interval is uniformly continuous and is integrable. The sums, products, composites of differentiable functions are differentiable, also the quotient when the denominator does not vanish.

Solution exercise 3
a)
$$\lim_{x\to 0+}\left(\frac{|x|}{x} + \sin x + \cos x\right) = 2 \quad \text{and} \quad \lim_{x\to 0-}\left(\frac{|x|}{x} + \sin x + \cos x\right) = 0.$$
b)
$$\limsup_{x\to+\infty} \frac{1}{2+\sin x} = 1, \qquad \liminf_{x\to+\infty} \frac{1}{2+\sin x} = \frac{1}{3},$$
and
$$\limsup_{x\to+\infty} \left(1+\frac{1}{x}\right)\sin x = 1, \qquad \liminf_{x\to+\infty} \left(1+\frac{1}{x}\right)\sin x = -1.$$

Solution exercise 4
The function f is continuous at 0 with $f(0) = 0$. We have $f(x) \geq 0$ for $x \geq 0$ and $f(x) \leq 0$ for $x \leq 0$. On the closed interval $[0, +\infty]$, the function
$$f(x) = \frac{x}{x+t}$$
is, continuous, differentiable with
$$f'(x) = \frac{t}{(x+t)^2} > 0,$$
hence the function is increasing and therefore
$$\sup_{x\geq 0} f(x) = \lim_{x\to+\infty} f(x) = 1, \quad \inf_{x\geq 0} f(x) = f(0) = 0.$$
Furthermore, f' is differentiable with
$$f''(x) = \frac{-2t}{(x+t)^3}\cdot$$

On the closed interval $(+\infty, 0]$, the function

$$f(x) = \frac{x}{-x+t}$$

is continuous, differentiable with

$$f'(x) = \frac{t}{(x-t)^2} > 0,$$

hence the function is increasing and therefore

$$\sup_{x \leq 0} f(x) = f(0) = 0, \quad \inf_{x \leq 0} f(x) = \lim_{x \to -\infty} f(x) = -1.$$

Furthermore, f' is differentiable with

$$f''(x) = \frac{-2t}{(x-t)^3}.$$

This shows that f is increasing on $(-\infty, +\infty)$,

$$\sup_{x \in \mathbf{R}} f(x) = \lim_{x \to +\infty} f(x) = 1, \quad \inf_{x \in \mathbf{R}} f(x) = \lim_{x \to -\infty} f(x) = -1.$$

Since $f'(0+) = f'(0-) = 1/t$ while $f''(0+) = -2/t^2 \neq 2/t^2 = f''(0-)$, the function f is k times differentiable for $k = 1$ but not for $k = 2$ (hence not for $k \geq 2$).

Solution exercise 5
Integration by part gives

$$\text{for} \quad I = \int_0^1 x^2 e^{-x} dx \quad \text{the value} \quad I = \frac{-1}{e} + 2J \quad \text{with} \quad J = \int_0^1 x e^{-x} dx.$$

Again, integrating by part gives

$$J = 1 - \frac{2}{e}\cdot \quad \text{Hence} \quad I = 2 - \frac{5}{e}\cdot$$

Solution exercise 6
A primitive of $x \cos x$ is $x \sin x + \cos x$. Hence

$$\int_0^t x \cos x dx = t \sin t + \cos t - 1.$$

One can also prove this by integrating by part.

Solution exercise 7
Let n be a relative integer. We have

$$\lim_{t \to +\infty} t^{n+2} e^{-t} = 0,$$

hence the improper integral

$$\int_1^{+\infty} t^n e^{-t} dt$$

is convergent.

Solution exercise 8
Since

$$\frac{|\sin x|}{1+x^2} \leq \frac{1}{1+x^2}$$

for all $x \in \mathbf{R}$, the integral

$$\int_{-\infty}^{+\infty} \frac{\sin x}{1+x^2} dx$$

is absolutely convergent, hence convergent. The function $f(x) = \sin x/(1+x^2)$ is odd: $f(-x) = -f(x)$, hence for all $R > 0$

$$\int_{-R}^{+R} \frac{\sin x}{1+x^2} dx = 0 \quad \text{and therefore} \quad \int_{-\infty}^{+\infty} \frac{\sin x}{1+x^2} dx = \lim_{R \to +\infty} \int_{-R}^{+R} \frac{\sin x}{1+x^2} dx = 0.$$

Solution exercise 9
a) We have

$$u_1 = \int_0^1 \frac{dt}{1+t} = \log 2.$$

b) For $n \geq 2$, a primitive of $(1+t)^{-n}$ is $(-1/(n-1))(1+t)^{-n+1}$, hence

$$u_n = \int_0^1 \frac{dt}{(1+t)^n} = \frac{1}{n-1} \left(1 - \frac{1}{2^{n-1}}\right).\!.$$

c) Since

$$\sum_{n \geq 2} \frac{1}{n-1} \frac{1}{2^{n-1}} \text{ is convergent and } \quad \sum_{n \geq 2} \frac{1}{n-1} \text{ diverges to } +\infty,$$

the series

$$\log 2 + \sum_{n \geq 2} \frac{1}{n-1} \left(1 - \frac{1}{2^{n-1}}\right)$$

diverges to $+\infty$.

Solution exercise 10
a) For $t \in \mathbf{R}$,

$$\lim_{n \to +\infty} \left(1 + \frac{t}{n}\right)^n = e^t.$$

This has been proved during the course, but we need (for the next question) to check that the limit is uniform on $[0, 1]$. This follows from the estimate

$$\sup_{0 \leq t \leq 1} \left| n \log\left(1 + \frac{t}{n}\right) - t \right| < \frac{2}{n} \quad \text{for sufficiently large } n$$

and the fact that the exponential function is continuous.
b) Since the limit is uniform on $[0, 1]$, we have

$$\lim_{n \to +\infty} \int_0^1 \left(1 + \frac{t}{n}\right)^n dt = \int_0^1 \lim_{n \to +\infty} \left(1 + \frac{t}{n}\right)^n dt = \int_0^1 e^t dt = e - 1.$$

`Remark.` *There is another solution for this exercise: since*

$$\frac{d}{dt} \left(1 + \frac{t}{n}\right)^{n+1} = \frac{n+1}{n} \left(1 + \frac{t}{n}\right)^n,$$

we have

$$\int_0^1 \left(1 + \frac{t}{n}\right)^n dt = \frac{n}{n+1} \left(\left(1 + \frac{1}{n}\right)^{n+1} - 1\right) = \left(1 + \frac{1}{n}\right)^n - \frac{n}{n+1}.$$

The conclusion follows from

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e \quad \textit{and} \quad \lim_{n\to+\infty}\frac{n}{n+1} = 1.$$

Solution exercise 11

For all integers k, the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

satisfies $x^k f(x) \to 0$ as $x \to 0$. It follows that for all $k \geq 0$, the function f is k times differentiable with $f^{(k)}(0) = 0$. The Taylor series of f is the power series with all coefficients 0. Hence f is not the sum of a power series.
Remark. $e^{-1/x} \to 0$ when $x \to 0+$ and $e^{-1/x} \to +\infty$ when $x \to 0-$, this is why one takes $1/x^2$ and not $1/x$.