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The algebraic independence of certain numbers to algebraic powers

by

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Dedicated to Professor Th. Schneider on the occasion of his 65th birthday

In 1949, A. O. Gelfond proved ([4], Theorem 1, pp. 132–133) that if α is an algebraic number ($\alpha \neq 0$, $\log\alpha \neq 0$) and β is a cubic irrational number, then the two numbers α^β and α^{β^2} are algebraically independent (over $\boldsymbol{Q}$). Shortly thereafter Gelfond and N. I. Feldman [5] gave a measure of algebraic independence of these two numbers. R. Wallisser has conjectured that, for β a cubic irrational, α^β and α^{β^2} are algebraically independent even when α is only well-approximated by algebraic numbers. In this paper, we establish Wallisser's conjecture when α is closely approximated by algebraic numbers of bounded degree. We wish to thank M. Mignotte for his helpful comments on an earlier draft of this paper.

THEOREM. *Let α be a complex number, $\alpha \neq 0$, $\log\alpha \neq 0$, and β a cubic irrational number. Let f: $\boldsymbol{N}\to\boldsymbol{R}$ with $f\nearrow\infty$ and let $d_0\in\boldsymbol{N}$. Assume that for infinitely many $T\in\boldsymbol{N}$, there exist algebraic numbers a_T of degree $\leqslant d_0$ satisfying*

$$\operatorname{log\,height} a_T \leqslant T,$$
$$\log|\alpha - a_T| < -e^{Tf(T)}.$$

Then the two numbers α^β and α^{β^2} are algebraically independent.

Remark 1. If α itself is algebraic, we let $\alpha = a_T$ for $T \geqslant \operatorname{log\,height}\alpha$.

Remark 2. If α is a complex number ($\alpha \neq 0$, $\log\alpha \neq 0$) and β a cubic irrational number, with α^β, α^{β^2} algebraically dependent, then for all $d_0\in\boldsymbol{N}$ there exist two positive constants

$$C = C(\alpha,\beta,d_0), \quad H = H(\alpha,\beta,d_0)$$

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such that, if a is an algebraic number of degree $\leqslant d_0$ and of height $\leqslant h$, with $h \geqslant H$, then

$$\log|a-a| > -h^C.$$

(In fact, C and H are effectively computable constants.)

EXAMPLE. For $\alpha = \sum_{n=0}^{\infty} (-1)^n 2^{-2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}} \Big\}{\scriptstyle 2n \text{ times}}$, α^β and α^{β^2} are algebraically independent for every cubic irrational number β.

Notations. We fix any determination of logarithm in the disk $|z-a| < a$ such that $\log a \neq 0$. When α belongs to this disk, we write α^β instead of $\exp(\beta\log\alpha)$.

When $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \lambda_2) \in \boldsymbol{Z}^3$, we write $\boldsymbol{\lambda\beta}$ instead of $\lambda_0+\lambda_1\beta+\lambda_2\beta^2$ and we define $|\lambda|$ to be $\max_{i=0,1,2} |\lambda_i|$. For convenience, any $\boldsymbol{\lambda}, \boldsymbol{\nu}$ will have non-negative coordinates unless expressly allowed to be negative (does not apply to $\boldsymbol{\mu}$).

The letters $c_1, c_2, c_3, \ldots$ will denote positive constants which are independent of T.

Auxiliary lemmas

LEMMA 1 (Siegel's Lemma). *Let R and S be positive integers, $2R < S$ and let $a_{ij} \in \boldsymbol{Z}[X]$, $1 \leqslant i \leqslant R$, $1 \leqslant j \leqslant S$, $A \geqslant 1$, satisfy*

$$\deg a_{ij} \leqslant \delta, \qquad \operatorname{height} a_{ij} \leqslant A.$$

Then there exist polynomials $f_1, \ldots, f_S \in \boldsymbol{Z}[X]$, not all zero, satisfying

$$\deg f_j \leqslant \delta, \qquad \operatorname{height} f_j \leqslant \big((1+\delta)^2 SA\big)^{2R/(S-2R)}$$

and

$$\sum_{j=1}^{S} a_{ij} f_j = 0 \quad \text{for} \quad 1 \leqslant i \leqslant R.$$

For a proof, see [2], Lemma 5.2.

LEMMA 2 (Gelfond). *Suppose $P, Q \in C[X]$. Then*

$$(\operatorname{height} PQ)\cdot e^{\deg PQ} \geqslant (\operatorname{height} P)(\operatorname{height} Q).$$

For a proof (of a more general result), see [4], Lemma 2, p. 135 or [6], Lemma 3, p. 149.

LEMMA 3 (Tijdeman). *Suppose $F(z) = \sum_{|\nu|<N} A_\nu e^{(\nu\cdot\beta)z}$, and set*

$$b = \max_{|\nu|<N} \{1, |(\boldsymbol{\nu}\cdot\boldsymbol{\beta})|\cdot\max(1, |\log\boldsymbol{a}|)\},$$

$$b_0 = \min_{|\mu|<N} \{1, |(\boldsymbol{\mu}\cdot\boldsymbol{\beta})|\cdot\min(1, |\log\boldsymbol{a}|)\},$$

$$E = \max_{\substack{|\lambda|<L \\ 0\leqslant p<P}} \big|F^{(p)}\big((\boldsymbol{\lambda}\cdot\boldsymbol{\beta})\log\boldsymbol{a}\big)\big|.$$

Then, if $L \leqslant N$ and $PL^3 \geqslant 2N^3+13b^2$, we have

$$\max|A_\nu| \leqslant L^3\sqrt{(N^3)!}\,e^{7b^2}\left(\frac{1}{2b_0b}\right)^{N^3}\left(\frac{72b}{b_0L^{3/2}}\right)^{PL^3} E.$$

([7], Theorem 3, pp. 87–88.)

LEMMA 4. *Let $f(X), g(X) \in \boldsymbol{Z}[X]$ have heights $|f|$ and $|g|$, respectively, and degrees m and n, respectively. Then $f(X)$ and $g(X)$ have a common divisor in $\boldsymbol{Z}[X]$ if and only if for some $\omega \in \boldsymbol{C}$,*

$$\max\{|f(\omega)|, |g(\omega)|\} \leqslant |f|^{-n}|g|^{-m}(m+n)^{-(m+n)}.$$

For a proof, see [4], Lemma V, pp. 145–146 or [1], Lemma 1, p. 14.

LEMMA 5. *Suppose $\omega \in \boldsymbol{C}$ and $P(X) \in \boldsymbol{Z}[X]$ satisfy $|P(\omega)| < e^{-\lambda d(h+d)}$, where $\lambda \geqslant 3$, $d = \deg P$, $\operatorname{height} P = e^h$. Then there is a factor $Q(X)$ of $P(X)$ which is a power of an irreducible polynomial in $\boldsymbol{Z}[X]$ such that*

$$\log|Q(\omega)| < -(\lambda-1)\,d(h+d).$$

For a proof, see [4], Lemma VI, p. 147 or [1], Lemma 3, pp. 15 –16

LEMMA 6. *Suppose $\omega \in \boldsymbol{C}$ is transcendental, $\xi \in \boldsymbol{C}$ is algebraic integral over $\boldsymbol{Z}[\omega]$, of degree δ, having a minimal polynomial (over $\boldsymbol{Z}[\omega]$) with coefficients of degree $\leqslant d$ and height $\leqslant e^h$. Let λ_1, λ_2 be two real numbers satisfying*

$$\lambda_1 > \lambda_2 > 6+2\log(\delta+1)+2\log(|\omega|+1).$$

If

$$-\lambda_1 d(h+d) \leqslant \log|\xi| \leqslant -\lambda_2 d(h+d),$$

then there exist an irreducible polynomial $P(\omega) \in \boldsymbol{Z}[\omega]$, and an integer $s \geqslant 1$, such that P^s divides the norm of ξ over $\boldsymbol{Q}(\omega)$, and that

$$-3\delta\lambda_1 d(h+d) \leqslant \log|P(\omega)| \leqslant -\frac{\lambda_2}{6s}\,d(h+d).$$

Proof of Lemma 6. The proof is an adaptation of Chudnovskii's arguments in [3].

Consider the minimal polynomial of ξ over $\boldsymbol{Z}[\omega]$:

$$\xi^\delta + u_{\delta-1}(\omega)\,\xi^{\delta-1} + \ldots + u_0(\omega) = 0,$$

with $u_i(\omega) \in \boldsymbol{Z}[\omega]$, $\deg u_i \leqslant d$, $\operatorname{height} u_i \leqslant e^h$ $(0 \leqslant i \leqslant \delta-1)$.

The norm of ξ over $\boldsymbol{Q}(\omega)$ is $u_0(\omega)$, and

$$u_0(\omega) = -\xi\sum_{i=1}^{\delta} u_i(\omega)\,\xi^{i-1} \qquad (\text{with } u_\delta = 1),$$

$$|u_0(\omega)| \leqslant |\xi|\,\delta(d+1)e^h\max\{1, |\omega|^d\},$$

since $|\xi| < 1$. Consequently

$$\log|u_0(\omega)| \leqslant -\lambda_2 d(h+d) + h + d\log\max\{1, |\omega|\} + \log\delta(d+1) < -\tfrac{1}{2}\lambda_2 d(h+d).$$

By Lemma 5, we get an irreducible polynomial $P(\omega) \in \boldsymbol{Z}[\omega]$, and an integer $s \geqslant 1$, such that $P^s(\omega)$ divides u_0, and

$$\log|P^s(\omega)| < -\left(\frac{\lambda_2}{2} - 1\right)d(h+d) < -\frac{\lambda_2}{6}d(h+d).$$

We show that

$$\log|P(\omega)| > -3\delta\lambda_1 d(h+d)$$

by deriving a contradiction from the contrary assumption.

Assume

$$\log|P(\omega)| \leqslant -3\delta\lambda_1 d(h+d).$$

We claim that for each j $(0 \leqslant j \leqslant \delta - 1)$,

$$\log|u_j(\omega)| \leqslant -2\delta\lambda_1 d(h+d).$$

The claim is true for $j = 0$: $P(\omega)$ divides $u_0(\omega)$, and the quotient has degree $\leqslant d$ and height $\leqslant e^{h+d}$ (using Lemma 2), hence has absolute value $\leqslant (d+1)e^{h+d}\max\{1, |\omega|^d\}$.

We now assume that the claim is true up to $j-1$, with $1 \leqslant j \leqslant \delta - 1$, and, under that assumption, prove it true for j. We can write

$$-u_j(\omega) = \xi^{\delta-j} + \sum_{i=0}^{j-1}\frac{u_i(\omega)}{\xi^{j-i}} + \xi\sum_{l=j+1}^{\delta-1}\xi^{l-j-1}u_l(\omega).$$

Using the induction hypothesis, and the lower bound for $|\xi|$, we deduce

$$\log\frac{|u_i(\omega)|}{|\xi|^{j-i}} \leqslant -(2\delta - j + i)\lambda_1 d(h+d) \leqslant -\delta\lambda_1 d(h+d),$$

for $0 \leqslant i \leqslant j-1$. Since

$$|u_l(\omega)| \leqslant (d+1)e^h\max\{1, |\omega|^d\}, \quad j+1 \leqslant l \leqslant \delta - 1,$$

we obtain

$$\log|u_j(\omega)| \leqslant -\tfrac{1}{2}\lambda_2 d(h+d).$$

Consequently, by Lemma 4, $u_j(\omega)$ and $P^s(\omega)$ have a common factor, i.e. $P(\omega)$ divides $u_j(\omega)$ in $\boldsymbol{Z}[\omega]$. As a result:

$$\log|u_j(\omega)| \leqslant -2\delta\lambda_1 d(h+d),$$

and the claim follows.

However, the claim gives

$$|\xi|^\delta \leqslant \sum_{j=0}^{\delta-1}|u_j(\omega)|\,|\xi|^{j-1} \leqslant \delta e^{-2\delta\lambda_1 d(h+d)},$$

and consequently

$$\log|\xi| \leqslant 1 - 2\lambda_1 d(h+d) < -\lambda_1 d(h+d),$$

contradicting the hypothesis.

This contradiction completes the proof of Lemma 6.

Proof of the theorem. The proof is based on the ideas of Gelfond and N. I. Feldman [5], and G. V. Chudnovskii [3]. It follows from Gelfond's transcendence measure ([4], Theorem III, p. 134) for numbers of the type a^b, where a and b are algebraic numbers with $\log a \neq 0$, b irrational, that both numbers a^β and a^{β^2} are transcendental. Suppose that the field $\boldsymbol{Q}(a^\beta, a^{\beta^2})$ has transcendence degree one. Then we can write $\boldsymbol{Q}(\beta, a^\beta, a^{\beta^2}) = \boldsymbol{Q}(\omega, \omega_1)$, where ω is transcendental (we can choose $\omega = a^\beta$), and ω_1 is integral over $\boldsymbol{Z}[\omega]$ of degree m. Let $v \in \boldsymbol{Z}[\omega]$, $v \neq 0$, such that va^β, va^{β^2}, $va^{-\beta}$ and $va^{-\beta^2} \in \boldsymbol{Z}[\omega, \omega_1]$. We may assume without loss of generality that β is an algebraic integer and that $f(T) \leqslant \log T$.

Let T be a sufficiently large positive integer with a_T an algebraic number satisfying the hypotheses of the theorem. Then select $\Delta \in \boldsymbol{N}$, $1 \leqslant \Delta \leqslant e^T$, such that Δa_T is an algebraic integer and set

$$N_0 = [\exp Tf(T)/7], \qquad N_1 = [N_0^2\log N_0].$$

It is easy to check that

$$N_1^3\log N_1 \leqslant e^{13Tf(T)/14}.$$

For $N \in \boldsymbol{N}$ with $N_0 \leqslant N \leqslant N_1$, we define

$$L_N = [N^{1/2}f(T)^{1/4}],$$
$$H_N = [N^{3/2}(\log N)f(T)^{-3/4}],$$
$$P_N = [c_1N^{3/2}f(T)^{-3/4}],$$

where $c_1 = 1/(4d_0m)$.

The inequality $NL_NT \leqslant 8H_N$ which follows from our assumption that $Tf(T) \leqslant 7\log N$, will be used repeatedly below, often without mention.

Step 1. We show that *there exist elements* $\varphi(\boldsymbol{v}) \in \boldsymbol{Z}[\omega]$, $|\boldsymbol{v}| < N$, *not all of which are zero and without a common divisor in* $\boldsymbol{Z}[\omega]$, *satisfying*

$$\log\text{height}\,\varphi(\boldsymbol{v}) \leqslant c_2H_N, \qquad \deg\varphi(\boldsymbol{v}) \leqslant c_3L_NN,$$

such that the function

$$F_N(z) = \sum_{|\boldsymbol{v}|<N}\varphi(\boldsymbol{v})\exp\big((\boldsymbol{v}\cdot\boldsymbol{\beta})z\big)$$

satisfies

$$\log|F_N(z)|_{|z|=N^{3/2}} \leqslant -c_4 N^3 \log N.$$

Proof. (i) Consider the numbers

$$\Phi_{p,\lambda} = \sum_{|\nu|<N} \varphi(\nu)(\nu\cdot\beta)^p a_T^{\mu_0} a^{\beta\mu_1} a^{\beta^2\mu_2} \qquad (0 \leqslant p < P_N),$$

where $\mu_0, \mu_1, \mu_2 \in \mathbf{Z}$ and $\mu_0 + \mu_1\beta + \mu_2\beta^2 = (\nu\cdot\beta)(\lambda\cdot\beta)$, for $|\lambda| < L_N$, and the $\varphi(\nu)$ satisfy

$$\log\operatorname{height}\varphi(\nu) \leqslant c_2 H_N, \qquad \deg\varphi(\nu) \leqslant c_3 L_N N.$$

The numbers

$$(v^{c_5}\Delta)^{NL_N}\Phi_{p,\lambda}$$

are polynomials in ω_1 and Δa_T with coefficients from $\mathbf{Z}[\omega]$. These coefficients themselves have degree $\leqslant c_6 NL_N$ and

$$\log\operatorname{height} \leqslant c_7(NL_N \log\operatorname{height} a_T + H_N) \leqslant c_8 H_N$$

by our choice of range of N. We want to choose the $\varphi(\nu)$ such that the coefficients of the at most $d_0 m$ monomials in ω_1 and Δa_T vanish for $0 \leqslant p < P_N$ and $|\lambda| < L_N$. The number of equations is at most

$$d_0 m\, P_N L_N^3 \leqslant N^3/4,$$

and the number of unknowns is N^3. Thus by Lemma 1 the system has a non-trivial solution with $\varphi_0(\nu) \in \mathbf{Z}[\omega]$ satisfying

$$\deg\varphi_0(\nu) \leqslant c_3 NL_N, \qquad \log\operatorname{height}\varphi_0(\nu) \leqslant c_9 H_N.$$

After dividing each $\varphi_0(\nu)$ by the greatest common divisor of all the $\varphi_0(\nu)$, Lemma 2 assures us that the quotients $\varphi(\nu)$ satisfy

$$\deg\varphi(\nu) \leqslant c_3 NL_N, \qquad \log\operatorname{height}\varphi(\nu) \leqslant c_9 H_N + c_3 NL_N \leqslant c_2 H_N,$$

as desired.

(ii) For $0 \leqslant p < P_N$ and $|\lambda| < L_N$, we have

$$\left|F_N^{(p)}\big((\lambda\cdot\beta)\log a\big) - \Phi_{p,\lambda}\right| \leqslant \sum_{\nu} |\varphi(\nu)|\,|\nu\cdot\beta|^p |a^\beta|^{\mu_1} |a^{\beta^2}|^{\mu_2} |a^{\mu_0} - a_T^{\mu_0}|.$$

So

$$\log\left|F_N^{(p)}\big((\lambda\cdot\beta)\log a\big)\right| \leqslant c_{10}(H_N + p\log N + NL_N T) + \log|a - a_T|.$$

But since

$$H_N + p\log N + NL_N T \leqslant 10 H_N \leqslant 10 H_{N_1} < N_1^3 \log N_1 \leqslant \exp\big(13Tf(T)/14\big),$$

we have

$$\log\left|F_N^{(p)}\big((\lambda\cdot\beta)\log a\big)\right| \leqslant -\tfrac{1}{2}\exp\big(Tf(T)\big) \leqslant -(\sqrt{N_0}/2)N^3\log N.$$

(iii) We now use Hermite's interpolation formula on the circles about the origin of radii $N^{3/2}$ and N^2. For $N^{3/2} \leqslant |z| < N^2$, we have

$$F_N(z) = \frac{1}{2\pi i}\int_{|\zeta|=N^2} \frac{F_N(\zeta)}{\zeta - z}\prod_{\lambda}\left(\frac{z-(\lambda\cdot\beta)\log a}{\zeta-(\lambda\cdot\beta)\log a}\right)^{P_N} d\zeta -$$

$$-\frac{1}{2\pi i}\sum_{\lambda}\sum_{p=0}^{P_N-1}\frac{F_N^{(p)}\big((\lambda\cdot\beta)\log a\big)}{p!} \times$$

$$\times \int_{|\zeta-\lambda\cdot\beta\log a|=b/2} \frac{(\zeta-\lambda\cdot\beta\log a)^p}{\zeta - z}\prod_{\lambda'}\left(\frac{z-\lambda'\cdot\beta\log a}{\zeta-\lambda'\cdot\beta\log a}\right)^{P_N} d\zeta$$

where the indices run over all λ, λ' with coordinates between 0 and $L_N - 1$, and where $b = |\log a|\min_{\lambda\neq\lambda'}|\lambda\cdot\beta - \lambda'\cdot\beta|$. To estimate

$$|F_N|_{N^{3/2}} = \max_{|z|=N^{3/2}}|F_N(z)|,$$

we use the following bounds:

$$\log|F_N|_{N^2} \leqslant c_{11}(H_N + N^3) < c_{12}N^3,$$

$$\log\sup_{\substack{|z|=N^{3/2}\\|\zeta|=N^2}}\prod_{\lambda}\left|\frac{z-\lambda\cdot\beta\log a}{\zeta-\lambda\cdot\beta\log a}\right|^{P_N} \leqslant -L_N^3 P_N\log\frac{N^2}{3N^{3/2}} \leqslant -\frac{c_1}{6}N^3\log N,$$

$$\log b \geqslant -c_{13}\log N,$$

$$\log\sup_{\substack{|\zeta-\lambda\cdot\beta\log a|=b/2\\|z|=N^{3/2}}}\prod_{\lambda'}\left|\frac{z-\lambda'\cdot\beta\log a}{\zeta-\lambda'\cdot\beta\log a}\right|^{P_N} \leqslant c_{14}L_N^3 P_N\log N \leqslant c_1 c_{14}N^3\log N.$$

Thus we obtain

$$\log|F_N|_{N^{3/2}} \leqslant -c_4 N^3\log N.$$

STEP 2. We now note that *there exist* $p_0 \in \mathbf{Z}$, $P_N \leqslant p_0 \leqslant \left[\frac{3}{c_1}\right]P_N - 1$ *and* $\lambda \in \mathbf{Z}^3$ *with* $|\lambda| < L_N$, *such that*

$$-c_{15}N^3\log N \leqslant \log|F_N^{(p_0)}(\lambda\cdot\beta\log a)| \leqslant -c_{16}N^3\log N.$$

The upper bound

$$\log|F_N^{(p_0)}(\lambda\cdot\beta\log a)| \leqslant -c_{16}N^3\log N$$

is a direct consequence of Step 1 and Cauchy's integral formula. Assume that, for every pair p_0, λ in the considered ranges,

$$\log|F_N^{(p_0)}(\lambda\cdot\beta\log a)| \leqslant -c_{15}N^3\log N$$

for some large c_{15}. Then, by Lemma 3,

$$\log\max_{\nu}|\varphi(\nu)| \leqslant -c_{17}N^3\log N, \quad \text{with } c_{17} > 0.$$

For each ν with $\varphi(\nu) \neq 0$, we choose by Lemma 5 a factor $q(\nu)$ of $\varphi(\nu)$ in $\boldsymbol{Z}[\omega]$ such that $q(\nu)$ is a power of an irreducible polynomial in $\boldsymbol{Z}[\omega]$ and

$$\log|q(\nu)| \leqslant -c_{18}N^3\log N,$$

$$\deg q(\nu) \leqslant c_3 NL_N, \quad \log\operatorname{height} q(\nu) \leqslant c_{19}H_N.$$

Since the $\varphi(\nu)$ do not all have a common factor in $\boldsymbol{Z}[\omega]$, at least two of the $q(\nu)$ must be powers of distinct irreducible polynomials, contradicting Lemma 4, since

$$2c_3c_{19}NL_NH_N + 2c_3NL_N\log(2c_3NL_N) \leqslant c_{20}N^3(\log N)f(T)^{-1/2}.$$

We now know that $\Phi_{p_0,\lambda}$ and hence $(v^{c_5}\Delta)^{c_{21}NL_N}\,\Phi_{p_0,\lambda}$ satisfy

$$-c_{22}N^3\log N \leqslant \log|w| \leqslant -c_{23}N^3\log N,$$

by the argument of Step 1 (ii).

STEP 3. The number $\xi_N = (v^{c_5}\Delta)^{c_{21}NL_N}\,\Phi_{p_0,\lambda}$ and its conjugates over $\boldsymbol{Q}(\omega)$ are polynomials in the conjugates of ω_1 and Δa_T over $\boldsymbol{Q}(\omega)$ (of degrees $\leqslant m$ and d_0 respectively), with coefficients in $\boldsymbol{Z}[\omega]$ having

$$\text{degree} \leqslant c_{24}NL_N, \quad \log\operatorname{height} \leqslant c_{25}H_N.$$

Using Lemma 6, we get an *irreducible polynomial* $R_N(\omega)$ *in* $\boldsymbol{Z}[\omega]$ *and an integer* $s_N \geqslant 1$, *such that* $R_N(\omega)$ *and* $Q_N(\omega) = R_N(\omega)^{s_N}$ *satisfy*

$$\deg Q_N \leqslant c_{24}NL_N; \quad \log\operatorname{height} Q_N \leqslant c_{26}H_N;$$

$$-N^3(\log N)f(T)^{1/4} < \log|R_N(\omega)|; \quad \log|Q_N(\omega)| \leqslant -c_{27}N^3\log N.$$

STEP 4. We apply Lemma 4 to the polynomials $Q_N(\omega)$ and $Q_{N+1}(\omega)$, for $N_0 \leqslant N < N_1$. We estimate

$$\deg Q_N\cdot\log\operatorname{height} Q_{N+1} + \deg Q_{N+1}\cdot\log\operatorname{height} Q_N + $$
$$+ (\deg Q_N + \deg Q_{N+1})\cdot\log(\deg Q_N + \deg Q_{N+1})$$
$$\leqslant c_{28}NL_NH_N \leqslant c_{28}N^3(\log N)f(T)^{-1/2}.$$

Consequently, by Lemma 4, $Q_N(\omega)$ and $Q_{N+1}(\omega)$ have a common factor $R_N(\omega) = R_{N+1}(\omega)$, since both polynomials are powers of irreducible polynomials in $\boldsymbol{Z}[\omega]$.

Consequently

$$R_{N_1} = R_{N_0} \quad \text{and} \quad Q_{N_1}(\omega) = R_{N_0}^{s_{N_1}}(\omega);$$

thus

$$-\log|Q_{N_1}(\omega)| = -s_{N_1}\log|R_{N_0}(\omega)| < c_{24}N_1^{3/2}f(T)^{1/4}N_0^3(\log N_0)f(T)^{1/4}$$
$$\leqslant c_{29}N_1^{3/2}f(T)^{1/4}N_1^{3/2}(\log N_1)^{-3/2}(\log N_1)f(T)^{1/4}$$
$$\leqslant c_{29}N_1^3(\log N_1)^{-1/2}f(T)^{1/2},$$

which contradicts the upper bound on $|Q_{N_1}(\omega)|$ found in Step 3.

The contradiction establishes the theorem.

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