

An introduction to irrationality and transcendence methods.

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¹ <http://www.math.jussieu.fr/~miw/articles/pdf/AWSabstract.pdf>

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