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An introduction to irrationality and transcendence methods.

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Lecture 1 [2]

1 Historical introduction to irrationality

1.1 Early history

1.1.1 Simple proofs of irrationality

The early history of irrationality goes back to the Greek mathematicians Hippasus of Metapontum (around 500 BC) and Theodorus of Cyrene, Eudoxus, Euclid. There are different early references in the Indian civilisation and the Sulba Sutras (around 800-500 BC).

Let us start with the irrationality of the number

$$\sqrt{2} = 1,414\,213\,562\,373\,095\,048\,801\,688\,724\,209\ \ldots$$

One of the most well known proofs is to argue by contradiction as follows: assume $\sqrt{2}$ is rational and write it as a/b where a and b are relatively prime positive rational integers. Then $a^2 = 2b^2$. It follows that a is even. Write $a = 2a'$. From $2a'^2 = b^2$ one deduces that b also is even, contradicting the assumption that a and b were relatively prime.

There are variants of this proof - a number of them are in the nice booklet [19]. For instance using the relation

$$\sqrt{2} = \frac{2-\sqrt{2}}{\sqrt{2}-1}$$

with $\sqrt{2} = a/b$ one deduces

$$\sqrt{2} = \frac{2b-a}{a-b}.$$

Now we have $1 < \sqrt{2} < 2$, hence $0 < a-b < b$, which shows that the denominator b of fraction $\sqrt{2} = a/b$ was not minimal.

This argument can be converted into a geometric proof: starting with an isosceles rectangle triangle with sides b and hypothenuse a, one constructs (using

[2] http://www.math.jussieu.fr/∼miw/articles/pdf/AWSLecture1.pdf

ruler and compass if one wishes) another similar triangle with smaller sides $a-b$ and hypothenuse $2b-a$. Such a proof of irrationality is reminiscent of the ancient Greek geometers constructions, and also of the infinite descent of Fermat.

A related but different geometric argument is to start with a rectangle having sides 1 and $1+\sqrt{2}$. We split it into two unit squares and a smaller rectangle. The length of this second rectangle is 1, its width is $\sqrt{2}-1$, hence its proportion is

$$\frac{1}{\sqrt{2}-1} = 1+\sqrt{2}.$$

Therefore the first and second rectangles have the same proportion. Now if we repeat the process and split the small rectangle into two squares (of sides $\sqrt{2}-1$) and a third tiny rectangle, the proportions of this third rectangle will again be $1+\sqrt{2}$. This means that the process will not end, each time we shall get two squares and a remaining smaller rectangle having the same proportion.

On the other hand if we start with a rectangle having integer side lengths, if we split it into several squares and if a small rectangle remains, then clearly the small rectangle while have integer side lengths. Therefore the process will not continue forever, it will stop when there is no remaining small rectangle. This proves again the irrationality of $\sqrt{2}$.

In algebraic terms the number $x = 1+\sqrt{2}$ satisfies

$$x = 2 + \frac{1}{x},$$

hence also

$$x = 2 + \frac{1}{2+\frac{1}{x}} = 2 + \frac{1}{2+\frac{1}{2+\frac{1}{x}}} = \cdots,$$

which yields the *(regular) continued fraction expansion* of $1+\sqrt{2}$. Here is the definition of the continued fraction expansion of a real number (basic references are Brezinski's references [4, 5].

Given a real number x, the Euclidean division in $\mathbb{R}$ of x by 1 yields a quotient $[x] \in \mathbb{Z}$ (the *integral part of* x) and a remainder $\{x\}$ in the interval $[0,1)$ (the *fractional part of* x) satisfying

$$x = [x] + \{x\}.$$

Set $a_0 = [x]$. Hence $a_0 \in \mathbb{Z}$. If x is an integer then $x = [x] = a_0$ and $\{x\} = 0$. In this case we just write $x = a_0$ with $a_0 \in \mathbb{Z}$. Otherwise we have $\{x\} > 0$ and we set $x_1 = 1/\{x\}$ and $a_1 = [x_1]$. Since $\{x\} < 1$ we have $x_1 > 1$ and $a_1 \geq 1$. Also

$$x = a_0 + \frac{1}{a_1 + \{x_1\}}.$$

Again, we consider two cases: if $x_1 \in \mathbb{Z}$ then $\{x_1\} = 0$, $x_1 = a_1$ and

$$x = a_0 + \frac{1}{a_1}$$

with two integers a_0 and a_1, with $a_1 \geq 2$ (recall $x_1 > 1$). Otherwise we can define $x_2 = 1/\{x_1\}$, $a_2 = [x_2]$ and go one step further:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \{x_2\}}}.$$

Inductively one obtains a relation

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \cfrac{}{a_{n-1} + \cfrac{1}{a_n + \{x_n\}}}}}}$$

with $0 \leq \{x_n\} < 1$. For ease of notation we write either

$$x = [a_0; a_1, a_1, a_2, \ldots, a_{n-1}, a_n + \{x_n\}]$$

or

$$x = a_0 + \frac{1\,|}{|a_1} + \frac{1\,|}{|a_2} + \cdots \frac{1\;\;|}{|a_{n-1}} + \frac{1\qquad|}{|a_n + \{x_n\}}.$$

This second notation is mainly used for *irregular* continued fractions[3].

The connection with the geometric proof of irrationality of $\sqrt{2}$ by means of rectangles and squares is now obvious: start with a positive real number x and consider a rectangle of sides 1 and x. Divide this rectangle into unit squares and a second rectangle. Then a_0 is the number of unit squares which occur, while the sides of the second rectangle are 1 and $\{x\}$. If x is not an integer, meaning $\{x\} > 0$, then we split the second rectangle into squares of sides $\{x\}$ plus a third rectangle. The number of squares is now a_1 and the third rectangle has sides $\{x\}$ and $1 - a_1\{x\}$. Going one in the same way, one checks that the number of squares we get at the n-th step is a_n.

This geometric point of view shows that the process stops after finitely many steps (meaning that some $\{x_n\}$ is zero, or equivalently that x_n is in $\mathbb{Z}$) if and only if x is rational.

Hence

$$x = [a_0; a_1, \ldots, a_n] \quad \text{or} \quad x = [a_0; a_1, \ldots, a_n, \ldots]$$

depending on whether $x_n \in \mathbb{Z}$ for some n or not. This is the *continued fraction expansion of* x. Notice that any irrational number has a unique infinite continued fraction expansion, while for rational numbers, the above construction

[3] A continued fraction expansion of the form

$$x = a_0 + \frac{b_1\,|}{|a_1} + \frac{b_2\,|}{|a_2} + \cdots \frac{b_{n-1}\,|}{|a_{n-1}} + \frac{b_n\qquad|}{|a_n + \{x_n\}}.$$

is called "irregular".

provides a unique well defined continued fraction which bears the restriction that the last a_n is ≥ 2. But we allow also the representation

$$[a_0; a_1, \dots, a_n - 1, 1].$$

For instance $11/3 = [3; 1, 2] = [3; 1, 1, 1]$.

We need a further notation for ultimately periodic continued fraction. Assume that x is irrational and that for some integers n_0 and $r > 0$ its continued fraction expansion $[a_0; a_1, \dots, a_n, \dots]$ satisfies

$$a_{n+r} = a_n \quad \text{for any } n \geq n_0.$$

Then we write

$$x = [a_0; a_1, \dots, a_{n_0-1}, \overline{a_{n_0}, a_{n_0+1}, \dots, a_{n_0+r-1}}].$$

For instance

$$\sqrt{2} = [1; 2, 2, 2, \dots] = [1; \overline{2}].$$

References on continued fractions are [10, 21, 15, 16, 6]. An interesting remark [19] on the continued fraction expansion of $\sqrt{2}$ is to relate the A4 paper format 21×29.7 to the fraction expansion

$$\frac{297}{210} = \frac{99}{70} = [1; 2, 2, 2, 2, 2].$$

There is nothing special with the square root of 2: most of the previous argument extend to the proof of irrationality of $\sqrt{n}$ when n is a positive integer which is not the square of an integer. For instance a proof of the irrationality of $\sqrt{n}$ when n is not the square of an integer runs as follows. Write $\sqrt{n} = a/b$ where b is the smallest positive integer such that $b\sqrt{n}$ is an integer. Further, denote by m the integral part of $\sqrt{n}$: this means that m is the positive integer such that $m < \sqrt{n} < m + 1$. The strict inequality $m < \sqrt{n}$ is the assumption that n is not a square. From $0 < \sqrt{n} - m < 1$ one deduces

$$0 < (\sqrt{n} - m)b < b.$$

Now the number $b' = (\sqrt{n} - m)b$ is a positive rational integer, the product $b'\sqrt{n}$ is an integer and $b' < b$, which contradicts the choice of b minimal.

Exercise 1.1. *Extend this proof to a proof of the irrationality of $\sqrt[k]{n}$, when n and k are positive integers and n is not the k-th power of an integer.*
`Hint.` *Assume that the number $x = \sqrt[k]{n}$ is rational. Then the numbers*

$$x^2, x^3, \dots, x^{k-1}$$

are also rational. Denote by d the least positive integer such that the numbers $dx, dx^2, \dots, dx^{k-1}$ are integers. Further, denote by m the integral part of x and consider the number $d' = (x - m)d$.

The irrationality of $\sqrt{5}$ is equivalent to the irrationality of the *Golden ratio* $\Phi = (1+\sqrt{5})/2$, root of the polynomial $X^2 - X - 1$, whose continued fraction expansion is

$$\Phi = [1; 1, 1, 1, 1, \dots] = [1, \overline{1}].$$

This expansion follows from the relation

$$\Phi = 1 + \frac{1}{\Phi}.$$

The geometric irrationality proof using rectangles that we described above for $1+\sqrt{2}$ works in a similar way for the Golden ratio: a rectangle of sides Φ and 1 splits into a square and a small rectangle of sides 1 and $\Phi - 1$, hence the first and the second rectangles have the same proportion

$$\Phi = \frac{1}{\Phi - 1}. \tag{1.2}$$

Therefore the process continues forever with one square and one smaller rectangle with the same proportion. Hence Φ and $\sqrt{5}$ are irrational numbers.

Exercise 1.3. *Perform the geometric construction starting with any rectangle of sides* 1 *and* x*: split it into a maximal number of squares of sides* 1*, and if a second smaller rectangle remains repeat the construction: split it into squares as much as possible and continue if a third rectangle remains.*
a) Prove that the number of squares in this process is the sequence of integers $(a_n)_{n\geq_0}$ *in the continued fraction expansion of* x*.*
b) Start with a unit square. Put on top of it another unit square: you get a rectangle with sides 1 *and* 2*. Next put on the right a square of sides* 2*, which produces a rectangle with sides* 2 *and* 3*. Continue the process as follows: when you reach a rectangle of small side* a *and large side* b*, complete it with a square of sides* b*, so that you get a rectangle with sides* b *and* $a+b$*.*
Which is the sequence of sides of the rectangles you obtain with this process? Generalizing this idea, deduce a geometrical construction of the rational number having continued fraction expansion

$$[a_0; a_1, \dots, a_k].$$

Another proof of the same result is to deduce from the equation (1.2) that a relation $\Phi = a/b$ with $0 < b < a$ yields

$$\Phi = \frac{b}{a-b},$$

hence a/b is not a rational fraction with minimal denominator.

Other numbers for which it is easy to prove the irrationality are quotients of logarithms: if m and n are positive integers such that $(\log m)/(\log n)$ is rational, say a/b, then $m^b = n^a$, which means that m *and* n *are multiplicatively dependent*. Recall that elements $x_1, \dots, x_r$ in an additive group are *linearly independent* if a relation $a_1x_1 + \cdots + a_rx_r = 0$ with rational integers $a_1, \dots, a_r$

implies $a_1 = \cdots = a_r = 0$. Similarly, elements $x_1, \ldots, x_r$ in a multiplicative group are *multiplicatively independent* if a relation $x_1^{a_1} \cdots x_r^{a_r} = 1$ with rational integers $a_1, \ldots, a_r$ implies $a_1 = \cdots = a_r = 0$. Therefore a quotient like $(\log 2)/\log 3$, and more generally $(\log m)/\log n$ where m and n are multiplicatively independent positive rational numbers, is irrational.

We have seen that *a real number is rational if and only if its continued fraction expansion is finite.* There is another criterion of irrationality using the g-adic expansion when g is an integer ≥ 2 (for $g = 10$ this is the decimal expansion, for $g = 2$ it is the diadic expansion). Indeed any real number x can be written

$$x = [x] + d_1 g^{-1} + d_2 g^{-2} + \cdots + d_n g^{-n} + \cdots$$

where the integers d_n (the digits of x) are in the range $0 \leq d_n < g$. Again there is unicity of such an expansion apart from the integer multiples of some g^{-n} which have two expansions, one where all sufficiently large digits vanish and one for which all sufficiently large digits are $g - 1$. This is due to the equation

$$g^{-n} = \sum_{k=0}^{n} (g-1) g^{-n-k-1}.$$

Here is the irrationality criterion using such expansions.

Lemma 1.4. *Let $g \geq 2$ be an integer and x a real number. Then x is rational if and only if the sequence $(d_n)_{n \geq 1}$ of digits of x in the expansion in basis g*

$$x = [x] + d_1 g^{-1} + d_2 g^{-2} + \cdots + d_n g^{-n} + \cdots \qquad (0 \leq d_n < g)$$

is ultimately periodic.

One might be tempted to conclude that it should be easy to decide whether a given real number is rational or not. However this is not the case with many constants from analysis, because most often one does not know any expansion, either in continued fraction or in any basis $b \geq 2$. And the fact is that for many such constants the answer is not known. For instance one does not know whether the *Euler–Mascheroni constant*

$$\begin{aligned} \gamma &= \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right) \\ &= 0,577\,215\,664\,901\,532\,860\,606\,512\,090\,082\ldots \end{aligned}$$

is rational or not: one expects that it is an irrational number (and even a transcendental number - see § 5.7). Other formulas for the same number are

$$\begin{aligned} \gamma &= \sum_{k=1}^{\infty} \left(\frac{1}{k} - \log\left(1 + \frac{1}{k} \right) \right) \\ &= \int_1^{\infty} \left(\frac{1}{[x]} - \frac{1}{x} \right) dx \\ &= -\int_0^1 \int_0^1 \frac{(1-x)dxdy}{(1-xy)\log(xy)}. \end{aligned}$$

J. Sondow uses (a generalization of) the last double integral in [24], he was inspired by F. Beukers' work on Apéry's proof of the irrationality of

$$\zeta(3) = \sum_{n\geq 1} \frac{1}{n^3} = 1,202\,056\,903\,159\,594\,285\,399\,738\,161\,511\ \ldots$$

in 1978. Recall that the values of the *Riemann zeta function*

$$\zeta(s) = \sum_{n\geq 1} n^{-s}$$

was considered by Euler for real s and by Riemann for complex s, the series being convergent for the real part of s greater than 1. Euler proved that the values $\zeta(2k)$ of this function at the even positive integers ($k \in \mathbb{Z}$, $k \geq 1$) are rational multiples of π^{2k}. For instance $\zeta(2) = \pi^2/6$. It is interesting to notice that Euler's proof relates the values $\zeta(2k)$ at the positive even integers with the values of the same function at the odd negative integers, namely $\zeta(1-2k)$. For Euler this involved divergent series[4], while Riemann defined $\zeta(s)$ for $s \in \mathbb{C}$, $s \neq 1$, by analytic continuation.

One might be tempted to guess that $\zeta(2k+1)/\pi^{2k+1}$ is a rational number when $k \geq 1$ is a positive integer. However the folklore conjecture is that this is not the case. In fact there are good reasons to conjecture that for any $k \geq 1$ and any non-zero polynomial $P \in \mathbb{Z}[X_0, X_1, \ldots, X_k]$, the number $P(\pi, \zeta(3), \zeta(5), \ldots, \zeta(2k+1))$ is not 0. But one does not know whether

$$\zeta(5) = \sum_{n\geq 1} \frac{1}{n^5} = 1,036\,927\,755\,143\,369\,926\,331\,365\,486\,457\ldots$$

is irrational or not. And there is no proof so far that $\zeta(3)/\pi^3 = 0.038768\ldots$ is irrational. According to T. Rivoal, among the numbers $\zeta(2n+1)$ with $n \geq 2$, infinitely many are irrational. And W. Zudilin proved that one at least of the four numbers

$$\zeta(5), \zeta(7), \zeta(9), \zeta(11)$$

is irrational. References with more information on this topic are given in the Bourbaki talk [13] by S. Fischler.

A related open question is the arithmetic nature of *Catalan's constant*

$$G = \sum_{n\geq 1} \frac{(-1)^n}{(2n+1)^2} = 0,915\,965\,594\,177\,219\,015\,0\ldots \tag{1.5}$$

Other open questions can be asked on the values of *Euler's Gamma fonction*

$$\Gamma(z) = e^{-\gamma z} z^{-1} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n} = \int_0^{\infty} e^{-t} t^z \cdot \frac{dt}{t}.$$

[4] For the theory of divergent series, including contributions by Euler, Laguerre, Stieltjes, Ramanujan, Poincaré and Padé, see [4] p. 8–9 et [5] p. 227 et 253–254.

As an example we do not know how to prove that the number

$$\Gamma(1/5)\cdots = 4,590\,843\,711\,998\,803\,053\,204\,758\,275\,929\,152\,0\ldots$$

is irrational.

The only rational values of z for which the answer is known (and in fact one knows the transcendence of the Gamma value in these cases) are

$$r \in \left\{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}\right\} \qquad (\text{mod } 1).$$

The number $\Gamma(1/n)$ appears when one computes *periods* of the Fermat curve $X^n + Y^n = Z^n$, and this curve is simpler (in technical terms it has genus ≤ 1) for $n = 2, 3, 4$ and 6. For $n = 5$ the genus is 2 and this is related with the fact that one is not able so far to give the answer for $\Gamma(1/5)$.

The collection of similar open problems is endless. For instance, is the number

$$e + \pi = 5,859\,874\,482\,048\,838\,473\,822\,930\,854\,632\ldots$$

rational or not? The answer is not yet known. And the same is true for any number in the following list (see also § 5.1)

$$\log \pi,\ 2^\pi,\ 2^e,\ \pi^e,\ e^e.$$

1.1.2 History of irrationality

The history of irrationality is closely connected with the history of continued fractions (see[4, 5]). (Even the first examples of transcendental numbers produced by Liouville in 1844 involved continued fractions, before he considered series).

The question of the irrationality of π was raised in India by Nīlakaṇṭha Somayājī, who was born around 1444 AD. In his comments on the work of Āryabhaṭa, (b. 476 AD) who stated that an approximation for π is $\pi \sim 3.1416$, Somayājī asks[5]:

> *Why then has an approximate value been mentioned here leaving behind the actual value? Because it (exact value) cannot be expressed.*

In 1767, H. Lambert [7] found the continued fraction expansion for the tangent function:

$$\tan x = \frac{x|}{|1} - \frac{x^2|}{|3} - \frac{x^2|}{|5} - \frac{x^2|}{|7} - \cdots - \frac{x^2\ \ |}{|2n+1} - \cdots$$

Here is how this irregular continued fraction occurs. Given two functions $A_0(x)$ and $A_1(x)$, deefine inductively

$$A_{n+1}(x) = (2n+1)A_n(x) + x^2 A_{n-1}(x) \quad (n \geq 1)$$

[5] K. Ramasubramanian, *The Notion of Proof in Indian Science*, 13th World Sanskrit Conference, 2006. http://www.iitb.ac.in/campus/diary/2006/august/tday2.htm

and set $u_n(x) = A_n(x)/A_{n-1}(x)$ $(n \geq 1)$. Hence

$$u_n(x) = \frac{A_n(x)}{A_{n-1}(x)} = \frac{x^2 A_n(x)}{(2n+1)A_n(x) - A_{n+1}} = \frac{x^2}{(2n+1) - u_{n+1}(x)}.$$

Therefore, for $k \geq 1$,

$$u_n(x) = \frac{x^2 \ \ |}{|2n+1} - \frac{x^2 \ \ |}{|2n+3} - \frac{x^2 \ \ |}{|2n+5} - \cdots - \frac{x^2 \ \ |}{|2n+2k+1} - u_{n+k+1}(x).$$

The main point is to see that the right hand side has a limit as $n \to \infty$ for a suitable choice of A_0 and A_1, namely

$$A_0(x) = \sin x, \quad A_1(x) = \sin x - x\cos x.$$

For this particular choice the sequence $(A_n)_{n\geq 0}$ is given by the integral formula

$$A_n(x) = \int_0^x tA_{n-1}(t)dt \quad (n \geq 1)$$

and there are sequences of polynomials f_n and g_n in $\mathbb{Z}[x]$ such that

$$A_n(x) = f_n(x)\sin x + g_n(x)\cos x.$$

The proof is similar for

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{x|}{|1} + \frac{x^2|}{|\,3} + \frac{x^2|}{|\,5} + \frac{x^2|}{|\,7} + \cdots + \frac{x^2 \ \ |}{|2n+1} + \cdots$$

involving the inductive relation

$$A_{n+1}(x) = (4n+2)A_n(x) + x^2 A_{n-1}(x) \quad (n \geq 1)$$

and the quotients $-A_n(x)/A_{n-1}(x)$. With the initial values

$$A_0(x) = e^x - 1, \quad A_1(x) = e^x(2-x) - 2 - x$$

the solution is

$$A_n(x) = \frac{x^{2n+1}}{n!}\int_0^1 e^{-tx}t^n(1-t)^{\cdot}dt.$$

Compare with Hermite's formulae in § 2.1. See also for instance [5] as well as [23, 26].

Lambert proved in his paper [7] that for x rational and non–zero, the number $\tan x$ cannot be rational. Since $\tan \pi/4 = 1$ it follows that π is irrational. Then he produced a continued fraction expansion for e^x and deduced that e^r is irrational when r is a non–zero rational number. This is equivalent to the fact that non–zero positive rational numbers have an irrational logarithm. A detailed description of Lambert's proof is given in [11].

Euler gave continued fractions expansions not only for e and e^2:

$$e = [2;\ \overline{1, 2j,\ 1}]_{j\geq 1} = [2;\ 1,\ 2,\ 1,\ 1,\ 4,\ 1,\ 1,\ 6,\ 1\dots], \tag{1.6}$$
$$e^2 = [7;\ \overline{3j-1,\ 1,\ 3j,\ 12j+6}]_{j\geq 1} = [7;\ 2,\ 1,\ 3,\ 18,\ 5,\ 1,\ 6,\ 30,\ 18\dots],$$

but also for $(e+1)/(e-1)$, for $(e^2+1)/(e^2-1)$, for $e^{1/n}$ with $n>1$, for $e^{2/n}$with odd $n>1$ and Hurwitz (1896) for $2e$ and $(e+1)/3$:

$$\frac{e+1}{e-1} = [\overline{2(2j+1)}]_{j\geq 0} = [2;\ 6,\ 10,\ 14\dots],$$
$$\frac{e^2+1}{e^2-1} = [\overline{2j+1}]_{j\geq 0} = [1;\ 3,\ 5,\ 7\dots],$$
$$e^{1/n} = [\overline{1,\ (2j+1)n-1,\ 1}]_{j\geq 0},$$
$$e^{2/n} = [\overline{1,\ (n-1)/2+3jn,\ 6n+12jn,\ (5n-1)/2+3jn,\ 1}]_{j\geq 0},$$
$$2e = [5,\ 2,\ \overline{3,\ 2j,\ 3,\ 1,\ 2j,\ 1}]_{j\geq 1},$$
$$\frac{e+1}{3} =$$
$$[1,\ 4,\ \overline{5,\ 4j-3,\ 1,\ 1,\ 36j-16,\ 1,\ 1,\ 4j-2,\ 1,\ 1,\ 36j-4,\ 1,\ 1,\ 4j-1,\ 1,\ 5,\ 4j,\ 1}]_{j\geq 1}.$$

Hermite proved the irrationality of π and π^2 (see [5] p. 207 and p. 247). Also A.M. Legendre proved, in 1794, by a modification of Lambert's proof, that π^2 is also an irrational number (see [5] p. 14.

There are not so many numbers for which one knows the irrationality but we don't know whether there are algebraic or transcendental. A notable exception is $\zeta(3)$, known to be irrational (Apéry, 1978) and expected to be transcendental.

1.2 Variation on a proof by Fourier (1815)

That e is not quadratic follows from the fact that the continued fraction expansion 1.6 of e, which was known by L. Euler in 1737 [10, 7, 23, 26, 4], is not periodic:

$$e = [2;\ 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]$$

Since this expansion is infinite we deduce that e is irrational. The fact that it is not ultimately periodic implies also that e is not a quadratic irrationality, as shown by Lagrange in 1770 – Euler knew already in 1737 that a number with an ultimately period continued fraction expansion is quadratic (see [10, 6, 21]).

Exercise 1.7. *a) Let b be a positive integer. Give the continued fraction expansion of the number*

$$\frac{-b+\sqrt{b^2+4}}{2}.$$

b) Let a and b be two positive integers. Write a degree 2 polynomial with integer coefficients having a root at the real number whose continued fraction expansion is

$$[0; \overline{a, b}].$$

c) Let a, b and c be positive integers. Write a degree 2 polynomial with integer coefficients having a root at the real number whose continued fraction expansion is

$$[0; \overline{a, b, c}].$$

The following easier and well known proof of the irrationality of e was given by J. Fourier in his course at the École Polytechnique in 1815. Later, in 1872 , C. Hermite proved that e is transcendental, while the work of F. Lindemann a dozen of years later led to a proof of the so-called Hermite–Lindemann Theorem: *for any nonzero algebraic number α the number e^α is transcendental.* However for this first section we study only weaker statements which are very easy to prove. We also show that Fourier's argument can be pushed a little bit further than what is usually done, as pointed out by J. Liouville in 1840.

1.2.1 Irrationality of e

We truncate the exponential series giving the value of e at some point N:

$$N!\, e - \sum_{n=0}^{N} \frac{N!}{n!} = \sum_{k\geq 1} \frac{N!}{(N+k)!}. \tag{1.8}$$

The right hand side of (1.8) is a sum of positive numbers, hence is positive (not zero). From the lower bound (for the binomial coefficient)

$$\frac{(N+k)!}{N!k!} \geq N+1 \quad \text{for } k \geq 1,$$

one deduces

$$\sum_{k\geq 1} \frac{N!}{(N+k)!} \leq \frac{1}{N+1} \sum_{k\geq 1} \frac{1}{k!} = \frac{e-1}{N+1}.$$

Therefore the right hand side of (1.8) tends to 0 when N tends to infinity. In the left hand side, $N!$ and $\sum_{n=0}^{N} N!/n!$ are integers. It follows that $N!e$ is never an integer, hence e is an irrational number.

1.2.2 The number e is not quadratic

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The fact that e is not a rational number implies that for each $m \geq 1$ the number $e^{1/m}$ is not rational. To prove that e^2 for instance is also irrational is not so easy (see the comment on this point in [1]).

The proof below is essentially the one given by J. Liouville in 1840 [17] which is quoted by Ch. Hermite ("ces travaux de l'illustre géomètre").

To prove that e does not satisfy a quadratic relation $ae^2 + be + c$ with a, b and c rational integers, not all zero, requires some new trick. Indeed if we just mimic the same argument we get

$$cN! + \sum_{n=0}^{N} (2^n a + b) \frac{N!}{n!} = -\sum_{k\geq 0} \left(2^{N+1+k} a + b\right) \frac{N!}{(N+1+k)!}.$$

The left hand side is a rational integer, but the right hand side tends to infinity (and not 0) with N, so we draw no conclusion.

Instead of this approach we write the quadratic relation as $ae+b+ce^{-1}=0$. This time it works:

$$bN!+\sum_{n=0}^{N}\left(a+(-1)^n c\right)\frac{N!}{n!}=-\sum_{k\geq 0}\left(a+(-1)^{N+1+k}c\right)\frac{N!}{(N+1+k)!}.$$

Again the left hand side is a rational integer, but now the right hand side tends to 0 when N tends to infinity, which is what we expected. However we need a little more work to conclude: we do not yet get the desired conclusion, we only deduce that both sides vanish. Now let us look more closely to the series in the right hand side. Write the two first terms A_N for $k=0$ and B_N for $k=1$:

$$\sum_{k\geq 0}\left(a+(-1)^{N+1+k}c\right)\frac{N!}{(N+1+k)!}=A_N+B_N+C_N$$

with

$$\begin{aligned}A_N&=\left(a-(-1)^N c\right)\frac{1}{N+1}\\ B_N&=\left(a+(-1)^N c\right)\frac{1}{(N+1)(N+2)}\\ C_N&=\sum_{k\geq 2}\left(a+(-1)^{N+1+k}c\right)\frac{N!}{(N+1+k)!}\end{aligned}$$

The above proof that the sum $A_N+B_N+C_N$ tends to zero as N tends to infinity shows more: each of the three sequences

$$A_N,\quad (N+1)B_N,\quad (N+1)(N+2)C_N$$

tends to 0 as N tends to infinity. Hence, from the fact that the sum $A_N+B_N+C_N$ vanishes for sufficiently large N, it easily follows that for sufficiently large N, each of the three terms A_N, B_N and C_N vanishes, hence $a-(-1)^N c$ and $a+(-1)^N c$ vanish, therefore $a=c=0$, and finally $b=0$.

Exercise 1.9. *Let $(a_n)_{n\geq 0}$ be a bounded sequence of rational integers.*
a) Prove that the following conditions are equivalent:
(i) The number

$$\vartheta_1=\sum_{n\geq 0}\frac{a_n}{n!}$$

is rational.
(ii) There exists $N_0>0$ such that $a_n=0$ for all $n\geq N_0$.
b) Prove that these properties are also equivalent to
(iii) The number

$$\vartheta_2=\sum_{n\geq 0}\frac{a_n 2^n}{n!}$$

is rational.

1.2.3 Irrationality of $e^{\sqrt{2}}$

We follow here a suggestion of D.M. Masser.

The trick here is to prove the stronger statement that $\vartheta = e^{\sqrt{2}} + e^{-\sqrt{2}}$ is an irrational number.

Summing the two series

$$e^{\sqrt{2}} = \sum_{n\geq 0} \frac{2^{n/2}}{n!} \quad \text{and} \quad e^{-\sqrt{2}} = \sum_{n\geq 0} (-1)^n \frac{2^{n/2}}{n!}$$

we deduce

$$\vartheta = 2 \sum_{m\geq 0} \frac{2^m}{(2m)!}.$$

It suffices to use the result of Exercise 1.9 with

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

– or to solve this exercise in this particular case as follows. Let N be a sufficiently large integer. Then

$$\frac{(2N)!}{2^N}\vartheta - 2\sum_{m=0}^{N} \frac{(2N)!}{2^{N-m}(2m)!} = 4\sum_{k\geq 0} \frac{2^k(2N)!}{(2N+2k+2)!}. \tag{1.10}$$

The right hand side of (1.10) is a sum of positive numbers, in particular it is not 0. Moreover the upper bound

$$\frac{(2N)!}{(2N+2k+2)!} \leq \frac{1}{(2N+2)(2k+1)!}$$

shows that the right hand side of (1.10) is bounded by

$$\frac{2}{N+1}\sum_{k\geq 0} \frac{2^k}{(2k+1)!} < \frac{\sqrt{2}e^{\sqrt{2}}}{N+1},$$

hence tends to 0 as N tends to infinity.

It remains to check that the coefficients $(2N)!/2^N$ and $(2N)!/2^{N-m}(2m)!$ $(0 \leq m \leq N)$ which occur in the left hand side of (1.10) are integers. The first one is nothing else than the special case $m = 0$ of the second one. Now for $0 \leq m \leq N$ the quotient

$$\frac{(2N)!}{(2m)!} = (2N)(2N-1)(2N-2)\cdots(2m+2)(2m+1)$$

is the product of $2N-2m$ consecutive integers, $N-m$ of which are even; hence it is a multiple of 2^{N-m}.

The same proof shows that the number $\sqrt{2}(e^{\sqrt{2}} - e^{-\sqrt{2}})$ is also irrational, but the argument does not seem to lead to the conclusion that $e^{\sqrt{2}}$ is not a quadratic number.

1.2.4 The number e^2 is not quadratic

The proof below is the one given by J. Liouville in 1840 [18] . See also [9].

We saw in § 1.2.2 that there was a difficulty to prove that e is not a quadratic number if we were to follow too closely Fourier's initial idea. Considering e^{-1} provided the clue. Now we prove that e^2 is not a quadratic number by truncating the series at carefully selected places. Consider a relation $ae^4+be^2+c=0$ with rational integer coefficients a, b and c. Write $ae^2+b+ce^{-2}=0$. Hence

$$\frac{N!b}{2^{N-1}}+\sum_{n=0}^{N}\left(a+(-1)^n c\right)\frac{N!}{2^{N-n-1}n!}=-\sum_{k\geq 0}\left(a+(-1)^{N+1+k}c\right)\frac{2^k N!}{(N+1+k)!}.$$

Like in § 1.2.2, the right hand side tends to 0 as N tends to infinity, and if the two first terms of the series vanish for some value of N, then we conclude $a=c=0$. What remains to be proved is that the numbers

$$\frac{N!}{2^{N-n-1}n!},\quad (0\leq n\leq N)$$

are integers. For $n=0$ this is the coefficient of b, namely $2^{-N+1}N!$. The fact that these numbers are integers is not true for all values of N, it is not true even for all sufficiently large N; but we do not need so much, it suffices that they are integers for infinitely many N, and that much is true.

The exponent $v_p(N!)$ of p in the prime decomposition of $N!$ is given by the (finite) sum (see for instance [14])

$$v_p(N!)=\sum_{j\geq 1}\left[\frac{N}{p^j}\right]. \tag{1.11}$$

Using the trivial upper bound $[m/p^j]\leq m/p^j$ we deduce the upper bound

$$v_p(n!)\leq\frac{n}{p-1}$$

for all $n\geq 0$. In particular $v_2(n!)\leq n$. On the other hand, when N is a power of p, say $N=p^t$, then (1.11) yields

$$v_p(N!)=p^{t-1}+p^{t-2}+\cdots+p+1=\frac{p^t-1}{p-1}=\frac{N-1}{p-1}.$$

Therefore when N is a power of 2 the number $N!$ is divisible by 2^{N-1} and we have, for $0\leq m\leq N$,

$$v_2(N!/n!)\geq N-n-1,$$

which means that the numbers $N!/2^{N-n-1}n!$ are integers.

1.2.5 The number $e^{\sqrt{3}}$ is irrational

Set $\vartheta = e^{\sqrt{3}} + e^{-\sqrt{3}}$. From the series expansion of the exponential function we derive

$$\frac{(2N)!}{3^{N-1}}\vartheta - 2\sum_{m=0}^{N}\frac{(2N)!}{(2m)!3^{N-m-1}} = 2\sum_{k\geq 0}\frac{3^k(2N)!}{(2N+2k+2)!}.$$

Take N of the form $(3^t+1)/2$ for some sufficiently large integer t. We deduce from (1.11) with $p=3$

$$v_3((2N)!) = \frac{3^t-1}{2} = N-1, \quad v_3((2m)!) \leq m, \quad (0 \leq m \leq N)$$

hence $v_3((2N)!/(2m)!) \geq N-m-1$.

1.2.6 Is-it possible to go further?

The same argument does not seem to yield the irrationality of e^3 (a proof using some particular continued fractions was given by Hurwitz in 1896 - see [4] p. 14–15). The range of applications of this method is limited. The main ideas allowing to go further have been introduced by Charles Hermite. These new ideas are basic for the development of transcendental number theory which we shall discuss in § 2.

1.2.7 A geometrical proof of the irrationality of e

The following proof of the irrationality of e is due to Jonathan Sondow [25]. Start with an interval I_1 of length 1. We are going to construct inductively a sequence of intervals $(I_n)_{n\geq 1}$, where for each n the interval I_n is obtained by splitting I_{n-1} into n intervals of the same length and keeping only one such piece. Hence the length of I_n will be $1/n!$.

In order to have the origin of I_n as

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

we start with $I_1 = [2,3]$. For $n \geq 2$, split I_{n-1} into n intervals and keep the second one: this is I_n. Hence

$$I_1 = \left[1+\frac{1}{1!}\,,\,1+\frac{2}{1!}\right] = [2,3],$$
$$I_2 = \left[1+\frac{1}{1!}+\frac{1}{2!}\,,\,1+\frac{1}{1!}+\frac{2}{2!}\right] = \left[\frac{5}{2!}\,,\,\frac{6}{2!}\right],$$
$$I_3 = \left[1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}\,,\,1+\frac{1}{1!}+\frac{1}{2!}+\frac{2}{3!}\right] = \left[\frac{16}{3!}\,,\,\frac{17}{3!}\right].$$

The origin of I_n is

$$1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} = \frac{a_n}{n!},$$

the length is $1/n!$, hence the endpoint of I_n is $(a_n + 1)/n!$. Also for $n \geq 1$ we have $a_{n+1} = (n+1)a_n + 1$.

The number e is the intersection of all these intervals[6], hence it lies in the interior of each I_n, and therefore it cannot be written as $a/n!$ with $a \in \mathbb{Z}$.

Since

$$\frac{p}{q} = \frac{(q-1)!\,p}{q!},$$

the irrationality of e follows.

As pointed out by Sondow in [25], the proof shows that for any integer $n > 1$,

$$\frac{1}{(n+1)!} < \min_{m \in \mathbb{Z}} \left| e - \frac{m}{n!} \right| < \frac{1}{n!}.$$

The *Smarandache function* is defined as follows: $S(q)$ is the least positive integer such that $S(q)!$ is a multiple of q:

$$S(1) = 1,\ S(2) = 2,\ S(3) = 3,\ S(4) = 4,\ S(5) = 5,\ S(6) = 3 \ldots$$

Hence $S(n) \leq n$ or all $n \geq 1$, $S(p) = p$ for p prime and $S(n!) = n$. From his proof Sondow [25] deduces an irrationality measure for e: *for any* $p/q \in \mathbb{Q}$ *with* $q \geq 2$,

$$\left| e - \frac{p}{q} \right| > \frac{1}{(S(q)+1)!}.$$

1.3 Irrationality Criteria

The main tool in Diophantine approximation is the basic property that *any non-zero integer has absolute value at least* 1. There are many corollaries of this fact. The first one we consider here is the following:
If ϑ *is a rational number, there is a positive constant* $c = c(\vartheta)$ *such that, for any rational number* p/q *with* $p/q \neq \vartheta$,

$$\left| \vartheta - \frac{p}{q} \right| \geq \frac{c}{q}. \tag{1.12}$$

This result is obvious: if $\vartheta = a/b$ then an admissible value for c is $1/b$, because the non-zero integer $aq - bp$ has absolute value at least 1.

This property is characteristic of rational numbers: a rational number cannot be well approximated by other rational numbers, while an irrational number can be well approximated by rational numbers.

We now give several such criteria. The first one was used implicitly in § 1.2.

[6] [25]; a more detailed proof that the intersection of the intervals I_n is e is given in Editor's Endnotes, Amer. Math. Monthly **114** (2007), 659.

1.3.1 First criterion

Lemma 1.13. *Let ϑ be a real number. The following conditions are equivalent*
(i) *ϑ is irrational.*
(ii) *For any $\epsilon > 0$ there exists $p/q \in \mathbb{Q}$ such that*

$$0 < \left|\vartheta - \frac{p}{q}\right| < \frac{\epsilon}{q}.$$

(iii) *For any real number $Q > 1$ there exists an integer q in the range $1 \leq q < Q$ and a rational integer p such that*

$$0 < \left|\vartheta - \frac{p}{q}\right| < \frac{1}{qQ}.$$

(iv) *There exist infinitely many $p/q \in \mathbb{Q}$ such that*

$$0 < \left|\vartheta - \frac{p}{q}\right| < \frac{1}{q^2}.$$

So far we needed only (ii)⇒(i), which is the easiest part, as we just checked in (1.12).

According to this implication, in order to prove that some number is irrational, it is sufficient (and in fact also necessary) to produce good rational approximations. Lemma 1.13 tells us that an irrational real number ϑ has very good *friends* among the rational numbers, the sharp inequality (iv) shows indeed that ϑ is well approximated by rational numbers (and a sharper version of (iv) will be proved in Lemma 1.18 below). Conversely, the proof we just gave shows that a rational number has *no good friend*, apart from himself. Hence in this world of rational approximation it suffices to have one good friend (not counting oneself) to guarantee that one has many very good friends.

1.3.2 Proof of Dirichlet's Theorem

The implications (iii)⇒(iv)⇒(ii)⇒(i) in Lemma 1.13 are easy. It only remains to prove (i)⇒(iii), which is a Theorem due to Dirichlet. For this we shall use the *box* or *pigeon hole* principle.

Proof of (i)⇒(iii). Let $Q > 1$ be given. Define $N = \lceil Q \rceil$: this means that N is the integer such that $N - 1 < Q \leq N$. Since $Q > 1$, we have $N \geq 2$.

For $x \in \mathbb{R}$ write $x = [x] + \{x\}$ with $[x] \in \mathbb{Z}$ (integral part of x) and $0 \leq \{x\} < 1$ (fractional part of x). Let $\vartheta \in \mathbb{R} \setminus \mathbb{Q}$. Consider the subset E of the unit interval $[0, 1]$ which consists of the $N + 1$ elements

$$0,\ \{\vartheta\},\ \{2\vartheta\},\ \{3\vartheta\},\ \ldots, \{(N-1)\vartheta\},\ 1.$$

Since ϑ is irrational, these $N+1$ elements are pairwise distinct. Split the interval $[0,1]$ into N intervals

$$I_j = \left[\frac{j}{N}\,,\,\frac{j+1}{N}\right] \quad (0 \le j \le N-1).$$

One at least of these N intervals, say I_{j_0}, contains at least two elements of E. Apart from 0 and 1, all elements $\{q\vartheta\}$ in E with $1 \le q \le N-1$ are irrational, hence belong to the union of the *open* intervals $(j/N,\ (j+1)/N)$ with $0 \le j \le N-1$.

If $j_0 = N-1$, then the interval

$$I_{j_0} = I_{N-1} = \left[1 - \frac{1}{N}\ ;\ 1\right]$$

contains 1 as well as another element of E of the form $\{q\vartheta\}$ with $1 \le q \le N-1$. Set $p = [q\vartheta]+1$. Then we have $1 \le q \le N-1 < Q$ and

$$p - q\vartheta = [q\vartheta] + 1 - [q\vartheta] - \{q\vartheta\} = 1 - \{q\vartheta\}, \quad \text{hence} \quad 0 < p - q\vartheta < \frac{1}{N} \le \frac{1}{Q}.$$

Otherwise we have $0 \le j_0 \le N-2$ and I_{j_0} contains two elements $\{q_1\vartheta\}$ and $\{q_2\vartheta\}$ with $0 \le q_1 < q_2 \le N-1$. Set

$$q = q_2 - q_1, \quad p = [q_2\vartheta] - [q_1\vartheta].$$

Then we have $0 < q = q_2 - q_1 \le N-1 < Q$ and

$$|q\vartheta - p| = |\{q_2\vartheta\} - \{q_1\vartheta\}| < 1/N \le 1/Q.$$

□

There are other proofs of (i)⇒(iii) – for instance one can use Minkowski's Theorem in the geometry of numbers, which is more powerful than Dirichlet's box principle. We shall come back to this point in section § 1.3.5.

Exercise 1.14. *This exercise extends the irrationality criterion Lemma 1.13 by replacing $\mathbb{Q}$ by $\mathbb{Q}(i)$. The elements in $\mathbb{Q}(i)$ are called the Gaussian numbers, the elements in $\mathbb{Z}(i)$ are called the Gaussian integers. The elements of $\mathbb{Q}(i)$ will be written p/q with $p \in \mathbb{Z}[i]$ and $q \in \mathbb{Z}$, $q > 0$.*

Let ϑ be a complex number. Check that the following conditions are equivalent.

(i) $\vartheta \notin \mathbb{Q}(i)$.

(ii) *For any $\epsilon > 0$ there exists $p/q \in \mathbb{Q}(i)$ such that*

$$0 < \left|\vartheta - \frac{p}{q}\right| < \frac{\epsilon}{q}.$$

(iii) *For any rational integer $N \geq 1$ there exists a rational integer q in the range $1 \leq q \leq N^2$ and a Gaussian integer p such that*

$$0 < \left|\vartheta - \frac{p}{q}\right| < \frac{\sqrt{2}}{qN}.$$

(iv) *There exist infinitely many Gaussian numbers $p/q \in \mathbb{Q}(i)$ such that*

$$\left|\vartheta - \frac{p}{q}\right| < \frac{\sqrt{2}}{q^{3/2}}.$$

The implication (ii)⇒(i) in Lemma 1.13 was used implicitely in § 1.1. We give here another illustration of this irrationality criterion to series studied by Liouville and Fredholm.

Several methods are available to investigate the arithmetic nature of numbers of the form

$$\sum_{n\geq 0} g^{-n^2} \quad \text{and} \quad \sum_{n\geq 0} g^{-2^n} \tag{1.15}$$

where g is a positive integer.

There is apparently a confusion in the litterature between these two series. The name *Fredholm series* is often wrongly attributed to the power series

$$\sum_{n\geq 0} z^{2^n}.$$

However Fredholm studied rather the series

$$\sum_{n\geq 0} z^{n^2}$$

(see the book [2] by Allouche & Shallit, Notes on chapter 13, page 403 as well as Shallit's paper [22]).

The series $\sum_{n\geq 0} z^{n^2}$ was explicitly quoted by Liouville (see for instance [12]). We shall come back to this question in section § 3.3.6 below (where we discuss Nesterenko's result in 1995 according to which this number is transcendental). Right now we only prove the irrationality of the numbers (1.15) for $a \in \mathbb{Z}$, $a \geq 2$ by means of Lemma 1.13. More generally we replace the sequences $(n^2)_{n\geq 0}$ and $(2^n)_{n\geq 0}$ by more general ones: one requires that they grow and tend to infinity sufficiently fast.

Lemma 1.16. *Let $g \geq 2$ be an integer, $(a_n)_{n\geq 0}$ a bounded sequence of rational integers and $(u_n)_{n\geq 0}$ an increasing sequence of positive integers. Assume there exists $c > 0$ and $n_0 \geq 0$ such that, for all $n \geq n_0$,*

$$u_{n+1} - u_n \geq cn.$$

Then the number

$$\vartheta = \sum_{n\geq 0} a_n g^{-u_n}$$

is irrational if and only if the support $\{n \geq 0\ ;\ a_n \neq 0\}$ *of the sequence* $(a_n)_{n\geq 0}$ *is infinite.*

Notice that Lemma 1.4 gives the result in the special case where the coefficients a_n satisfy $0 \leq a_n < g$.

Proof. Obviously if the support of the sequence $(a_n)_{n\geq 0}$ is finite then the number ϑ is rational. Assume the support is infinite and let N be a sufficiently large integer with $a_{N+1} \neq 0$. Let A be an upper bound for $|a_n|$. Set $q_N = g^{u_N}$,

$$p_N = \sum_{n=0}^{N} a_n g^{u_N - u_n}, \quad r_N = a_{N+1} g^{u_N - u_{N+1}} \quad \text{and} \quad R_N = \sum_{k=2}^{\infty} a_{N+k} g^{u_N - u_{N+k}},$$

so that

$$q_N \vartheta - p_N - r_N = R_N.$$

Then p_N and q_N are rational integers. By induction on $k \geq 1$ one checks

$$u_{N+k} \geq u_N + ckN + v_k \quad \text{where} \quad v_k := c\frac{k(k-1)}{2}.$$

Therefore, for sufficiently large N,

$$|R_N| < g^{u_N - u_{N+1}} \leq |r_N] \leq A g^{u_N - u_{N+1}}.$$

It follows that $q_N\vartheta - p_N$ does not vanish and tends to 0 as N tends to infinity. Lemma 1.13 shows that ϑ is irrational. □

1.3.3 Irrationality of at least one number

Lemma 1.13 is a criterion for irrationality of one number, we extend it to a criterion for the irrationality of at least one number in a given set. There are far reaching generalizations (especially due to Yu. V. Nesterenko) of such results to quantitative statements, yielding irrationality measures or even measures of linear independence.

Lemma 1.17. *Let* $\vartheta_1, \ldots, \vartheta_m$ *be real numbers. The following conditions are equivalent*
(i) *One at least of* $\vartheta_1, \ldots, \vartheta_m$ *is irrational.*
(ii) *For any* $\epsilon > 0$ *there exist* $p_1, \ldots, p_m, q$ *in* $\mathbb{Z}$ *with* $q > 0$ *such that*

$$0 < \max_{1\leq i\leq m} \left|\vartheta_i - \frac{p_i}{q}\right| < \frac{\epsilon}{q}.$$

(iii) *For any integer* $Q > 1$ *there exists* $p_1, \ldots, p_m, q$ *in* $\mathbb{Z}$ *such that* $1 \leq q \leq Q^m$ *and*

$$0 < \max_{1\leq i\leq m} \left|\vartheta_i - \frac{p_i}{q}\right| \leq \frac{1}{qQ}.$$

(iv) *There is an infinite set of $q \in \mathbb{Z}$, $q > 0$, for which there there exist $p_1, \dots, p_m$ in $\mathbb{Z}$ satisfying*

$$0 < \max_{1 \le i \le m} \left| \vartheta_i - \frac{p_i}{q} \right| < \frac{1}{q^{1+1/m}}.$$

Proof. The proofs of (iii)⇒(iv)⇒(ii)⇒(i) are easy.

For (i)⇒(iii) we use Dirichlet's box principle like in the proof of Lemma 1.13. Consider the $Q^m + 1$ elements

$$\xi_q = \big(\{q\vartheta_1\}, \dots, \{q\vartheta_m\}\big) \quad (q = 0, 1, \dots, Q^m)$$

in the unit cube $[0, 1)^m$ of $\mathbb{R}^m$. Split this unit cube into Q^m cubes having sides of lengths $1/Q$. One at least of these small cubes contains at least two ξ_q, say ξ_{q_1} and ξ_{q_2}, with $0 \le q_2 < q_1 \le Q^m$. Set $q = q_1 - q_2$ and take for p_i the nearest integer to ϑ_i, $1 \le i \le m$. This completes the proof of Lemma 1.17.

□

An alternative arguments relies on geometry of numbers - see section § 1.3.5 and W.M. Schmidt's Lecture Notes [21] Chap. II, § 1 - it follows that it is not necessary to assume Q to be an integer, and the strict inequality $q < Q^m$ can be achieved.

1.3.4 Hurwitz Theorem

The following result improves the implication (i)⇒(iv) of Lemma 1.13.

Lemma 1.18. *Let ϑ be a real number. The following conditions are equivalent*
(i) *ϑ is irrational.*
(ii) *There exist infinitely many $p/q \in \mathbb{Q}$ such that*

$$0 < \left| \vartheta - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}.$$

Of course the implication (ii)⇒(i) in Lemma 1.18 is weaker than the implication (iv)⇒(i) in Lemma 1.13. What is new is the converse.

Classical proofs of the equivalence between (i) and (ii) in Lemma 1.18 involve either continued fractions or Farey series. We give here a proof which does not involve continued fractions, but they occur implicitly.

Lemma 1.19. *Let ϑ be a real irrational number. Then there exists infinitely many pairs $(p/q, r/s)$ of irreducible fractions such that*

$$\frac{p}{q} < \vartheta < \frac{r}{s} \quad \textit{and} \quad qr - ps = 1.$$

In this statement and the next ones it is sufficient to prove inequalities $\le$ in place of $<$: the strict inequalities are plain from the irrationality of ϑ.

Proof. Let H be a positive integer. Among the irreducible rational fractions a/b with $1 \leq b \leq H$, select one for which $|\vartheta - a/b|$ is minimal. If $a/b < \vartheta$ rename a/b as p/q, while if $a/b > \vartheta$, then rename a/b as r/s.

First consider the case where $a/b < \vartheta$, hence $a/b = p/q$. Since $\gcd(p, q) = 1$, using Euclidean's algorithm, one deduces (Bézout's Theorem) that there exist $(r, s) \in \mathbb{Z}^2$ such that $qr - sp = 1$ with $1 \leq s < q$ and $|r| < |p|$. Since $1 \leq s < q \leq H$, from the choice of a/b it follows that

$$\left|\vartheta - \frac{p}{q}\right| \leq \left|\vartheta - \frac{r}{s}\right|$$

hence r/s does not belong to the interval $[p/q, \vartheta]$. Since $qr - sp > 0$ we also have $p/q < r/s$, hence $\vartheta < r/s$.

In the second case where $a/b > \vartheta$ and $r/s = a/b$ we solve $qr - sp = 1$ by Euclidean algorithm with $1 \leq q < s$ and $|p| < r$, and the argument is similar.

We now complete the proof of infinitely many such pairs. Once we have a finite set of such pairs $(p/q, r/s)$, we use the fact that there is a rational number m/n closer to ϑ than any of these rational fractions. We use the previous argument with $H \geq n$. This way we produce a new pair $(p/q, r/s)$ of rational numbers which is none of the previous ones (because one at least of the two rational numbers p/q, r/s is a better approximation than the previous ones). Hence this construction yields infinitely many pairs, as claimed.

□

Lemma 1.20. *Let ϑ be a real irrational number. Assume $(p/q, r/s)$ are irreducible fractions such that*

$$\frac{p}{q} < \vartheta < \frac{r}{s} \quad \textit{and} \quad qr - ps = 1.$$

Then

$$\min\left\{q^2\left(\vartheta - \frac{p}{q}\right)\ ,\ s^2\left(\frac{r}{s} - \vartheta\right)\right\} < \frac{1}{2}.$$

Proof. Define

$$\delta = \min\left\{q^2\left(\vartheta - \frac{p}{q}\right)\ ,\ s^2\left(\frac{r}{s} - \vartheta\right)\right\}.$$

From

$$\frac{\delta}{q^2} \leq \vartheta - \frac{p}{q} \quad \text{and} \quad \frac{\delta}{s^2} \leq \frac{r}{s} - \vartheta$$

with $qr - ps = 1$ one deduces that the number $t = s/q$ satisfies

$$t + \frac{1}{t} \leq \frac{1}{\delta}.$$

Since the minimum of the function $t \mapsto t + 1/t$ is 2 and since $t \neq 1$, we deduce $\delta < 1/2$.

□

Remark. *The inequality $t + (1/t) \geq 2$ for all $t > 0$ with equality if and only if $t = 1$ is equivalent to the arithmetico-geometric inequality*

$$\sqrt{xy} \leq \frac{x+y}{2},$$

when x and y are positive real numbers, with equality if and only if $x = y$. The correspondance between both estimates is $t = \sqrt{x/y}$.

From Lemmas 1.19 and 1.20 it follows that for $\vartheta \in \mathbb{R}\backslash\mathbb{Q}$, there exist infinitely many $p/q \in \mathbb{Q}$ such that

$$0 < \left|\vartheta - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

A further step is required in order to complete the proof of Lemma 1.18.

Lemma 1.21. *Let ϑ be a real irrational number. Assume $(p/q, r/s)$ are irreducible fractions such that*

$$\frac{p}{q} < \vartheta < \frac{r}{s} \quad \textit{and} \quad qr - ps = 1.$$

Define $u = p + r$ and $v = q + s$. Then

$$\min\left\{q^2\left(\vartheta - \frac{p}{q}\right)\ ,\ s^2\left(\frac{r}{s} - \vartheta\right)\ ,\ v^2\left|\vartheta - \frac{u}{v}\right|\right\} < \frac{1}{\sqrt{5}}.$$

Proof. First notice that $qu - pv = 1$ and $rv - su = 1$. Hence

$$\frac{p}{q} < \frac{u}{v} < \frac{r}{s}.$$

We repeat the proof of Lemma 1.20 ; we distinguish two cases according to whether u/v is larger or smaller than ϑ. Since both cases are quite similar, let us assume $\vartheta < u/v$. The proof of Lemma 1.20 shows that

$$\frac{s}{q} + \frac{q}{s} \leq \frac{1}{\delta} \quad \text{and} \quad \frac{v}{q} + \frac{q}{v} \leq \frac{1}{\delta}.$$

Hence each of the four numbers s/q, q/s, v/q, q/v satisfies $t + 1/t \leq 1/\delta$. Now the function $t \mapsto t + 1/t$ is decreasing on the interval $(0, 1)$ and increasing on the interval $(1, +\infty)$. It follows that our four numbers all lie in the interval $(1/x, x)$, where x is the root > 1 of the equation $x + 1/x = 1/\delta$. The two roots x and $1/x$ of the quadratic polynomial $X^2 - (1/\delta)X + 1$ are at a mutual distance equal to the square root of the discriminant $\Delta = (1/\delta)^2 - 4$ of this polynomial. Now

$$\frac{v}{q} - \frac{s}{q} = 1,$$

hence the length $\sqrt{\Delta}$ of the interval $(1/x, x)$ is ≥ 1 and therefore $\delta \leq 1/\sqrt{5}$. This completes the proof of Lemma 1.21. □

We now show that Lemma 1.18 is optimal.

Denote again by $\Phi = 1.6180339887499\ldots$ the Golden ratio, which is the root > 1 of the polynomial $X^2 - X - 1$. The discriminant of this polynomial is 5. Recall also the definition of the Fibonacci sequence $(F_n)_{n\geq 0}$:

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \quad (n \geq 2).$$

Lemma 1.22. *For any $q \geq 1$ and any $p \in \mathbb{Z}$,*

$$\left|\Phi - \frac{p}{q}\right| > \frac{1}{\sqrt{5}q^2 + (q/2)}.$$

On the other hand

$$\lim_{n\to\infty} F_{n-1}^2 \left|\Phi - \frac{F_n}{F_{n-1}}\right| = \frac{1}{\sqrt{5}}.$$

Proof. It suffices to prove the lower bound when p is the nearest integer to $q\Phi$. From $X^2 - X - 1 = (X - \Phi)(X + \Phi^{-1})$ we deduce

$$p^2 - pq - q^2 = q^2 \left(\frac{p}{q} - \Phi\right)\left(\frac{p}{q} + \Phi^{-1}\right).$$

The left hand side is a non-zero rational integer, hence has absolute value at least 1. We now bound the absolute value of the right hand side from above. Since $p < q\Phi + (1/2)$ and $\Phi + \Phi^{-1} = \sqrt{5}$ we have

$$\frac{p}{q} + \Phi^{-1} \leq \sqrt{5} + \frac{1}{2q}.$$

Hence

$$1 \leq q^2 \left|\frac{p}{q} - \Phi\right| \left(\sqrt{5} + \frac{1}{2q}\right)$$

The first part of Lemma 1.22 follows.

The real vector space of sequences $(v_n)_{n\geq 0}$ satisfying $v_n = v_{n-1} + v_{n-2}$ has dimension 2, a basis is given by the two sequences $(\Phi^n)_{n\geq 0}$ and $((-\Phi^{-1})^n)_{n\geq 0}$. From this one easily deduces the formula

$$F_n = \frac{1}{\sqrt{5}}(\Phi^n - (-1)^n \Phi^{-n})$$

due to A. De Moivre (1730), L. Euler (1765) and J.P.M. Binet (1843). It follows that F_n is the nearest integer to

$$\frac{1}{\sqrt{5}}\Phi^n,$$

hence the sequence $(u_n)_{n\geq 2}$ of quotients of Fibonacci numbers

$$u_n = F_n/F_{n-1}$$

satisfies $\lim_{n\to\infty} u_n = \Phi$.

By induction one easily checks

$$F_n^2 - F_n F_{n-1} - F_{n-1}^2 = (-1)^{n-1}$$

for $n \geq 1$. The left hand side is $F_{n-1}^2(u_n - \Phi)(u_n + \Phi^{-1})$, as we already saw. Hence

$$F_{n-1}^2|\Phi - u_n| = \frac{1}{\Phi^{-1} + u_n},$$

and the limit of the right hand side is $1/(\Phi + \Phi^{-1}) = 1/\sqrt{5}$. The result follows. □

Remark. *The sequence $u_n = F_n/F_{n-1}$ is also defined by*

$$u_2 = 2,\ u_n = 1 + \frac{1}{u_{n-1}},\quad (n \geq 3).$$

Hence

$$u_n = 1 + \frac{1}{1 + \dfrac{1}{u_{n-2}}} = 1 + \frac{1}{1 + \dfrac{1}{1 + \dfrac{1}{u_{n-3}}}} = \cdots$$

Exercise 1.23. *Set $G_0 = 0$, $G_1 = 1$, and by induction define $G_n = 2G_{n-1} + G_{n-2}$ for $n \geq 2$.*
a) Check, for all $n \geq 1$,

$$G_n^2 - 2G_n G_{n-1} - G_{n-1}^2 = (-1)^{n-1}.$$

b) Show that the sequence $(G_n/G_{n-1})_{n\geq 2}$ converges for $n \to \infty$. What is the limit?
c) Show that there is a sequence $(p_n/q_n)_{n\geq 1}$ of rational numbers such that

$$\lim_{n\to\infty} q_n \left| q_n\sqrt{2} - p_n \right| = \frac{1}{2\sqrt{2}}.$$

d) Show that for any $\kappa > 2\sqrt{2}$, there are only finitely many rational numbers $p/q \in \mathbb{Q}$ satisfying

$$\left| \sqrt{2} - \frac{p}{q} \right| \leq \frac{1}{\kappa q^2}.$$

e) Let Δ be a positive real number. Give an example of a homogeneous quadratic polynomial $f(X,Y) = aX^2 + bXY + cY^2$ of degree 2 with discriminant $b^2 - 4ac = \Delta$ such that

$$\min\big\{|f(x,y)|\ ;\ (x,y) \in \mathbb{Z}\times\mathbb{Z},\ (x,y) \neq (0,0)\big\} = \sqrt{\Delta/5}$$

and give another example with

$$\min\big\{|f(x,y)|\ ;\ (x,y) \in \mathbb{Z}\times\mathbb{Z},\ (x,y) \neq (0,0)\big\} = \sqrt{\Delta/8}.$$

Remark. *It is known (see for instance [21] p. 25) that if k is a positive integer, if an irrational real number ϑ has a continued fraction expansion $[a_0; a_1, a_2, \dots]$ with $a_n \geq k$ for infinitely many n, then*

$$\liminf_{q\to\infty} q^2 \left|\vartheta - \frac{p}{q}\right| \leq \frac{1}{\sqrt{4+k^2}}.$$

This proof of Lemma 1.22 can be extended by replacing $X^2 - X - 1$ by any irreducible polynomial with integer coefficients. Recall that the ring $\mathbb{Z}[X]$ is factorial, its irreducible elements of positive degree are the non-constant polynomials with integer coefficients which are irreducible in $\mathbb{Q}[X]$ (i.e. not a product of two non-constant polynomials in $\mathbb{Q}[X]$) and have content 1. The *content* of a polynomial in $\mathbb{Z}[X]$ is the greatest common divisor of its coefficients.

The *minimal polynomial* of an algebraic number α is the unique irreducible polynomial $P \in \mathbb{Z}[X]$ which vanishes at α and has a positive leading coefficient.

The next lemma ([21] p. 6 Lemma 2E) is a variant of Liouville's inequality that we shall study more throughly in § 2.3.1.

Lemma 1.24. *Let α be a real algebraic number of degree $d \geq 2$ and minimal polynomial $P \in \mathbb{Z}[X]$. Define $c = |P'(\alpha)|$. Let $\epsilon > 0$. Then there exists an integer q_0 such that, for any $p/q \in \mathbb{Q}$ with $q \geq q_0$,*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{(c+\epsilon)q^d}.$$

Proof. Let q be a sufficiently large positive integer and let p be the nearest integer to $q\alpha$. In particular

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{2}.$$

Denote a_0 the leading coefficient of P and by $\alpha_1, \dots, \alpha_d$ its the roots with $\alpha_1 = \alpha$. Hence

$$P(X) = a_0(X-\alpha_1)(X-\alpha_2)\cdots(X-\alpha_d)$$

and

$$q^d P(p/q) = a_0 q^d \prod_{i=1}^{d} \left(\frac{p}{q} - \alpha_i\right). \tag{1.25}$$

Also

$$P'(\alpha) = a_0 \prod_{i=2}^{d} (\alpha - \alpha_i).$$

The left hand side of (1.25) is a rational integer. It is not zero because P is irreducible of degree ≥ 2. For $i \geq 2$ we use the estimate

$$\left|\alpha_i - \frac{p}{q}\right| \leq |\alpha_i - \alpha| + \frac{1}{2q}.$$

We deduce

$$1 \leq q^d a_0 \left|\alpha - \frac{p}{q}\right| \prod_{i=2}^{d} \left(|\alpha_i - \alpha| + \frac{1}{2q}\right).$$

For sufficiently large q the right hand side is bounded from above by

$$q^d \left|\alpha - \frac{p}{q}\right| \left(|P'(\alpha)| + \epsilon\right).$$

□

If α is a real root of a quadratic polynomial $P(X) = aX^2 + bX + c$, then $P'(\alpha) = 2a\alpha + b$ is a square root of the discriminant of P. So Hurwitz Lemma 1.18 is optimal for all quadratic numbers having a minimal polynomial of discriminant 5. Incidentally, this shows that 5 is the smallest positive discriminant of an irreducible quadratic polynomial in $\mathbb{Z}[X]$ (of course it is easily checked directly that if a, b, c are three rational integers with $a > 0$ and $b^2 - 4ac$ positive and not a perfect square in $\mathbb{Z}$, then $b^2 - 4ac \geq 5$).

It follows that for the numbers of the form $(a\Phi + b)/(c\Phi + d)$ with integers a, b, c, d having $ad - bc = \pm 1$, one cannot replace in Lemma 1.18 the number $\sqrt{5}$ by a larger number.

If one omits these irrational numbers in the field generated by the Golden ratio, then Hurwitz showed that one can replace $\sqrt{5}$ by $2\sqrt{2}$, and again this is optimal. This is the beginning of the so-called *Markoff* [7] *spectrum* $\sqrt{5}$, $\sqrt{8}$, $\sqrt{221}/5$, $\sqrt{1517}/13$, ... which tends to $1/3$ and is obtained as follows. First consider the set of integers m for which the *Markoff equation*

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2$$

has a solution in positive integers (m_1, m_2) with $0 < m_1 \leq m_2 \leq m$. The infinite increasing sequence of these integers m starts with

$$1,\ 2,\ 5,\ 13,\ 29,\ 34,\ 89,\ 169,\ 194,\ 233,\ 433,\ 610,\ 985,\ 1325,\ 1597, \ldots \quad (1.26)$$

and there is an easy and well known algorithm to construct it (see for instance [8] Chap. 7 and [27]): apart from $(1, 1, 1)$ and $(2, 1, 1)$, for any solution (m, m_1, m_2) there are three exactly solutions sharing two components with (m, m_1, m_2), namely

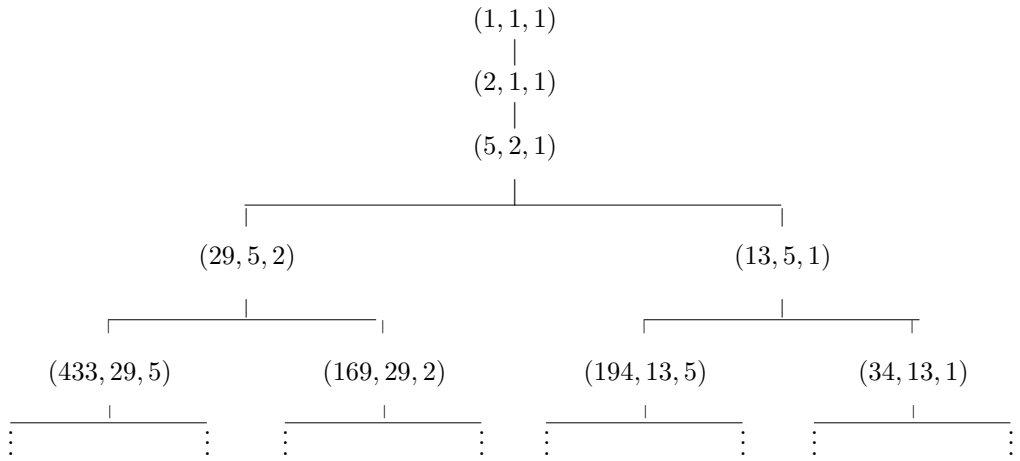
$$(m', m_1, m_2), \quad (m, m_1', m_2), \quad (m, m_1, m_2'),$$

where

$$m' = 3m_1m_2 - m, \quad m_1' = 3mm_2 - m_1, \quad m_2' = 3mm_1 - m_2.$$

[7] His name is spelled *Markov* in probability theory.

This produces the *Markoff tree*



For each m in the Markoff sequence (1.26), we define

$$\mu_m = \frac{\sqrt{9m^2 - 4}}{m}.$$

Then there is an explicit quadratic form $f_m(x, y)$ such that $f_m(x, 1) = 0$ and there is a root α_m of f_m for which

$$\limsup_{q \in \mathbb{Z},\ q \to \infty} (q \|q\alpha_m\|) = \frac{1}{\mu_m},$$

where $\|\cdot\|$ denotes the distance too the nearest integer:

$$\|x\| = \min_{m \in \mathbb{Z}} |x - m| = \min\big\{\{x\}\,;\, 1 - \{x\}\big\}.$$

The sequence of $(m, f_m, \alpha_m, \mu_m)$ starts as follows,

m	1	2	5	13
$f_m(x, 1)$	$x^2 + x - 1$	$x^2 + 2x - 1$	$5x^2 + 11x - 5$	$13x^2 + 29x - 13$
α_m	$[0; \overline{1}]$	$[0; \overline{2}]$	$[0; \overline{2211}]$	$[0; \overline{221111}]$
μ_m	$\sqrt{5}$	$\sqrt{8}$	$\sqrt{221}/5$	$\sqrt{1517}/13$

The third row gives the continued fraction expansion for α_m.

Exercise 1.27. *Check that any solution (m, m_1, m_2) of Markoff's equation (1.26) is in Markoff's tree.*

We conclude this section with a further irrationality criterion related to Lemmas 1.20 and 1.21.

Lemma 1.28. *Let ϑ be a real number. The following conditions are equivalent*
(i) *ϑ is irrational.*
(ii) *For any $\epsilon > 0$ there exists p/q and r/s in $\mathbb{Q}$ such that*

$$\frac{p}{q} < \vartheta < \frac{r}{s}, \quad qr - ps = 1$$

and

$$\max\{q\vartheta - p\ ;\ r - s\vartheta\} < \epsilon.$$

(iii) *There exist infinitely many pairs $(p/q, r/s)$ of rational numbers such that*

$$\frac{p}{q} < \vartheta < \frac{r}{s}, \quad qr - ps = 1$$

and

$$\max\{q(q\vartheta - p)\ ;\ s(r - s\vartheta)\} < 1.$$

Proof. The implications (iii)⇒(ii)⇒(i) are easy. For (i)⇒(iii) we use the arguments in the proof of Lemma 1.19, but we use also an auxiliary result from the theory of continued fractions.

Since ϑ is irrational, Hurwitz Lemma 1.18 shows that there are infinitely many p/q such that

$$\left|\vartheta - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

We shall use the fact that such a p/q is a so-called *best approximation to ϑ*: this means that for any $a/b \in \mathbb{Q}$ with $1 \le b \le q$ and $a/b \ne p/q$, we have

$$\left|\vartheta - \frac{a}{b}\right| > \left|\vartheta - \frac{p}{q}\right|.$$

Assume first $p/q < \vartheta$. Let r/s be defined by $qr - ps = 1$ and $1 \le s < q$, $|r| < |p|$. We have

$$0 < \frac{r}{s} - \vartheta < \frac{r}{s} - \frac{p}{q} = \frac{1}{qs} \le \frac{1}{s^2}.$$

Next assume $p/q > \vartheta$. In this case rename it r/s and define p/q by $qr - ps = 1$ and $1 \le q < s$, $|p| < |r|$.

Finally repeat the argument in the proof of Lemma 1.19 to get an infinite set of approximations. Lemma 1.28 follows. □

1.3.5 A criterion for linear independence

We first state a criterion for linear independence which will be used in § 2 for the proof by Hermite of the transcendence of e. This is a generalisation (from personal notes of Michel Laurent after a course he gave in Marseille) of one of Lemma 1.28. Most often in mathematics there is sort of an entropy: when a statement provides a necessary and sufficient condition, and when one of the two implication is easy while the other requires more work, then it is the difficult part which is most useful. Here we have a counterexample to this claim (which does not belong to mathematics but rather to social science): in the Criterion 1.29 below, one of the implications is easy while the other is deeper; but it turns out that it is the easy one which is required in transcendence proofs.

Let $\vartheta_1, \dots, \vartheta_m$ be real numbers and a_0, $a_1, \dots, a_m$ rational integers, not all of which are 0. Our goal is to prove that the number

$$L = a_0 + a_1\vartheta_1 + \cdots + a_m\vartheta_m$$

is not 0.

The idea is to approximate simultaneously $\vartheta_1, \dots, \vartheta_m$ by rational numbers $p_1/q, \dots, p_m/q$ with the same denominator $q > 0$.

Let $q, p_1, \dots, p_m$ be rational integers with $q > 0$. For $1 \le k \le m$ set

$$\epsilon_k = q\vartheta_k - p_k.$$

Then $qL = M + R$ with

$$M = a_0q + a_1p_1 + \cdots + a_mp_m \in \mathbb{Z} \quad \text{and} \quad R = a_1\epsilon_1 + \cdots + a_m\epsilon_m \in \mathbb{R}.$$

If $M \neq 0$ and $|R| < 1$ we deduce $L \neq 0$.

One of the main difficulties is often to check $M \neq 0$. This question gives rise to the so-called *zero estimates* or *non-vanishing lemmas*. In the present situation, we wish to find a $m+1$–tuple $(q, p_1, \dots, p_m)$ giving a simultaneous rational approximation to $(\vartheta_1, \dots, \vartheta_m)$, but we also require that it lies outside the hyperplane $a_0x_0 + a_1x_1 + \cdots + a_mp_m = 0$ of $\mathbb{Q}^{m+1}$. Since this needs to be checked for all hyperplanes, the solution is to construct not only one tuple $(q, p_1, \dots, p_m)$ in $\mathbb{Z}^{m+1} \setminus \{0\}$, but $m+1$ such tuples which are linearly independent. This yields $m+1$ pairs (M_k, R_k), $k = 0, \dots, m$ in place of a single pair (M, R), and from $(a_0, \dots, a_m) \neq 0$ one deduces that one at least of $M_0, \dots, M_m$ is not 0.

It turns out that nothing is lossed by using such arguments: existence of linearly independent simultaneous rational approximations for $\vartheta_1, \dots, \vartheta_m$ are characteristic of linearly independent numbers $1, \vartheta_1, \dots, \vartheta_m$. As we just said earlier, we shall use only the easy part of the next Lemma 1.29.

Lemma 1.29. *Let $\underline{\vartheta} = (\vartheta_1, \dots, \vartheta_m) \in \mathbb{R}^m$. Then the following conditions are equivalent.*
(i) The numbers $1, \vartheta_1, \dots, \vartheta_m$ are linearly independent over $\mathbb{Q}$.

(ii) For any $\epsilon > 0$ there exist $m+1$ linearly independent elements $\underline{b}_0, \underline{b}_1, \ldots, \underline{b}_m$ in $\mathbb{Z}^{m+1}$, say

$$\underline{b}_i = (q_i, p_{1i}, \ldots, p_{mi}), \quad (0 \le i \le m)$$

with $q_i > 0$, such that

$$\max_{1 \le k \le m} \left| \vartheta_k - \frac{p_{ki}}{q_i} \right| \le \frac{\epsilon}{q_i}, \quad (0 \le i \le m). \tag{1.30}$$

In (ii) there is no non-vanishing condition. For $m = 1$ this criterion is not identical to the irrationality criterion: in Lemma 1.13, we required for each ϵ one approximation p/q distinct from θ. Here we need two linearly independent approximations: hence, if θ is rational, one at least of them is not the trivial one.

The condition on linear independence of the elements $\underline{b}_0, \underline{b}_1, \ldots, \underline{b}_m$ means that the determinant

$$\begin{vmatrix} q_0 & p_{10} & \cdots & p_{m0} \\ \vdots & \vdots & \ddots & \vdots \\ q_m & p_{1m} & \cdots & p_{mm} \end{vmatrix}$$

is not 0.

For $0 \le i \le m$, set

$$\underline{r}_i = \left(\frac{p_{1i}}{q_i}, \ldots, \frac{p_{mi}}{q_i} \right) \in \mathbb{Q}^m.$$

Further define, for $\underline{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$

$$|\underline{x}| = \max_{1 \le i \le m} |x_i|.$$

Also for $\underline{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $\underline{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ set

$$\underline{x} - \underline{y} = (x_1 - y_1, \ldots, x_m - y_m),$$

so that

$$|\underline{x} - \underline{y}| = \max_{1 \le i \le m} |x_i - y_i|.$$

Then the relation (1.30) in Lemma 1.29 can be written

$$|\underline{\vartheta} - \underline{r}_i| \le \frac{\epsilon}{q_i}, \quad (0 \le i \le m).$$

We shall prove a more explicit version of (ii)⇒(i): we check that *any tuple $(q, p_1, \ldots, p_m) \in \mathbb{Z}^{m+1}$ producing a tuple $(p_1/q, \ldots, p_m/q) \in \mathbb{Q}^m$ of sufficiently good rational approximations to $\underline{\vartheta}$ satisfies the same linear dependence relations as $1, \vartheta_1, \ldots, \vartheta_m$.*

Lemma 1.31. *Let $\vartheta_1, \dots, \vartheta_m$ be real numbers. Assume that the numbers $1, \vartheta_1, \dots, \vartheta_m$ are linearly dependent over $\mathbb{Q}$: let $a, b_1, \dots, b_m$ be rational integers, not all of which are zero, satisfying*

$$a + b_1\vartheta_1 + \cdots + b_m\vartheta_m = 0.$$

Let $\epsilon > 0$ satisfy $\sum_{k=1}^m |b_k] < 1/\epsilon$. Assume further that $(q, p_1, \dots, p_m) \in \mathbb{Z}^{m+1}$ satisfies $q > 0$ and

$$\max_{1\le k\le m} |q\vartheta_k - p_k| \le \epsilon.$$

Then

$$aq + b_1p_1 + \cdots + b_mp_m = 0.$$

Proof. In the relation

$$qa + \sum_{k=1}^m b_kp_k = -\sum_{k=1}^m b_k(q\vartheta_k - p_k),$$

the right hand side has absolute value less than 1 and the left hand side is a rational integer, so it is 0.

□

Proof of (ii)⇒(i) in Lemma 1.29. By assumption (ii) we have $m+1$ linearly independent elements $\underline{b}_i \in \mathbb{Z}^{m+1}$ such that the corresponding rational approximation satisfy the assumptions of Lemma 1.31. Consider a non–zero linear form

$$aX_0 + b_1X_1 + \cdots + b_mX_m = 0.$$

Since $L \neq 0$, one at least of the $L(\underline{b}_i)$ is not 0. For this $\underline{b}_i$ the conclusion of Lemma 1.31 is not satisfied, hence

$$a + b_1\vartheta_1 + \cdots + b_m\vartheta_m \neq 0.$$

□

Proof of (i)⇒(ii) in Lemma 1.29. Let $\epsilon > 0$. Assume (i) holds. By Dirichlet's box principle (Lemma 1.7), there exists $\underline{b} = (q, p_1, \dots, p_m) \in \mathbb{Z}^{m+1}$ with $q > 0$ such that

$$\max_{1\le k\le m} \left|\vartheta_k - \frac{p_k}{q}\right| \le \frac{\epsilon}{q}.$$

Consider the subset $E_\epsilon \subset \mathbb{Z}^{m+1}$ of these tuples. We show that the $\mathbb{Q}$-vector subspace V_ϵ of $\mathbb{Q}^{m+1}$ spanned by E_ϵ is $\mathbb{Q}^{m+1}$. It will follow that there are $m+1$ linearly independent elements in E_ϵ.

If $V_\epsilon \neq \mathbb{Q}^{m+1}$, then there is a hyperplane $a_0z_0 + a_1z_1 + \cdots + a_mz_m = 0$ containing E_ϵ. Any $\underline{b} = (q, p_1, \dots, p_m)$ in E_ϵ has

$$a_0q + a_1p_1 + \cdots + a_mp_m = 0.$$

For each $n \geq 1/\epsilon$, let $\underline{b}_n = (q_n, p_{1n}, \ldots, p_{mn}) \in E_\epsilon$ satisfy

$$\max_{1\leq k\leq m} \left|\vartheta_k - \frac{p_{kn}}{q_n}\right| \leq \frac{1}{nq_n}.$$

Then

$$-a_0 + a_1\theta_1 + \cdots + a_m\theta_m = \sum_{k=1}^{m} a_k \left(\theta_k - \frac{p_{kn}}{q_n}\right).$$

Hence

$$|-a_0 + a_1\theta_1 + \cdots + a_m\theta_m| \leq \frac{1}{nq_n} \sum_{k=1}^{m} |a_k|.$$

The right hand side tends to 0 as n tends to infinity, hence the left hand side vanishes, and $1, \vartheta_1, \ldots, \vartheta_m$ are $\mathbb{Q}$–linearly dependent, which contradicts (i).

□

Appendix: Geometry of numbers: subgroups of $\mathbb{R}^n$.

In Lemma 1.17 of § 1.3.3, we used Dirichlet's box principle. We show how to use Minkowski's geometry of numbers to produce an alternative argument. References for this section are [3, 14, 21].

Lemma 1.32. *A subgroup G of $\mathbb{R}^n$ is discrete in $\mathbb{R}^n$ if and only if there exists an open subset U of $\mathbb{R}^n$ containing 0 such that $G \cap U$ is discrete.*

Exercise 1.33. *1. Check that a non discrete subgroup of $\mathbb{R}$ is dense in $\mathbb{R}$*
2. Give the list of closed subgroups of $\mathbb{R}$.
3. Let G be a finitely generated subgroup of $\mathbb{R}$. Give a necessary and sufficient condition on the rank of G for G to be dense in $\mathbb{R}$.
4. Let $\vartheta \in \mathbb{R}$. Give a necessary and sufficient condition on ϑ for the subgroup $\mathbb{Z} + \mathbb{Z}\vartheta$ to be dense in $\mathbb{R}$.

Proposition 1.34. *Let G be a discrete subgroup of $\mathbb{R}^n$. There exists an integer t in the interval $0 \leq t \leq n$ and there exist elements $e_1, \ldots, e_t$ in G, which are linearly independent over $\mathbb{R}$, such that $G = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_t$.*

In particular $e_1, \ldots, e_t$ are linearly independent over $\mathbb{Z}$, hence G is free of rank t. The integer t is the dimension of the $\mathbb{R}$–subspace of $\mathbb{R}^n$ spanned by G.

Exercise 1.35. *From Proposition 1.34, deduce that in a discrete subgroup of $\mathbb{R}^n$, linearly independent elements over $\mathbb{Z}$ are linearly independent over $\mathbb{R}$.*

Definition. *A discrete subgroup of $\mathbb{R}^n$ of maximal rank n is called a lattice) of $\mathbb{R}^n$.*

Proof of Proposition 1.34. Denote by V the vector subspace of $\mathbb{R}^n$ over $\mathbb{R}$ spanned by G, by t its dimension and let $\{f_1, \ldots, f_t\}$ be a maximal subset of G which is free over $\mathbb{R}$: it is a basis of V over $\mathbb{R}^n$ and $G' = \mathbb{Z}f_1 + \cdots + \mathbb{Z}f_t$ is a subgroup

of G. We show that G' has finite index in G, which means that there are only finitely many classes of G modulo G'.

Let K be the compact subset of $\mathbb{R}^n$ defined by

$$\{u_1 f_1 + \cdots + u_t f_t \ ; \ 0 \leq u_i \leq 1 \ (1 \leq i \leq t)\}.$$

Since G is discrete, $G \cap K$ is finite.

Let $x \in G$. Then $x \in V$, hence we can write $x = x_1 f_1 + \cdots + x_t f_t$ with $x_i \in \mathbb{R}$. Let $m_i = [x_i]$ be the integral part of x_i:

$$m_i \in \mathbb{Z}, \quad 0 \leq x_i - m_i < 1 \qquad (1 \leq i \leq n).$$

Set $x' = m_1 f_1 + \cdots + m_t f_t$. Then $x' \in G'$ and $x - x' \in G \cap K$. Therefore there are only finitely many classes of G modulo G', which means that G' has finite index in G.

Denote by s the order of the finite group G/G' and set $f_i' = f_i/s$ $(1 \leq i \leq t)$. We have

$$G' = \mathbb{Z} f_1 + \cdots + \mathbb{Z} f_t \subset G \subset \mathbb{Z} f_1' + \cdots + \mathbb{Z} f_t',$$

and the conclusion follows from the classical structure theorem on modules on principal rings.

□

Theorem 1.36 (Structure of subgroups of $\mathbb{R}^n$)**.** *Let G be an additive subgroup of $\mathbb{R}^n$. There exists a maximal vector subspace V of $\mathbb{R}^n$ over $\mathbb{R}$ which is contained in the topological closure of G. Let d be the dimension of V and $d + t$ the dimension of the vector space spanned by G over $\mathbb{R}$. Set $G' = G \cap V$. Then G' is dense in V and there exists a discrete subgroup G'' of G, of rank t, such that G is the direct sum of G' and G''.*

Exercise 1.37. *Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Consider the subgroup*

$$G = \mathbb{Z}^n + \mathbb{Z}\mathbf{x} = \{(a_1 + a_0 x_1, \ldots, a_n + a_0 x_n) \ ; \ (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}\}$$

of $\mathbb{R}^n$.

1. *Show that G is discrete in $\mathbb{R}^n$ if and only if $\mathbf{x} \in \mathbb{Q}^n$.*

2. *Deduce that the following properties are equivalent.*

(i) 0 is an accumulation point of G.

(ii) For any $\epsilon > 0$, there exist integers $p_1, \ldots, p_n$, q, with $q > 0$, such that

$$0 < \max_{1 \leq i \leq n} |q x_i - p_i| < \epsilon.$$

(iii) A least one of the n numbers $x_1, \ldots, x_n$ is irrational.

3. *Check that G is dense in $\mathbb{R}^n$ if and only if the numbers $1, x_1, \ldots, x_n$ are linearly independent over $\mathbb{Q}$.*

Deduce that for any $(\xi_1, \xi_2) \in \mathbb{R}^2$ and for any $\epsilon > 0$, there exist rational integers p_1, p_2 and q with

$$|\xi_1 - p_1 - q\sqrt{2}| \leq \epsilon \quad \textit{and} \quad |\xi_2 - p_1 - q\sqrt{3}| \leq \epsilon.$$

Let G be a lattice in $\mathbb{R}^n$. For each basis $\mathbf{e} = \{e_1, \dots, e_n\}$ of G the parallelogram

$$P_{\mathbf{e}} = \{x_1 e_1 + \cdots + x_n e_n \ ; \ 0 \leq x_i < 1 \ (1 \leq i \leq n)\}$$

is a *fundamental domain* for G, which means a complete system of representative of classes modulo G. We get a partition of $\mathbb{R}^n$ as

$$\mathbb{R}^n = \bigcup_{g \in G} (P_{\mathbf{e}} + g) \tag{1.38}$$

A change of bases of G is obtained with a matrix with integer coefficients having determinant ± 1, hence the Lebesgue measure $\mu(P_{\mathbf{e}})$ of $P_{\mathbf{e}}$ does not depend on $\mathbf{e}$: this number is called the *volume* of the lattice G and denoted by $v(G)$.

Here is an example of results obtained by H. Minkowski in the XIX–th century as an application of his *geometry of numbers.*

Theorem 1.39 (Minkowski)**.** *Let G be a lattice in $\mathbb{R}^n$ and B a measurable subset of $\mathbb{R}^n$. Set $\mu(B) > v(G)$. Then there exist $x \neq y$ in B such that $x - y \in G$.*

Proof. From (1.38) we deduce that B is the disjoint union of the $B \cap (P_{\mathbf{e}} + g)$ with g running over G. Hence

$$\mu(B) = \sum_{g \in G} \mu\left(B \cap (P_{\mathbf{e}} + g)\right).$$

Since Lebesgue measure is invariant under translation

$$\mu\left(B \cap (P_{\mathbf{e}} + g)\right) = \mu\left((-g + B) \cap P_{\mathbf{e}}\right).$$

The sets $(-g + B) \cap P_{\mathbf{e}}$ are all contained in $P_{\mathbf{e}}$ and the sum of their measures is $\mu(B) > \mu(P_{\mathbf{e}})$. Therefore they are not all pairwise disjoint – this is one of the versions of the *Dirichlet box principle*). There exists $g \neq g'$ in G such that

$$(-g + B) \cap (-g' + B) \neq \emptyset.$$

Let x and y in B satisfy $-g + x = -g' + y$. Then $x - y = g - g' \in G \setminus \{0\}$.

□

Corollary 1.40. *Let G be a lattice in $\mathbb{R}^n$ and let B be a measurable subset of $\mathbb{R}^n$, convex and symmetric with respect to the origin, such that $\mu(B) > 2^n v(G)$. Then $B \cap G \neq \{0\}$.*

Proof. We use Theorem 1.39 with the set

$$B' = \frac{1}{2} B = \{x \in \mathbb{R}^n \ ; \ 2x \in B\}.$$

We have $\mu(B') = 2^{-n} \mu(B) > v(G)$, hence by Theorem 1.39 there exists $x \neq y$ in B' such that $x - y \in G$. Now $2x$ and $2y$ are in B, and since B is symmetric $-2y \in B$. Finally B is convex, hence $(2x - 2y)/2 = x - y \in G \cap B \setminus \{0\}$.

□

Remark. *With the notations of Corollary 1.40, if B is also compact in $\mathbb{R}^n$, then the weaker inequality $\mu(B) \geq 2^n v(G)$ suffices to reach the conclusion. This is obtained by applying Corollary 1.40 with $(1+\epsilon)B$ for $\epsilon \to 0$.*

Exercise 1.41. *Let m and n be positive integers.*
a) Let t_{ij} for $1 \leq i,j \leq n$ be n^2 real numbers with determinant ± 1. Let $A_1, \ldots, A_n$ be positive real numbers with $A_1 \cdots A_n = 1$. Show that there exists an non–zero element $(x_1, \ldots, x_n)$ in $\mathbb{Z}^n$ such that

$$|x_1 t_{i1} + \cdots + x_n t_{in}| < A_i \quad \text{for} \quad 1 \leq i \leq n-1$$

and

$$|x_1 t_{1n} + \cdots + x_n t_{nn}| \leq A_n.$$

`Hint`. *First solve the system with the weaker inequality $<$ in place of $<$*

$$|x_1 t_{i1} + \cdots + x_n t_{in}| \leq A_i \quad \text{for} \quad 1 \leq i \leq n$$

by using Corollary 1.40. Next use the same method but with A_n replaced with $A_n + \epsilon$ for a sequence of ϵ which tends to 0.
b) Deduce the following result. Let ϑ_{ij} $(1 \leq i \leq n,\ 1 \leq j \leq m)$ be mn real numbers. Let $Q > 1$ be a real number. Show that there exists rational integers $q_1, \ldots, q_m$, $p_1, \ldots, p_n$ with

$$1 \leq \max\{|q_1|, \ \ldots, \ |q_m|\} < Q^{n/m}$$

and

$$\max_{1 \leq i \leq n} |\vartheta_{i1} q_1 + \cdots + \vartheta_{im} q_m - p_i| \leq \frac{1}{Q}.$$

`Hint`. *Use a) with n replaced by $n+m$ and for a triangular matrix $(t_{ij})_{1 \leq i,j \leq m+n}$ with 1 on the diagonal.*
c) Deduce that if $\vartheta_1, \ldots, \vartheta_m$ are real numbers and H a real number > 1, then there exists a tuple $(a_0, a_1, \ldots, a_m)$ of rational integers such that

$$0 < \max_{1 \leq i \leq m} |a_i| < H \quad \text{and} \quad |a_0 + a_1 \vartheta_1 + \cdots + a_m \vartheta_m| \leq H^{-m}.$$

d) Let ϑ be a real number with $|\vartheta| \leq 1/2$, d a positive integer and H a positive integer. Show that there exists a non–zero polynomial $P \in \mathbb{Z}[X]$ of degree $\leq d$ and coefficients in the interval $[-H, H]$ such that

$$|P(\vartheta)| \leq H^{-d}.$$

Exercise 1.42. *Let m and n be positive integers and ϑ_{ij} $(1 \leq i \leq n,\ 1 \leq j \leq m$ be mn real numbers. Let $Q \geq 1$ be a positive integer. Show that there exists rational integers $q_1, \ldots, q_m$, $p_1, \ldots, p_n$ with*

$$1 \leq \max\{|q_1|, \ \ldots, \ |q_m|\} < Q^{n/m}$$

and

$$\max_{1\le i\le n} |\vartheta_{i1}q_1 + \cdots + \vartheta_{im}q_m - p_i| \le \frac{1}{Q}\cdot$$

Deduce that if $\vartheta_1, \ldots, \vartheta_m$ are real numbers and H a positive integer, then there exists a tuple $(a_0, a_1, \ldots, a_m)$ of rational integers such that

$$0 < \max_{1\le i\le m} |a_i| \le H \quad \text{and} \quad |a_0 + a_1\vartheta_1 + \cdots + a_m\vartheta_m| \le H^{-m}.$$

We conclude this section with the definition of a *rational subspace*. Let $k \subset K$ be a field extension and n a positive integer. For a K-vector subspace V of K^n, the two following properties are equivalent:
(i) There exists a basis of V which consists of elements in k^n.
(ii) There exist linear forms $L_1, \ldots, L_m$ with coefficients in k such that V is the intersection of the hyperplans $L_i = 0$, $(1 \le i \le m)$.
When there properties are satisfied the subspace V is called *rational over k*.

Exercise 1.43. *Let $\vartheta_1, \ldots, \vartheta_m$ be real numbers. Assume that $1, \vartheta_1, \ldots, \vartheta_m$ are linearly independent over $\mathbb{Q}$. Let V be a vector subspace of $\mathbb{R}^{m+1}$ which is rational over $\mathbb{Q}$ and has dimension $\le m$.*
a) Check that the intersection of V with the real line $\mathbb{R}(1, \vartheta_1, \ldots, \vartheta_m)$ is $\{0\}$.
b) Deduce that

$$\|(x_0, x_1, \ldots, x_m)\| = \max_{1\le i\le m} |x_0\vartheta_j - x_j|$$

defines a norm on V.

References

[1] M. Aigner & G.M. Ziegler – *Proofs from THE BOOK*, Springer (2001).

[2] J.-P. Allouche & J. Shallit – *Automatic sequences, Theory, applications, generalizations*, Cambridge University Press, Cambridge, 2003,

[3] N. Bourbaki – *Eléments de Mathématique*, Topologie Générale, Herman 1974, Chap. VII, § 1, N°1, Prop. 2;

[4] Brezinski, C. – *The long history of continued fractions and Padé approximants.* Padé approximation and its applications, Amsterdam 1980, pp. 1–27, Lecture Notes in Math., **888**, Springer, Berlin-New York, 1981.

[5] Brezinski, C. – *History of continued fractions and Padé approximants.* Springer Series in Computational Mathematics, **12**. Springer-Verlag, Berlin, 1991.

[6] Y. Bugeaud – *Approximation by algebraic numbers*, Cambridge Tracts in Mathematics, vol. 160, Cambridge University Press, Cambridge, 2004.

[7] H. Cohn – *A short proof of the simple continued fraction expansion of e*, Math Monthly **113** January 2006, 57–62 .
http://fr.arXiv.org/abs/math.NT/061660

[8] J.H. Conway & R.K. Guy – *The book of numbers*, Copernicus Books, Springer Science + Business Media, 2006.

[9] J. Cosgrave – *New Proofs of the Irrationality of e^2 and e^4*, unpublished . http://services.spd.dcu.ie/johnbcos/esquared.htm

[10] L. Euler – *De fractionibus continuis dissertatio*, Commentarii Acad. Sci. Petropolitanae, 9 (1737), 1744, p. 98–137; Opera Omnia Ser. I vol. 14, Commentationes Analyticae, p. 187–215.

[11] P. Eymard & J.P. Lafon – *Autour du nombre π*, Hermann 2000. *The number π*, AMS 2004.

[12] N. I. Fel′dman & Yu. V. Nesterenko – *Transcendental numbers*, in *Number Theory, IV*, Encyclopaedia Math. Sci., vol. **44**, Springer, Berlin, 1998, p. 1–345.

[13] S. Fischler – *Irrationalité de valeurs de zêta, (d'après Apéry, Rivoal, ...)*, Sém. Bourbaki 2002-2003, N° 910 (Novembre 2002). Astérisque **294** (2004), 27-62.
http://www.math.u-psud.fr/∼fischler/publi.html

[14] G.H. Hardy & A.M. Wright, – *An Introduction to the Theory of Numbers*, Oxford Sci. Publ., 1938.

[15] A. Ya. Khinchine – *Continued fractions*, Dover Publications Inc., third edition (1997).

[16] H.W. Lenstra Jr – *Solving the Pell Equation*, Notices of the A.M.S. **49** (2) (2002) 182–192.

[17] J. Liouville – *Sur l'irrationalité du nombre $e = 2,718\ldots$*, J. Math. Pures Appl. (1) **5** (1840), p. 192.

[18] J. Liouville – *Addition à la note sur l'irrationalité du nombre e*, J. Math. Pures Appl. (1) **5** (1840), p. 193–194.

[19] B. Rittaud – *Le fabuleux destin de $\sqrt{2}$*, Éditions Le Pommier (2006).

[20] T. Rivoal – *Applications arithmétiques de l'interpolation lagrangienne*, IJNT, to appear.

[21] W. M. Schmidt – *Diophantine approximation*, Lecture Notes in Mathematics, vol. 785, Springer-Verlag, Berlin, 1980.

[22] J. Shallit – *Real numbers with bounded partial quotients: a survey*, L'Enseignement Mathématique, **38** (1992), 151-187.

[23] S.A. Shirali – *Continued fraction for e*, Resonance, vol. **5** N°1, Jan. 2000, 14–28.
http://www.ias.ac.in/resonance/

[24] J. Sondow – *Criteria for irrationality of Euler's constant*, Proc. Amer. Math. Soc. **131** (2003), 3335–3344
http://xxx.lanl.gov/pdf/math.NT/0209070
http://home.earthlink.net/∼jsondow/

[25] J. Sondow – *A geometric proof that e is irrational and a new measure of its irrationality,* Amer. Math. Monthly **113** (2006), 637–641
http://arxiv.org/pdf/0704.1282v1

[26] B. Sury – *Bessels contain continued fractions of progressions*, Resonance, vol. **10** N°3, March 2005, 80–87.
http://www.ias.ac.in/resonance/

[27] M. Waldschmidt, *Open Diophantine Problems*, Moscow Mathematical Journal **4** N°1, 2004, 245–305.