Exceptional Sets and Such

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Defintion

Let f be an entire function. We define *an exceptional set for f* to be

$$S_f = \{\alpha \in \overline{\mathbb{Q}} | f(\alpha) \in \overline{\mathbb{Q}}\}.$$

Examples

Arbitrary finite subsets of algebraic numbers are easily seen to be exceptional. For instance, if

$$g(z) = e^{(z-\alpha_1)\cdots(z-\alpha_k)},$$

then $S_g = \{\alpha_1, \ldots, \alpha_k\}$.

Set up

We can also look at the Taylor series centered at a point in S_f and require that the coefficients lie in $\overline{\mathbb{Q}}$. We conjectured that every subset of $\overline{\mathbb{Q}}$ is an exceptional set in this more restrictive sense. We generalized this statement to the following theorem.

The Big Theorem

Fix $A \subset \mathbb{C}$ with A countable. For each integer $s \geq 0$ and each $\alpha \in A$, fix a dense subset $E_{\alpha,s} \subset \mathbb{C}$. Then we can find an entire function f such that $f^{(s)}(\alpha) \in E_{\alpha,s}$.

We will show that this theorem holds for infinite subsets A, but a similar proof will show that it holds for finite A as well.

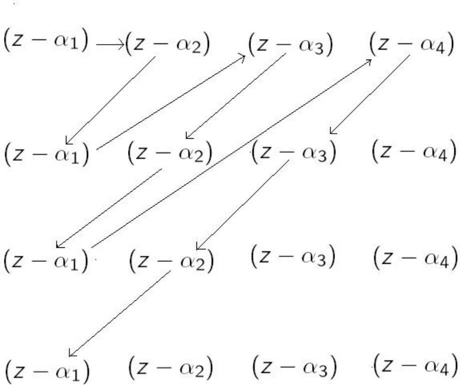
Proof

First we enumerate $A \subset \{\alpha_1, \alpha_2, \ldots\}$. We construct a sequence of polynomials as follows.

$$\begin{aligned}
P_0(z) &= 1 \\
P_1(z) &= (z-\alpha_1) \\
P_2(z) &= (z-\alpha_1)(z-\alpha_2) \\
P_3(z) &= (z-\alpha_1)^2(z-\alpha_2) \\
P_4(z) &= (z-\alpha_1)^2(z-\alpha_2)(z-\alpha_3) \\
P_5(z) &= (z-\alpha_1)^2(z-\alpha_2)^2(z-\alpha_3) \\
P_6(z) &= (z-\alpha_1)^3(z-\alpha_2)^2(z-\alpha_3) \\
P_7(z) &= (z-\alpha_1)^3(z-\alpha_2)^2(z-\alpha_3)(z-\alpha_4)
\end{aligned}$$

Proof (cont.)

The pattern can be seen by following the arrows and picking up the corresponding term at each node:



Proof (cont.)

Now we define $f(z) = \sum_{n=0}^{\infty} a_n P_n(z)$ where we will define the a_n's recursively in order to satisfy these two conditions:

1. The a_n's must decrease sufficiently fast for the function to converge.
2. The a_n must be constructed to ensure the desired conditions on f.

Proof (cont.)

First we will place restrictions on a_n to make f entire. f will converge absolutely when

$$\limsup_{n\to\infty} \frac{|a_{n+1}||P_{n+1}(z)|}{|a_n||P_n(z)|} < 1.$$

Note that from the construction of $P_n(z)$, we have that $\frac{P_{n+1}(z)}{P_n(z)} = z - \alpha_{m(n)}$ where $m(n) < n$.

Proof (cont.)

Define

$$r_n = \min\left(\frac{1}{n}, \frac{1}{2\max_{m\le n}|\alpha_m|}\right)$$

If $\frac{|a_{n+1}|}{|a_n|} < r_n$, then

$$\begin{aligned}\frac{|a_{n+1}|}{|a_n|}|z-\alpha_{m(n)}| &\le \frac{|a_{n+1}|}{|a_n|}|z| + \frac{|a_{n+1}|}{|a_n|}\max_{m\le n}|\alpha_m| \\ &\le \frac{1}{n}|z| + \frac{1}{2}\end{aligned}$$

Proof (cont.)

Then

$$\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}|z-\alpha_{m(n)}| \leq \frac{1}{2}$$

Thus, if $\frac{|a_{n+1}|}{|a_n|} < r_n$, then f will converge absolutely.

Proof (cont.)

Now we will fix the a_i's recursively. To this end, we will adopt the following notation:

$$f(\alpha_1) = \beta_0, f(\alpha_2) = \beta_1, f'(\alpha_1) = \beta_2, f(\alpha_3) = \beta_3, f'(\alpha_2) = \beta_4, \ldots, \text{and}$$

$$E_{\alpha_1,0} = E_0, E_{\alpha_2,0} = E_1, E_{\alpha_1,1} = E_2, E_{\alpha_3,0} = E_3, E_{\alpha_2,1} = E_4, \ldots$$

We will fix the a_i's so that $\beta_k \in E_k$.

Proof (cont.)

First, $\beta_0 = f(\alpha_1) = a_0$, so we pick a_0 to be any nonzero element of E_0. Now

$$\beta_1 = f(\alpha_2) = a_0 + a_1(\alpha_2 - \alpha_1).$$

To ensure convergence, we need $|a_1| < r_0|a_0|$. This forces us to pick a_1 in the open ball centered at a_0 with radius $\frac{r_0|a_0|}{\alpha_2 - \alpha_1}$. Since E_1 is dense, we can pick such a $a_1 \neq 0$ with $\beta_1 \in E_1$.

Proof (cont.)

As a slightly less trivial example, we will look at the how to compute a_5. We have that

$$\begin{aligned} f(z) &= a_0 + a_1(z-\alpha_1) + a_2(z-\alpha_1)(z-\alpha_2) \\ &+ a_3(z-\alpha_1)^2(z-\alpha_2) + a_4(z-\alpha_1)^2(z-\alpha_2)(z-\alpha_3) \\ &+ a_5(z-\alpha_1)^2(z-\alpha_2)^2(z-\alpha_3) + (z-\alpha_1)^3 g(x) \end{aligned}$$

Direct computation shows us that

$$\begin{aligned} f''(\alpha_1) &= 2a_2 + 2a_3(\alpha_1-\alpha_2) + 2a_4(\alpha_1-\alpha_2)(\alpha_1-\alpha_3) \\ &+ 2a_5(\alpha_1-\alpha_2)^2(\alpha_1-\alpha_3) + 0. \end{aligned}$$

We have already picked a_2, a_3, and a_4. We determine the open subset that must contain a_5 (based on convergence requirements, and then we pick a_5 in this subset so that $\beta_5 = f''(\alpha_1) \in E_5 = E_{\alpha_1,3}$ (which is dense).

Proof (cont.)

In general, $\beta_k = P(\alpha_i - \alpha_1, \ldots, \alpha_i - \alpha_{m(k)})$ (with coefficients only depending on $a_0, \ldots, a_k$). We have already picked $a_0, \ldots, a_{k-1}$. Since E_k is dense in $\mathbb{C}$, we can pick $a_k \neq 0$, so that $\beta_k \in E_k$ and a_k is small enough to ensure convergence.

Exceptional sets

Now suppose that $B \subset A = \overline{\mathbb{Q}}$ (with B and $A \backslash B$ both infinite), we can enumerate $\overline{\mathbb{Q}} = \{\alpha_1, \alpha_2, \ldots\}$ where $\alpha_{2n+1} \in B$ and $\alpha_{2n+2} \notin B$. Now from our theorem, we can construct an entire function f with $E_{\alpha_{2n+1},s} = \overline{\mathbb{Q}}$ and $E_{\alpha_{2n+2},s} = \mathbb{C} \backslash \overline{\mathbb{Q}}$ for all $n, s \geq 0$. Thus, all derivatives of f at α_{2n+1} are algebraic, and hence we have the Taylor series

$$f(z) = \sum_{k=0}^{\infty} c_k (z - \alpha_{2n+1})^k$$

where $c_k \in \overline{\mathbb{Q}}$, and $S_f = B$.
This shows that any infinite subset of $\overline{\mathbb{Q}}$ is exceptional.
Furthermore, the same construction would work if we took the coefficients to be in $\mathbb{Q}(i)$!

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