Some Consequences of Schanuel's Conjecture

March 19, 2008

Conjecture and Corollaries

Conjecture (Schanuel): Let $x_1, \ldots, x_n$ be $\mathbb{Q}$-linearly independent complex numbers. Then the transcendence degree over $\mathbb{Q}$ of the field $\mathbb{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n})$ is at least n.

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Corollaries:

Algebraic independence of π and e over $\mathbb{Q}$.

$\pi, \log \pi, \log \log \pi, \ldots$ are algebraically independent over $\overline{\mathbb{Q}}$.

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- $L = \bigcup_{n \geq 0} L_n$.

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More generally:

$$E \text{ and } L \text{ are linearly disjoint over } \overline{\mathbb{Q}}.$$

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Proof: by induction on n, $\pi \notin E_n$.

Base case: $\pi \notin E_0 = \overline{\mathbb{Q}}$ is clear.

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- π is algebraic over $\mathbb{Q}(e^x : x \in E_{n-1})$.

Therefore π is algebraic over $\mathbb{Q}(\exp(A_{n-1}))$ for some finite $A_{n-1} \subseteq E_{n-1}$.

Key Construction

Following similarly:

- A_{n-1} is algebraic over $\mathbb{Q}(\exp(A_{n-2}))$ for some finite $A_{n-2} \subseteq E_{n-2}$.

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Following similarly:

- A_{n-1} is algebraic over $\mathbb{Q}(\exp(A_{n-2}))$ for some finite $A_{n-2} \subseteq E_{n-2}$.
- A_{n-2} is algebraic over $\mathbb{Q}(\exp(A_{n-3}))$ for some finite $A_{n-3} \subseteq E_{n-3}$.

$\vdots$

- A_1 is algebraic over $\mathbb{Q}(\exp(A_0))$ for some finite $A_0 \subseteq E_0 = \overline{\mathbb{Q}}$.

End of proof

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- Take $B \subseteq A$ such that $\exp(B)$ is a transcendence basis of $\mathbb{Q}(\exp(A))$.
- $\{i\pi\} \cup B$ are $\mathbb{Q}$-linearly independent.
- By Schanuel's Conjecture $trdeg_{\mathbb{Q}}\mathbb{Q}(i\pi, B, \exp(B)) \geq |B| + 1$.

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- But $trdeg_{\mathbb{Q}}\mathbb{Q}(i\pi, B, \exp(B)) = trdeg_{\mathbb{Q}}\mathbb{Q}(i\pi, B, \exp(A)) = trdeg_{\mathbb{Q}}\mathbb{Q}(\exp(A)) = trdeg_{\mathbb{Q}}\mathbb{Q}(\exp(B)) \leq |B|$.

Main result

We say K_1 and K_2 are linearly disjoint over k iff:

$\{x_1, \ldots, x_n\} \subseteq K_1$ linearly independent over $k \Rightarrow$ linearly independent over K_2.

Theorem: Schanuel's Conjecture implies E and L are linearly disjoint over $\overline{\mathbb{Q}}$.

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- $e, e^e, e^{e^e}, \ldots$ are algebraically independent over L.

The Proof

Let's prove E_m and L_n are linearly disjoint.
Take $\{l_i\} \subseteq L_n$ linearly independent over $\overline{\mathbb{Q}}$ and $\{e_i\} \subseteq E_m$ such that $\sum l_i e_i = 0$.

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Proceeding as before:
$\exists$ finite $A \subseteq E_{m-1}$ such that $A \cup \{e_i\}$ algebraic over $\mathbb{Q}(\exp(A))$.
$\exists$ finite $C \subseteq L_n$ finite such that $\exp(C) \cup \{l_i\}$ algebraic over $\mathbb{Q}(C)$.

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$\exists$ finite $A \subseteq E_{m-1}$ such that $A \cup \{e_i\}$ algebraic over $\mathbb{Q}(\exp(A))$.
$\exists$ finite $C \subseteq L_n$ finite such that $\exp(C) \cup \{l_i\}$ algebraic over $\mathbb{Q}(C)$.

Take $B \subseteq A$ such that $\exp(B)$ is a transcendence basis of $\mathbb{Q}(\exp(A))$.
Take $D \subseteq C$ such that D is a transcendence basis of $\mathbb{Q}(C)$.

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We have $B \cup D$ linearly independent over $\mathbb{Q}$.

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By Schanuel's Conjecture
$trdeg_{\mathbb{Q}}\mathbb{Q}(B, D, \exp(B), \exp(D)) \geq |B| + |D|$.

The Proof

We have $B \cup D$ linearly independent over $\mathbb{Q}$.

By Schanuel's Conjecture
$trdeg_{\mathbb{Q}}\mathbb{Q}(B, D, \exp(B), \exp(D)) \geq |B| + |D|$.

However

$$trdeg_{\mathbb{Q}}\mathbb{Q}(B, D, \exp(B), \exp(D)) = trdeg_{\mathbb{Q}}\mathbb{Q}(\exp(B), D) \leq |B| + |D|.$$

The Proof

Therefore $\mathbb{Q}(\exp(B))$ and $\mathbb{Q}(D)$ are free over $\overline{\mathbb{Q}}$, and the same is true for $\overline{\mathbb{Q}(\exp(B))}$ and $\overline{\mathbb{Q}(D)}$.

Since $\overline{\mathbb{Q}}$ is algebraically closed, $\overline{\mathbb{Q}(\exp(B))}$ and $\overline{\mathbb{Q}(D)}$ are linearly independent over $\overline{\mathbb{Q}}$ (see Lang's Algebra).

References

1. S. Lang, *Algebra*, Addison Wesley 1995.

2. Michel Waldschmidt, *An introduction to irrationality and transcendence methods*, Lecture Notes AWS 2008.