Academia Sinica, Taipei October 30, 2003

Hopf algebras and Diophantine problems

Michel Waldschmidt

miw@math.jussieu.fr http://www.math.jussieu.fr/∼miw/

Hopf algebras (commutative, cocommutative, of finite type)

Algebraic groups (commutative, linear, over $\overline{\mathbf{Q}}$)

Exponential polynomials

Transcendence of values of exponential polynomials

Algebra of multizeta values

Algebras (over $k = \mathbf{C}$ or $k = \overline{\mathbf{Q}}$)

A k-**algebra** (A, m, η) is a k-vector space A with a **product** $m : A \otimes A \to A$ and a **unit** $\eta : k \longrightarrow A$ which are k-linear maps such that the following diagrams commute:

(Associativity)

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes \mathrm{Id}} & A \otimes A \\ \mathrm{Id} \otimes m \downarrow & & \downarrow m \\ A \otimes A & \xrightarrow[m]{} & A \end{array}$$

(Unit)

$$\begin{array}{ccccc} k \otimes A & \xrightarrow{\eta \otimes \mathrm{Id}} & A \otimes A & \xleftarrow{\mathrm{Id} \otimes \eta} & A \otimes k \\ \downarrow & & \downarrow m & & \downarrow \\ A & = & A & = & A \end{array}$$

Commutative algebras

A k-algebra is $commutative$ if the diagram

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\tau} & A \otimes A \\ m \downarrow & & \downarrow m \\ A & = & A \end{array}$$

commutes. Here $\tau(x \otimes y) = y \otimes x$.

Coalgebras

A k-**coalgebra** (A, Δ, ϵ) is a k-vector space A with a **coproduct** $\Delta : A \to A \otimes A$ and a **counit** $\epsilon : A \longrightarrow k$ which are k-linear maps such that the following diagrams commute:

(Coassociativity)

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \Delta \downarrow & & \downarrow \Delta \otimes \mathrm{Id} \\ A \otimes A & \xrightarrow[\mathrm{Id} \otimes \Delta]{} & A \otimes A \otimes A \end{array}$$

(Counit)

$$\begin{array}{ccccc} A & = & A & = & A \\ \downarrow & & \downarrow \Delta & & \downarrow \\ k \otimes A & \xleftarrow[\epsilon \otimes \mathrm{Id}]{} & A \otimes A & \xrightarrow[\mathrm{Id} \otimes \epsilon]{} & A \otimes k \end{array}$$

Commutative coalgebras

A k-coalgebra is *commutative* if the diagram

$$\begin{array}{ccc} A & = & A \\ \Delta \downarrow & & \downarrow \Delta \\ A \otimes A & \underset{\tau}{\longleftarrow} & A \otimes A \end{array}$$

commutes.

Bialgebras

A **bialgebra** $(A, m, \eta, \Delta, \epsilon)$ is a k-algebra (A, m, η) together with a coalgebra structure (A, Δ, ϵ) which is *compatible*: Δ and ϵ are algebra morphisms

$$\Delta(xy) = \Delta(x)\Delta(y), \quad \epsilon(xy) = \epsilon(x)\epsilon(y).$$

Hopf Algebras

A **Hopf algebra** $(H, m, \eta, \Delta, \epsilon, S)$ is a bialgebra $(H, m, \eta, \Delta, \epsilon)$ with an *antipode* $S : H \to H$ which is a k-linear map such that the following diagram commutes:

$$
\begin{array}{ccccc}
H \otimes H & \stackrel{\Delta}{\longleftarrow} & H & \stackrel{\Delta}{\longrightarrow} & H \otimes H \\
{\scriptstyle \mathrm{Id}\otimes S}\downarrow & & {\scriptstyle \eta\circ\epsilon}\downarrow & & \downarrow{\scriptstyle S\otimes\mathrm{Id}} \\
H \otimes H & \underset{m}{\longrightarrow} & H & \underset{m}{\longleftarrow} & H \otimes H
\end{array}
$$

In a Hopf Algebra the *primitive* elements

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

satisfy $\epsilon(x) = 0$ and $S(x) = -x$; they form a Lie algebra for the bracket

$$[x, y] = xy - yx.$$

In a Hopf Algebra the *primitive* elements

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

satisfy $\epsilon(x) = 0$ and $S(x) = -x$; they form a Lie algebra for the bracket

$$[x, y] = xy - yx.$$

The *group-like* elements

$$\Delta(x) = x \otimes x, \quad x \neq 0$$

are invertible, they satisfy $\epsilon(x) = 1$, $S(x) = x^{-1}$ and form a multiplicative group.

Example 1.

Let G be a finite multiplicative group, kG the algebra of G over k which is a k vector-space with basis G. The mapping

$$m : kG \otimes kG \to kG$$

extends the product

$$(x, y) \mapsto xy$$

of G by linearity. The unit

$$\eta : k \to kG$$

maps 1 to 1_G.

Define a coproduct and a counit

$$\Delta : kG \to kG \otimes kG \text{ and } \epsilon : kG \to k$$

by extending

$$\Delta(x) = x \otimes x \text{ and } \epsilon(x) = 1 \text{ for } x \in G$$

by linearity. The antipode

$$S : kG \to kG$$

is defined by

$$S(x) = x^{-1} \text{ for } x \in G.$$

Since $\Delta(x) = x \otimes x$ for $x \in G$ this Hopf algebra kG is cocommutative.

It is a commutative algebra if and only if G is commutative.

The set of group like elements is G: one recovers G from kG.

Example 2.

Again let G be a finite multiplicative group. Consider the k-algebra k^G of mappings $G \to k$, with basis $\delta_g \quad (g \in G)$, where

$$\delta_g(g') = \begin{cases} 1 & \text{for } g' = g, \\ 0 & \text{for } g' \neq g. \end{cases}$$

Define m by

$$m(\delta_g \otimes \delta_{g'}) = \delta_g \delta_{g'}.$$

Hence m is commutative and $m(\delta_g \otimes \delta_g) = \delta_g$ for $g \in G$. The unit $\eta : k \to k^G$ maps 1 to $\sum_{g \in G} \delta_g$.

Define a coproduct $\Delta : k^G \to k^G \otimes k^G$ and a counit $\epsilon : k^G \to k$ by

$$\Delta(\delta_g) = \sum_{g'g''=g} \delta_{g'} \otimes \delta_{g''} \quad \text{and} \quad \epsilon(\delta_g) = \delta_g(1_G).$$

The coproduct Δ is cocommutative if and only if the group G is commutative.

Define an antipode S by

$$S(\delta_g) = \delta_{g^{-1}}.$$

Remark. One may identify $k^G \otimes k^G$ and $k^{G\times G}$ with

$$\delta_g \otimes \delta_{g'} = \delta_{g,g'}.$$

Duality of Hopf Algebras

The Hopf algebras kG from example 1 and k^G from example 2 are *dual* from each other:

$$
\begin{array}{ccc}
kG \times k^G & \longrightarrow & k \\
(g_1, \delta_{g_2}) & \longmapsto & \delta_{g_2}(g_1)
\end{array}
$$

The basis G of kG is dual to the basis $(\delta_g)_{g \in G}$ of k^G.

Example 3.

Let G be a topological compact group over $\mathbf{C}$. Denote by $\mathfrak{R}(G)$ the set of continuous functions $f : G \to \mathbf{C}$ such that the translates $f_t : x \mapsto f(tx)$, for $t \in G$, span a finite dimensional vector space.

Define a coproduct Δ, a counit ϵ and an antipode S on $\mathfrak{R}(G)$ by

$$\Delta f(x, y) = f(xy), \quad \epsilon(f) = f(1), \quad Sf(x) = f(x^{-1})$$

for x, $y \in G$.

Hence $\mathfrak{R}(G)$ is a commutative Hopf algebra.

Example 4.

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{U}(\mathfrak{g})$ its universal envelopping algebra, namely $\mathfrak{T}(\mathfrak{g})/\mathfrak{I}$ where $\mathfrak{T}(\mathfrak{g})$ is the tensor algebra of $\mathfrak{g}$ and $\mathfrak{I}$ the two sided ideal generated by $XY - YX - [X, Y]$.

Define a coproduct Δ, a counit ϵ and an antipode S on $\mathfrak{U}(\mathfrak{g})$ by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \epsilon(x) = 0, \quad S(x) = -x$$

for $x \in \mathfrak{g}$.

Hence $\mathfrak{U}(\mathfrak{g})$ is a cocommutative Hopf algebra.

The set of primitive elements is $\mathfrak{g}$: one recovers $\mathfrak{g}$ from $\mathfrak{U}(\mathfrak{g})$.

Duality of Hopf Algebras *(again)*

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Then the two Hopf algebras $\mathfrak{R}(G)$ and $\mathfrak{U}(\mathfrak{g})$ are dual from each other.

Bicommutative Hopf algebras of finite type

1.

$$H = k[X], \;\; \Delta(X) = X \otimes 1 + 1 \otimes X, \;\; \epsilon(X) = 0, \;\; S(X) = -X.$$

Bicommutative Hopf algebras of finite type

1.

$H = k[X],\ \ \Delta(X) = X \otimes 1 + 1 \otimes X,\ \ \epsilon(X) = 0,\ \ S(X) = -X.$

$$k[X] \otimes k[X] \simeq k[T_1, T_2], \quad X \otimes 1 \mapsto T_1, \quad 1 \otimes X \mapsto T_2$$

$$\Delta P(X) = P(T_1 + T_2), \quad \epsilon P(X) = P(0), \quad SP(X) = P(-X).$$

Bicommutative Hopf algebras of finite type

1.

$H = k[X],\ \ \Delta(X) = X \otimes 1 + 1 \otimes X,\ \ \epsilon(X) = 0,\ \ S(X) = -X.$

$$k[X] \otimes k[X] \simeq k[T_1, T_2], \quad X \otimes 1 \mapsto T_1, \quad 1 \otimes X \mapsto T_2$$

$$\Delta P(X) = P(T_1 + T_2), \quad \epsilon P(X) = P(0), \quad SP(X) = P(-X).$$

$$\mathbf{G}_a(K) = \mathrm{Hom}_k(k[X], K), \quad k[\mathbf{G}_a] = k[X]$$

$k[\mathbf{G}_a]$ is a bicommutative Hopf algebra of finite type.

Bicommutative Hopf algebras of finite type

2.

$$H = k[Y, Y^{-1}], \quad \Delta(Y) = Y \otimes Y, \quad \epsilon(Y) = 1, \quad S(Y) = Y^{-1}.$$

Bicommutative Hopf algebras of finite type

2.

$$H = k[Y, Y^{-1}], \quad \Delta(Y) = Y \otimes Y, \quad \epsilon(Y) = 1, \quad S(Y) = Y^{-1}.$$

$$H \otimes H \simeq k[T_1, T_1^{-1}, T_2, T_2^{-1}], \quad Y \otimes 1 \mapsto T_1, \quad 1 \otimes Y \mapsto T_2$$

$$\Delta P(Y) = P(T_1 T_2), \quad \epsilon P(Y) = P(1), \quad SP(Y) = P(Y^{-1}).$$

Bicommutative Hopf algebras of finite type

2.

$$H = k[Y, Y^{-1}], \quad \Delta(Y) = Y \otimes Y, \quad \epsilon(Y) = 1, \quad S(Y) = Y^{-1}.$$

$$H \otimes H \simeq k[T_1, T_1^{-1}, T_2, T_2^{-1}], \quad Y \otimes 1 \mapsto T_1, \quad 1 \otimes Y \mapsto T_2$$

$$\Delta P(Y) = P(T_1 T_2), \quad \epsilon P(Y) = P(1), \quad SP(Y) = P(Y^{-1}).$$

$$\mathbf{G}_m(K) = \mathrm{Hom}_k(k[Y, Y^{-1}], K), \quad k[\mathbf{G}_m] = k[Y, Y^{-1}],$$

$k[\mathbf{G}_m]$ is a bicommutative Hopf algebra of finite type.

Bicommutative Hopf algebras of finite type

3.

$$H = k[X_1, \ldots, X_{d_0}, Y_1, Y_1^{-1}, \ldots, Y_{d_1}, Y_{d_1}^{-1}]$$
$$\simeq k[X]^{\otimes d_0} \otimes k[Y, Y^{-1}]^{\otimes d_1}$$

Primitive elements: k-space $kX_1 + \cdots + kX_{d_0}$,
dimension d_0.

Group-like elements: multiplicative group $\langle Y_1, \ldots, Y_{d_1} \rangle$,
rank d_1.

$$G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1}$$
$$k[G] = H, \quad G(K) = \mathrm{Hom}_k(H, K).$$

Bicommutative Hopf algebras of finite type

3.

$$H = k[X_1, \ldots, X_{d_0}, Y_1, Y_1^{-1}, \ldots, Y_{d_1}, Y_{d_1}^{-1}]$$

$$\simeq k[X]^{\otimes d_0} \otimes k[Y, Y^{-1}]^{\otimes d_1}$$

The category of commutative linear algebraic groups over k $G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1}$ *is anti-equivalent to the category of Hopf algebras of finite type which are bicommutative (commutative and cocomutative)*

$$H = k[G].$$

The vector space of primitive elements has dimension d_0 while the rank of the group-like elements is d_1.

Other examples

If W is a k-vector space of dimension ℓ_0, $\mathrm{Sym}(W)$ is a bicommutative Hopf algebra of finite type, anti-isomorphic to $k[\mathbf{G}_a^{\ell_0}]$:

For a basis $\partial_1, \ldots, \partial_{\ell_0}$ of W, $\mathrm{Sym}(W) \simeq k[\partial_1, \ldots, \partial_{\ell_0}]$.

Other examples

If W is a k-vector space of dimension ℓ_0, $\mathrm{Sym}(W)$ is a bicommutative Hopf algebra of finite type, anti-isomorphic to $k[\mathbf{G}_a^{\ell_0}]$.

If Γ is a torsion free finitely generated $\mathbf{Z}$-module of rank ℓ_1, then the group algebra $k\Gamma$ is again a bicommutative Hopf algebra of finite type, anti-isomorphic to $k[\mathbf{G}_m^{\ell_1}]$:

For a basis $\gamma_1, \ldots, \gamma_{\ell_1}$ of Γ, $k\Gamma \simeq k[\gamma_1, \gamma_1^{-1}, \ldots, \gamma_{\ell_1}, \gamma_{\ell_1}^{-1}]$.

Other examples

If W is a k-vector space of dimension ℓ_0, $\mathrm{Sym}(W)$ is a bicommutative Hopf algebra of finite type, anti-isomorphic to $k[\mathbf{G}_a^{\ell_0}]$.

If Γ is a torsion free finitely generated $\mathbf{Z}$-module of rank ℓ_1, then the group algebra $k\Gamma$ is again a bicommutative Hopf algebra of finite type, anti-isomorphic to $k[\mathbf{G}_m^{\ell_1}]$.

The category of bicommutative Hopf algebras of finite type is equivalent to the category of pairs (W, Γ) where W is a k-vector space and Γ is a finitely generated $\mathbf{Z}$-module:

$$H = \mathrm{Sym}(W) \otimes k\Gamma.$$

Commutative linear algebraic groups over $\overline{\mathbf{Q}}$

$G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \qquad d = d_0 + d_1$

$$G(\overline{\mathbf{Q}}) = \overline{\mathbf{Q}}^{d_0} \times (\overline{\mathbf{Q}}^{\times})^{d_1}$$

$$(\beta_1, \ldots, \beta_{d_0}, \alpha_1, \ldots, \alpha_{d_1})$$

Commutative linear algebraic groups over $\overline{\mathbf{Q}}$

$$G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \qquad d = d_0 + d_1$$

$$G(\overline{\mathbf{Q}}) = \overline{\mathbf{Q}}^{d_0} \times (\overline{\mathbf{Q}}^{\times})^{d_1}$$

$$\exp_G : T_e(G) = \mathbf{C}^d \longrightarrow G(\mathbf{C}) = \mathbf{C}^{d_0} \times (\mathbf{C}^{\times})^{d_1}$$

$$(z_1, \ldots, z_d) \longmapsto (z_1, \ldots, z_{d_0}, e^{z_{d_0+1}}, \ldots, e^{z_d})$$

Commutative linear algebraic groups over $\overline{\mathbf{Q}}$

$G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1} \qquad d = d_0 + d_1$

$$G(\overline{\mathbf{Q}}) = \overline{\mathbf{Q}}^{d_0} \times (\overline{\mathbf{Q}}^{\times})^{d_1}$$

$$\exp_G : T_e(G) = \mathbf{C}^d \longrightarrow G(\mathbf{C}) = \mathbf{C}^{d_0} \times (\mathbf{C}^{\times})^{d_1}$$

$$(z_1, \ldots, z_d) \longmapsto (z_1, \ldots, z_{d_0}, e^{z_{d_0+1}}, \ldots, e^{z_d})$$

For α_j and β_i in $\overline{\mathbf{Q}}$,

$$\exp_G(\beta_1, \ldots, \beta_{d_0}, \log \alpha_1, \ldots, \log \alpha_{d_1}) \in G(\overline{\mathbf{Q}})$$

Baker's Theorem. *If*

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n = 0$$

with algebraic β_i *and* α_j, *then*

1. $\beta_0 = 0$

2. If $(\beta_1, \ldots, \beta_n) \neq (0, \ldots, 0)$, *then* $\log \alpha_1, \ldots, \log \alpha_n$ *are* $\mathbf{Q}$-*linearly dependent.*

3. If $(\log \alpha_1, \ldots, \log \alpha_n) \neq (0, \ldots, 0)$, *then* $\beta_1, \ldots, \beta_n$ *are* $\mathbf{Q}$-*linearly dependent.*

Example: $(3-2\sqrt{5})\log 3+\sqrt{5}\log 9-\log 27=0.$

Example: $(3-2\sqrt{5})\log 3+\sqrt{5}\log 9-\log 27=0.$

Corollaries.

1. *Hermite-Lindemann* $(n=1)$: transcendence of

$$e,\quad \pi,\quad \log 2,\quad e^{\sqrt{2}}.$$

Example: $(3-2\sqrt{5})\log 3+\sqrt{5}\log 9-\log 27=0.$

Corollaries.

1. *Hermite-Lindemann* $(n=1)$: transcendence of

$$e,\quad \pi,\quad \log 2,\quad e^{\sqrt{2}}.$$

2. *Gel'fond-Schneider* $(n=2,\ \beta_0=0)$: transcendence of

$$2^{\sqrt{2}},\quad \log 2/\log 3,\quad e^{\pi}.$$

Example: $(3-2\sqrt{5})\log 3+\sqrt{5}\log 9-\log 27=0.$

Corollaries.

1. *Hermite-Lindemann* $(n=1)$: transcendence of

$$e,\quad \pi,\quad \log 2,\quad e^{\sqrt{2}}.$$

2. *Gel'fond-Schneider* $(n=2,\ \beta_0=0)$: transcendence of

$$2^{\sqrt{2}},\quad \log 2/\log 3,\quad e^{\pi}.$$

3. *Example with* $n=2,\ \beta_0\neq 0$: transcendence of

$$\int_0^1\frac{dx}{1+x^3}=\frac{1}{3}\log 2+\frac{\pi}{3\sqrt{3}}\cdot$$

Values of exponential polynomials

Proof of Baker's Theorem. Assume

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} = \log \alpha_n$$

(B_1) (Gel'fond–Baker's Method)

Functions: $z_0,\ e^{z_1},\ \ldots,\ e^{z_{n-1}},\ e^{\beta_0 z_0 + \beta_1 z_1 + \cdots + \beta_{n-1} z_{n-1}}$

Points: $\mathbf{Z}(1, \log \alpha_1, \ldots, \log \alpha_{n-1}) \in \mathbf{C}^n$

Derivatives: $\partial/\partial z_i$, $(0 \le i \le n-1)$.

Values of exponential polynomials

Proof of Baker's Theorem. Assume

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} = \log \alpha_n$$

(B_1) (Gel'fond–Baker's Method)

Functions: $z_0,\ e^{z_1},\ \ldots,\ e^{z_{n-1}},\ e^{\beta_0 z_0 + \beta_1 z_1 + \cdots + \beta_{n-1} z_{n-1}}$

Points: $\mathbf{Z}(1, \log \alpha_1, \ldots, \log \alpha_{n-1}) \in \mathbf{C}^n$

Derivatives: $\partial/\partial z_i$, $(0 \leq i \leq n-1)$.

$n+1$ functions, n variables, 1 point, n derivatives

Another proof of Baker's Theorem. Assume again

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} = \log \alpha_n$$

(B_2) (Generalization of Schneider's method)

Functions: $z_0, \ z_1, \ \ldots, z_{n-1},$

$$e^{z_0} \alpha_1^{z_1} \cdots \alpha_{n-1}^{z_{n-1}} = \exp\{z_0 + z_1 \log \alpha_1 + \cdots + z_{n-1} \log \alpha_{n-1}\}$$

Points: $\{0\} \times \mathbf{Z}^{n-1} + \mathbf{Z}(\beta_0, , \ldots, \beta_{n-1}) \in \mathbf{C}^n$

Derivative: $\partial / \partial z_0$.

Another proof of Baker's Theorem. Assume again

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} = \log \alpha_n$$

(B_2) (Generalization of Schneider's method)

Functions: $z_0, \ z_1, \ \ldots, z_{n-1},$

$$e^{z_0}\alpha_1^{z_1}\cdots\alpha_{n-1}^{z_{n-1}} = \exp\{z_0 + z_1 \log \alpha_1 + \cdots + z_{n-1} \log \alpha_{n-1}\}$$

Points: $\{0\} \times \mathbf{Z}^{n-1} + \mathbf{Z}(\beta_0, , \ldots, \beta_{n-1}) \in \mathbf{C}^n$
Derivative: $\partial/\partial z_0$.

$n+1$ functions, n variables, n points, 1 derivative

Six Exponentials Theorem. *If* x_1, x_2 *are two complex numbers which are* $\mathbf{Q}$*-linearly independent and if* y_1, y_2, y_3 *are three complex numbers which are* $\mathbf{Q}$*-linearly independent, then one at least of the six numbers*

$$e^{x_i y_j} \quad (i = 1, 2,\ j = 1, 2, 3)$$

is transcendental.

Proof of the six exponentials Theorem

Assume $x_1, \ldots, x_a$ are $\mathbf{Q}$-linearly independent numbers and $y_1, \ldots, y_b$ are $\mathbf{Q}$-linearly independent numbers such that

$$e^{x_i y_j} \in \overline{\mathbf{Q}} \quad \text{for} \quad i = 1, \ldots, a,, \ j = 1, \ldots, b$$

with $ab > a + b$.

Functions: $e^{x_i z} \quad (1 \leq i \leq a)$

Points: $y_j \in \mathbf{C} \quad (1 \leq j \leq b)$

a functions, 1 variable, b points, 0 derivative

Linear Subgroup Theorem

$G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1}, \qquad d = d_0 + d_1.$

$W \subset T_e(G)$ a $\mathbf{C}$-subspace which is rational over $\overline{\mathbf{Q}}$. Let ℓ_0 be its dimension.

$Y \subset T_e(G)$ a finitely generated subgroup with $\Gamma = \exp(Y)$ contained in $G(\overline{\mathbf{Q}}) = \overline{\mathbf{Q}}^{d_0} \times (\overline{\mathbf{Q}}^{\times})^{d_1}$. Let ℓ_1 be the $\mathbf{Z}$-rank of Γ.

$V \subset T_e(G)$ a $\mathbf{C}$-subspace containing both W and Y. Let n be the dimension of V.

Hypothesis:

$$\boxed{n(\ell_1 + d_1) < \ell_1 d_1 + \ell_0 d_1 + \ell_1 d_0}$$

$$\boxed{n(\ell_1 + d_1) < \ell_1 d_1 + \ell_0 d_1 + \ell_1 d_0}$$

$d_0 + d_1$ is the number of functions

d_0 are linear

d_1 are exponential

n is the number of variables

ℓ_0 is the number of derivatives

ℓ_1 is the number of points

	d_0	d_1	ℓ_0	ℓ_1	n
Baker B_1	1	n	n	1	n
Baker B_2	n	1	1	n	n
Six exponentials	0	a	0	b	1

	d_0	d_1	ℓ_0	ℓ_1	n
Baker B_1	1	n	n	1	n
Baker B_2	n	1	1	n	n
Six exponentials	0	a	0	b	1

Baker:

$$n(\ell_1 + d_1) = n^2 + n$$

$$\ell_1 d_1 + \ell_0 d_1 + \ell_1 d_0 = n^2 + n + 1$$

Six exponentials: $a + b < ab$

$$n(\ell_1 + d_1) = a + b$$

$$\ell_1 d_1 + \ell_0 d_1 + \ell_1 d_0 = ab$$

duality:

$$(d_0, d_1, \ell_0, \ell_1) \longleftrightarrow (\ell_0, \ell_1, d_0, d_1)$$

$$\left(\frac{d}{dz}\right)^s \left(z^t e^{xz}\right)_{z=y} = \left(\frac{d}{dz}\right)^t \left(z^s e^{yz}\right)_{z=x}.$$

Fourier-Borel duality:

$$(d_0, d_1, \ell_0, \ell_1) \longleftrightarrow (\ell_0, \ell_1, d_0, d_1)$$

$$\left(\frac{d}{dz}\right)^s \left(z^t e^{xz}\right)_{z=y} = \left(\frac{d}{dz}\right)^t \left(z^s e^{yz}\right)_{z=x}.$$

$$\mathsf{L}_{sy} : f \longmapsto \left(\frac{d}{dz}\right)^s f(y).$$

$$f_\zeta(z) = e^{z\zeta}, \qquad \mathsf{L}_{sy}(f_\zeta) = \zeta^s e^{y\zeta}.$$

$$\mathsf{L}_{sy}(z^t f_\zeta) = \left(\frac{d}{d\zeta}\right)^t \mathsf{L}_{sy}(f_\zeta).$$

For $\underline{v} = (v_1, \ldots, v_n) \in \mathbf{C}^n$, set

$$D_{\underline{v}} = v_1 \frac{\partial}{\partial z_1} + \cdots + v_n \frac{\partial}{\partial z_n} \cdot$$

Let $\underline{w}_1, \ldots, \underline{w}_{\ell_0}$, $\underline{u}_1, \ldots, \underline{u}_{d_0}$, $\underline{x}$ and $\underline{y}$ in $\mathbf{C}^n$, $\underline{t} \in \mathbf{N}^{d_0}$ and $\underline{s} \in \mathbf{N}^{\ell_0}$. For $\underline{z} \in \mathbf{C}^n$, write

$$(\mathbf{u}\underline{z})^{\underline{t}} = (\underline{u}_1\underline{z})^{t_1} \cdots (\underline{u}_{d_0}\underline{z})^{t_{d_0}} \quad \text{and} \quad D_{\mathbf{w}}^{\underline{s}} = D_{\underline{w}_1}^{s_1} \cdots D_{\underline{w}_{\ell_0}}^{s_{\ell_0}}.$$

Then

$$\boxed{D_{\mathbf{w}}^{\underline{s}}\Big((\mathbf{u}\underline{z})^{\underline{t}} e^{\underline{x}\underline{z}}\Big)\Big|_{\underline{z}=\underline{y}} = D_{\mathbf{u}}^{\underline{t}}\Big((\mathbf{w}\underline{z})^{\underline{s}} e^{\underline{y}\underline{z}}\Big)\Big|_{\underline{z}=\underline{x}}}$$

Interpretation of the duality in terms of Hopf algebras

following Stéphane Fischler

Let $\mathfrak{C}_1$ be the category with

objects: (G, W, Γ) where $G = \mathbf{G}_a^{d_0} \times \mathbf{G}_m^{d_1}$, $W \subset T_e(G)$ is rational over $\overline{\mathbf{Q}}$ and $\Gamma \in G(\overline{\mathbf{Q}})$ is finitely generated

morphisms: $f : (G_1, W_1, \Gamma_1) \to (G_2, W_2, \Gamma_2)$ where $f : G_1 \to G_2$ is a morphism of algebraic groups such that $f(\Gamma_1) \subset \Gamma_2$ and f induces a morphism

$$df : T_e(G_1) \longrightarrow T_e(G_2)$$

such that $df(W_1) \subset W_2$.

Let H be a bicommutative Hopf algebra over $\overline{\mathbf{Q}}$ of finite type. Denote by d_0 the dimension of the $\overline{\mathbf{Q}}$-vector space of primitive elements and by d_1 the rank of the group of group-like elements.

Let H' be also a bicommutative Hopf algebra over $\overline{\mathbf{Q}}$ of finite type, ℓ_0 the dimension of the space of primitive elements and ℓ_1 the rank of the group-like elements.

Let $\langle\cdot\rangle : H \times H' \longrightarrow \overline{\mathbf{Q}}$ be a bilinear product such that

$$\langle x, yy'\rangle = \langle \Delta x, y \otimes y'\rangle \quad \text{and} \quad \langle xx', y\rangle = \langle x \otimes x', \Delta y\rangle.$$

Let $\mathfrak{C}_2$ be the category with

objects: $(H, H', \langle\cdot\rangle)$ pair of Hopf algebras with a bilinear product as above.

morphisms: $(f, g) : (H_1, H'_1, \langle\cdot\rangle_1) \longrightarrow (H_2, H'_2, \langle\cdot\rangle_2)$ where $f : H_1 \longrightarrow H_2$ and $g : H'_2 \longrightarrow H'_1$ are Hopf algebras morphisms such that

$$\langle x_1, g(x'_2)\rangle_1 = \langle f(x_1), x'_2\rangle_2.$$

Stéphane Fischler: *The categories* $\mathfrak{C}_1$ *and* $\mathfrak{C}_2$ *are equivalent. Further, Fourier-Borel duality amounts to permute* H *and* H'.

Consequence: interpolation lemmas are equivalent to zero estimates.

Stéphane Fischler: *The categories $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are equivalent.*

For $R \in \mathbf{C}[G]$, $\partial_1, \ldots, \partial_k \in W$ and $\gamma \in \Gamma$, set

$$\langle R, \gamma \otimes \partial_1 \cdot \ldots \cdot \partial_k \rangle = \partial_1 \cdot \ldots \cdot \partial_k R(\gamma).$$

Conversely, for $H_1 = \mathbf{C}[G]$ and $H_2 = \mathrm{Sym}(W) \otimes k\Gamma$, consider

$$\begin{array}{ccc} \Gamma & \longrightarrow & G(\mathbf{C}) \\ \gamma & \longmapsto & \big(R \mapsto \langle R, \gamma \rangle\big) \end{array}$$

and

$$\begin{array}{ccc} W & \longrightarrow & T_e(G) \\ \partial & \longmapsto & \big(R \mapsto \langle R, \partial \rangle\big) \end{array}$$

Stéphane Fischler: *The categories* $\mathfrak{C}_1$ *and* $\mathfrak{C}_2$ *are equivalent. Further, Fourier-Borel duality amounts to permute* H *and* H'*.*

Open Problems:

- Define n associated with (G, Γ, W) in terms of $(H, H', \langle\cdot\rangle)$

Stéphane Fischler: *The categories* $\mathfrak{C}_1$ *and* $\mathfrak{C}_2$ *are equivalent. Further, Fourier-Borel duality amounts to permute* H *and* H'.

Open Problems:

• Define n associated with (G, Γ, W) in terms of $(H, H', \langle\cdot\rangle)$

• Extend to non linear commutative algebraic groups (elliptic curves, abelian varieties, and generally semi-abelian varieties)

Stéphane Fischler: *The categories $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are equivalent. Further, Fourier-Borel duality amounts to permute H and H'.*

Open Problems:

- Define n associated with (G, Γ, W) in terms of $(H, H', \langle\cdot\rangle)$
- Extend to non linear commutative algebraic groups (elliptic curves, abelian varieties, and generally semi-abelian varieties)
- Extend to non bicommutative Hopf algebras (of finite type to start with)

Stéphane Fischler: *The categories* $\mathfrak{C}_1$ *and* $\mathfrak{C}_2$ *are equivalent. Further, Fourier-Borel duality amounts to permute* H *and* H'*.*

Open Problems:

- Define n associated with (G, Γ, W) in terms of $(H, H', \langle\cdot\rangle)$
- Extend to non linear commutative algebraic groups (elliptic curves, abelian varieties, and generally semi-abelian varieties)
- Extend to non bicommutative Hopf algebras (of finite type to start with)
- (?) Transcendence results on non commutative algebraic groups