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Lidstone interpolation II. Two variables

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According to Lidstone's interpolation theory, an entire function of a single variable of exponential type $< \pi$ is determined by it derivatives of even order at 0 and 1. In a previous paper, we gave a survey of this classical univariate theory. Here we generalize it to two variables. Multivariate Lidstone interpolation will be the topic of a forthcoming paper.

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1. Introduction

In [7], we gave a survey of Lidstone theory for analytic functions of single variable. Here we extend this theory to two variables.

In § 1 we give references to earlier papers on bivariate Lidstone polynomials. In § 2 we introduce our generalization to two variables of the univariate theory. The existence of the bivariate polynomials follows from Theorem 2.1. They give rise to four new sequences of polynomials - they correspond to the two sequences $\big(\Lambda_k(z)\big)_{k\geq 0}$ and $\big(\Lambda_k(1-z)\big)_{k\geq 0}$ for the univariate case. In Theorem 4.1 we give a closed formula for these polynomials in terms of the univariate Lidstone polynomials.

In § 5 we give a characterization of these polynomials by means of a system of partial differential equations (Proposition 5.1). In § 6, we give explicitly the generating series of these sequences (Theorem 6.1) involving the generating series $\frac{\sinh(\zeta z)}{\sinh \zeta}$ of the sequence of univariate Lidstone polynomials. In § 7 we give a two dimensional generalization of the theorem of Poritsky and Whittaker on the expansion of entire functions of exponential type $< \pi$ (Theorem 7.1) – we start the proof by explaining how these polynomials and their generating series were identified. We also prove 2–dimensional analogs of the results of Buck and Schoenberg for entire functions of two variables of finite exponential type (Theorem 8.1 and Corollary 8.1).

This paper is an introduction to a forthcoming paper [8] where we extend the theory to several variables.

We work with two complex variables $\underline{z} = (z_1, z_2)$. We write $|\underline{z}|$ for $\max\{|z_1|, |z_2|\}$. For an analytic function f of two variables, we use the notation

$$|f|_r = \sup_{|\underline{z}|=r} |f(\underline{z})|.$$

For $\underline{t} = (t_1, t_2) \in \mathbb{N}^2$, we set $\|\underline{t}\| = t_1 + t_2$, $\underline{t}! = t_1!t_2!$ and we define

$$D^{\underline{t}} = \left(\frac{\partial}{\partial z_1}\right)^{t_1} \left(\frac{\partial}{\partial z_2}\right)^{t_2}.$$

We also write $\underline{z}^{\underline{t}} = z_1^{t_1} z_2^{t_2}$. For $\underline{\zeta} = (\zeta_1, \zeta_2)$ and $\underline{z} = (z_1, z_2)$ in $\mathbb{C}^2$, we write $\underline{\zeta}\underline{z} = \zeta_1 z_1 + \zeta_2 z_2$. We use the Kronecker symbol

$$\delta_{ab} = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}$$

According to Lidstone interpolation in one variable, any polynomial $f \in \mathbb{C}[z]$ has a finite expansion

$$f(z) = \sum_{k \geq 0} f^{(2k)}(0)\Lambda_k(1-z) + \sum_{k \geq 0} f^{(2k)}(1)\Lambda_k(z).$$

We keep the notations of [7] for the Lidstone polynomials in one variable: for $k \geq 0$,

$$\Lambda_{2k,1}(z) = \Lambda_k(z), \qquad \Lambda_{2k,0}(z) = \Lambda_k(1-z),$$

so that any polynomial $f \in \mathbb{C}[z]$ in a single variable can be written

$$f(z) = \sum_{t \in 2\mathbb{N}} f^{(t)}(0)\Lambda_{t,0}(z) + \sum_{t \in 2\mathbb{N}} f^{(t)}(1)\Lambda_{t,1}(z).$$

Let $f \in \mathbb{C}[z_1, z_2]$ be a polynomial in two variables. Using Lidstone polynomials in a single variable, one deduces

$$\begin{aligned} f(z_1, z_2) = \sum_{t_1 \in 2\mathbb{N}} \sum_{t_2 \in 2\mathbb{N}} \Big(&(D^{(t_1,t_2)} f)(0,0)\Lambda_{t_1,0}(z_1)\Lambda_{t_2,0}(z_2) + \\ &(D^{(t_1,t_2)} f)(1,0)\Lambda_{t_1,1}(z_1)\Lambda_{t_2,0}(z_2) + (D^{(t_1,t_2)} f)(0,1)\Lambda_{t_1,0}(z_1)\Lambda_{t_2,1}(z_2) + \\ &(D^{(t_1,t_2)} f)(1,1)\Lambda_{t_1,1}(z_1)\Lambda_{t_2,1}(z_2)\Big). \end{aligned}$$

This produces an expansion involving the derivatives $D^{(t_1,t_2)}f$ at the four points $\underline{e}_0 = (0,0)$, $\underline{e}_1 = (1,0)$, $\underline{e}_2 = (0,1)$ and $\underline{e}_1 + \underline{e}_2 = (1,1)$ with t_1 and t_2 both even. If we know $(D^{\underline{t}} f)(\underline{e}_i)$ for t_1 and t_2 both even at these four points, we know the polynomial f. However, if f has degree D, this expansion involves polynomials $\Lambda_{t_1,i}(z_1)\Lambda_{t_2,j}(z_2)$ (with $i, j \in \{0,1\}$) of degree $D+1$.

Let $T \in 2\mathbb{N}$. For a polynomial f of total degree $\leq T+1$, we have $D^{\underline{t}} f = 0$ as soon as $\|\underline{t}\| > T$ is even. The dimension of the space $\mathbb{C}[z_1, z_2]_{\leq T+1}$ of polynomials of total degree $\leq T+1$ is $\frac{1}{2}(T+2)(T+3)$. The number of $\underline{t} \in \mathbb{N}^2$ with $\|\underline{t}\|$ even and

$\|\underline{t}\| \leq T$ is $\frac{1}{4}(T+2)^2$. The number of $\underline{t} \in \mathbb{N}^2$ with t_1 and t_2 both even and $\|\underline{t}\| \leq T$ is $\frac{1}{8}(T+2)(T+4)$. We notice that

$$\frac{1}{2}(T+2)(T+3) = \frac{1}{4}(T+2)^2 + \frac{1}{4}(T+2)(T+4).$$

The generalization to functions of two variables of Lidstone univariate theory that we develop in this paper involves giving up the symmetry among the three points $\underline{e}_0$, $\underline{e}_1$, $\underline{e}_2$. Our conditions at $\underline{e}_0$ involve all $D^{\underline{t}}$ with t_1+t_2 even, while the conditions at $\underline{e}_1$ and $\underline{e}_2$ involve only the $D^{\underline{t}}$ with both t_1 and t_2 even. Consequently, we introduce the following subset of $\mathbb{N}^2 \times \{0,1,2\}$:

$$\mathcal{T} = \left\{(\underline{t}, 0) \;\mid\; \|\underline{t}\| \in 2\mathbb{N}\right\} \bigcup \left\{(\underline{t}, i) \in \mathbb{N}^2 \times \{1,2\} \;\mid\; t_1 \text{ and } t_2 \in 2\mathbb{N}\right\}.$$

Our approach is different from the one in [3], where the authors use the univariate theory to cover the triangle with corners $\underline{e}_0, \underline{e}_1, \underline{e}_2$: they write the expansion of a function on each segment $[t\underline{e}_1, t\underline{e}_2]$, $0 \leq t \leq 1$, by means of Lidstone interpolation in a single variable. They produce explicit formulae for the main terms and also for the remainder. They study uniform convergence and investigate computational aspects.

These interpolation formulae of [3] are combined with bivariate Shepard operators in [1,2]. The point of view of [3] is also used in [4] where the authors obtain a new class of embedded boundary-type cubature formulae on the simplex.

2. The sequences of bivariate polynomials

Here is the corresponding generalization of [7, Lemma 1].

Lemma 2.1. *Let $f \in \mathbb{C}[\underline{z}]$ be a polynomial satisfying*

$$D^{\underline{t}} f(\underline{e}_i) = 0 \text{ for all } (\underline{t}, i) \in \mathcal{T}. \tag{2.1}$$

Then $f = 0$.

We will give two proofs of this lemma.

Proof. [First proof of Lemma 2.1] This first proof relies on [7, Lemma 1] (Lidstone interpolation for polynomials in a single variable).

Let

$$f(z_1, z_2) = \sum_{k_1 \geq 0} \sum_{k_2 \geq 0} a_{k_1,k_2} z_1^{k_1} z_2^{k_2} \in \mathbb{C}[z_1, z_2]$$

satisfy

$$(D^{\underline{t}} f)(\underline{e}_i) = 0 \text{ for all } (\underline{t}, i) \in \mathcal{T}.$$

For $k_2 \geq 0$, define a polynomial f_{k_2} of a single variable by setting

$$f_{k_2}(z_1) = \sum_{k_1 \geq 0} a_{k_1,k_2} z_1^{k_1} = \frac{1}{k_2!} D^{(0,k_2)} f(z_1, 0),$$

so that

$$f(z_1,z_2)=\sum_{k_2\ge 0} f_{k_2}(z_1)z_2^{k_2}.$$

Let $k_2\in 2\mathbb{N}$. For each t_1, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}z_1}\right)^{t_1} f_{k_2}(z_1)=\frac{1}{k_2!}(D^{(t_1,k_2)}f)(z_1,0).$$

If t_1 is even and $i\in\{0,1\}$, then $((t_1,k_2),i)\in\mathcal{T}$. Using [7, Lemma 1], we deduce $f_{k_2}=0$, hence $a_{k_1,k_2}=0$ for all k_2 even and all $k_1\ge 0$. In the same way, fixing $k_1\ge 0$, considering the polynomial

$$\sum_{k_2\ge 0} a_{k_1,k_2}z_2^{k_2}=\frac{1}{k_1!}(D^{(k_1,0)}f)(0,z_2)$$

and using [7, Lemma 1], we deduce $a_{k_1,k_2}=0$ for all $k_1\in 2\mathbb{N}$ and all $k_2\ge 0$. Therefore the condition $a_{k_1,k_2}\ne 0$ implies that k_1 and k_2 are both odd - this implies that k_1+k_2 is even. But the hypothesis $(D^{(k_1,k_2)}f)(0,0)=0$ for all $(k_1,k_2)\in\mathbb{N}^2$ with $k_1+k_2\in 2\mathbb{N}$ implies $a_{k_1,k_2}=0$ for all $(k_1,k_2)\in\mathbb{N}^2$ with k_1 and k_2 both odd, hence $a_{k_1,k_2}=0$ for all $(k_1,k_2)\in\mathbb{N}^2$, and therefore $f=0$. □

We have seen in § 1 that the dimension of the space $\mathbb{C}[\underline{z}]_{\le T+1}$, which is $\frac{1}{2}(T+2)(T+3)$, is also the number of $(\underline{t},i)\in\mathcal{T}$ which satisfy $\|\underline{t}\|\le T$. Therefore Lemma 2.1 can be stated as follows:

Lemma 2.2. *For $T\in 2\mathbb{N}$, the map $f\mapsto\big((D^{\underline{t}}f)(\underline{e}_i)\big)_{\substack{(\underline{t},i)\in\mathcal{T}\\ \|\underline{t}\|\le T}}$ is an isomorphism from the space of polynomials of total degree $\le T+1$ to the space of complex tuples $\big(a_{\underline{t},i}\big)_{(\underline{t},i)\in\mathcal{T},\|\underline{t}\|\le T}$.*

For T an even nonnegative integer, there is a natural bijective map between the set of $\underline{k}\in\mathbb{N}^2$ with $\|\underline{k}\|\le T+1$ and the set $\{(\underline{t},i)\in\mathcal{T}\ \mid\ \|\underline{t}\|\le T\}$: the image of (k_1,k_2) with k_1+k_2 even is $((k_1,k_2),0)$, the image of (k_1,k_2) with k_1 odd and k_2 even is $((k_1-1,k_2),1)$, and finally the image of (k_1,k_2) with k_1 even and k_2 odd is $((k_1,k_2-1),2)$. For the inverse bijective map, the image of $((t_1,t_2),0)$ is (t_1,t_2), the image of $((t_1,t_2),1)$ is (t_1+1,t_2) and the image of $((t_1,t_2),2)$ is (t_1,t_2+1).

For $(k_1,k_2)\in\mathbb{N}^2$ and $(t_1,t_2)\in\mathbb{N}^2$, we have

$$D^{(t_1,t_2)}(z_1^{k_1}z_2^{k_2})=\begin{cases}\frac{k_1!}{(k_1-t_1)!}\frac{k_2!}{(k_2-t_2)!}z_1^{k_1-t_1}z_2^{k_2-t_2} & \text{if } k_1\ge t_1 \text{ and } k_2\ge t_2,\\ 0 & \text{otherwise.}\end{cases}$$

Hence

$$D^{(t_1,t_2)}(z_1^{k_1}z_2^{k_2})(\underline{e}_0)=\underline{t}!\delta_{\underline{t},\underline{k}}.$$

Also for $\{i,j\}=\{1,2\}$, we have

$$D^{(t_1,t_2)}(z_1^{k_1}z_2^{k_2})(\underline{e}_i)=\begin{cases}\frac{k_i!k_j!}{(k_i-t_i)!} & \text{if } k_i\ge t_i \text{ and } k_j=t_j,\\ 0 & \text{otherwise.}\end{cases}$$

Proof. [Second proof of Lemma 2.1] We proceed by induction on the even integer $T \geq 0$ to prove that the map of Lemma 2.2 is injective on $\mathbb{C}[\underline{z}]_{\leq T+1}$: if a polynomial f of total degree $\leq T+1$ satisfies $(D^{\underline{t}}f)(\underline{e}_i) = 0$ for all $(\underline{t}, i) \in \mathcal{T}$ with $\|\underline{t}\| \leq T$, then $f = 0$. This is true for $T = 0$: a polynomial f of total degree ≤ 1 is of the form $a_{00} + a_{10}z_1 + a_{01}z_2$, there are 3 elements $(\underline{t}, i) \in \mathcal{T}$ with $\|\underline{t}\| \leq 1$, which are $((0,0),0)$, $((0,0),1)$, $((0,0),2)$, and the conditions $f(\underline{e}_0) = f(\underline{e}_1) = f(\underline{e}_2) = 0$ imply $a_{00} = a_{10} = a_{01} = 0$.

Assume that $T \geq 2$ is even and that the result is true for $T-2$. Let $f \in \mathbb{C}[z_1, z_2]$ have total degree $\leq T+1$:

$$f(z_1, z_2) = \sum_{k_1+k_2 \leq T+1} a_{k_1,k_2} z_1^{k_1} z_2^{k_2}.$$

Let k_1, k_2 satisfy $k_1 + k_2 = T$. Using the assumption $(D^{\underline{t}}f)(\underline{e}_i) = 0$ with $(t_1, t_2) = (k_1, k_2)$ and $i = 0$, we deduce $a_{k_1,k_2} = 0$.

Next, let k_1, k_2 satisfy $k_1 + k_2 = T + 1$. If k_1 is odd (hence k_2 is even) we use the assumption $(D^{\underline{t}}f)(\underline{e}_i) = 0$ with $(t_1, t_2) = (k_1 - 1, k_2)$ and $i = 1$. If k_1 is even (hence k_2 is odd) we use the assumption $(D^{\underline{t}}f)(\underline{e}_i) = 0$ with $(t_1, t_2) = (k_1, k_2 - 1)$ and $i = 2$. In both cases we deduce $a_{k_1,k_2} = 0$. Hence f has total degree $\leq T - 2$ and we conclude thanks to the induction hypothesis. □

From Lemma 2.2 we deduce:

Theorem 2.1. *For each $(\underline{t}, i) \in \mathcal{T}$, there exists a unique polynomial $\Lambda_{\underline{t},i}$ satisfying, for all $(\underline{\tau}, j) \in \mathcal{T}$,*

$$(D^{\underline{\tau}}\Lambda_{\underline{t},i})(\underline{e}_j) = \delta_{\underline{\tau},\underline{t}}\delta_{ij}.$$

The polynomial $\Lambda_{\underline{t},i}$ has degree at most $\|\underline{t}\| + 1$.

An equivalent formulation is the following:

Corollary 2.1. *Any polynomial $f \in \mathbb{C}[z_1, z_2]$ can be expanded as a finite sum*

$$f(z_1, z_2) = \sum_{(\underline{t},i) \in \mathcal{T}} (D^{\underline{t}}f)(\underline{e}_i)\Lambda_{\underline{t},i}(z_1, z_2).$$

This formula can be written

$$f(z_1, z_2) = \sum_{\|\underline{t}\| \in 2\mathbb{N}} (D^{\underline{t}}f)(0,0)\Lambda_{\underline{t},0}(z_1, z_2) + \sum_{t_1,t_2 \in 2\mathbb{N}} (D^{\underline{t}}f)(1,0)\Lambda_{\underline{t},1}(z_1, z_2) + \sum_{t_1,t_2 \in 2\mathbb{N}} (D^{\underline{t}}f)(0,1)\Lambda_{\underline{t},2}(z_1, z_2).$$

3. Recurrence formulae

From Corollary 2.1 one deduces, for t_1 and t_2 even,

$$\frac{z_1^{t_1+1}}{(t_1+1)!}\frac{z_2^{t_2}}{t_2!} = \sum_{\substack{0\le\tau_1\le t_1\\ \tau_1\in 2\mathbb{N}}} \frac{1}{(t_1-\tau_1+1)!}\Lambda_{(\tau_1,t_2),1}(\underline{z}),$$

$$\frac{z_1^{t_1}}{t_1!}\frac{z_2^{t_2+1}}{(t_2+1)!} = \sum_{\substack{0\le\tau_2\le t_2\\ \tau_2\in 2\mathbb{N}}} \frac{1}{(t_2-\tau_2+1)!}\Lambda_{(t_1,\tau_2),2}(\underline{z}),$$

$$\frac{z_1^{t_1}}{t_1!}\frac{z_2^{t_2}}{t_2!} = \Lambda_{\underline{t},0}(\underline{z}) + \sum_{\substack{0\le\tau_1\le t_1\\ \tau_1\in 2\mathbb{N}}} \frac{1}{(t_1-\tau_1)!}\Lambda_{(\tau_1,t_2),1}(\underline{z}) + \sum_{\substack{0\le\tau_2\le t_2\\ \tau_2\in 2\mathbb{N}}} \frac{1}{(t_2-\tau_2)!}\Lambda_{(t_1,\tau_2),2}(\underline{z}),$$

while for t_1 and t_2 odd we have

$$\frac{z_1^{t_1}}{t_1!}\frac{z_2^{t_2}}{t_2!} = \Lambda_{(t_1,t_2),0}(\underline{z}).$$

This yields recurrence formulae producing the polynomials $\Lambda_{\underline{t},i}$ by induction on $\|\underline{t}\|$:

Lemma 3.1. *For t_1 and t_2 even, we have*

$$\Lambda_{\underline{t},1}(\underline{z}) = \frac{z_1^{t_1+1}}{(t_1+1)!}\frac{z_2^{t_2}}{t_2!} - \sum_{\substack{0\le\tau_1\le t_1-2\\ \tau_1\in 2\mathbb{N}}} \frac{1}{(t_1-\tau_1+1)!}\Lambda_{(\tau_1,t_2),1}(\underline{z}),$$

$$\Lambda_{\underline{t},2}(\underline{z}) = \frac{z_1^{t_1}}{t_1!}\frac{z_2^{t_2+1}}{(t_2+1)!} - \sum_{\substack{0\le\tau_2\le t_2-2\\ \tau_2\in 2\mathbb{N}}} \frac{1}{(t_2-\tau_2+1)!}\Lambda_{(t_1,\tau_2),2}(\underline{z}),$$

$$\Lambda_{\underline{t},0}(\underline{z}) = \frac{z_1^{t_1}}{t_1!}\frac{z_2^{t_2}}{t_2!} - \sum_{\substack{0\le\tau_1\le t_1\\ \tau_1\in 2\mathbb{N}}} \frac{1}{(t_1-\tau_1)!}\Lambda_{(\tau_1,t_2),1}(\underline{z}) - \sum_{\substack{0\le\tau_2\le t_2\\ \tau_2\in 2\mathbb{N}}} \frac{1}{(t_2-\tau_2)!}\Lambda_{(t_1,\tau_2),2}(\underline{z}).$$

For t_1 and t_2 odd, we have

$$\Lambda_{\underline{t},0}(\underline{z}) = \frac{z_1^{t_1}}{t_1!}\frac{z_2^{t_2}}{t_2!}.$$

4. Explicit formulae

The Lidstone polynomials in a single variable $\Lambda_{t,1}(z)$ and $\Lambda_{t,0}(z) = \Lambda_{t,1}(1-z)$, ($t$ even, $z \in \mathbb{C}$) introduced in [7, §2] enable us to give explicit formulae for the polynomials $\Lambda_{\underline{t},i}$ –in § 7, before proving Theorem 7.1, we will mimic the argument of the proof of [7, §6] to explain how these formulae were found.

Theorem 4.1. *For t_1 and t_2 both even, we have*

$$\Lambda_{(t_1,t_2),1}(z_1,z_2) = \Lambda_{t_1,1}(z_1)\frac{z_2^{t_2}}{t_2!},$$
$$\Lambda_{(t_1,t_2),2}(z_1,z_2) = \frac{z_1^{t_1}}{t_1!}\Lambda_{t_2,1}(z_2),$$
$$\Lambda_{(t_1,t_2),0}(z_1,z_2) = \frac{z_1^{t_1}}{t_1!}\Lambda_{t_2,0}(z_2) + \Lambda_{t_1,0}(z_1)\frac{z_2^{t_2}}{t_2!} - \frac{z_1^{t_1}z_2^{t_2}}{t_1!t_2!}.$$

For t_1 and t_2 both odd, we have

$$\Lambda_{(t_1,t_2),0}(z_1,z_2) = \frac{z_1^{t_1}}{t_1!}\frac{z_2^{t_2}}{t_2!}.$$

Proof. The last formula of Theorem 4.1, for t_1 and t_2 both odd and $i = 0$, follows from Lemma 3.1.

Suppose now that both t_1 and t_2 are even. It is not difficult to check by brute force that the right hand sides of each of the three first formulae satisfy the properties of Theorem 2.1 which give a characterization of $\Lambda_{\underline{t},i}$.

Here is an alternative argument, which relies on Lemma 3.1. We first check the formula of Theorem 4.1 for $\Lambda_{\underline{t},i}$ with $i = 1$ by induction on t_1. For $t_1 = 0$ Lemma 3.1 gives

$$\Lambda_{(0,t_2),1}(z_1,z_2) = z_1\frac{z_2^{t_2}}{t_2!};$$

recall $\Lambda_{0,1}(z_1) = z_1$. Assuming that the formula of Theorem 4.1 for $i = 1$ holds for $0 \le \tau_1 < t_1$, we deduce it for t_1 by means of Lemma 3.1 and of the recurrence formula in one variable [7, Equation (4)]. The proof of the formula for $i = 2$ is similar.

Once we know the two first formulae (for $i = 1$ and $i = 2$), we deduce the third one (for $i = 0$) thanks to [7, Equation (5)] in one variable and to Lemma 3.1. □

Example. Here is the formula for the expansion of a polynomial in 2 variables of total degree ≤ 3, involving the univariate Lidstone polynomials $\Lambda_{0,1}(z) = z$, $\Lambda_{0,0}(z) = 1 - z$, $\Lambda_{2,1}(z) = \frac{1}{6}(z^3 - z)$ and $\Lambda_{2,0}(z) = \Lambda_{2,1}(1 - z)$:

$$\begin{aligned}f(z_1,z_2) &= f(0,0)(1 - z_1 - z_2) + f(1,0)z_1 + f(0,1)z_2 + (D^{(1,1)}f)(0,0)z_1z_2\\ &+(D^{(2,0)}f)(0,0)\left(\Lambda_{2,0}(z_1) - \frac{1}{2}z_1^2z_2\right) + (D^{(2,0)}f)(1,0)\Lambda_{2,1}(z_1) + \frac{1}{2}(D^{(2,0)}f)(0,1)z_1^2z_2\\ &+(D^{(0,2)}f)(0,0)\left(\Lambda_{2,0}(z_2) - \frac{1}{2}z_1z_2^2\right) + (D^{(0,2)}f)(0,1)\Lambda_{2,1}(z_2) + \frac{1}{2}(D^{(0,2)}f)(1,0)z_1z_2^2.\end{aligned}$$

For t_1 and t_2 both even and $i \in \{0,1,2\}$, the total degree of $\Lambda_{(t_1,t_2),i}(z_1,z_2)$ is t_1+t_2+1, the homogeneous component of highest degree is

$$\begin{cases} -\dfrac{z_1^{t_1+1}}{(t_1+1)!}\dfrac{z_2^{t_2}}{t_2!} - \dfrac{z_1^{t_1}}{t_1!}\dfrac{z_2^{t_2+1}}{(t_2+1)!} & \text{for } i=0,\\ \dfrac{z_1^{t_1+1}}{(t_1+1)!}\dfrac{z_2^{t_2}}{t_2!} & \text{for } i=1,\\ \dfrac{z_1^{t_1}}{t_1!}\dfrac{z_2^{t_2+1}}{(t_2+1)!} & \text{for } i=2. \end{cases}$$

From Theorem 4.1 and [7, Equation (15)], one deduces, for any $(\underline{t},i) \in \mathcal{T}$ and any $\underline{z} \in \mathbb{C}^2$,

$$\begin{cases} |\Lambda_{\underline{t},1}(\underline{z})| \le 2\pi^{-t_1}\mathrm{e}^{3\pi|z_1|/2}\dfrac{|z_2|^{t_2}}{t_2!},\\ |\Lambda_{\underline{t},2}(\underline{z})| \le 2\dfrac{|z_1|^{t_1}}{t_1!}\pi^{-t_2}\mathrm{e}^{3\pi|z_2|/2},\\ |\Lambda_{\underline{t},0}(\underline{z})| \le \dfrac{|z_1|^{t_1}|z_2|^{t_2}}{t_1!t_2!} + 2\mathrm{e}^{3\pi/2}\left(\dfrac{|z_1|^{t_1}}{t_1!}\pi^{-t_2}\mathrm{e}^{3\pi|z_2|/2} + \pi^{-t_1}\mathrm{e}^{3\pi|z_1|/2}\dfrac{|z_2|^{t_2}}{t_2!}\right). \end{cases} \tag{4.1}$$

Example. From Theorem 4.1, using equation [7, Equation (3)] for $\Lambda_{4,0}$, we deduce

$$\Lambda_{(4,4),0}(z_1,z_2) = \frac{z_1^4 z_2^4}{4!^2} - \left(\frac{z_1^5}{5!} + \frac{z_1^3}{18} - \frac{z_1}{45}\right)\frac{z_2^4}{4!} - \frac{z_1^4}{4!}\left(\frac{z_2^5}{5!} + \frac{z_2^3}{18} - \frac{z_2}{45}\right);$$

using the numerical values $4!^2 = 576$, $4!5! = 2880$, $4!18 = 432$, $4!45 = 1080$ we deduce

$$\begin{aligned} \Lambda_{(4,4),0}(z_1,z_2) = &-\frac{1}{2880}z_1^5z_2^4 - \frac{1}{2880}z_1^4z_2^5 + \frac{1}{576}z_1^4z_2^4\\ &-\frac{1}{432}z_1^4z_2^3 - \frac{1}{432}z_1^3z_2^4 + \frac{1}{1080}z_1z_2^4 + \frac{1}{1080}z_1^4z_2. \end{aligned} \tag{4.2}$$

5. Differential equation

The following result is an analog in two variables of [7, Lemma 2] for the family of polynomials $\Lambda_{\underline{t},0}$ ($\underline{t} = (t_1,t_2)$, t_1, t_2 both even). There are three further similar statements, that we will not need nor prove, for the three other families of polynomials

$$\begin{aligned} &\Lambda_{(t_1,t_2),0},\ t_1 \text{ and } t_2 \text{ both odd} \ge 1,\\ &\Lambda_{(t_1,t_2),1},\ t_1 \text{ and } t_2 \text{ both even} \ge 0,\\ &\Lambda_{(t_1,t_2),2},\ t_1 \text{ and } t_2 \text{ both even} \ge 0. \end{aligned}$$

Proposition 5.1. *The family of polynomials $\Lambda_{\underline{t},0}$ ($\underline{t} = (t_1,t_2)$, t_1, t_2 both even) is the unique solution of the system of differential equations*

$$\begin{cases} D^{(2,0)}L_{t_1,t_2} = L_{t_1-2,t_2} & \text{when both } t_1 \text{ and } t_2 \text{ are even with } t_1 \ge 2\\ D^{(0,2)}L_{t_1,t_2} = L_{t_1,t_2-2} & \text{when both } t_1 \text{ and } t_2 \text{ are even with } t_2 \ge 2 \end{cases} \tag{5.1}$$

satisfying $L_{t_1,0} = \Lambda_{(t_1,0),0}$ *for* $t_1 \geq 0$ *even and* $L_{0,t_2} = \Lambda_{(0,t_2),0}$ *for* $t_2 \geq 0$ *even, with the four initial conditions*

$$L_{t_1,t_2}(\underline{e}_0) = L_{t_1,t_2}(\underline{e}_1) = L_{t_1,t_2}(\underline{e}_2) = D^{(1,1)}L_{t_1,t_2}(\underline{e}_0) = 0 \quad \textit{for all } t_1 \textit{ and } t_2 \textit{ both even with } \|\underline{t}\| \geq 2.$$

Notice that $\Lambda_{(t_1,0),0}(z_1, z_2) = \Lambda_{(0,t_1),0}(z_2, z_1)$.

The families of polynomials, for $t_1, t_2 \in 2\mathbb{N}$,

$$\Lambda_{t_1+t_2,1}(1 - z_1 - z_2), \quad \frac{z_1^{t_1+1}}{(t_1+1)!}\frac{z_2^{t_2}}{t_2!}, \quad \frac{z_1^{t_1}}{t_1!}\frac{z_2^{t_2+1}}{(t_2+1)!}, \quad \frac{z_1^{t_1}}{t_1!}\frac{z_2^{t_2}}{t_2!},$$

satisfy the system of differential equations (5.1), but not the other assumptions.

Notice also that $D^{(1,1)}\Lambda_{(1,1),0}(z_1, z_2) = 1$, while $\Lambda_{(0,0),0}(z_1, z_2) = 1 - z_1 - z_2$.

Proof. [Proof of Proposition 5.1] The fact that the family of polynomials $\Lambda_{\underline{t},0}$ ($\underline{t} = (t_1, t_2)$, t_1, t_2 both even), satisfies these conditions follows from Theorem 2.1.

Conversely, let $L_{\underline{t}}$ be a solution. We prove $L_{t_1,t_2} = \Lambda_{(t_1,t_2),0}$ by induction on $t_1 + t_2$. This is true by assumption for $t_1 = 0$ and also for $t_2 = 0$. Assume $t_1 \geq 2$ and $t_2 \geq 2$. Define $g = L_{t_1,t_2} - \Lambda_{(t_1,t_2),0}$. Using the two differential equations

$$D^{(2,0)}L_{t_1,t_2} = L_{t_1-2,t_2} \text{ and } D^{(0,2)}L_{t_1,t_2} = L_{t_1,t_2-2}$$

together with

$$D^{(2,0)}\Lambda_{(t_1,t_2),0} = \Lambda_{(t_1-2,t_2),0} \text{ and } D^{(0,2)}\Lambda_{(t_1,t_2),0} = \Lambda_{(t_1,t_2-2),0}$$

and with the induction hypothesis

$$L_{t_1-2,t_2} = \Lambda_{(t_1-2,t_2),0} \text{ and } L_{t_1,t_2-2} = \Lambda_{(t_1,t_2-2),0},$$

we deduce $D^{(2,0)}g = D^{(0,2)}g = 0$; therefore g has degree ≤ 1 in z_1 and z_2: it is of the form $g(\underline{z}) = a_{00} + a_{10}z_1 + a_{01}z_2 + a_{11}z_1z_2$. Such a polynomial is completely determined by the four numbers $g(\underline{e}_0)$, $g(\underline{e}_1)$, $g(\underline{e}_2)$, $(D^{(1,1)}g)(\underline{e}_0)$. From the four initial conditions we deduce $g = 0$. The result follows. □

6. Generating series

Following [5, p. 27], we will say that a series of functions $\sum_\alpha a_\alpha(z)$ converges *normally* in an open subset Ω of $\mathbb{C}^n$ if $\sum_\alpha \sup_K |a_\alpha(z)|$ converges for every compact set $K \subset \Omega$. For instance [5, Theorem 2.2.6] an analytic function in a polydisc $\{\underline{z} \in \mathbb{C}^n \mid |z_j| < r_j,\ j = 1, \ldots, n\}$ is the sum of its Taylor expansion at the origin with normal convergence in this polydisc.

Denote by M_0, $\widetilde{M}_0$, M_1, M_2 the four generating series

$$M_0(\underline{\zeta},\underline{z}) = \sum_{t_1,t_2 \text{ both even}} \Lambda_{(t_1,t_2),0}(z_1,z_2)\zeta_1^{t_1}\zeta_2^{t_2},$$

$$\widetilde{M}_0(\underline{\zeta},\underline{z}) = \sum_{t_1,t_2 \text{ both odd}} \Lambda_{(t_1,t_2),0}(z_1,z_2)\zeta_1^{t_1}\zeta_2^{t_2},$$

$$M_1(\underline{\zeta},\underline{z}) = \sum_{t_1,t_2 \text{ both even}} \Lambda_{(t_1,t_2),1}(z_1,z_2)\zeta_1^{t_1}\zeta_2^{t_2},$$

$$M_2(\underline{\zeta},\underline{z}) = \sum_{t_1,t_2 \text{ both even}} \Lambda_{(t_1,t_2),2}(z_1,z_2)\zeta_1^{t_1}\zeta_2^{t_2}.$$

Using the generating series [7, Equation (6)] of Lidstone polynomials in a single variable

$$\sum_{t\in 2\mathbb{N}} \Lambda_{t,1}(z)\zeta^t = \frac{\sinh(\zeta z)}{\sinh(\zeta)} \tag{6.1}$$

combined with Theorem 4.1, we deduce:

Theorem 6.1. *These four generating series are normally convergent in the domain*

$$\{(\zeta_1,\zeta_2,z_1,z_2)\in\mathbb{C}^4 \mid |\zeta_1|<\pi,\ |\zeta_2|<\pi\}$$

where they define analytic functions of 4 *variables, namely*

$$\begin{aligned}M_0(\underline{\zeta},\underline{z}) &= \cosh(\zeta_1 z_1)\frac{\sinh(\zeta_2(1-z_2))}{\sinh(\zeta_2)} + \frac{\sinh(\zeta_1(1-z_1))}{\sinh(\zeta_1)}\cosh(\zeta_2 z_2)\\ &\qquad - \cosh(\zeta_1 z_1)\cosh(\zeta_2 z_2),\\ \widetilde{M}_0(\underline{\zeta},\underline{z}) &= \sinh(\zeta_1 z_1)\sinh(\zeta_2 z_2),\\ M_1(\underline{\zeta},\underline{z}) &= \frac{\sinh(\zeta_1 z_1)}{\sinh(\zeta_1)}\cosh(\zeta_2 z_2),\\ M_2(\underline{\zeta},\underline{z}) &= \cosh(\zeta_1 z_1)\frac{\sinh(\zeta_2 z_2)}{\sinh(\zeta_2)}.\end{aligned}$$

For the proof of the normal convergence of the series (6.1) in $\{(\zeta,z)\in\mathbb{C}^2 \mid |\zeta|<\pi\}$, let us introduce the entire function of a single variable

$$\varphi(z) = \begin{cases}\frac{\sinh(z)}{z} & \text{for } z\neq 0,\\ 1 & \text{for } z=0.\end{cases}$$

Since the function $1/\varphi(\zeta)$ is analytic in $|\zeta|<\pi$, the function of two variables

$$F(\zeta,z) = z\frac{\varphi(\zeta z)}{\varphi(\zeta)}$$

is analytic in the domain $\{(\zeta,z)\in\mathbb{C}^2 \mid |\zeta|<\pi,\ z\in\mathbb{C}\}$; hence its Taylor expansion is normally convergent in this domain. Finally, for $z\in\mathbb{C}$, we have

$$F(\zeta,z) = \begin{cases}\frac{\sinh(\zeta z)}{\sinh(\zeta)} & \text{for } \zeta\neq 0,\\ z & \text{for } \zeta=0.\end{cases}$$

7. Expansion of entire functions of two variables

Let f be an entire function of two complex variables. We define the exponential type $\tau(f) \in [0, \infty]$ as

$$\tau(f) = \limsup_{|z_1|+|z_2|\to\infty} \frac{1}{|z_1| + |z_2|} \log |f(z_1, z_2)|.$$

If f has an exponential type $\leq \tau$, then for each $z_1 \in \mathbb{C}$, the function $z_2 \mapsto f(z_1, z_2)$ has exponential type $\leq \tau$ and for each $z_2 \in \mathbb{C}$, the function $z_1 \mapsto f(z_1, z_2)$ has exponential type $\leq \tau$:

$$\limsup_{r\to\infty} \frac{1}{r} \log \sup_{|z_2|\leq r} |f(z_1, z_2)| \leq \tau \text{ for all } z_1 \in \mathbb{C}$$

and

$$\limsup_{r\to\infty} \frac{1}{r} \log \sup_{|z_1|\leq r} |f(z_1, z_2)| \leq \tau \text{ for all } z_2 \in \mathbb{C}.$$

As pointed out by Damien Roy, the function

$$\sum_{k\geq 0} \frac{(z_1 z_2)^k}{k!^2}$$

has exponential type 1; however, for each $\epsilon > 0$, for each $z_1 \in \mathbb{C}$, the function $z_2 \mapsto f(z_1, z_2)$ has exponential type $\leq \epsilon$ and for each $z_2 \in \mathbb{C}$, the function $z_1 \mapsto f(z_1, z_2)$ has exponential type $\leq \epsilon$.

Lemma 7.1. *For any $\underline{z}_0 \in \mathbb{C}^2$, we have*

$$\limsup_{k_1+k_2\to\infty} |D^{(k_1,k_2)} f(\underline{z}_0)|^{\frac{1}{k_1+k_2}} = \tau(f).$$

Proof. Since $f(\underline{z})$ and $f(\underline{z}_0 + \underline{z})$ have the same exponential type, it suffices to prove the result for $\underline{z}_0 = 0$. We write the Taylor expansion of f at the origin

$$f(\underline{z}) = \sum_{k_1\geq 0} \sum_{k_2\geq 0} c_{k_1,k_2} z_1^{k_1} z_2^{k_2}$$

with

$$c_{k_1,k_2} = \frac{1}{k_1! k_2!} D^{(k_1,k_2)} f(0).$$

We first prove the upper bound for $\tau(f)$. Let $\tau > 0$. Assume

$$|c_{k_1,k_2}| \leq \frac{\tau^{k_1+k_2}}{k_1! k_2!}$$

for all sufficiently large $k_1 + k_2$, say $k_1 + k_2 > K$. Then, for $\underline{z} \in \mathbb{C}^2$,

$$|f(\underline{z})| \leq \sum_{k_1+k_2\leq K} |c_{k_1,k_2}||z_1|^{k_1}|z_2|^{k_2} + \sum_{k_1+k_2>K} \frac{\tau^{k_1+k_2}|z_1|^{k_1}|z_2|^{k_2}}{k_1! k_2!}$$

with

$$\sum_{k_1+k_2>K} \frac{\tau^{k_1+k_2}|z_1|^{k_1}|z_2|^{k_2}}{k_1!k_2!} \le \mathrm{e}^{\tau(|z_1|+|z_2|)}.$$

Hence $\tau(f) \le \tau$.

We now prove the lower bound for $\tau(f)$. From Parseval's relation

$$\frac{1}{(2\pi)^2}\int_0^\pi \int_0^\pi |f(r_1\mathrm{e}^{i\theta_1}, r_2\mathrm{e}^{i\theta_2})|^2 \mathrm{d}\theta_1\mathrm{d}\theta_2 = \sum_{k_1\ge 0}\sum_{k_2\ge 0} |c_{k_1,k_2}|^2 r_1^{2k_1} r_2^{2k_2}$$

for $r_1 > 0$ and $r_2 > 0$, we deduce Cauchy's inequalities [5, Theorem 2.2.7]:

$$|c_{k_1,k_2}| r_1^{k_1} r_2^{k_2} \le \sup_{\substack{|z_1|=r_1\\ |z_2|=r_2}} |f(z_1,z_2)|. \tag{7.1}$$

Assume $\tau(f) < \infty$; let $\tau = \tau(f)$. For any $\epsilon > 0$ we have

$$|f(z_1,z_2)| \le \mathrm{e}^{(\tau+\epsilon)(|z_1|+|z_2|)}$$

for sufficiently large $|z_1| + |z_2|$ depending on ϵ. We apply this upper bound for $|z_1| = r_1 = k_1/\tau$ and $|z_2| = r_2 = k_2/\tau$ using Cauchy's inequalities (7.1):

$$|D^{(k_1,k_2)}f(0)| \le \frac{k_1!\tau^{k_1}}{k_1^{k_1}} \cdot \frac{k_2!\tau^{k_2}}{k_2^{k_2}} \mathrm{e}^{(1+(\epsilon/\tau))(k_1+k_2)}$$

for sufficiently large $k_1 + k_2$. From Stirling's formula (for $N \ge 1$)

$$N! < N^N \mathrm{e}^{-N}\sqrt{2\pi N}\mathrm{e}^{1/(12N)}$$

we conclude

$$\limsup_{k_1+k_2\to\infty} |D^{(k_1,k_2)}f(0)|^{1/(k_1+k_2)} \le \tau. \qquad \square$$

From Lemma 7.1 se deduce:

Corollary 7.1. *Let f be an entire function of two variables of exponential type $\le \tau$. Then the* Laplace transform *of f, viz. the function of two complex variables*

$$F(\zeta_1,\zeta_2) = \sum_{k_1\ge 0}\sum_{k_2\ge 0} D^{(k_1,k_2)}f(0)\zeta_1^{-k_1-1}\zeta_2^{-k_2-1},$$

is analytic in the domain $\bigl\{(\zeta_1,\zeta_2)\in\mathbb{C}^2 \;\mid\; |\zeta_1|>\tau,\ |\zeta_2|>\tau\bigr\}$.

Theorem 7.1. *Let f be an entire function in $\mathbb{C}^2$ having exponential type $< \pi$. Then*

$$f(\underline{z}) = \sum_{(\underline{t},i)\in\mathcal{T}} (D^{\underline{t}}f)(\underline{e}_i)\Lambda_{\underline{t},i}(\underline{z}),$$

where the series is normally convergent in $\mathbb{C}^2$.

Before starting with the proof of Theorem 7.1, let us look at what it means for the function $e^{\underline{\zeta}\underline{z}}$ when $\underline{\zeta} \in \mathbb{C}^2$ satisfies $0 < |\zeta_1| < \pi$ and $0 < |\zeta_2| < \pi$; this function has exponential type $\max\{|\zeta_1|, |\zeta_2|\} < \pi$; from Theorem 7.1 we deduce:

$$e^{\underline{\zeta}\underline{z}} = \sum_{\|\underline{t}\| \in 2\mathbb{N}} \Lambda_{\underline{t},0}(\underline{z})\underline{\zeta}^{\underline{t}} + e^{\zeta_1} \sum_{t_1,\, t_2 \in 2\mathbb{N}} \Lambda_{\underline{t},1}(\underline{z})\underline{\zeta}^{\underline{t}} + e^{\zeta_2} \sum_{t_1,\, t_2 \in 2\mathbb{N}} \Lambda_{\underline{t},2}(\underline{z})\underline{\zeta}^{\underline{t}},$$

which can be written

$$e^{\underline{\zeta}\underline{z}} = \sum_{(\underline{t},i) \in \mathcal{T}} \Lambda_{\underline{t},i}(\underline{z}) e^{\zeta_i} \underline{\zeta}^{\underline{t}} \tag{7.2}$$

by setting $\zeta_0 = 0$.

We wish to use this formula by replacing ζ_1 with $-\zeta_1$ and/or ζ_2 with $-\zeta_2$. However the first sum does not behave well under these substitutions. This is why we splitted it into two parts in § 6, so that

$$\sum_{(\underline{t},i) \in \mathcal{T}} \Lambda_{\underline{t},i}(\underline{z}) e^{\zeta_i} \underline{\zeta}^{\underline{t}} = M_0(\underline{\zeta}, \underline{z}) + \widetilde{M}_0(\underline{\zeta}, \underline{z}) + M_1(\underline{\zeta}, \underline{z}) e^{\zeta_1} + M_2(\underline{\zeta}, \underline{z}) e^{\zeta_2}.$$

Hence (7.2) yields

$$\begin{cases} e^{\zeta_1 z_1 + \zeta_2 z_2} = M_0(\underline{\zeta}, \underline{z}) + \widetilde{M}_0(\underline{\zeta}, \underline{z}) + M_1(\underline{\zeta}, \underline{z}) e^{\zeta_1} + M_2(\underline{\zeta}, \underline{z}) e^{\zeta_2}, \\ e^{-\zeta_1 z_1 + \zeta_2 z_2} = M_0(\underline{\zeta}, \underline{z}) - \widetilde{M}_0(\underline{\zeta}, \underline{z}) + M_1(\underline{\zeta}, \underline{z}) e^{-\zeta_1} + M_2(\underline{\zeta}, \underline{z}) e^{\zeta_2}, \\ e^{\zeta_1 z_1 - \zeta_2 z_2} = M_0(\underline{\zeta}, \underline{z}) - \widetilde{M}_0(\underline{\zeta}, \underline{z}) + M_1(\underline{\zeta}, \underline{z}) e^{\zeta_1} + M_2(\underline{\zeta}, \underline{z}) e^{-\zeta_2}, \\ e^{-\zeta_1 z_1 - \zeta_2 z_2} = M_0(\underline{\zeta}, \underline{z}) + \widetilde{M}_0(\underline{\zeta}, \underline{z}) + M_1(\underline{\zeta}, \underline{z}) e^{-\zeta_1} + M_2(\underline{\zeta}, \underline{z}) e^{-\zeta_2}. \end{cases} \tag{7.3}$$

Let U, U_1, U_2, V_1, V_2 be five variables. Introduce the matrix

$$P = \begin{pmatrix} U & U & U_1 & U_2 \\ U & -U & V_1 & U_2 \\ U & -U & U_1 & V_2 \\ U & U & V_1 & V_2 \end{pmatrix}.$$

Its determinant is $4U^2(U_1 - V_1)(U_2 - V_2)$, hence is not 0. Denote by R the matrix P specialized at $(U, U_1, V_1, U_2, V_2) = (1, e^{\zeta_1}, e^{-\zeta_1}, e^{\zeta_2}, e^{-\zeta_2})$, and by Q the matrix P specialized at $(U, U_1, V_1, U_2, V_2) = (1, 1, -1, 1, -1)$:

$$R = \begin{pmatrix} 1 & 1 & e^{\zeta_1} & e^{\zeta_2} \\ 1 & -1 & e^{-\zeta_1} & e^{\zeta_2} \\ 1 & -1 & e^{\zeta_1} & e^{-\zeta_2} \\ 1 & 1 & e^{-\zeta_1} & e^{-\zeta_2} \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}.$$

Their determinants are not 0, thanks to our assumption $0 < |\zeta_1|, |\zeta_2| < \pi$. From (7.3) we deduce

$$\begin{pmatrix} e^{\zeta_1 z_1 + \zeta_2 z_2} \\ e^{-\zeta_1 z_1 + \zeta_2 z_2} \\ e^{\zeta_1 z_1 - \zeta_2 z_2} \\ e^{-\zeta_1 z_1 - \zeta_2 z_2} \end{pmatrix} = R \begin{pmatrix} M_0(\underline{\zeta}, \underline{z}) \\ \widetilde{M}_0(\underline{\zeta}, \underline{z}) \\ M_1(\underline{\zeta}, \underline{z}) \\ M_2(\underline{\zeta}, \underline{z}) \end{pmatrix}.$$

We also have

$$QR = 4\begin{pmatrix} 1 & 0 & \cosh(\zeta_1) & \cosh(\zeta_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sinh(\zeta_1) & 0 \\ 0 & 0 & 0 & \sinh(\zeta_2) \end{pmatrix},$$

hence

$$QR\begin{pmatrix} M_0(\underline{\zeta},\underline{z}) \\ \widetilde{M}_0(\underline{\zeta},\underline{z}) \\ M_1(\underline{\zeta},\underline{z}) \\ M_2(\underline{\zeta},\underline{z}) \end{pmatrix} = 4\begin{pmatrix} M_0(\underline{\zeta},\underline{z}) + \cosh(\zeta_1)M_1(\underline{\zeta},\underline{z}) + \cosh(\zeta_2)M_2(\underline{\zeta},\underline{z}) \\ \widetilde{M}_0(\underline{\zeta},\underline{z}) \\ \sinh(\zeta_1)M_1(\underline{\zeta},\underline{z}) \\ \sinh(\zeta_2)M_2(\underline{\zeta},\underline{z}) \end{pmatrix}$$

$$= 4\begin{pmatrix} \cosh(\zeta_1 z_1)\cosh(\zeta_2 z_2) \\ \sinh(\zeta_1 z_1)\sinh(\zeta_2 z_2) \\ \sinh(\zeta_1 z_1)\cosh(\zeta_2 z_2) \\ \cosh(\zeta_1 z_1)\sinh(\zeta_2 z_2). \end{pmatrix}.$$

Therefore (7.3) implies

$$\begin{aligned} \widetilde{M}_0(\underline{\zeta},\underline{z}) &= \sinh(\zeta_1 z_1)\sinh(\zeta_2 z_2), \\ M_1(\underline{\zeta},\underline{z}) &= \frac{\sinh(\zeta_1 z_1)\cosh(\zeta_2 z_2)}{\sinh(\zeta_1)}, \\ M_2(\underline{\zeta},\underline{z}) &= \frac{\cosh(\zeta_1 z_1)\sinh(\zeta_2 z_2)}{\sinh(\zeta_2)} \end{aligned}$$

and

$$\begin{aligned} M_0(\underline{\zeta},\underline{z}) = \cosh(\zeta_1 z_1)\cosh(\zeta_2 z_2) &- \frac{\sinh(\zeta_1 z_1)\cosh(\zeta_2 z_2)}{\tanh(\zeta_1)} \\ &- \frac{\cosh(\zeta_1 z_1)\sinh(\zeta_2 z_2)}{\tanh(\zeta_2)}. \end{aligned} \tag{7.4}$$

From

$$\sinh(\zeta(1-z)) = \sinh(\zeta)\cosh(\zeta z) - \sinh(\zeta z)\cosh(\zeta) \tag{7.5}$$

we see that (7.4) is equivalent to the formula for M_0 in Theorem 6.1.

In conclusion, from (7.2), which is a special case of Theorem 7.1, we deduce (7.3), which yields the results of Theorem 6.1. One easily deduces Theorem 4.1 from Theorem 6.1, and this is how we discovered the explicit formulae for the polynomials $\Lambda_{\underline{t},i}$.

We are now ready to proceed to the proof of Theorem 7.1.

Proof. [Proof of Theorem 7.1]

We start by using elementary hyperbolic trigonometric formulae:

$$\begin{aligned} e^{\zeta_1 z_1 + \zeta_2 z_2} &= \cosh(\zeta_1 z_1 + \zeta_2 z_2) + \sinh(\zeta_1 z_1 + \zeta_2 z_2), \\ \cosh(\zeta_1 z_1 + \zeta_2 z_2) &= \cosh(\zeta_1 z_1)\cosh(\zeta_2 z_2) + \sinh(\zeta_1 z_1)\sinh(\zeta_2 z_2), \\ \sinh(\zeta_1 z_1 + \zeta_2 z_2) &= \sinh(\zeta_1 z_1)\cosh(\zeta_2 z_2) + \cosh(\zeta_1 z_1)\sinh(\zeta_2 z_2). \end{aligned}$$

Using Theorem 6.1 (with the formula for M_0 given by (7.4)) and writing

$$\mathrm{e}^{\zeta_1} M_1(\underline{\zeta},\underline{z}) = \left(1+\frac{\cosh(\zeta_1)}{\sinh(\zeta_1)}\right)\sinh(\zeta_1 z_1)\cosh(\zeta_2 z_2),$$
$$\mathrm{e}^{\zeta_2} M_2(\underline{\zeta},\underline{z}) = \left(1+\frac{\cosh(\zeta_2)}{\sinh(\zeta_2)}\right)\sinh(\zeta_2 z_2)\cosh(\zeta_1 z_1),$$

we deduce the first formula of (7.3), which implies the three other ones. As a consequence, we obtain (7.2), which is the special case of Theorem 7.1 for the functions $\underline{z} \mapsto \mathrm{e}^{\underline{\zeta}\underline{z}}$ when $\underline{\zeta} \in \mathbb{C}^2$ has $|\zeta_1| < \pi$ and $|\zeta_2| < \pi$. As a matter of fact, the results we established before the present proof show that (7.2) is equivalent to the formulae of Theorem 6.1.

From this special case we deduce the general case of Theorem 7.1 by means of Laplace transform in two variables, as follows. Let

$$f(z_1,z_2) = \sum_{k_1\geq 0}\sum_{k_2\geq 0} \frac{a_{k_1,k_2}}{k_1!k_2!} z_1^{k_1} z_2^{k_2}$$

be an entire function in $\mathbb{C}^2$ of exponential type $\leq \tau$. Corollary 7.1 shows that the Laplace transform of f

$$F(\zeta_1,\zeta_2) = \sum_{k_1\geq 0}\sum_{k_2\geq 0} a_{k_1,k_2}\zeta_1^{-k_1-1}\zeta_2^{-k_2-1},$$

is analytic in the domain $\bigl\{(\zeta_1,\zeta_2) \in \mathbb{C}^2 \ \mid\ |\zeta_1| > \tau,\ |\zeta_2| > \tau\bigr\}$. For $r > \tau$, it follows from Cauchy's residue Theorem [7, Equation (11)] and from the uniform convergence of the series for F on $|\zeta_1| = |\zeta_2| = r$ that we have

$$f(z_1,z_2) = \frac{1}{(2\pi \mathrm{i})^2}\iint_{|\zeta_1|=|\zeta_2|=r} \mathrm{e}^{\zeta_1 z_1+\zeta_2 z_2} F(\zeta_1,\zeta_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2 \tag{7.6}$$

and

$$D^{(t_1,t_2)} f(z_1,z_2) = \frac{1}{(2\pi \mathrm{i})^2}\iint_{|\zeta_1|=|\zeta_2|=r} \zeta_1^{t_1}\zeta_2^{t_2}\mathrm{e}^{\zeta_1 z_1+\zeta_2 z_2} F(\zeta_1,\zeta_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2. \tag{7.7}$$

Assume $\tau < \pi$. Let r satisfy $\tau < r < \pi$. In (7.6) we replace $\mathrm{e}^{\zeta_1 z_1+\zeta_2 z_2}$ by the right hand side of (7.2):

$$f(z_1,z_2) = \sum_{(\underline{t},i)\in\mathcal{T}} \Lambda_{\underline{t},i}(z_1,z_2)\frac{1}{(2\pi \mathrm{i})^2}\iint_{|\zeta_1|=|\zeta_2|=r} \zeta_1^{t_1}\zeta_2^{t_2}\mathrm{e}^{\zeta_i} F(\zeta_1,\zeta_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2.$$

Using (7.7), we deduce

$$f(z_1,z_2) = \sum_{(\underline{t},i)\in\mathcal{T}} (D^{\underline{t}} f)(\underline{e}_i)\Lambda_{\underline{t},i}(z_1,z_2).$$

The fact that this series is normally convergent in $\mathbb{C}^2$ follows from Lemma 7.1 and the upper bounds (4.1) : given $r > 0$ and κ with $\tau(f) < \kappa < \pi$, there exists $c > 0$ such that, for $(\underline{t},i) \in \mathcal{T}$ with sufficiently large $\|\underline{t}\|$, we have

$$|D^{\underline{t}} f(\underline{e}_i)| \leq \kappa^{\|\underline{t}\|} \quad\text{and}\quad \sup_{\|\underline{z}\|\leq r} |\Lambda_{\underline{t},i}(\underline{z})| \leq \frac{c}{\pi^{\|\underline{t}\|}},$$

hence the series

$$\sum_{(\underline{t},i)\in\mathcal{T}} |(D^{\underline{t}}f)(\underline{e}_i)| \sup_{\|\underline{z}\|\le r} |\Lambda_{\underline{t},i}(\underline{z})|$$

converges. This completes the proof of Theorem 7.1. □

8. Functions of finite exponential type

Let K be a nonnegative integer and $h_{1,1},\dots,h_{K,1}$ and $h_{1,2},\dots,h_{K,2}$ be even entire functions of a single variable with exponential type $\le K\pi$. Then the function

$$f(z_1,z_2)=\sum_{k=1}^{K}\left(h_{k,1}(z_2)\sin(k\pi z_1)+h_{k,2}(z_1)\sin(k\pi z_2)\right)$$

is an entire function of exponential type $\le K\pi$ which satisfies $(D^{\underline{t}}f)(\underline{e}_i)=0$ for all $(\underline{t},i)\in\mathcal{T}$. Corollary 8.1 below shows that any entire function having exponential type $\le K\pi$ which satisfies $(D^{\underline{t}}f)(\underline{e}_i)=0$ for all $(\underline{t},i)\in\mathcal{T}$ is of this form.

Here is the two–variable analog of the result of Buck's [7, Proposition 3] for a single variable:

Theorem 8.1. *Let K be a nonnegative integer. Let f be an entire function in $\mathbb{C}^2$ of finite exponential type $\le\tau$, with $\tau<(K+1)\pi$. Then for $\underline{z}\in\mathbb{C}^2$ we have*

$$f(\underline{z})=\sum_{(\underline{t},i)\in\mathcal{T}}(D^{\underline{t}}f)(\underline{e}_i)g_{\underline{t},i}(\underline{z})+\sum_{k=1}^{K}\left(h_{k,1}(z_2)\sin(k\pi z_1)+h_{k,2}(z_1)\sin(k\pi z_2)\right),$$

where the functions $g_{\underline{t},i}(\underline{z})$ are entire functions in $\mathbb{C}^2$, the series is normally convergent in $\mathbb{C}^2$ and $h_{k,1}$, $h_{k,2}$ $(k=1,2,\dots,K)$ are even entire functions of a single variable of exponential type $\le\tau$.

Proof. Theorem 7.1 proves the formula of Theorem 8.1 for $K=0$. Assume $K\ge 1$. We extend the proof of [7, § 8] to two variables: in the formula of the Laplace transform (7.6), we will expand $\mathrm{e}^{\zeta_1 z_1+\zeta_2 z_2}$ using the first formula of (7.3) together with Theorem 6.1.

For $M_1(\underline{\zeta},\underline{z})$ and $M_2(\underline{\zeta},\underline{z})$ we use [7, Equation (17)] in the form

$$\frac{\sinh(\zeta z)}{\sinh(\zeta)}=A_K(\zeta,z)+G_K(\zeta,z)$$

with

$$A_K(\zeta,z)=2\pi\sum_{k=1}^{K}\frac{(-1)^{k+1}k\sin(k\pi z)}{\zeta^2+k^2\pi^2}$$

and G_K an analytic function in the domain $\left\{(\zeta,z)\in\mathbb{C}^2 \;\mid\; |\zeta|<(K+1)\pi\right\}$. For $M_0(\underline{\zeta},\underline{z})$ we use [7, Equation (20)] in the form

$$\sinh(\zeta z)\coth(\zeta)=B_K(\zeta,z)+H_K(\zeta,z)$$

with

$$B_K(\zeta, z) = -2\pi \sum_{k=1}^{K} \frac{k\sin(k\pi z)}{\zeta^2 + k^2\pi^2}$$

and H_K an analytic function in the domain $\big\{(\zeta, z) \in \mathbb{C}^2 \ \mid\ |\zeta| < (K+1)\pi\big\}$. From Theorem 6.1 we derive

$$\begin{aligned} M_1(\underline{\zeta}, \underline{z}) &= \big(A_K(\zeta_1, z_1) + G_K(\zeta_1, z_1)\big)\cosh(\zeta_2 z_2),\\ M_2(\underline{\zeta}, \underline{z}) &= \cosh(\zeta_1 z_1)\big(A_K(\zeta_2, z_2) + G_K(\zeta_2, z_2)\big) \end{aligned}$$

and from (7.4) we get

$$\begin{aligned} M_0(\underline{\zeta}, \underline{z}) = \cosh(\zeta_1 z_1)\cosh(\zeta_2 z_2) &- \big(B_K(\zeta_1, z_1) + H_K(\zeta_1, z_1)\big)\cosh(\zeta_2 z_2)\\ &- \cosh(\zeta_1 z_1)\ \big(B_K(\zeta_2, z_2) + H_K(\zeta_2, z_2)\big). \end{aligned}$$

We define $g_{\underline{t},1}(\underline{z})$ and $g_{\underline{t},2}(\underline{z})$ for t_1 and t_2 in $2\mathbb{N}$ by writing the Taylor series

$$\begin{aligned} G_K(\zeta_1, z_1)\cosh(\zeta_2 z_2) &= \sum_{(\underline{t},1)\in\mathcal{T}} g_{\underline{t},1}(z_1, z_2)\underline{\zeta}^{\underline{t}},\\ \cosh(\zeta_1 z_1) G_K(\zeta_2, z_2) &= \sum_{(\underline{t},2)\in\mathcal{T}} g_{\underline{t},2}(z_1, z_2)\underline{\zeta}^{\underline{t}}, \end{aligned}$$

and so

$$g_{t_1,t_2,2}(z_1, z_2) = g_{t_2,t_1,1}(z_2, z_1).$$

Next we define $g_{\underline{t},0}(\underline{z})$ for $\|\underline{t}\| \in 2\mathbb{N}$ as the coefficients of the Taylor series

$$\begin{aligned} \cosh(\zeta_1 z_1 + \zeta_2 z_2) - H_K(\zeta_1, z_1)\cosh(\zeta_2 z_2) - \cosh(\zeta_1 z_1) H_K(\zeta_2, z_2)&\\ = \sum_{(\underline{t},0)\in\mathcal{T}} g_{\underline{t},0}(z_1, z_2)\underline{\zeta}^{\underline{t}}&. \end{aligned}$$

Finally we set

$$\begin{aligned} \chi(\underline{\zeta}, \underline{z}) = \big(A_K(\zeta_1, z_1)\mathrm{e}^{\zeta_1} - B_K(\zeta_1, z_1)\big)\cosh(\zeta_2 z_2)&\\ + \big(A_K(\zeta_2, z_2)\mathrm{e}^{\zeta_2} - B_K(\zeta_2, z_2)\big)\cosh(\zeta_1 z_1)&. \end{aligned}$$

Using the first formula of (7.3) we obtain

$$\mathrm{e}^{\underline{\zeta}\underline{z}} = \sum_{(\underline{t},i)\in\mathcal{T}} g_{\underline{t},i}(\underline{z})\mathrm{e}^{\zeta_i}\underline{\zeta}^{\underline{t}} + \chi(\underline{\zeta}, \underline{z}), \tag{8.1}$$

where for each $z \in \mathbb{C}$ the series is normally convergent in $\{\underline{\zeta} \in \mathbb{C}^2 \ \mid\ |\underline{\zeta}| < (K+1)\pi\}$. Further, we have

$$\chi(\underline{\zeta}, \underline{z}) = \sum_{k=1}^{K} \big(\chi_{k,1}(\underline{\zeta}, z_2)\sin(k\pi z_1) + \chi_{k,2}(\underline{\zeta}, z_1)\sin(k\pi z_2)\big)\,, \tag{8.2}$$

where

$$\chi_{k,1}(\underline{\zeta}, z) = \frac{2\pi k}{\zeta_1^2 + k^2\pi^2}\big((-1)^{k+1}\mathrm{e}^{\zeta_1} + 1\big)\cosh(\zeta_2 z)$$

and

$$\chi_{k,2}(\underline{\zeta},z)=\frac{2\pi k}{\zeta_2^2+k^2\pi^2}\bigl((-1)^{k+1}\mathrm{e}^{\zeta_2}+1\bigr)\cosh(\zeta_1 z).$$

For $k=1,\dots,K$ and $j=1,2$, the function $\chi_{k,j}(\underline{\zeta},z)$ is analytic in the domain

$$\bigl\{(\underline{\zeta},z)\in\mathbb{C}^3 \;\mid\; K\pi<|\zeta_1|<(K+1)\pi,\; K\pi<|\zeta_2|<(K+1)\pi\bigr\}$$

and the map $z\mapsto\chi_{k,j}(\underline{\zeta},z)$ is even.

Let f be an entire function in $\mathbb{C}^2$ of finite exponential type $\le\tau$, with $K\pi\le\tau<(K+1)\pi$. The assumption $\tau\ge K\pi$ is no loss of generality [a] for the proof of Theorem 8.1. Let r satisfy $\tau<r<(K+1)\pi$. We denote by F the Laplace transform of f. Combining (7.6) with (8.1) we deduce

$$\begin{aligned}f(z_1,z_2)&=\frac{1}{(2\pi\mathrm{i})^2}\iint_{|\zeta_1|=|\zeta_2|=r}\left(\sum_{(\underline{t},i)\in\mathcal{T}}\mathrm{e}^{\zeta_i}g_{\underline{t},i}(\underline{z})\underline{\zeta}^{\underline{t}}+\chi(\underline{\zeta},\underline{z})\right)F(\zeta_1,\zeta_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2\\&=\sum_{(\underline{t},i)\in\mathcal{T}}g_{\underline{t},i}(\underline{z})\frac{1}{(2\pi\mathrm{i})^2}\iint_{|\zeta_1|=|\zeta_2|=r}\mathrm{e}^{\zeta_i}\underline{\zeta}^{\underline{t}}F(\zeta_1,\zeta_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2\\&\quad+\frac{1}{(2\pi\mathrm{i})^2}\iint_{|\zeta_1|=|\zeta_2|=r}\chi(\underline{\zeta},\underline{z})F(\zeta_1,\zeta_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2.\end{aligned}$$

From (7.7) we deduce

$$\frac{1}{(2\pi\mathrm{i})^2}\iint_{|\zeta_1|=|\zeta_2|=r}\mathrm{e}^{\zeta_i}\underline{\zeta}^{\underline{t}}F(\zeta_1,\zeta_2)\mathrm{d}\zeta_1\mathrm{d}\zeta_2=(D^{\underline{t}}f)(\underline{e}_i).$$

Finally using (8.2) we deduce the formula of Theorem 8.1 with

$$h_{k,j}(z)=\frac{1}{(2\pi\mathrm{i})^2}\iint_{|\zeta_1|=|\zeta_2|=r}\chi_{k,j}(\underline{\zeta},z)F(\underline{\zeta})\mathrm{d}\zeta_1\mathrm{d}\zeta_2$$

for any r with $\tau<r<(K+1)\pi$. This function $h_{k,j}$ has exponential type $\le r$. Since this is true for all r in the range $\tau<r<(K+1)\pi$, and since the exponential type is defined with a lim sup, the exponential type is $\le\tau$.

Let κ satisfy $\tau(f)<\kappa<r$ (recall $r<(K+1)\pi$) and let $R>0$. Lemma 7.1 implies that for $(\underline{t},i)\in\mathcal{T}$ with sufficiently large $\|\underline{t}\|$, we have

$$|D^{\underline{t}}f(\underline{e}_i)|\le\kappa^{\|\underline{t}\|}.$$

Since the three functions $G_K(\zeta_1,z_1)\cosh(\zeta_2z_2)$, $\cosh(\zeta_1z_1)G_K(\zeta_2,z_2)$ and

$$\cosh(\zeta_1z_1+\zeta_2z_2)-H_K(\zeta_1,z_1)\cosh(\zeta_2z_2)-\cosh(\zeta_1z_1)H_K(\zeta_2,z_2)$$

are analytic in the domain

$$\{(\zeta_1,\zeta_2,z_1,z_2)\in\mathbb{C}^4 \;\mid\; |\zeta_1|<(K+1)\pi,\;|\zeta_2|<(K+1)\pi\}$$

[a] In [7, Proposition 3], the condition $\tau(f)<r<(K+1)\pi$ should be replaced with $\max\{\tau(f),K\pi\}<r<(K+1)\pi$.

and since $r < (K+1)\pi$, it follows from Cauchy's inequalities (7.1) that there exists $c > 0$ (depending on r and R) such that, for $(\underline{t}, i) \in \mathcal{T}$, we have

$$\sup_{\|\underline{z}\| \le R} |g_{\underline{t},i}(\underline{z})| \le \frac{c}{r^{\|\underline{t}\|}}.$$

Hence the series

$$\sum_{(\underline{t},i)\in\mathcal{T}} |(D^{\underline{t}} f)(\underline{e}_i)| \sup_{\|\underline{z}\| \le R} |g_{\underline{t},i}(\underline{z})|$$

converges.

This completes the proof of Theorem 8.1. □

A consequence is the following analog in two variables of Schoenberg's result [7, Corollary 2]:

Corollary 8.1. *Let f be an entire function having exponential type $\le \tau$, with $\tau < (K+1)\pi$. Assume $(D^{\underline{t}} f)(\underline{e}_i) = 0$ for all $(\underline{t}, i) \in \mathcal{T}$. Then there exist even entire functions of a single variable $h_{k,1}$ and $h_{k,2}$ $(k = 1, 2, \ldots, K)$ having exponential type $\le \tau$ such that*

$$f(z_1, z_2) = \sum_{k=1}^{K} \big(h_{k,1}(z_2)\sin(k\pi z_1) + h_{k,2}(z_1)\sin(k\pi z_2)\big).$$

9. Further bivariate interpolation theories

In the present paper we extended to two variables the univariate Lidstone interpolation theory by considering

$(D^{(t_1,t_2)} f)(e_0)$ with t_1 and t_2 even, $\quad (D^{(t_1,t_2)} f)(e_0)$ with t_1 and t_2 odd,
$(D^{(t_1,t_2)} f)(e_1)$ with t_1 and t_2 even, $\quad (D^{(t_1,t_2)} f)(e_2)$ with t_1 and t_2 even.

Our solution is not the unique one for which the theory can be developed. Indeed, there is a potential similar story for each choice of 4 triples (ν_1, ν_2, i) among the 12 elements in $(\mathbb{Z}/2\mathbb{Z})^2 \times \{0, 1, 2\}$. For a polynomial or an entire function f, one considers the set of values

$$(D^{(t_1,t_2)} f)(e_i) \text{ with } t_1 \in \nu_1 \text{ and } t_2 \in \nu_2$$

for the four selected triples (ν_1, ν_2, i). Our choice is

$$(0,0,0), \quad (1,1,0), \quad (0,0,1), \quad (0,0,2).$$

For each $i \in \{0, 1, 2\}$, the choice of the four triples

$$(0,0,i), \quad (1,1,i), \quad (1,0,i), \quad (0,1,i)$$

corresponds to the Taylor expansion at $\underline{e}_i$. There are $\binom{12}{4} = 495$ choices for the four triples (ν_1, ν_2, i). However, this number can be reduced by using symmetries. Furthermore, not all of these sets of four triples are admissible.

9.1. *Admissible sets of four triples*

Let $\mathcal{S}$ be a set of four triples $(\nu_1, \nu_2, i) \in (\mathbb{Z}/2\mathbb{Z})^2 \times \{0,1,2\}$. We denote by $\mathfrak{T}_S$ the set of $(\underline{t}, i) \in \mathbb{N}^2 \times \{0,1,2\}$ such that $t_1 \in \nu_1$ and $t_2 \in \nu_2$ for all $(\nu_1, \nu_2, i) \in \mathcal{S}$.

Definition: A set $\mathcal{S}$ of four triples (ν_1, ν_2, i) is called *admissible* if the map from $\mathbb{C}[z_1, z_2]$ to $\mathbb{C}^{\mathfrak{T}_S}$ which sends f to the tuple $(D^{\underline{t}} f)(e_i)$, $(\underline{t}, i) \in \mathfrak{T}_S$ is an isomorphism.

For instance the choice

$$\{(0,0,0), \quad (1,1,0), \quad (0,0,1), \quad (1,1,1)\}$$

is not admissible: if g is an odd function of a single variable, the function $f(z_1, z_2) = g(z_2)$ satisfies

$$(D^{\underline{t}} f)(\underline{e}_0) = (D^{\underline{t}} f)(\underline{e}_1) = 0$$

for all $\underline{t}$ with $\|\underline{t}\|$ even. Also, if, for the four triples, the values of ν_1 are all 1, then the existence and unicity of the interpolation polynomials is not guaranteed; this situation is similar to the question of interpolation of an univariate function using derivatives of odd order at two points [6, §1.2] (*even Lidstone–type sequences*); for such an interpolation problem, instead of interpolating $f(z_1, z_2)$ using the four triples $(1, \nu_2, i)$, we interpolate $(\partial/\partial z_1) f(z_1, z_2)$ using the corresponding four triples $(0, \nu_2, i)$, assuming this second set is admissible.

Lemma 2.1 shows that our set of four triples

$$\{(0,0,0), \quad (1,1,0), \quad (0,0,1), \quad (0,0,2)\}$$

is admissible.

9.2. *Bivariate Whittaker interpolation*

Two other admissible choices for the four triples (ν_1, ν_2, i) are

$$\{(0,0,0), \quad (1,1,0), \quad (1,0,1), \quad (0,1,2)\}$$

and

$$\{(0,0,0), \quad (1,1,0), \quad (0,1,1), \quad (1,0,2)\};$$

one may compare with Whittaker's expansion of a function of a single variable using derivatives of odd order at one point and even order at the other [6, §6].

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