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Families of Thue equations associated with a rank one subgroup of the unit group of a number field

by

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Abstract. Let K be an algebraic number field of degree $d \geq 3$, $\sigma_1, \sigma_2, \ldots, \sigma_d$ the embeddings of K into $\mathbb{C}$, α a nonzero element in K, $a_0 \in \mathbb{Z}$, $a_0 > 0$ and

$$F_0(X,Y) = a_0 \prod_{i=1}^{d} (X - \sigma_i(\alpha)Y).$$

Let υ be a unit in K. For $a \in \mathbb{Z}$, we twist the binary form $F_0(X,Y) \in \mathbb{Z}[X,Y]$ by the powers υ^a ($a \in \mathbb{Z}$) of υ by setting

$$F_a(X,Y) = a_0 \prod_{i=1}^{d} (X - \sigma_i(\alpha\upsilon^a)Y).$$

Given $m \geq 0$, our main result is an effective upper bound for the size of solutions $(x,y,a) \in \mathbb{Z}^3$ of the Diophantine inequalities

$$0 < |F_a(x,y)| \leq m$$

for which $xy \neq 0$ and $\mathbb{Q}(\alpha\upsilon^a) = K$. Our estimate is explicit in terms of its dependence on m, the regulator of K and the heights of F_0 and of υ; it also involves an effectively computable constant depending only on d.

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1 Introduction and the main results

Let $d \geq 3$ be a given integer. We denote by $\kappa_1, \kappa_2, \ldots, \kappa_{38}$ positive effectively computable constants which depend only on d.

Let K be a number field of degree d. Denote by $\sigma_1, \sigma_2, \ldots, \sigma_d$ the embeddings of K into $\mathbb{C}$ and by R the regulator of K. Let $\alpha \in K$, $\alpha \neq 0$, and let $a_0 \in \mathbb{Z}$, $a_0 > 0$, be such that the coefficients of the polynomial

$$f_0(X) = a_0 \prod_{i=1}^{d} \bigl(X - \sigma_i(\alpha)\bigr)$$

are in $\mathbb{Z}$. Let υ be a unit in K, not a root of unity. For $a \in \mathbb{Z}$, define the polynomial $f_a(X)$ in $\mathbb{Z}[X]$ and the binary form $F_a(X, Y)$ in $\mathbb{Z}[X, Y]$ by

$$f_a(X) = a_0 \prod_{i=1}^{d} \bigl(X - \sigma_i(\alpha\upsilon^a)\bigr)$$

and

$$F_a(X, Y) = Y^d f_a(X/Y) = a_0 \prod_{i=1}^{d} \bigl(X - \sigma_i(\alpha\upsilon^a)Y\bigr).$$

Define

$$\lambda_0 = a_0 \prod_{i=1}^{d} \max\{1, |\sigma_i(\alpha)|\} \quad \text{and} \quad \lambda = \prod_{i=1}^{d} \max\{1, |\sigma_i(\upsilon)|\}.$$

Let $m \in \mathbb{Z}$, $m > 0$. We consider the family of Diophantine inequalities

$$0 < |F_a(x, y)| \le m, \tag{1}$$

where the unknowns (x, y, a) take their values in the set of elements in $\mathbb{Z}^3$ such that $xy \neq 0$ and $\mathbb{Q}(\alpha\upsilon^a) = K$. It follows from the results in [4] that the set of solutions is finite. However, the proof in [4] relies on Schmidt's subspace theorem, which is not effective. Here by using lower bounds for linear forms in logarithms, we give an upper bound for $\max\{|x|, |y|, |a|\}$, which is explicit in terms of m, R, λ_0 and λ and which involves an effectively computable constant depending only on d.

For $x \in \mathbb{R}$, $x > 0$, we use the notation $\log^* x$ to denote $\max\{1, \log x\}$. Here is our main result.

Theorem 1. *There exists an effectively computable constant $\kappa_1 > 0$, depending only on d, such that any solution $(x, y, a) \in \mathbb{Z}^3$ of (1), which verifies $xy \neq 0$ and $\mathbb{Q}(\alpha\upsilon^a) = K$, satisfies*

$$|a| \le \kappa_1 \lambda^{d^2(d+2)/2} (R + \log m + \log \lambda_0) R \log^\star R.$$

Under the assumptions of Theorem 1, with the help of the upper bound

$$\mathrm{H}(F_a) \le 2^d \lambda_0 \, \lambda^{|a|}$$

for the (usual) height of the form F_a (namely the maximum of the absolute values of the coefficients of the form), it follows from the bound (3.2) in [1, Theorem 3] (see also [2, Th. 9.6.2]) that

$$\log \max\{|x|, |y|\} \le \kappa_2 \bigl(R + \log^\star m + |a| \log \lambda + \log \lambda_0\bigr) R (\log^\star R) \tag{2}$$

where κ_2 is an explicit constant depending only on d. Combining this upper bound with our Theorem 1 provides an effective upper bound for $\max\{|x|,|y|,|a|\}$.

Corollary 2. *Under the assumptions of Theorem* 1*, there exists an effectively computable constant κ_3 depending only on d such that*

$$\max\{|x|,|y|,|a|\} \leq \kappa_3 \lambda^{d^2(d+2)/2}(\log\lambda)(R+\log m+\log\lambda_0)R^2(\log^\star R)^2.$$

Our proof of Theorem 1 actually gives a much stronger estimate for $|a|$; see Theorem 3 below. It involves an extra parameter $\mu>1$ that we now define.

For $i=1,\dots,d$, set $\upsilon_i=\sigma_i(\upsilon)$ and assume

$$|\upsilon_1|\leq|\upsilon_2|\leq\cdots\leq|\upsilon_d|.$$

Define

$$\mu=\begin{cases}\lambda & \text{if } |\upsilon_1|=|\upsilon_{d-1}| \text{ or } |\upsilon_2|=|\upsilon_d|,\\ \min\left\{\dfrac{|\upsilon_{d-1}|}{|\upsilon_1|},\dfrac{|\upsilon_d|}{|\upsilon_2|}\right\} & \text{if } |\upsilon_1|<|\upsilon_2|=|\upsilon_{d-1}|<|\upsilon_d|,\\ \dfrac{|\upsilon_{d-1}|}{|\upsilon_2|} & \text{if } |\upsilon_2|<|\upsilon_{d-1}|.\end{cases}$$

Notice that the condition $|\upsilon_1|=|\upsilon_{d-1}|$ means $|\upsilon_1|=|\upsilon_2|=\cdots=|\upsilon_{d-1}|$ and that the condition $|\upsilon_2|=|\upsilon_d|$ means $|\upsilon_2|=|\upsilon_3|=\cdots=|\upsilon_d|$; using Lemma 12, we deduce that each of these two conditions implies that d is odd, hence that the field K is almost totally imaginary (namely, with a single real embedding) – compare with [9].

Theorem 3. *There exists a positive effectively computable constant κ_4, depending only on d, with the following property. Let $(x,y,a)\in\mathbb{Z}^3$ satisfy*

$$xy\neq 0,\quad [\mathbb{Q}(\alpha\upsilon^a):\mathbb{Q}]=d \quad\textit{and}\quad 0<|F_a(x,y)|\leq m.$$

Then

$$|a|\leq\kappa_4\frac{\log\lambda}{\log\mu}(R+\log m+\log\lambda_0+\log\lambda)R\log\left(R\frac{(\log\lambda)^2}{\log\mu}\right). \tag{3}$$

On the one hand, using Lemma 13 (§3.6), we will prove in §5 that

$$\log\mu\geq\kappa_5\lambda^{-d^2(d+2)/2}(\log\lambda)^2,$$

which will enable us to deduce Theorem 1 from Theorem 3. On the other hand, thanks to (7), we have $\mu\leq\lambda^2$. Hence the largest possible value of

μ is λ^{κ_6} with a positive constant κ_6 depending only on d. For the units υ satisfying such an estimate, the conclusion of Theorem 3 becomes

$$|a| \le \kappa_7(R + \log m + \log \lambda_0 + \log \lambda) R(\log R + \log^\star \log^\star \lambda) \tag{4}$$

with a positive effectively computable constant κ_7 depending only on d. In §2, we give a few examples where this last bound is valid.

In Theorem 1, the hypothesis that υ is not a root of unity cannot be omitted. Here is an example with $\alpha = a_0 = m = 1$. Let $\Phi_n(X)$ be the cyclotomic polynomial of index n and degree $\varphi(n)$ (Euler totient function). Let ζ_n be a primitive n–th root of unity. Set $f_0 = \Phi_n$ and $u = \zeta_n$. For $a \in \mathbb{Z}$ with $\gcd(a, n) = 1$, the irreducible polynomial f_a of ζ_n^a is nothing else than f_0. Hence, if the equation

$$F_0(x, y) = \pm 1$$

has a solution $(x, y) \in \mathbb{Z}^2$ with $xy \neq 0$, then for infinitely many $a \in \mathbb{Z}$ the twisted Thue equation $F_a(x, y) = \pm 1$ has also the solution (x, y), since $F_a = F_0$. For instance, when $n = 12$, we have $\Phi_{12}(X) = X^4 - X^2 + 1$ and the equation

$$x^4 - x^2y^2 + y^4 = 1$$

has the solutions $(1, 1)$, $(-1, 1)$, $(1, -1)$, $(-1, -1)$.

Let us compare the results of the present paper with our previous work.

The main result of [5], which deals only with non totally real cubic equations, is a special case of Theorem 3; the "constants" in [5] depend on α and υ, while here they depend only on d. The main result of [6] deals with Thue equations twisted by a set of units which is not supposed to be a group of rank 1, but it involves an assumption (namely that at least two of the conjugates of υ have a modulus as large as a positive power of $\lceil \upsilon \rceil$) which we do not need here. Our Theorem 3 also improves the main result of [7]: we remove the assumption that the unit is totally real (besides, the result of [7] is not explicit in terms of the heights and regulator). We also notice that part (iii) of Theorem 1.1 of [8] follows from our Theorem 3. The main result of [9] does not assume that the twists are done by a group of units of rank 1, but it needs a strong assumption which does not occur here, namely that the field K has at most one real embedding.

2 Examples

The lower bound $\mu \ge \lambda^{\kappa_6}$ quoted in Section 1 is true
• when $d = 3$ and the cubic field K is not totally real;
• for the Salem numbers;
• for the roots of the polynomials in the families giving the simplest fields of degree 3 (see [8]), and also the simplest fields of degrees 4 and 6;

• when $|\upsilon_1| = |\upsilon_2|$ and $|\upsilon_{d-1}| = |\upsilon_d|$ with $d \geq 4$. In particular when $-\upsilon$ is a Galois conjugate of υ (which means that the irreducible polynomial of υ is in $\mathbb{Z}[X^2]$).

Here is an example of this last situation. Let ϵ be an algebraic unit, not a root of unity, of degree $\ell \geq 2$ and conjugates $\epsilon_1, \epsilon_2, \ldots, \epsilon_\ell$. Let $h \geq 2$ and let $d = \ell h$. For $a \in \mathbb{Z}$, define

$$F_a(X,Y) = \prod_{i=1}^{\ell}(X^h - \epsilon_i^a Y^h). \tag{5}$$

Let R be the regulator of the field $\mathbb{Q}(\epsilon^{1/h})$.

From Theorem 3 we deduce the following corollary.

Corollary 4. *Let $m \geq 1$. If the form F_a in (5) is irreducible and if there exists $(x,y) \in \mathbb{Z}^2$ with $xy \neq 0$ and $|F_a(x,y)| \leq m$, then*

$$|a| \leq \kappa_8(R + \log m + \log \overline{|\epsilon|}) R \log^\star(R \log \overline{|\epsilon|}).$$

Proof. Without loss of generality, assume $|\epsilon_1| \leq |\epsilon_2| \leq \cdots \leq |\epsilon_\ell|$, so that $|\epsilon_\ell| = \overline{|\epsilon|}$. Let ζ be a primitive h-th root of unity. Let $\upsilon = \epsilon^{1/h}$. We apply Theorem 3 with $\alpha = \zeta$, $a_0 = 1$, $\lambda_0 = 1$, $\lambda \leq \overline{|\epsilon|}^\ell$, $F_0(X,Y) = (X^h - Y^h)^\ell$ and

$$\upsilon_{ih+j} = \zeta^{j-1}\epsilon_{i+1}^{1/h} \quad (0 \leq i \leq \ell - 1,\ 1 \leq j \leq h).$$

From $|\upsilon_1| = |\upsilon_2| = |\epsilon_1|^{1/h} < 1$ and $|\upsilon_{d-1}| = |\upsilon_d| = |\epsilon_\ell|^{1/h}$ we deduce

$$\mu = \left|\frac{\epsilon_\ell}{\epsilon_1}\right|^{1/h} = \left|\frac{\upsilon_d}{\upsilon_1}\right|$$

and using (7) we conclude

$$\log \mu \geq \frac{2}{d-1} \log \lambda. \qquad \square$$

A variant of this proof is to take $\alpha = 1$, $\lambda_0 = 1$, $F_0(X,Y) = (X-Y)^d$, and to use the fact that ζ^a is also a primitive h-th root of unity since F_a is irreducible.

Remark. There are cases where μ is very small when compared to λ. Let D be an integer ≥ 2. Consider the algebraic number field $K = \mathbf{Q}(\omega)$ where $\omega = \sqrt[d]{D^d - 1}$. The number $\upsilon = D - \omega$ is a *Bernstein-Hasse* unit of K. When d is fixed, λ is larger than $\kappa_9 D^{d-1}$, while μ is bounded above by κ_{10}. In this example, when d is odd, the field K is almost totally imaginary in the sense of [9] and our proof yields the estimate (4). However, when d is even, we are not able to prove the estimate (4) : the estimate (3) has one extra factor $\log \lambda$.

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3 Auxiliary results

3.1 Mahler measure, house and height

When f is a polynomial in one variable of degree d with coefficients in $\mathbb{Z}$ and leading coefficient $c_0 > 0$, the Mahler measure of f is

$$\mathrm{M}(f) = c_0 \prod_{i=1}^{d} \max\{1, |\gamma_i|\},$$

where $\gamma_1, \gamma_2, \dots, \gamma_d$ are the roots of f in $\mathbb{C}$.

Let us recall[1] that the logarithmic height $\mathrm{h}(\gamma)$ of an algebraic number γ of degree d is $\frac{1}{d} \log \mathrm{M}(\gamma)$ where $\mathrm{M}(\gamma)$ is the Mahler measure of the irreducible polynomial of γ. We have

$$\mathrm{M(f)} \le \sqrt{\mathrm{d}+1}\,\mathrm{H(f)} \quad \text{and} \quad \mathrm{H(f)} \le 2^{\mathrm{d}}\mathrm{M(f)} \tag{6}$$

(see [12], Annex to Chapter 3, *Inequalities Between Different Heights of a Polynomial,* pp. 113–114; see also [2, §1.9]). The second upper bound in (6) could be replaced by the sharper one

$$\mathrm{H}(f) \le \binom{d}{\lfloor d/2 \rfloor} \mathrm{M}(f),$$

but we shall not need it.

Let υ be a unit of degree d and conjugates $\upsilon_1, \dots, \upsilon_d$ with

$$|\upsilon_1| \le |\upsilon_2| \le \cdots \le |\upsilon_d|,$$

so that $\lceil\overline{\upsilon}\rceil = |\upsilon_d|$. Let $\lambda = \mathrm{M}(\upsilon)$ and let s be an index in $\{1, \dots, d-1\}$ such that

$$|\upsilon_1| \le |\upsilon_2| \le \cdots \le |\upsilon_s| \le 1 \le |\upsilon_{s+1}| \le \cdots \le |\upsilon_d|.$$

We have

$$\lambda = \mathrm{M}(\upsilon) = |\upsilon_{s+1} \cdots \upsilon_d| \le |\upsilon_d|^{d-s} \le |\upsilon_d|^{d-1}$$

and

$$\mathrm{M}(\upsilon^{-1}) = |\upsilon_1 \cdots \upsilon_s|^{-1} = \mathrm{M}(\upsilon) = \lambda$$

with

$$\lambda \le |\upsilon_1|^{-s} \le |\upsilon_1|^{-(d-1)}.$$

Therefore we have

$$\lambda^{1/(d-1)} \le |\upsilon_d| \le \lambda \quad \text{and} \quad \lambda^{-1} \le |\upsilon_1| \le \lambda^{-1/(d-1)}. \tag{7}$$

[1] Our h is the same as in [2], it corresponds to the logarithm of the h in [1].

3.2 An elementary result

For the convenience of the reader, we include the following elementary result – similar arguments are often used without explicit mention in the literature.

Lemma 5. *Let U and V be positive numbers satisfying $U \le V \log^\star U$. Then $U < 2V \log^\star V$.*

Proof. If $\log U \le 1$, the assumption is $U \le V$ and the conclusion follows. Assume $\log U > 1$. Then $\log U \le \sqrt{U}$, hence the hypothesis of the lemma implies $U \le V\sqrt{U}$ and therefore we have $U \le V^2$. We deduce

$$\log U \le 2 \log V,$$

hence

$$U \le V \log U \le 2V \log V. \qquad \square$$

3.3 Diophantine tool

In this section only, the positive integer d is not restricted to $d \ge 3$.

The main tool is the following Diophantine estimate ([6, Proposition 2], [12, Theorem 9.1] or [2, Th. 3.2.4]), the proof of which uses transcendental number theory.

Proposition 6. *Let s and D be two positive integers. There exists an effectively computable positive constant $\kappa(s, D)$, depending only upon s and D, with the following property. Let $\eta_1, \ldots, \eta_s$ be nonzero algebraic numbers generating a number field of degree $\le D$. Let $c_1, \ldots, c_s$ be rational integers and let $H_1, \ldots, H_s$ be real numbers ≥ 1 satisfying*

$$H_i \ge \mathrm{h}(\eta_i) \quad (1 \le i \le s).$$

Let C be a real number with $C \ge 2$. Suppose that one of the following two statements is true:

(i) $C \ge \max_{1\le j\le s} |c_j|$

or

(ii) $H_j \le H_s$ *for* $1 \le j \le s$ *and*

$$C \ge \max_{1\le j\le s} \left\{ \frac{H_j}{H_s} |c_j| \right\}.$$

Suppose also $\eta_1^{c_1} \cdots \eta_s^{c_s} \ne 1$. Then

$$|\eta_1^{c_1} \cdots \eta_s^{c_s} - 1| > \exp\{-\kappa(s, D) H_1 \cdots H_s \log C\}.$$

The statement (ii) of Proposition 6 implies the statement (i) by permuting the indices so that $H_j \le H_s$ for $1 \le j \le s$; however, we find it more convenient to use part (i) so that we can use the estimate without permuting the indices.

We will use Proposition 6 several times. Here is a first consequence.

Corollary 7. *Let $d \ge 1$. There exists an effectively computable constant κ_{11}, which depends only on d, with the following property. Let K be a number field of degree d. Let α_1, α_2, υ_1, υ_2 be nonzero elements in K and let a be a nonzero integer. Set $\gamma_1 = \alpha_1\upsilon_1^a$ and $\gamma_2 = \alpha_2\upsilon_2^a$. Let λ_0 and λ satisfy*

$$\max\{\mathrm{h}(\alpha_1), \mathrm{h}(\alpha_2)\} \le \log \lambda_0, \quad \max\{\mathrm{h}(\upsilon_1), \mathrm{h}(\upsilon_2)\} \le \log \lambda$$

and assume $\gamma_1 \ne \gamma_2$. Define

$$\chi = (\log^\star \lambda_0)(\log^\star \lambda)\log^\star\left(|a|\min\left\{1,\ \frac{\log^\star \lambda}{\log^\star \lambda_0}\right\}\right).$$

Then

$$|\gamma_1 - \gamma_2| \ge \max\{|\gamma_1|, |\gamma_2|\}\, e^{-\kappa_{11}\chi}.$$

Proof. By symmetry, without loss of generality, we may assume $|\gamma_2| \ge |\gamma_1|$. Set

$$s = 2, \quad \eta_1 = \frac{\upsilon_1}{\upsilon_2}, \quad \eta_2 = \frac{\alpha_1}{\alpha_2}, \quad c_1 = a, \quad c_2 = 1,$$

$$H_1 = 2\log^\star \lambda, \quad H_2 = 2\log^\star \lambda_0, \quad C = \max\left\{2, |a|\min\left\{1, \frac{H_1}{H_2}\right\}\right\}.$$

The conclusion of Corollary 7 follows from Proposition 6 (via part (i) if $H_1 \ge H_2$, via part (ii) otherwise), thanks to the relation

$$|\eta_1^{c_1}\eta_2^{c_2} - 1| = |\gamma_2|^{-1}|\gamma_1 - \gamma_2|.$$

□

3.4 Lower bound for the height and the regulator

For the record, we quote Kronecker's Theorem and its effective improvement.

Lemma 8. (a) *If a nonzero algebraic integer α has all its conjugates in the closed unit disc $\{z \in \mathbb{C} \mid |z| \le 1\}$, then α is a root of unity.*

(b) *More precisely, given $d \ge 1$, there exists an effectively computable positive constant κ_{12}, depending only on d, such that, if α is a nonzero algebraic integer of degree d satisfying $\mathrm{h}(\alpha) < \kappa_{12}$, then α is a root of unity.*

Proof. Voutier (1996) refined an earlier estimate due to Dobrowolski (1979) by proving that the conclusion of part (b) in Lemma 8 holds with

$$\kappa_{12} = \begin{cases} \log 2 & \text{if } d = 1, \\ \dfrac{2}{d(\log(3d))^3} & \text{if } d \geq 2. \end{cases}$$

See for instance [2, Prop. 3.2.9] and [12, §3.6]. □

Lemma 9. *There exists an explicit absolute constant $\kappa_{13} > 0$ such that the regulator R of any number field of degree ≥ 2 satisfies $R > \kappa_{13}$.*

Proof. According to a result of Friedman (1989 – see [2, (1.5.3)]) the conclusion of Lemma 9 holds with $\kappa_{13} = 0.2052$. □

3.5 A basis of units of an algebraic number field

Here is Lemma 1 of [1]. See also [2, Proposition 4.3.9]. The result is essentially due to C.L. Siegel [11].

Proposition 10. *Let d be a positive integer with $d \geq 3$. There exist effectively computable constants $\kappa_{14}, \kappa_{15}, \kappa_{16}$ depending only on d, with the following property. Let K be a number field of degree d, with unit group of rank r. Let R be the regulator of this field. Denote by $\varphi_1, \varphi_2, \ldots, \varphi_r$ a set of r embeddings of K into $\mathbb{C}$ containing the real embeddings and no pair of conjugate embeddings. Then there exists a fundamental system of units $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_r\}$ of K which satisfies the following:*

(i) $\displaystyle\prod_{1 \leq i \leq r} \mathrm{h}(\epsilon_i) \leq \kappa_{14} R$;

(ii) $\displaystyle\max_{1 \leq i \leq r} \mathrm{h}(\epsilon_i) \leq \kappa_{15} R$;

(iii) *The absolute values of the entries of the inverse matrix of*

$$\big(\log|\varphi_j(\epsilon_i)|\big)_{1 \leq i,j \leq r}$$

do not exceed κ_{16}.

The next result is [10, Lemma A.15].

Lemma 11. *Let ϵ_1, ϵ_2, ..., ϵ_r be an independent system of units for K satisfying the condition (ii) of Proposition 10. Let $\beta \in \mathbb{Z}_K$ with $\mathrm{N}_{K/\mathbb{Q}}(\beta) = m \neq 0$. Then there exist $b_1, b_2, \ldots, b_r$ in $\mathbb{Z}$ and $\tilde{\beta} \in \mathbb{Z}_K$ with conjugates $\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_d$, satisfying*

$$\beta = \tilde{\beta} \epsilon_1^{b_1} \epsilon_2^{b_2} \cdots \epsilon_r^{b_r}$$

and

$$|m|^{1/d} e^{-\kappa_{17} R} \leq |\tilde{\beta}_j| \leq |m|^{1/d} e^{\kappa_{17} R} \quad \textit{for} \quad j = 1, \ldots, d.$$

The conclusion of Lemma 11 can be written

$$\left|\log\left(|m|^{-1/d}|\tilde{\beta}_j|\right)\right| \le \kappa_{17}R \quad \text{for} \quad j=1,\ldots,d.$$

3.6 Estimates for the conjugates

Lemma 12. *Let γ be an algebraic number of degree $d \ge 3$. Let γ_1, γ_2, ..., γ_d be the conjugates of γ with $|\gamma_1| \le |\gamma_2| \le \cdots \le |\gamma_d|$.*

(a) *If $|\gamma_1| < |\gamma_2|$ and $\gamma_2 \in \mathbb{R}$, then $|\gamma_2| < |\gamma_3|$.*

(b) *If $|\gamma_{d-1}| < |\gamma_d|$ and $\gamma_{d-1} \in \mathbb{R}$, then $|\gamma_{d-2}| < |\gamma_{d-1}|$.*

Proof. (a) The conditions $|\gamma_1| < |\gamma_2| \le |\gamma_i|$ for $3 \le i \le d$ imply that γ_1 is real and that $-\gamma_1$ is not a conjugate of γ_1. Hence the minimal polynomial of γ is not a polynomial in X^2. Assume $|\gamma_2| = |\gamma_3|$. Since $-\gamma_2$ is not a conjugate of γ_2, we deduce $\gamma_3 \not\in \mathbb{R}$, hence $d \ge 4$. We may assume $\gamma_4 = \overline{\gamma_3}$. Let σ be an automorphism of $\overline{\mathbb{Q}}$ which maps γ_2 to γ_1; via σ, let γ_j be the image of γ_3 and γ_k the image of γ_4. From

$$\gamma_2^2 = \gamma_3\gamma_4$$

we deduce $\gamma_1^2 = \gamma_j\gamma_k$ and $|\gamma_1|^2 = |\gamma_j\gamma_k|$. This is not possible since $|\gamma_j| > |\gamma_1|$ and $|\gamma_k| > |\gamma_1|$.

(b) We deduce (b) from (a), by using $\gamma \mapsto 1/\gamma$ (or by repeating the proof, *mutatis mutandis*). □

Remark. Here is an example showing that the assumptions of Lemma 12 are sharp. The polynomial $X^4 - 4X^2 + 1$ is irreducible, its roots are

$$\upsilon_1 = \sqrt{2-\sqrt{3}}, \quad \upsilon_2 = -\upsilon_1, \quad \upsilon_3 = 1/\upsilon_1 = \sqrt{2+\sqrt{3}}, \quad \upsilon_4 = -\upsilon_3$$

with

$$\upsilon_1 = |\upsilon_2| < \upsilon_3 = |\upsilon_4|.$$

More generally, if $h \ge 2$ is a positive integer and ϵ is a quadratic unit with Galois conjugate ϵ' and if $\epsilon^{1/h}$ has degree $2h$, then it has h conjugates of absolute value $|\epsilon|^{1/h}$ and h conjugates of absolute value $|\epsilon'|^{1/h}$. See also §2.

Lemma 13. *Let υ be an algebraic unit of degree $d \ge 3$. Set $\lambda = \mathrm{M}(\upsilon)$. Let υ' and υ'' be two conjugates of υ with $|\upsilon'| < |\upsilon''|$. Then*

$$\log\frac{|\upsilon''|}{|\upsilon'|} \ge \kappa_{18}\lambda^{-(d^3+2d^2-d+2)/2}.$$

We will deduce Lemma 13 from Theorem 1 of [3] which[2] states the following.

[2]This reference was kindly suggested to us by Yann Bugeaud.

Lemma 14 (X. Gourdon and B. Salvy [3])**.** *Let P be a polynomial of degree $d \geq 2$ with integer coefficients and with Mahler measure* $\mathrm{M}(P)$*. If α' and α'' are two roots of P with $|\alpha'| < |\alpha''|$, then*

$$|\alpha''| - |\alpha'| \geq \kappa_{19} M(P)^{-d(d^2+2d-1)/2}$$

with

$$\kappa_{19} = \frac{\sqrt{3}}{2}\bigl(d(d+1)/2\bigr)^{-d(d+1)/4-1}.$$

Proof of Lemma 13. We apply Lemma 14 to the minimal polynomial of υ. To conclude the proof of Lemma 13, we use the bounds $|\upsilon'| \leq \lambda$ and

$$\log(1+x) \geq \frac{x}{2} \quad \text{for} \quad 0 \leq x \leq 1 \quad \text{with} \quad x = \frac{|\upsilon''|}{|\upsilon'|} - 1. \qquad \square$$

4 Proof of Theorem 3

To prove Theorem 3 with the assumption $|F_a(x,y)| \leq m$, it suffices to consider the equation $F_a(x,y) = m$ with $m \neq 0$.

Let $(a,x,y,m) \in \mathbb{Z}^4$ satisfy $m \neq 0$, $xy \neq 0$, $[\mathbb{Q}(\alpha\upsilon^a) : \mathbb{Q}] = d$ and

$$F_a(x,y) = m.$$

Without loss of generality, we may restrict (a,y) to $a \geq 0$ (otherwise, replace υ by υ^{-1}) and to $y > 0$ (otherwise replace $F_a(X,Y)$ by $F_a(X,-Y)$).

The form $\tilde{F}_a(X,Y) = a_0^{d-1}F_a(X,Y)$ has coefficients in $\mathbb{Z}$, and if we set $\tilde{x} = a_0 x$, $\tilde{y} = y$, $\tilde{m} = a_0^{d-1}m$ we have $\tilde{F}_a(\tilde{x},\tilde{y}) = \tilde{m}$ with $(\tilde{x},\tilde{y}) \in \mathbb{Z}^2$. Therefore, there is no loss of generality to assume $a_0 = 1$.

Theorem 3 includes the assumption that υ is not a root of unity, hence $\lambda > 1$. More precisely, it follows from part (b) of Lemma 8 that

$$\log\lambda \geq \kappa_{12}.$$

In particular, we have

$$\log^\star \lambda \leq \max\left\{1, \frac{1}{\kappa_{12}}\right\} \log\lambda,$$

an inequality which can be written

$$\log^\star \lambda \leq \kappa_{20} \log\lambda \tag{8}$$

with an effectively computable constant $\kappa_{20} > 0$.

From Lemma 9, we deduce that $R > \kappa_{13}$. Therefore, there is no loss of generality to assume that, for a sufficiently large constant κ_{21}, we have

$$a \geq \kappa_{21}\bigl(\log|m| + (\log^\star \lambda_0)\log^\star\log^\star\lambda\bigr). \tag{9}$$

This hypothesis will frequently be used, sometimes without explicit mention.

By assumption, $\mathbb{Q}(\alpha\upsilon^a) = K$. If some conjugate $\sigma_j(\alpha\upsilon^a)$ of $\alpha\upsilon^a$ is real, then it follows that $\sigma_j(K) \subset \mathbb{R}$, hence the embedding σ_j is real, and α_j and υ_j are both real. We also notice that if $\sigma_j(\upsilon) = -\sigma_i(\upsilon)$ with $i \neq j$, then it follows that υ and $-\upsilon$ are conjugate, hence the irreducible polynomial of υ belongs to $\mathbb{Z}[X^2]$.

Recall that $\upsilon_i = \sigma_i(\upsilon)$ $(i = 1, \dots, d)$ and that

$$|\upsilon_1| \leq |\upsilon_2| \leq \cdots \leq |\upsilon_d|.$$

Let us write α_i for $\sigma_i(\alpha)$ $(i = 1, \dots, d)$. Let

$$\gamma = \alpha\upsilon^a \quad \text{and} \quad \beta = x - \gamma y.$$

Since $a_0 = 1$, it follows that α, β and γ are algebraic integers in K. For $j = 1, 2, \dots, d$, define γ_j and β_j by

$$\gamma_j = \sigma_j(\gamma) = \alpha_j\upsilon_j^a, \quad \beta_j = \sigma_j(\beta) = x - \alpha_j\upsilon_j^a y = x - \gamma_j y.$$

The assumption $F_a(x, y) = m$ yields $\beta_1\beta_2 \cdots \beta_d = m$. Let $i_0 \in \{1, 2, \dots, d\}$ be an index such that

$$|\beta_{i_0}| = \min_{1 \leq i \leq d} |\beta_i|.$$

We define $\Psi_1, \Psi_2, \dots, \Psi_d$ by the following conditions:

$$\beta_i = \begin{cases} \gamma_{i_0} y \Psi_i & \text{for } 1 \leq i < i_0, \\ \gamma_i y \Psi_i & \text{for } i_0 < i \leq d \end{cases}$$

and

$$\beta_{i_0} = \frac{m}{y^{d-1}} \cdot \frac{\gamma_1\gamma_2 \cdots \gamma_{i_0-1}}{\gamma_{i_0}^{i_0-2}} \Psi_{i_0}.$$

We split the proof into several steps.

`Step 1.` We start by proving that

$$|x| \leq 2\lambda_0\lambda^a y \tag{10}$$

and that there exists an effectively computable positive constant κ_{22} depending only on d such that

$$e^{-\kappa_{22}\chi} \leq |\Psi_i| \leq e^{\kappa_{22}\chi} \qquad (i = 1, 2, \dots, d) \tag{11}$$

with

$$\chi = (\log^\star \lambda_0)(\log \lambda) \log\left(a \min\left\{1, \frac{\log \lambda}{\log^\star \lambda_0}\right\}\right).$$

From the estimate (11) we will deduce

$$|\beta_{i_0}| < |\beta_i|$$

for $i \neq i_0$, which implies $\alpha_{i_0} \in \mathbb{R}$ and $\upsilon_{i_0} \in \mathbb{R}$.

`Remark.` The estimate (11) can be written as follows:

$$\left|\log\left(|\beta_i||y^{-1}| \max\{|\gamma_i^{-1}|, |\gamma_{i_0}^{-1}|\}\right)\right| \leq \kappa_{22}\chi$$

for $i \neq i_0$ and

$$\left|\log\left(|\beta_{i_0}|\frac{y^{d-1}}{|m|}\left|\gamma_1^{-1}\cdots\gamma_{i_0-1}^{-1}\gamma_{i_0}^{i_0-2}\right|\right)\right| \leq \kappa_{22}\chi.$$

`Proof` of (10) and (11). We have

$$|x| = |\beta_{i_0} + \gamma_{i_0} y| \leq |\beta_{i_0}| + |\gamma_{i_0}|y. \tag{12}$$

From $|\beta_{i_0}| \leq |\beta_i|$ for $i = 1, 2, \ldots, d$ and $\beta_1 \cdots \beta_d = m$, we deduce $|\beta_{i_0}| \leq |m|^{1/d}$, hence

$$|x| \leq |m|^{1/d} + |\gamma_{i_0}|y \leq |m|^{1/d} + \lambda_0\lambda^a y.$$

Using the assumption (9), we check $|m|^{1/d} \leq \lambda_0\lambda^a y$, whereupon the inequality (10) is secured.

We also have

$$|\beta_{i_0}|^{d-1} \max_{1\leq i\leq d} |\beta_i| \leq |m|. \tag{13}$$

For $i = 1, 2, \ldots, d$, we write

$$\beta_i = \beta_{i_0} + y(\gamma_{i_0} - \gamma_i). \tag{14}$$

We have

$$|\alpha_1\alpha_2\cdots\alpha_d| \geq 1$$

(recall $a_0 = 1$), hence

$$\frac{1}{\lambda_0} \leq |\alpha_i| \leq \lambda_0 \quad \text{for } i = 1, 2, \ldots, d.$$

We choose an index $j_0 \neq i_0$ as follows:

- If $|\upsilon_{i_0}| \leq \lambda^{1/(2(d-1))}$, we take $j_0 = d$ so that, with the help of (7), we have $|\upsilon_{j_0}| \geq \lambda^{1/(d-1)}$, whereupon with the help of (9) we obtain

$$\left|\frac{\gamma_{i_0}}{\gamma_{j_0}}\right| < \frac{1}{2}.$$

- If $|\upsilon_{i_0}| > \lambda^{1/(2(d-1))}$, we take $j_0 = 1$ so that, again with the help of (7), we have $|\upsilon_{j_0}| \leq \lambda^{-1/(d-1)}$, whereupon with the help of (9) we obtain

$$\left|\frac{\gamma_{j_0}}{\gamma_{i_0}}\right| < \frac{1}{2}.$$

In both cases, we deduce

$$|\gamma_{j_0}-\gamma_{i_0}| \geq \frac{1}{2}\max\{|\gamma_{j_0}|,|\gamma_{i_0}|\} \geq \frac{\lambda^{a/(2(d-1))}}{2\lambda_0}$$

and therefore, using (9) again together with (13) and (14), we obtain

$$|\beta_{j_0}| \geq |\gamma_{j_0}-\gamma_{i_0}|y - |\beta_{i_0}| \geq \frac{\lambda^{a/(2(d-1))}y}{2\lambda_0} - |m|^{1/d} \geq \lambda^{a/(2d)}y.$$

Since $\max_{1\leq i\leq d}|\beta_i| \geq \lambda^{a/(2d)}y$, from (13) we deduce

$$|\beta_{i_0}| \leq \left(\frac{|m|}{y\lambda^{a/(2d)}}\right)^{1/(d-1)}. \tag{15}$$

In particular, thanks to (9), we have

$$|\beta_{i_0}| \leq \frac{1}{2}. \tag{16}$$

Using the assumption $|x|\geq 1$ together with (12), we deduce

$$\frac{|x|}{2} \leq |\gamma_{i_0}|y \leq |x| + |\beta_{i_0}| \leq \frac{3|x|}{2}. \tag{17}$$

Let $i \neq i_0$. The upper bound

$$|\gamma_i-\gamma_{i_0}| \leq 2\max\{|\gamma_{i_0}|,|\gamma_i|\}$$

is trivial, while the lower bound

$$|\gamma_i-\gamma_{i_0}| \geq \max\{|\gamma_{i_0}|,|\gamma_i|\}e^{-\kappa_{23}\chi} \tag{18}$$

follows from (8) and from Corollary 7. We first use the lower bound

$$|\gamma_i-\gamma_{i_0}| \geq |\gamma_{i_0}|e^{-\kappa_{23}\chi}.$$

Using (17), we obtain

$$|\gamma_i-\gamma_{i_0}| \geq \frac{1}{2y}e^{-\kappa_{23}\chi} \geq \frac{2}{y}e^{-\kappa_{24}\chi} \tag{19}$$

with $\kappa_{24}>0$. Using the contrapositive of Lemma 5 with

$$U = a\frac{\log^\star\lambda}{\log^\star\lambda_0}, \quad V = \frac{1}{\kappa_{25}}\log^\star\lambda,$$

we deduce from (9) that

$$\chi \leq \kappa_{25}a\log^\star\lambda.$$

Recall that κ_{21} is sufficiently large, hence κ_{25} is sufficiently small. Now from (15), the inequality $|m| \leq e^{|a|/\kappa_{21}}$ and (19) we deduce

$$|\beta_{i_0}| \leq |m|^{1/(d-1)} \lambda^{-a/(2d(d-1))} \leq \lambda^{-\kappa_{26} a} \leq e^{-\kappa_{24}\chi} \leq \frac{1}{2} y |\gamma_i - \gamma_{i_0}|.$$

Therefore, for $i \neq i_0$, using (14), we deduce

$$\frac{1}{2} y |\gamma_i - \gamma_{i_0}| \leq |\beta_i| \leq \frac{3}{2} y |\gamma_i - \gamma_{i_0}|.$$

Using once more (18), we obtain (11) for $i \neq i_0$. We also deduce

$$|\beta_i| > \lambda^{-a/(2d)} \quad \text{for} \quad i \neq i_0. \tag{20}$$

Recall

$$\mathrm{N}(\gamma) = \gamma_1 \gamma_2 \cdots \gamma_d = \mathrm{N}(\alpha)\mathrm{N}(\upsilon)^a = \pm \mathrm{N}(\alpha) \quad \text{and} \quad \mathrm{N}(\beta) = \beta_1 \beta_2 \cdots \beta_d = m.$$

The estimate (11) for $i = i_0$ follows from the relations

$$\Psi_1 \Psi_2 \cdots \Psi_d \mathrm{N}(\gamma) = 1,$$

$$\frac{m}{\beta_{i_0}} = \prod_{i \neq i_0} \beta_i = y^{d-1} \gamma_{i_0}^{i_0-1} \gamma_{i_0+1} \cdots \gamma_d \prod_{i \neq i_0} \Psi_i$$

and

$$\frac{\mathrm{N}(\gamma)}{\gamma_{i_0}^{i_0-1} \gamma_{i_0+1} \cdots \gamma_d} = \frac{\gamma_1 \cdots \gamma_{i_0-1}}{\gamma_{i_0}^{i_0-2}}.$$

From (9) and (11), we deduce

$$|\beta_{i_0}| < \frac{|m|}{y^{d-1}} |\gamma_1| e^{\kappa_{22}\chi} < \lambda^{-a/(2d)},$$

hence from (20) we infer $|\beta_{i_0}| < |\beta_i|$ for $i \neq i_0$. It follows that β_{i_0} is real, and therefore γ_{i_0}, α_{i_0} and υ_{i_0} also. □

`Step 2`. Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_r$ be a basis of the group of units of K given by Proposition 10. From Lemma 11, it follows that there exists $\tilde{\beta} \in \mathbb{Z}_K$ and $b_1, b_2, \ldots, b_r$ in $\mathbb{Z}$ with

$$\beta = \tilde{\beta} \epsilon_1^{b_1} \epsilon_2^{b_2} \cdots \epsilon_r^{b_r}$$

and

$$|m|^{1/d} e^{-\kappa_{17} R} \leq |\tilde{\beta}_i| \leq |m|^{1/d} e^{\kappa_{17} R} \quad \text{for} \quad i = 1, 2, \ldots, d.$$

We set

$$B = \kappa_{27}(R + a \log \lambda + \log y) \tag{21}$$

with a sufficiently large constant κ_{27}. We want to prove that

$$\max_{1 \leq i \leq r} |b_i| \leq B.$$

`Proof.` We consider the system of d linear forms in r variables with real coefficients

$$L_i(X_1, X_2, \dots, X_r) = \sum_{j=1}^{r} X_j \log|\sigma_i(\epsilon_j)|, \quad (i = 1, 2, \dots, d).$$

The rank is r. By Proposition 10(ii),

$$\log|\sigma_i(\epsilon_j)| \leq \kappa_{28} R.$$

For $i = 1, 2, \dots, d$, define $e_i = L_i(b_1, b_2, \dots, b_r)$. We have

$$e_i = \log|\sigma_i(\beta/\tilde{\beta})| = \log|\beta_i/\tilde{\beta}_i|,$$

hence, using the inequality $|m| \leq e^{|a|/\kappa_{21}}$ and (11), we deduce

$$|e_i| \leq \kappa_{29}(R + a\log\lambda + \log y).$$

Computing $b_1, b_2, \dots, b_r$ by means of the system of linear equations

$$L_i(b_1, b_2, \dots, b_r) = e_i \quad (i = 1, 2, \dots, d)$$

and using Proposition 10(iii), we deduce

$$\max_{1 \leq j \leq r} |b_j| \leq \kappa_{30} \max_{1 \leq i \leq d} |e_i| \leq B. \qquad \square$$

`Step 3`. From the inequality (3.2) in [1, Theorem 3] (see also [2, Th. 9.6.2]), thanks to (9), we deduce the following upper bound for $|x|$ and $|y|$ in terms of a, λ, λ_0, m and R: there exists a positive effectively computable constant κ_{31} depending only on d such that

$$\log\max\{|x|, y\} \leq \kappa_{31} R(\log^\star R)\bigl(R + a\log\lambda\bigr). \tag{22}$$

`Step 4`. Assume $c\gamma_i\beta_j \neq \gamma_k\beta_\ell$ for some indices i, j, k, ℓ in $\{1, \dots, d\}$ and some $c \in \{1, -1\}$. Then there exists $\kappa_{32} > 0$ such that

$$\left| c\frac{\gamma_i\beta_j}{\gamma_k\beta_\ell} - 1 \right| \geq \exp\Biggl\{ -\kappa_{32}(\log\lambda)(R + \log|m| + \log\lambda_0 + \log\lambda)R \\ \times \log\left(\frac{Ra\log\lambda}{R + \log|m| + \log\lambda_0 + \log\lambda} \right) \Biggr\}.$$

`Proof.` This lower bound follows from Proposition 6(ii) with

$$\frac{c\gamma_i\beta_j}{\gamma_k\beta_\ell} = \eta_1^{c_1}\eta_2^{c_2}\cdots\eta_s^{c_s},$$

where $s = r + 2$ and

$$\eta_t = \frac{\sigma_j(\epsilon_t)}{\sigma_\ell(\epsilon_t)} \; (1 \le t \le r), \quad \eta_{r+1} = \frac{\sigma_i(\upsilon)}{\sigma_k(\upsilon)}, \quad \eta_{r+2} = \frac{c\sigma_j(\tilde{\beta})\sigma_i(\alpha)}{\sigma_\ell(\tilde{\beta})\sigma_k(\alpha)},$$

$$c_t = b_t \; (1 \le t \le r), \quad c_{r+1} = a, \quad c_{r+2} = 1,$$

$$H_t = \max\{1, 2\mathrm{h}(\epsilon_t)\} \; (1 \le t \le r),$$

$$H_{r+1} = \max\{1, 2\log\lambda\}, \quad H_{r+2} = \kappa_{33}(R + \log|m| + \log\lambda_0 + \log\lambda),$$

$$C = 2 + \frac{2a\log\lambda + RB}{H_{r+2}}\cdot$$

Using Proposition 10(i) together with part (b) of Lemma 8, we deduce

$$H_1 H_2 \cdots H_r \le \kappa_{34} R.$$

Finally we deduce from the steps 2 and 3 that

$$\log C \le \kappa_{35} \log\left(\frac{Ra\log\lambda}{R + \log|m| + \log\lambda_0 + \log\lambda}\right),$$

and this secures the lower bound for $\left|c\frac{\gamma_i\beta_j}{\gamma_k\beta_l} - 1\right|$ announced above. □

`Step 5.` We will prove Theorem 3 by assuming

$$\max_{1\le i\le d} \max\{|\Psi_i|, |\Psi_i|^{-1}\} > \mu^{a/4}.$$

Using (11), we deduce from our assumption

$$\frac{a}{4}\log\mu < \kappa_{22}\chi,$$

hence

$$a \le \frac{4\kappa_{22}}{\log\mu}(\log^\star\lambda_0)(\log^\star\lambda)\log^\star\left(a\frac{\log^\star\lambda}{\log^\star\lambda_0}\right).$$

With

$$U = \frac{a\log^\star\lambda}{\log^\star\lambda_0} \quad \text{and} \quad V = \frac{4\kappa_{22}(\log^\star\lambda)^2}{\log\mu},$$

we have $U \le V\log^\star U$, and we conclude that we can use Lemma 5 to deduce

$$a \le \frac{8\kappa_{22}(\log^\star\lambda_0)(\log^\star\lambda)}{\log\mu}\log\left(\frac{4\kappa_{22}(\log^\star\lambda)^2}{\log\mu}\right),$$

and the conclusion of Theorem 3 follows.

In the rest of the paper, we assume

$$\max_{1\le i\le d} \max\{|\Psi_i|, |\Psi_i|^{-1}\} \le \mu^{a/4}. \tag{23}$$

`Step 6.` Our next goal is to prove the following results.

(a) Assume $1 \le i_0 \le d-2$ and

$$\frac{|\upsilon_{d-1}|}{|\upsilon_{i_0}|} \ge \sqrt{\mu}.$$

Then

$$0 < \left| \frac{\gamma_{d-1}\beta_d}{\gamma_d\beta_{d-1}} - 1 \right| \le 4\lambda_0^2\mu^{-a/4}.$$

(b) Assume $3 \le i_0 \le d$ and

$$\frac{|\upsilon_{i_0}|}{|\upsilon_2|} \ge \sqrt{\mu}.$$

Then

$$0 < \left| \frac{\beta_1}{\beta_2} - 1 \right| \le 2\lambda_0^2\mu^{-a/4}.$$

(c) Assume $2 \le i_0 \le d-1$ and

$$\min\left\{ \frac{|\upsilon_{i_0}|}{|\upsilon_1|}, \frac{|\upsilon_d|}{|\upsilon_{i_0}|} \right\} \ge \mu.$$

Then

$$\left| \frac{\gamma_{i_0}\beta_d}{\gamma_d\beta_1} + 1 \right| \le 4|m|\lambda_0^4\mu^{-a/2}.$$

`Proof` (a) We approximate β_d by $-\gamma_d y$, β_{d-1} by $-\gamma_{d-1}y$ and we eliminate y. Since γ has degree d, we have

$$\beta_d\gamma_{d-1} - \beta_{d-1}\gamma_d = x(\gamma_{d-1} - \gamma_d) \ne 0.$$

From (17) we deduce $|x| \le 2|\gamma_{i_0}y|$ and

$$|\beta_d\gamma_{d-1} - \beta_{d-1}\gamma_d| \le 2|\gamma_{i_0}|(|\gamma_d| + |\gamma_{d-1}|)y.$$

Using $\beta_{d-1} = \gamma_{d-1}y\Psi_{d-1}$ together with the assumption

$$|\upsilon_d| \ge |\upsilon_{d-1}| \ge \sqrt{\mu}|\upsilon_{i_0}|,$$

we deduce

$$\left| \frac{\gamma_{d-1}\beta_d}{\gamma_d\beta_{d-1}} - 1 \right| \le \frac{2|\gamma_{i_0}|(|\gamma_{d-1}| + |\gamma_d|)}{|\gamma_{d-1}\gamma_d|}|\Psi_{d-1}|^{-1} \le 4\lambda_0^2\mu^{-a/2}|\Psi_{d-1}|^{-1}.$$

The conclusion of (a) follows from (23).

(b) We approximate β_1 and β_2 by x and we eliminate x. Since $\gamma_1 \ne \gamma_2$, we have

$$|\beta_1 - \beta_2| = |(\gamma_2 - \gamma_1)y| \ne 0.$$

From $\beta_2 = \gamma_{i_0} y \Psi_2$ and the assumption

$$|\upsilon_1| \le |\upsilon_2| \le \mu^{-1/2} |\upsilon_{i_0}|,$$

we deduce

$$\left| \frac{\beta_1}{\beta_2} - 1 \right| \le \frac{|\gamma_2| + |\gamma_1|}{|\gamma_{i_0}|} |\Psi_2|^{-1} \le 2\lambda_0^2 \mu^{-a/2} |\Psi_2|^{-1}.$$

Again, the conclusion of (b) follows from (23).

(c) We approximate β_1 by x, β_d by $-y\gamma_d$ and x by $y\gamma_{i_0}$, then we eliminate x and y. More precisely we have

$$\beta_1\gamma_d + \beta_d\gamma_{i_0} = (\gamma_d + \gamma_{i_0})\beta_{i_0} + \gamma_{i_0}^2 y - \gamma_1\gamma_d y.$$

Hence

$$\frac{\gamma_{i_0}\beta_d}{\gamma_d\beta_1} + 1 = \frac{(\gamma_d + \gamma_{i_0})\beta_{i_0}}{\gamma_d\beta_1} + \frac{\gamma_{i_0}^2 y}{\gamma_d\beta_1} - \frac{\gamma_1 y}{\beta_1}.$$

We have $\beta_1 = \gamma_{i_0} y \Psi_1$. Therefore we have

$$\frac{|\gamma_{i_0}|^2 y}{|\gamma_d\beta_1|} = \frac{|\gamma_{i_0}|}{|\gamma_d|} |\Psi_1|^{-1} \le \lambda_0^2 \left| \frac{\upsilon_{i_0}}{\upsilon_d} \right|^a |\Psi_1|^{-1}$$

and

$$\frac{|\gamma_1| y}{|\beta_1|} = \frac{|\gamma_1|}{|\gamma_{i_0}|} |\Psi_1|^{-1} \le \lambda_0^2 \left| \frac{\upsilon_1}{\upsilon_{i_0}} \right|^a |\Psi_1|^{-1}.$$

Finally, from

$$|\beta_{i_0}| \le \frac{|m|}{y^{d-1}} |\gamma_1 \Psi_{i_0}|$$

we deduce

$$\begin{aligned} \frac{(|\gamma_d| + |\gamma_{i_0}|)|\beta_{i_0}|}{|\gamma_d\beta_1|} &\le (1 + \lambda_0^2) \left| \frac{\beta_{i_0}}{\beta_1} \right| \le (1 + \lambda_0^2) \frac{|m|}{y^d} \frac{|\gamma_1 \Psi_{i_0}|}{|\gamma_{i_0}\Psi_1|} \\ &\le \lambda_0^2 (1 + \lambda_0^2) \frac{|m|}{y^d} \left| \frac{\upsilon_1}{\upsilon_{i_0}} \right|^a \frac{|\Psi_{i_0}|}{|\Psi_1|}. \end{aligned}$$

Hence from the assumptions

$$|\upsilon_1| \le \mu^{-1} |\upsilon_{i_0}| \quad \text{and} \quad |\upsilon_{i_0}| \le \mu^{-1} |\upsilon_d|,$$

we deduce

$$\left| \frac{\gamma_{i_0}\beta_d}{\gamma_d\beta_1} + 1 \right| \le 4|m|\lambda_0^4 \mu^{-a} \frac{|\Psi_{i_0}|}{|\Psi_1|}.$$

The conclusion of (c) follows from (23). □

`Step 7.` (a) Assume $|\upsilon_{i_0}| = |\upsilon_1|$. Since $\upsilon_{i_0} \in \mathbb{R}$, we deduce from Lemma 12 that

$$|\upsilon_1| < |\upsilon_{d-1}|.$$

If $|\upsilon_2| < |\upsilon_{d-1}|$, then

$$\frac{|\upsilon_{d-1}|}{|\upsilon_{i_0}|} \geq \frac{|\upsilon_{d-1}|}{|\upsilon_2|} = \mu$$

and we are in the case (a) of the step 6.

If $|\upsilon_2| = |\upsilon_{d-1}|$, then $i_0 = 1$, we have

$$\frac{|\upsilon_{d-1}|}{|\upsilon_1|} \geq \mu$$

and again we are in the case (a) of the step 6.

(b) Assume $|\upsilon_{i_0}| = |\upsilon_d|$. Using Lemma 12, we deduce

$$|\upsilon_d| > |\upsilon_2|.$$

If $|\upsilon_2| < |\upsilon_{d-1}|$, then

$$\frac{|\upsilon_{i_0}|}{|\upsilon_1|} \geq \frac{|\upsilon_{d-1}|}{|\upsilon_2|} = \mu$$

and we are in the case (b) of the step 6.

If $|\upsilon_2| = |\upsilon_{d-1}|$, then $i_0 = d$, we have

$$\frac{|\upsilon_d|}{|\upsilon_2|} \geq \mu$$

and again we are in the case (b) of the step 6.

(c) Assume finally $|\upsilon_1| < |\upsilon_{i_0}| < |\upsilon_d|$. In particular we have $2 \leq i_0 \leq d-1$. Assume that we are neither in the case (a) nor in the case (b) of the step 6. From

$$\frac{|\upsilon_{d-1}|}{|\upsilon_{i_0}|} < \sqrt{\mu} \quad \text{and} \quad \frac{|\upsilon_{i_0}|}{|\upsilon_2|} < \sqrt{\mu}$$

we deduce

$$\frac{|\upsilon_{d-1}|}{|\upsilon_2|} < \mu.$$

Given the definition of μ, it follows that we have $|\upsilon_2| = |\upsilon_{d-1}|$. Since υ_{i_0} is real, Lemma 12 implies $d = 3$ and therefore $i_0 = 2$, $|\upsilon_1| < |\upsilon_2| < |\upsilon_3|$ and

$$\mu = \min\left\{\frac{|\upsilon_3|}{|\upsilon_2|}, \frac{|\upsilon_2|}{|\upsilon_1|}\right\}.$$

From

$$|\gamma_1| \leq \lambda_0 |\upsilon_1|^a \leq \lambda_0 \lambda^{-a/2} < 1, \qquad |\beta_2| = |\beta_{i_0}| < 1$$

and

$$|\gamma_2 \beta_3| = |\gamma_2 \gamma_3 \Psi_3| y \geq \frac{y|\Psi_3|}{|\gamma_1|} \geq \lambda_0^{-1} \lambda^{a/2} |\Psi_3| > 1,$$

we deduce $|\gamma_1\beta_2| < 1 < |\gamma_2\beta_3|$, hence

$$\gamma_1\beta_2 + \gamma_2\beta_3 \neq 0.$$

There is an element of the Galois group of the Galois closure of the cubic field $\mathbb{Q}(\upsilon)$ which maps υ_1 to υ_2, υ_2 to υ_3, υ_3 to υ_1. Therefore,

$$\gamma_2\beta_3 + \gamma_3\beta_1 \neq 0.$$

From part (c) of the step 6 we deduce

$$0 < \left| \frac{\gamma_2\beta_3}{\gamma_3\beta_1} + 1 \right| \leq 4m\lambda_0^4\mu^{-a/2}.$$

`Step 8`. Combining the steps 6 and 7 with the step 4 where we choose

$$\begin{cases} i = \ell = d-1,\ j = k = d,\ c = 1 & \text{in the case (a)}, \\ i = k = i_0,\ j = 1,\ \ell = d,\ c = 1 & \text{in the case (b)}, \\ i = i_0,\ j = k = d,\ \ell = 1,\ c = -1 & \text{in the case (c)}. \end{cases}$$

we deduce

$$a \log \mu \leq \kappa_{36} R(R + \log|m| + \log\lambda_0 + \log\lambda)(\log\lambda) \times \log\left(\frac{Ra\log\lambda}{R + \log|m| + \log\lambda_0 + \log\lambda}\right)\cdot$$

For

$$U = \frac{Ra\log\lambda}{R + \log|m| + \log\lambda_0 + \log\lambda} \quad \text{and} \quad V = \kappa_{37}\frac{R^2(\log\lambda)^2}{\log\mu},$$

we have $U \leq V \log^\star U$. Therefore we use Lemma 5 to obtain the conclusion of Theorem 3.

5 Proofs of Theorem 1 and of Corollary 2

Proof of Theorem 1. Since $d \geq 3$, under the assumptions of Lemma 13 we have

$$\log\frac{|\upsilon''|}{|\upsilon'|} \geq \frac{\kappa_{18}(\log\lambda)^2}{\lambda^{d^2(d+2)/2}}\cdot$$

From Lemma 12, we deduce that under the assumptions of Theorem 1 and with the notations of Theorem 3, we have

$$\log\mu \geq \frac{\kappa_{38}(\log\lambda)^2}{\lambda^{d^2(d+2)/2}}\cdot$$

Hence Theorem 3 implies Theorem 1. □

Proof of Corollary 2. The conclusion of Corollary 2 follows from Theorem 1 thanks to the upper bound (2). □

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