Families of Thue equations associated with a rank one subgroup of the unit group of a number field

by

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Abstract. Let $K$ be an algebraic number field of degree $d \geq 3$, $\sigma_1, \sigma_2, \ldots, \sigma_d$ the embeddings of $K$ into $\mathbb{C}$, $\alpha$ a nonzero element in $K$, $a_0 \in \mathbb{Z}$, $a_0 > 0$ and

$$F_0(X,Y) = a_0 \prod_{i=1}^{d} (X - \sigma_i(\alpha)Y).$$

Let $v$ be a unit in $K$. For $a \in \mathbb{Z}$, we twist the binary form $F_0(X,Y) \in \mathbb{Z}[X,Y]$ by the powers $v^a$ ($a \in \mathbb{Z}$) of $v$ by setting

$$F_a(X,Y) = a_0 \prod_{i=1}^{d} (X - \sigma_i(\alpha v^a)Y).$$

Given $m \geq 0$, our main result is an effective upper bound for the size of solutions $(x,y,a) \in \mathbb{Z}^3$ of the Diophantine inequalities

$$0 < |F_a(x,y)| \leq m$$

for which $xy \neq 0$ and $\mathbb{Q}(\alpha v^a) = K$. Our estimate is explicit in terms of its dependence on $m$, the regulator of $K$ and the heights of $F_0$ and of $v$; it also involves an effectively computable constant depending only on $d$.

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1 Introduction and the main results

Let $d \geq 3$ be a given integer. We denote by $\kappa_1, \kappa_2, \ldots, \kappa_{d^3}$ positive effectively computable constants which depend only on $d$.

Let $K$ be a number field of degree $d$. Denote by $\sigma_1, \sigma_2, \ldots, \sigma_d$ the embeddings of $K$ into $\mathbb{C}$ and by $R$ the regulator of $K$. Let $\alpha \in K$, $\alpha \neq 0$, and
let \( a_0 \in \mathbb{Z}, a_0 > 0 \), be such that the coefficients of the polynomial

\[
f_0(X) = a_0 \prod_{i=1}^{d}(X - \sigma_i(\alpha))
\]

are in \( \mathbb{Z} \). Let \( \upsilon \) be a unit in \( K \), not a root of unity. For \( a \in \mathbb{Z} \), define the polynomial \( f_a(X) \) in \( \mathbb{Z}[X] \) and the binary form \( F_a(X,Y) \) in \( \mathbb{Z}[X,Y] \) by

\[
f_a(X) = a_0 \prod_{i=1}^{d}(X - \sigma_i(\alpha \upsilon^a))
\]

and

\[
F_a(X,Y) = Y^d f_a(X/Y) = a_0 \prod_{i=1}^{d}(X - \sigma_i(\alpha \upsilon^a) Y).
\]

Define

\[
\lambda_0 = a_0 \prod_{i=1}^{d} \max\{1, |\sigma_i(\alpha)|\} \quad \text{and} \quad \lambda = \prod_{i=1}^{d} \max\{1, |\sigma_i(\upsilon)|\}.
\]

Let \( m \in \mathbb{Z}, m > 0 \). We consider the family of Diophantine inequalities

\[
0 < |F_a(x,y)| \leq m,
\]

where the unknowns \((x,y,a)\) take their values in the set of elements in \( \mathbb{Z}^3 \) such that \( xy \neq 0 \) and \( \mathbb{Q}(\alpha \upsilon^a) = K \). It follows from the results in [4] that the set of solutions is finite. However, the proof in [4] relies on Schmidt’s subspace theorem, which is not effective. Here by using lower bounds for linear forms in logarithms, we give an upper bound for \( \max\{|x|, |y|, |a|\} \), which is explicit in terms of \( m, R, \lambda_0 \) and \( \lambda \) and which involves an effectively computable constant depending only on \( d \).

For \( x \in \mathbb{R}, x > 0 \), we use the notation \( \log^* x \) to denote \( \max\{1, \log x\} \).

Here is our main result.

**Theorem 1.** There exists an effectively computable constant \( \kappa_1 > 0 \), depending only on \( d \), such that any solution \((x,y,a)\) in \( \mathbb{Z}^3 \) of (1), which verifies \( xy \neq 0 \) and \( \mathbb{Q}(\alpha \upsilon^a) = K \), satisfies

\[
|a| \leq \kappa_1 \lambda^d (d+2)/2 (R + \log m + \log \lambda_0) R \log^* R.
\]

Under the assumptions of Theorem [1] with the help of the upper bound

\[
H(F_a) \leq 2^d \lambda_0 |a|
\]

for the (usual) height of the form \( F_a \) (namely the maximum of the absolute values of the coefficients of the form), it follows from the bound (3.2) in [1] Theorem 3] (see also [2 Th. 9.6.2]) that

\[
\log \max\{|x|, |y|\} \leq \kappa_2 (R + \log^* m + |a| \log \lambda + \log \lambda_0) R (\log^* R)
\]
where $\kappa_2$ is an explicit constant depending only on $d$. Combining this upper bound with our Theorem 1 provides an effective upper bound for $\max\{\log |x|, \log |y|, |a|\}$.

Corollary 2. Under the assumptions of Theorem 1, there exists an effectively computable constant $\kappa_3$ depending only on $d$ such that

$$\max\{\log |x|, \log |y|, |a|\} \leq \kappa_3 \lambda^{d(d+2)/2} \lambda R^2 \log^2 \mu.$$

Our proof of Theorem 1 actually gives a much stronger estimate for $|a|$; see Theorem 3 below. It involves an extra parameter $\mu > 1$ that we now define.

For $i = 1, \ldots, d$, set $\upsilon_i = \sigma_i(v)$ and assume

$$|\upsilon_1| \leq |\upsilon_2| \leq \cdots \leq |\upsilon_d|.$$

Define

$$\mu = \begin{cases} 
\lambda & \text{if } |\upsilon_1| = |\upsilon_{d-1}| \text{ or } |\upsilon_2| = |\upsilon_d|, \\
\min \left\{ \frac{|\upsilon_{d-1}|}{|\upsilon_1|}, \frac{|\upsilon_d|}{|\upsilon_2|} \right\} & \text{if } |\upsilon_1| < |\upsilon_2| = |\upsilon_{d-1}| < |\upsilon_d|, \\
\frac{|\upsilon_{d-1}|}{|\upsilon_2|} & \text{if } |\upsilon_2| < |\upsilon_{d-1}|.
\end{cases}$$

Notice that the condition $|\upsilon_1| = |\upsilon_{d-1}|$ means $|\upsilon_1| = |\upsilon_2| = \cdots = |\upsilon_{d-1}|$ and that the condition $|\upsilon_2| = |\upsilon_d|$ means $|\upsilon_2| = |\upsilon_3| = \cdots = |\upsilon_d|$; using Lemma 2 we deduce that each of these two conditions implies that $d$ is odd, hence that the field $K$ is almost totally imaginary (namely, with a single real embedding) – compare with 9.

Theorem 3. There exists a positive effectively computable constant $\kappa_4$, depending only on $d$, with the following property. Let $(x, y, a) \in \mathbb{Z}^3$ satisfy

$$xy \neq 0, \quad [\mathbb{Q}(\alpha v^a) : \mathbb{Q}] = d \quad \text{and} \quad 0 < |F_a(x, y)| \leq m.$$

Then

$$|a| \leq \frac{\log \lambda}{\log \mu} (R + \log m + \log \lambda_0 + \log \lambda) R \log \left( \frac{\log \lambda^2}{\log \mu} \right).$$

On the one hand, using Lemma 13 (§3.4), we will prove in §5 that

$$\log \mu \geq \kappa_5 \lambda^{-d^2(d+2)/2} (\log \lambda)^2,$$

which will enable us to deduce Theorem 1 from Theorem 3. On the other hand, thanks to (7), we have $\mu \leq \lambda^2$. Hence the largest possible value of
\( \mu \) is \( \lambda^{\kappa_6} \) with a positive constant \( \kappa_6 \) depending only on \( d \). For the units \( \upsilon \) satisfying such an estimate, the conclusion of Theorem 3 becomes

\[
|a| \leq \kappa_7 R + \log m + \log \lambda_0 + \log \lambda) R(\log R + \log \lambda^* \log \lambda)
\]

with a positive effectively computable constant \( \kappa_7 \) depending only on \( d \). In §2 we give a few examples where this last bound is valid.

In Theorem 1, the hypothesis that \( \upsilon \) is not a root of unity cannot be omitted. Here is an example with \( \alpha = a_0 = m = 1 \). Let \( \Phi_n(X) \) be the cyclotomic polynomial of index \( n \) and degree \( \varphi(n) \) (Euler totient function). Let \( \zeta_n \) be a primitive \( n \)-th root of unity. Set \( f_0 = \Phi_n \) and \( u = \zeta_n \). For \( a \in \mathbb{Z} \) with \( \gcd(a, n) = 1 \), the irreducible polynomial \( f_a \) of \( \zeta_n^a \) is nothing else than \( f_0 \). Hence, if the equation

\[
F_0(x, y) = \pm 1
\]

has a solution \( (x, y) \in \mathbb{Z}^2 \) with \( xy \neq 0 \), then for infinitely many \( a \in \mathbb{Z} \) the twisted Thue equation \( F_a(x, y) = \pm 1 \) has also the solution \( (x, y) \), since \( F_a = F_0 \). For instance, when \( n = 12 \), we have \( \Phi_{12}(X) = X^4 - X^2 + 1 \) and the equation

\[
x^4 - x^2 y^2 + y^4 = 1
\]

has the solutions \((1, 1), (-1, 1), (1, -1), (-1, -1)\).

Let us compare the results of the present paper with our previous work.

The main result of [5], which deals only with non totally real cubic equations, is a special case of Theorem 3: the “constants” in [5] depend on \( \alpha \) and \( \upsilon \), while here they depend only on \( d \). The main result of [6] deals with Thue equations twisted by a set of units which is not supposed to be a group of rank 1, but it involves an assumption (namely that at least two of the conjugates of \( \upsilon \) have a modulus as large as a positive power of \( \upsilon \)) which we do not need here. Our Theorem 3 also improves the main result of [7]: we remove the assumption that the unit is totally real (besides, the result of [7] is not explicit in terms of the heights and regulator). We also notice that part (iii) of Theorem 1.1 of [8] follows from our Theorem 3. The main result of [9] does not assume that the twists are done by a group of units of rank 1, but it needs a strong assumption which does not occur here, namely that the field \( K \) has at most one real embedding.

2 Examples

The lower bound \( \mu \geq \lambda^{\kappa_6} \) quoted in Section 1 is true

- when \( d = 3 \) and the cubic field \( K \) is not totally real;
- for the Salem numbers;
- for the roots of the polynomials in the families giving the simplest fields of degree 3 (see [8]), and also the simplest fields of degrees 4 and 6;
• when $|v_1| = |v_2|$ and $|v_{d-1}| = |v_d|$ with $d \geq 4$. In particular when $-v$ is a Galois conjugate of $v$ (which means that the irreducible polynomial of $v$ is in $\mathbb{Z}[X^2]$).

Here is an example of this last situation. Let $\epsilon$ be an algebraic unit, not a root of unity, of degree $\ell \geq 2$ and conjugates $\epsilon_1, \epsilon_2, \ldots, \epsilon_\ell$. Let $h \geq 2$ and let $d = \ell h$. For $a \in \mathbb{Z}$, define

\begin{equation}
F_a(X, Y) = \prod_{i=1}^\ell (X^h - \epsilon_i^a Y^h).
\end{equation}

Let $R$ be the regulator of the field $\mathbb{Q}(\epsilon^{1/h})$.

From Theorem 3 we deduce the following corollary.

**Corollary 4.** Let $m \geq 1$. If the form $F_a$ in (5) is irreducible and if there exists $(x, y) \in \mathbb{Z}^2$ with $xy \neq 0$ and $|F_a(x, y)| \leq m$, then

$$|a| \leq \kappa_8 (R + \log m + \log |\tau|)R \log^* (R \log |\tau|).$$

**Proof.** Without loss of generality, assume $|\epsilon_1| \leq |\epsilon_2| \leq \cdots \leq |\epsilon_\ell|$, so that $|\epsilon_\ell| = |\tau|$. Let $\zeta$ be a primitive $h$-th root of unity. Let $v = \epsilon^{1/h}$. We apply Theorem 3 with $\alpha = \zeta$, $a_0 = 1$, $\lambda_0 = 1$, $\lambda \leq |\tau|^\ell$, $F_0(X, Y) = (X^h - Y^h)^\ell$ and $v_{ih+j} = \zeta^{j-1} \epsilon_{i+1}^{1/h} \quad (0 \leq i \leq \ell - 1, 1 \leq j \leq h)$.

From $|v_1| = |v_2| = |\epsilon_1|^{1/h} < 1$ and $|v_{d-1}| = |v_d| = |\epsilon_\ell|^{1/h}$ we deduce

$$\mu = \frac{|\epsilon_\ell|^{1/h}}{|\epsilon_1|^{1/h}} = \frac{|v_d|}{|v_1|}$$

and using (7) we conclude

$$\log \mu \geq \frac{2}{d-1} \log \lambda. \quad \Box$$

A variant of this proof is to take $\alpha = 1$, $\lambda_0 = 1$, $F_0(X, Y) = (X - Y)^d$, and to use the fact that $\zeta^a$ is also a primitive $h$-th root of unity since $F_a$ is irreducible.

**Remark.** There are cases where $\mu$ is very small when compared to $\lambda$. Let $D$ be an integer $\geq 2$. Consider the algebraic number field $K = \mathbb{Q}(\omega)$ where $\omega = \sqrt[4]{D^d - 1}$. The number $v = D - \omega$ is a Bernstein-Hasse unit of $K$. When $d$ is fixed, $\lambda$ is larger than $\kappa_9 D^{d-1}$, while $\mu$ is bounded above by $\kappa_{10}$. In this example, when $d$ is odd, the field $K$ is almost totally imaginary in the sense of [9] and our proof yields the estimate (4). However, when $d$ is even, we are not able to prove the estimate (4) : the estimate (3) has one extra factor $\log \lambda$.


3 Auxiliary results

3.1 Mahler measure, house and height

When $f$ is a polynomial in one variable of degree $d$ with coefficients in $\mathbb{Z}$ and leading coefficient $c_0 > 0$, the Mahler measure of $f$ is

$$M(f) = c_0 \prod_{i=1}^{d} \max\{1, |\gamma_i|\},$$

where $\gamma_1, \gamma_2, \ldots, \gamma_d$ are the roots of $f$ in $\mathbb{C}$.

Let us recall\(^\dagger\) that the logarithmic height $h(\gamma)$ of an algebraic number $\gamma$ of degree $d$ is $\frac{1}{d} \log M(\gamma)$ where $M(\gamma)$ is the Mahler measure of the irreducible polynomial of $\gamma$. We have

$$M(f) \leq \sqrt{d+1} H(f) \quad \text{and} \quad H(f) \leq 2^d M(f)$$

(see \cite{12}, Annex to Chapter 3, *Inequalities Between Different Heights of a Polynomial*, pp. 113–114; see also \cite{2} §1.9). The second upper bound in (6) could be replaced by the sharper one

$$H(f) \leq \left(\frac{d}{\lfloor d/2 \rfloor}\right) M(f),$$

but we shall not need it.

Let $\upsilon$ be a unit of degree $d$ and conjugates $\upsilon_1, \ldots, \upsilon_d$ with

$$|\upsilon_1| \leq |\upsilon_2| \leq \cdots \leq |\upsilon_d|,$$

so that $|\upsilon| = |\upsilon_d|$. Let $\lambda = M(\upsilon)$ and let $s$ be an index in $\{1, \ldots, d-1\}$ such that

$$|\upsilon_1| \leq |\upsilon_2| \leq \cdots \leq |\upsilon_s| \leq 1 \leq |\upsilon_{s+1}| \leq \cdots \leq |\upsilon_d|.$$

We have

$$\lambda = M(\upsilon) = |\upsilon_{s+1} \cdots \upsilon_d| \leq |\upsilon_d|^{d-s} \leq |\upsilon_d|^{d-1}$$

and

$$M(\upsilon^{-1}) = |\upsilon_1 \cdots \upsilon_s|^{-1} = M(\upsilon) = \lambda$$

with

$$\lambda \leq |\upsilon_1|^{-s} \leq |\upsilon_1|^{-(d-1)}.$$

Therefore we have

$$\lambda^{1/(d-1)} \leq |\upsilon_d| \leq \lambda \quad \text{and} \quad \lambda^{-1} \leq |\upsilon_1| \leq \lambda^{-1/(d-1)}.$$

\(^\dagger\)Our $h$ is the same as in \cite{2}, it corresponds to the logarithm of the $h$ in \cite{11}.
3.2 An elementary result

For the convenience of the reader, we include the following elementary result—similar arguments are often used without explicit mention in the literature.

**Lemma 5.** Let $U$ and $V$ be positive numbers satisfying $U \leq V \log^* U$. Then $U < 2V \log^* V$.

**Proof.** If $\log U \leq 1$, the assumption is $U \leq V$ and the conclusion follows. Assume $\log U > 1$. Then $\log U \leq \sqrt{U}$, hence the hypothesis of the lemma implies $U \leq V \sqrt{U}$ and therefore we have $U \leq V^2$. We deduce

$$\log U \leq 2 \log V,$$

hence

$$U \leq V \log U \leq 2V \log V.$$ \hfill \Box

3.3 Diophantine tool

In this section only, the positive integer $d$ is not restricted to $d \geq 3$.

The main tool is the following Diophantine estimate ([6, Proposition 2], [12, Theorem 9.1] or [2, Th. 3.2.4]), the proof of which uses transcendental number theory.

**Proposition 6.** Let $s$ and $D$ be two positive integers. There exists an effectively computable positive constant $\kappa(s, D)$, depending only upon $s$ and $D$, with the following property. Let $\eta_1, \ldots, \eta_s$ be nonzero algebraic numbers generating a number field of degree $\leq D$. Let $c_1, \ldots, c_s$ be rational integers and let $H_1, \ldots, H_s$ be real numbers $\geq 1$ satisfying

$$H_i \geq h(\eta_i) \quad (1 \leq i \leq s).$$

Let $C$ be a real number with $C \geq 2$. Suppose that one of the following two statements is true:

(i) $C \geq \max_{1 \leq j \leq s} |c_j|$

or

(ii) $H_j \leq H_s$ for $1 \leq j \leq s$ and

$$C \geq \max_{1 \leq j \leq s} \left\{ \frac{H_j}{H_s} |c_j| \right\}.$$

Suppose also $\eta_1^{c_1} \cdots \eta_s^{c_s} \neq 1$. Then

$$|\eta_1^{c_1} \cdots \eta_s^{c_s} - 1| > \exp\{-\kappa(s, D)H_1 \cdots H_s \log C\}.$$
The statement (ii) of Proposition 6 implies the statement (i) by permuting the indices so that $H_j \leq H_s$ for $1 \leq j \leq s$; however, we find it more convenient to use part (i) so that we can use the estimate without permuting the indices.

We will use Proposition 6 several times. Here is a first consequence.

**Corollary 7.** Let $d \geq 1$. There exists an effectively computable constant $\kappa_{11}$, which depends only on $d$, with the following property. Let $K$ be a number field of degree $d$. Let $\alpha_1, \alpha_2, v_1, v_2$ be nonzero elements in $K$ and let $a$ be a nonzero integer. Set $\gamma_1 = \alpha_1 v_1^a$ and $\gamma_2 = \alpha_2 v_2^a$. Let $\lambda_0$ and $\lambda$ satisfy

$$\max\{h(\alpha_1), h(\alpha_2)\} \leq \log \lambda_0, \quad \max\{h(v_1), h(v_2)\} \leq \log \lambda$$

and assume $\gamma_1 \neq \gamma_2$. Define

$$\chi = (\log^* \lambda_0)(\log^* \lambda) \log^* \left(|a| \min \left\{1, \frac{\log^* \lambda}{\log^* \lambda_0}\right\}\right).$$

Then

$$|\gamma_1 - \gamma_2| \geq \max \{|\gamma_1|, |\gamma_2|\} e^{-C}.$$

**Proof.** By symmetry, without loss of generality, we may assume $|\gamma_2| \geq |\gamma_1|$. Set

$$s = 2, \quad \eta_1 = \frac{v_1}{v_2}, \quad \eta_2 = \frac{\alpha_1}{\alpha_2}, \quad c_1 = a, \quad c_2 = 1,$$

$$H_1 = 2 \log^* \lambda, \quad H_2 = 2 \log^* \lambda_0, \quad C = \max \left\{2, |a| \min \left\{1, \frac{H_1}{H_2}\right\}\right\}.$$  

The conclusion of Corollary 7 follows from Proposition 6 (via part (i) if $H_1 \geq H_2$, via part (ii) otherwise), thanks to the relation

$$|\eta_1^{c_1} \eta_2^{c_2} - 1| = |\gamma_2|^{-1}|\gamma_1 - \gamma_2|.$$  

3.4 Lower bound for the height and the regulator

For the record, we quote Kronecker’s Theorem and its effective improvement.

**Lemma 8.** (a) If a nonzero algebraic integer $\alpha$ has all its conjugates in the closed unit disc $\{z \in \mathbb{C} \mid |z| \leq 1\}$, then $\alpha$ is a root of unity.

(b) More precisely, given $d \geq 1$, there exists an effectively computable positive constant $\kappa_{12}$, depending only on $d$, such that, if $\alpha$ is a nonzero algebraic integer of degree $d$ satisfying $h(\alpha) < \kappa_{12}$, then $\alpha$ is a root of unity.
Proof. Voutier (1996) refined an earlier estimate due to Dobrowolski (1979) by proving that the conclusion of part (b) in Lemma 8 holds with

$$\kappa_{12} = \begin{cases} 
\log 2 & \text{if } d = 1, \\
\frac{2}{d(\log(3d))^3} & \text{if } d \geq 2.
\end{cases}$$

See for instance [2, Prop. 3.2.9] and [12, §3.6]. \qed

Lemma 9. There exists an explicit absolute constant $\kappa_{13} > 0$ such that the regulator $R$ of any number field of degree $\geq 2$ satisfies $R > \kappa_{13}$. \qed

Proof. According to a result of Friedman (1989 – see [2, (1.5.3)]) the conclusion of Lemma 9 holds with $\kappa_{13} = 0$.\qed

3.5 A basis of units of an algebraic number field

Here is Lemma 1 of [1]. See also [2, Proposition 4.3.9]. The result is essentially due to C.L. Siegel [11].

Proposition 10. Let $d$ be a positive integer with $d \geq 3$. There exist effectively computable constants $\kappa_{14}, \kappa_{15}, \kappa_{16}$ depending only on $d$, with the following property. Let $K$ be a number field of degree $d$, with unit group of rank $r$. Let $R$ be the regulator of this field. Denote by $\varphi_1, \varphi_2, \ldots, \varphi_r$ a set of $r$ embeddings of $K$ into $C$ containing the real embeddings and no pair of conjugate embeddings. Then there exists a fundamental system of units $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_r\}$ of $K$ which satisfies the following:

(i) $\prod_{1 \leq i \leq r} h(\epsilon_i) \leq \kappa_{14} R$;

(ii) $\max_{1 \leq i \leq r} h(\epsilon_i) \leq \kappa_{15} R$;

(iii) The absolute values of the entries of the inverse matrix of $$(\log |\varphi_j(\epsilon_i)|)_{1 \leq i,j \leq r}$$ do not exceed $\kappa_{16}$.

The next result is [10, Lemma A.15].

Lemma 11. Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_r$ be an independent system of units for $K$ satisfying the condition (ii) of Proposition 10. Let $\beta \in \mathbb{Z}_K$ with $N_{K/\mathbb{Q}}(\beta) = m \neq 0$. Then there exist $b_1, b_2, \ldots, b_r$ in $\mathbb{Z}$ and $\tilde{\beta} \in \mathbb{Z}_K$ with conjugates $\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_d$, satisfying

$$\beta = \tilde{\beta} \epsilon_1^{b_1} \epsilon_2^{b_2} \cdots \epsilon_r^{b_r}$$

and

$$|m|^{1/d} e^{-\kappa_{17} R} \leq |\tilde{\beta}_j| \leq |m|^{1/d} e^{\kappa_{17} R} \quad \text{for} \quad j = 1, \ldots, d.$$
The conclusion of Lemma 11 can be written
\[ \left| \log \left( \left| m \right|^{-1/d} \tilde{\beta}_j \right) \right| \leq \kappa_1 R \quad \text{for} \quad j = 1, \ldots, d. \]

3.6 Estimates for the conjugates

**Lemma 12.** Let \( \gamma \) be an algebraic number of degree \( d \geq 3 \). Let \( \gamma_1, \gamma_2, \ldots, \gamma_d \) be the conjugates of \( \gamma \) with \( |\gamma_1| \leq |\gamma_2| \leq \cdots \leq |\gamma_d| \).

(a) If \( |\gamma_1| < |\gamma_2| \) and \( \gamma_2 \in \mathbb{R} \), then \( |\gamma_2| < |\gamma_3| \).

(b) If \( |\gamma_{d-1}| < |\gamma_d| \) and \( \gamma_{d-1} \in \mathbb{R} \), then \( |\gamma_{d-2}| < |\gamma_{d-1}| \).

**Proof.** (a) The conditions \( |\gamma_1| < |\gamma_2| \leq \cdots \leq |\gamma_i| \) for \( 3 \leq i \leq d \) imply that \( \gamma_1 \) is real and that \( -\gamma_1 \) is not a conjugate of \( \gamma_1 \). Hence the minimal polynomial of \( \gamma_1 \) is not a polynomial in \( X^2 \). Assume \( |\gamma_2| = |\gamma_3| \). Since \( -\gamma_2 \) is not a conjugate of \( \gamma_2 \), we deduce \( \gamma_3 \notin \mathbb{R} \), hence \( d \geq 4 \). We may assume \( \gamma_4 = \overline{\gamma_3} \).

Let \( \sigma \) be an automorphism of \( \mathbb{Q} \) which maps \( \gamma_2 \) to \( \gamma_1 \); via \( \sigma \), let \( \gamma_j \) be the image of \( \gamma_3 \) and \( \gamma_k \) the image of \( \gamma_4 \). From \( \gamma_2^2 = \gamma_3 \gamma_4 \) we deduce \( \gamma_1^2 = \gamma_j \gamma_k \) and \( |\gamma_1|^2 = |\gamma_j \gamma_k| \). This is not possible since \( |\gamma_j| > |\gamma_1| \) and \( |\gamma_k| > |\gamma_1| \).

(b) We deduce (b) from (a), by using \( \gamma \mapsto 1/\gamma \) (or by repeating the proof, \textit{mutatis mutandis}). \( \square \)

**Remark.** Here is an example showing that the assumptions of Lemma 12 are sharp. The polynomial \( X^4 - 4X^2 + 1 \) is irreducible, its roots are
\[ v_1 = \sqrt{2 - \sqrt{3}}, \quad v_2 = -v_1, \quad v_3 = 1/v_1 = \sqrt{2 + \sqrt{3}}, \quad v_4 = -v_3 \]
with
\[ v_1 = |v_2| < v_3 = |v_4|. \]

More generally, if \( h \geq 2 \) is a positive integer and \( \epsilon \) is a quadratic unit with Galois conjugate \( \epsilon' \) and if \( \epsilon^{1/h} \) has degree \( 2h \), then it has \( h \) conjugates of absolute value \( |\epsilon|^{1/h} \) and \( h \) conjugates of absolute value \( |\epsilon'|^{1/h} \). See also [2].

**Lemma 13.** Let \( \nu \) be an algebraic unit of degree \( d \geq 3 \). Set \( \lambda = M(\nu) \).
Let \( \nu' \) and \( \nu'' \) be two conjugates of \( \nu \) with \( |\nu'| < |\nu''| \). Then
\[ \log \left| \frac{\nu''}{\nu'} \right| \geq \kappa_1 \lambda^{-(d^3 + 2d^2 - d + 2)/2}. \]

We will deduce Lemma 13 from Theorem 1 of [3] which states the following.

[2] This reference was kindly suggested to us by Yann Bugeaud.
Lemma 14 (X. Gourdon and B. Salvy [3]). Let $P$ be a polynomial of degree $d \geq 2$ with integer coefficients and with Mahler measure $M(P)$. If $\alpha'$ and $\alpha''$ are two roots of $P$ with $|\alpha'| < |\alpha''|$, then

$$|\alpha''| - |\alpha'| \geq \kappa_{19} M(P)^{-d(d^2 + 2d - 1)/2}$$

with

$$\kappa_{19} = \frac{\sqrt{3}}{2} \left( d(d + 1)/2 \right)^{-d(d+1)/4}.$$

Proof of Lemma 14. We apply Lemma 14 to the minimal polynomial of $\upsilon$. To conclude the proof of Lemma 13, we use the bounds $|\upsilon'| \leq \lambda$ and

$$\log(1 + x) \geq \frac{x}{2} \text{ for } 0 \leq x \leq 1 \text{ with } x = \frac{|\upsilon''|}{|\upsilon'|} - 1.$$

4 Proof of Theorem 3

In order to prove Theorem 3 with the assumption $|F_a(x, y)| \leq m$, it suffices to consider the equation $F_a(x, y) = m$ with $m \neq 0$.

Let $(a, x, y, m) \in \mathbb{Z}^4$ satisfy $m \neq 0$, $xy \neq 0$, $[\mathbb{Q}(\alpha^a) : \mathbb{Q}] = d$ and

$$F_a(x, y) = m.$$

Without loss of generality, we may restrict $(a, y)$ to $a \geq 0$ (otherwise, replace $\upsilon$ by $\upsilon^{-1}$) and to $y > 0$ (otherwise replace $F_a(X, Y)$ by $F_a(X, -Y)$).

The form $\tilde{F}_a(X, Y) = a_0^{d-1} F_a(X, Y)$ has coefficients in $\mathbb{Z}$, and if we set $\tilde{x} = a_0 x$, $\tilde{y} = y$, $\tilde{m} = a_0^{d-1} m$ we have $\tilde{F}_a(\tilde{x}, \tilde{y}) = \tilde{m}$ with $(\tilde{x}, \tilde{y}) \in \mathbb{Z}^2$. Therefore, there is no loss of generality to assume $a_0 = 1$.

Theorem 3 includes the assumption that $\upsilon$ is not a root of unity, hence $\lambda > 1$. More precisely, it follows from part (b) of Lemma 8 that

$$\log \lambda \geq \kappa_{12}.$$

In particular, we have

$$\log^* \lambda \leq \max \left\{ 1, \frac{1}{\kappa_{12}} \right\} \log \lambda,$$

an inequality which can be written

(8) $$\log^* \lambda \leq \kappa_{20} \log \lambda$$

with an effectively computable constant $\kappa_{20} > 0$.

From Lemma 9, we deduce that $R > \kappa_{13}$. Therefore, there is no loss of generality to assume that, for a sufficiently large constant $\kappa_{21}$, we have

(9) $$a \geq \kappa_{21} \left( \log |m| + (\log^* \lambda_0) \log \log^* \lambda \right).$$
This hypothesis will frequently be used, sometimes without explicit mention.

By assumption, $\mathbb{Q}(\alpha\nu^a) = K$. If some conjugate $\sigma_j(\alpha\nu^a)$ of $\alpha\nu^a$ is real, then it follows that $\sigma_j(K) \subset \mathbb{R}$, hence the embedding $\sigma_j$ is real, and $\alpha_j$ and $\nu_j$ are both real. We also notice that if $\sigma_j(\nu) = -\sigma_i(\nu)$ with $i \neq j$, then it follows that $\nu$ and $-\nu$ are conjugate, hence the irreducible polynomial of $\nu$ belongs to $\mathbb{Z}[X^2]$.

Recall that $\nu_i = \sigma_i(\nu)$ ($i = 1, \ldots, d$) and that

$$|\nu_1| \leq |\nu_2| \leq \cdots \leq |\nu_d|.$$  

Let us write $\alpha_i$ for $\sigma_i(\alpha)$ ($i = 1, \ldots, d$). Let

$$\gamma = \alpha\nu^a \quad \text{and} \quad \beta = x - \gamma y.$$  

Since $a_0 = 1$, it follows that $\alpha$, $\beta$ and $\gamma$ are algebraic integers in $K$. For $j = 1, 2, \ldots, d$, define $\gamma_j$ and $\beta_j$ by

$$\gamma_j = \sigma_j(\gamma) = \alpha_j \nu_j^a, \quad \beta_j = \sigma_j(\beta) = x - \alpha_j \nu_j^a y = x - \gamma_j y.$$  

The assumption $F_a(x, y) = m$ yields $\beta_1 \beta_2 \cdots \beta_d = m$. Let $i_0 \in \{1, 2, \ldots, d\}$ be an index such that

$$|\beta_{i_0}| = \min_{1 \leq i \leq d} |\beta_i|.$$  

We define $\Psi_1, \Psi_2, \ldots, \Psi_d$ by the following conditions:

$$\beta_i = \begin{cases} \gamma_{i_0} y \Psi_i & \text{for } 1 \leq i < i_0, \\ \gamma_i y \Psi_i & \text{for } i_0 < i \leq d \end{cases}$$  

and

$$\beta_{i_0} = \frac{m}{y^{d-1}} \cdot \frac{\gamma_1 \gamma_2 \cdots \gamma_{i_0-1}}{\gamma_{i_0-2}} \Psi_{i_0}.$$  

We split the proof into several steps.

**Step 1.** We start by proving that

$$(10) \quad |x| \leq 2\lambda_0 \lambda^a y$$  

and that there exists an effectively computable positive constant $\kappa_{22}$ depending only on $d$ such that

$$(11) \quad e^{-\kappa_{22} \chi} \leq |\Psi_i| \leq e^{\kappa_{22} \chi} \quad (i = 1, 2, \ldots, d)$$  

with

$$\chi = (\log^* \lambda_0)(\log \lambda) \log \left( \min \left\{ 1, \frac{\log \lambda}{\log^* \lambda_0} \right\} \right).$$  

From the estimate (11) we will deduce

$$|\beta_{i_0}| < |\beta_i|.$$  

12
for \( i \neq i_0 \), which implies \( \alpha_{i_0} \in \mathbb{R} \) and \( \nu_{i_0} \in \mathbb{R} \).

**Remark.** The estimate (11) can be written as follows:

\[
\left| \log \left( \frac{\beta_i}{y-1} \max \{ \gamma_{i}, \gamma_{i}^{-1} \} \right) \right| \leq \kappa_{22}
\]

for \( i \neq i_0 \) and

\[
\left| \log \left( \frac{\beta_{i_0}}{y^{d-1}} \gamma_{i_0}^{-1} \gamma_{i_0}^{-1} \gamma_{i_0}^{-2} \right) \right| \leq \kappa_{22}.
\]

**Proof of (10) and (11).** We have

(12)

\[
|x| = |\beta_{i_0} + \gamma_{i_0} y| \leq |\beta_{i_0}| + |\gamma_{i_0}| y.
\]

From \( |\beta_{i_0}| \leq |\beta_i| \) for \( i = 1, 2, \ldots, d \) and \( \beta_1 \cdots \beta_d = m \), we deduce

\[
|x| \leq |m|^{1/d} + |\gamma_{i_0}| y \leq |m|^{1/d} + \lambda_0 \lambda^a y.
\]

Using the assumption (9), we check \( |m|^{1/d} \leq \lambda_0 \lambda^a y \), whereupon the inequality (10) is secured.

We also have

(13)

\[
|\beta_{i_0}|^{d-1} \max_{1 \leq i \leq d} |\beta_i| \leq |m|.
\]

For \( i = 1, 2, \ldots, d \), we write

(14)

\[
\beta_i = \beta_{i_0} + y(\gamma_{i_0} - \gamma_i).
\]

We have

\[
|\alpha_1 \alpha_2 \cdots \alpha_d| \geq 1
\]

(recall \( a_0 = 1 \)), hence

\[
\frac{1}{\lambda_0} \leq |\alpha_i| \leq \lambda_0 \quad \text{for} \quad i = 1, 2, \ldots, d.
\]

We choose an index \( j_0 \neq i_0 \) as follows:

- If \( |\nu_{i_0}| \leq \lambda^{1/(2(d-1))} \), we take \( j_0 = d \) so that, with the help of (7), we have \( |v_{j_0}| \geq \lambda^{1/(d-1)} \), whereupon with the help of (9) we obtain

\[
\frac{|\gamma_{j_0}|}{\gamma_{i_0}} < \frac{1}{2}.
\]

- If \( |\nu_{i_0}| > \lambda^{1/(2(d-1))} \), we take \( j_0 = 1 \) so that, again with the help of (7), we have \( |v_{j_0}| \leq \lambda^{-1/(d-1)} \), whereupon with the help of (9) we obtain

\[
\frac{|\gamma_{j_0}|}{\gamma_{i_0}} < \frac{1}{2}.
\]

13
In both cases, we deduce

\[ |\gamma_j - \gamma_{i_0}| \geq \frac{1}{2} \max\{|\gamma_j|,|\gamma_{i_0}|\} \geq \frac{\lambda^{a/(2(d-1))}}{2\lambda_0} \]

and therefore, using (9) again together with (13) and (14), we obtain

\[ |\beta_j| \geq |\gamma_j - \gamma_{i_0}|y - |\beta_{i_0}| \geq \frac{\lambda^{a/(2(d-1))}y}{2\lambda_0} - |m|^{1/d} \geq \lambda^{a/(2d)}y. \]

Since \( \max_{1 \leq i \leq d} |\beta_i| \geq \lambda^{a/(2d)}y \), from (13) we deduce

\[ |\beta_{i_0}| \leq \left( \frac{|m|}{y^{\lambda^{a/(2d)}}} \right)^{1/(d-1)}. \]

In particular, thanks to (9), we have

\[ |\beta_{i_0}| \leq \frac{1}{2}. \]

Using the assumption |\( x \)| \( \geq 1 \) together with (12), we deduce

\[ \frac{|x|}{2} \leq |\gamma_{i_0}|y \leq |x| + |\beta_{i_0}| \leq \frac{3|x|}{2}. \]

Let \( i \neq i_0 \). The upper bound

\[ |\gamma_i - \gamma_{i_0}| \leq 2 \max\{|\gamma_{i_0}|,|\gamma_i|\} \]

is trivial, while the lower bound

\[ |\gamma_i - \gamma_{i_0}| \geq \max\{|\gamma_{i_0}|,|\gamma_i|\} e^{-\kappa_{23}x} \]

follows from (8) and from Corollary 7. We first use the lower bound

\[ |\gamma_i - \gamma_{i_0}| \geq |\gamma_{i_0}| e^{-\kappa_{23}x}. \]

Using (17), we obtain

\[ |\gamma_i - \gamma_{i_0}| \geq \frac{1}{2y} e^{-\kappa_{23}x} \geq \frac{2}{y} e^{-\kappa_{24}x} \]

with \( \kappa_{24} > 0 \). Using the contrapositive of Lemma 5 with

\[ U = \frac{\log^* \lambda}{\log^* \lambda_0}, \quad V = \frac{1}{\kappa_{25}} \log^* \lambda, \]

we deduce from (9) that

\[ \chi \leq \kappa_{25} \log^* \lambda. \]
Recall that $\kappa_{21}$ is sufficiently large, hence $\kappa_{25}$ is sufficiently small. Now from (15), the inequality $|m| \leq e^{a/\kappa_{21}}$ and (19) we deduce

$$|\beta_{i_0}| \leq |m|^{1/(d-1)}\lambda^{-a/(2(d-1))} \leq \lambda^{-\kappa_{26}} \leq e^{-\kappa_{23}} \leq \frac{1}{2}y|\gamma_i - \gamma_{i_0}|.$$  

Therefore, for $i \neq i_0$, using (14), we deduce

$$\frac{1}{2}y|\gamma_i - \gamma_{i_0}| \leq |\beta_i| \leq \frac{3}{2}y|\gamma_i - \gamma_{i_0}|.$$  

Using once more (18), we obtain (11) for $i \neq i_0$. We also deduce

(20)  

$$|\beta_i| > \lambda^{-a/(2d)} \quad \text{for} \quad i \neq i_0.$$  

Recall

$$N(\gamma) = \gamma_1 \gamma_2 \cdots \gamma_d = N(\alpha) N(v)^a = \pm N(\alpha) \quad \text{and} \quad N(\beta) = \beta_1 \beta_2 \cdots \beta_d = m.$$  

The estimate (11) for $i = i_0$ follows from the relations

$$\Psi_1 \Psi_2 \cdots \Psi_d N(\gamma) = 1,$$

$$\frac{m}{\beta_{i_0}} = \prod_{i \neq i_0} \beta_i = y^{d-1} \gamma_{i_0}^{-1} \gamma_{i_0+1} \cdots \gamma_d \prod_{i \neq i_0} \Psi_i$$

and

$$\frac{N(\gamma)}{\gamma_{i_0}^{\gamma_{i_0-1}} \gamma_{i_0+1} \cdots \gamma_d} = \frac{\gamma_1 \cdots \gamma_{i_0-1}}{\gamma_{i_0-2}}.$$

From (9) and (11), we deduce

$$|\beta_{i_0}| < \frac{|m|}{y^{d-1}} |\gamma_1| e^{\kappa_{23}} < \lambda^{-a/(2d)},$$

hence from (20) we infer $|\beta_{i_0}| < |\beta_i|$ for $i \neq i_0$. It follows that $\beta_{i_0}$ is real, and therefore $\gamma_{i_0}, \alpha_{i_0}$ and $v_{i_0}$ also. \hfill \Box

Step 2. Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_r$ be a basis of the group of units of $K$ given by Proposition 10. From Lemma 11 it follows that there exists $\tilde{\beta} \in \mathbb{Z}_K$ and $b_1, b_2, \ldots, b_r$ in $\mathbb{Z}$ with

$$\beta = \tilde{\beta} \epsilon_{b_1} \epsilon_{b_2} \cdots \epsilon_{b_r}$$

and

$$|m|^{1/d} e^{-\kappa_{26}R} \leq |\tilde{\beta}| \leq |m|^{1/d} e^{-\kappa_{26}R} \quad \text{for} \quad i = 1, 2, \ldots, d.$$  

We set

(21)  

$$B = \kappa_{27}(R + a \log \lambda + \log y)$$  

with a sufficiently large constant $\kappa_{27}$. We want to prove that

$$\max_{1 \leq i \leq r} |b_i| \leq B.$$
Proof. We consider the system of $d$ linear forms in $r$ variables with real coefficients

$$L_i(X_1, X_2, \ldots, X_r) = \sum_{j=1}^{r} X_j \log |\sigma_i(\epsilon_j)|, \quad (i = 1, 2, \ldots, d).$$

The rank is $r$. By Proposition 10(ii),

$$\log |\sigma_i(\epsilon_j)| \leq \kappa_{28} R.$$ 

For $i = 1, 2, \ldots, d$, define $e_i = L_i(b_1, b_2, \ldots, b_r)$. We have

$$e_i = \log |\sigma_i(\beta/\tilde{\beta})| = \log |\beta_i/\tilde{\beta}_i|,$$

hence, using the inequality $|m| \leq e^{\theta a}/\kappa_{28}$ and (11), we deduce

$$|e_i| \leq \kappa_{29}(R + a \log \lambda + \log y).$$

Computing $b_1, b_2, \ldots, b_r$ by means of the system of linear equations

$$L_i(b_1, b_2, \ldots, b_r) = e_i \quad (i = 1, 2, \ldots, d)$$

and using Proposition 10(iii), we deduce

$$\max_{1 \leq j \leq r} |b_j| \leq \kappa_{30} \max_{1 \leq i \leq d} |e_i| \leq B. \quad \square$$

Step 3. From the inequality (3.2) in [1, Theorem 3] (see also [2, Th. 9.6.2]), thanks to (9), we deduce the following upper bound for $|x|$ and $|y|$ in terms of $a, \lambda, \lambda_0, m$ and $R$: there exists a positive effectively computable constant $\kappa_{31}$ depending only on $d$ such that

$$(22) \log \max\{|x|, y\} \leq \kappa_{31} R (\log^* R)(R + a \log \lambda).$$

Step 4. Assume $c_{\gamma_i\beta_j} \neq \gamma_k\beta_\ell$ for some indices $i, j, k, \ell$ in $\{1, \ldots, d\}$ and some $c \in \{1, -1\}$. Then there exists $\kappa_{32} > 0$ such that

$$\left| \frac{c_{\gamma_i\beta_j}}{\gamma_k\beta_\ell} - 1 \right| \geq \exp \left\{-\kappa_{32} (\log \lambda)(R + \log |m| + \log \lambda_0 + \log \lambda)R \right\} \times \log \left( \frac{Ra \log \lambda}{R + \log |m| + \log \lambda_0 + \log \lambda} \right).$$

Proof. This lower bound follows from Proposition 6(ii) with

$$\frac{c_{\gamma_i\beta_j}}{\gamma_k\beta_\ell} = \eta_1^{c_1} \eta_2^{c_2} \cdots \eta_s^{c_s},$$

16
where \( s = r + 2 \) and

\[
\eta_t = \frac{\sigma_j(\epsilon_t)}{\sigma_\ell(\epsilon_t)} \quad (1 \leq t \leq r), \quad \eta_{r+1} = \frac{\sigma_i(v)}{\sigma_k(v)}, \quad \eta_{r+2} = \frac{c \cdot \sigma_j(\beta)}{\sigma_\ell(\beta)},
\]

\[
c_t = b_t \quad (1 \leq t \leq r), \quad c_{r+1} = a, \quad c_{r+2} = 1,
\]

\[
H_t = \max\{1, 2h(\epsilon_t)\} \quad (1 \leq t \leq r),
\]

\[
H_{r+1} = \max\{1, 2 \log \lambda\}, \quad H_{r+2} = \kappa_{33}(R + \log |m| + \log \lambda_0 + \log \lambda),
\]

\[
C = 2 + \frac{2a \log \lambda + RB}{H_{r+2}}.
\]

Using Proposition \( \[10] \) together with part (b) of Lemma \( \[8] \) we deduce

\[
H_1 H_2 \cdots H_r \leq \kappa_{34} R.
\]

Finally we deduce from the steps 2 and 3 that

\[
\log C \leq \kappa_{35} \log \left( \frac{R a \log \lambda}{R + \log |m| + \log \lambda_0 + \log \lambda} \right),
\]

and this secures the lower bound for \( |c_\gamma \beta_j \gamma_k \beta_\ell - 1| \) announced above. \( \square \)

**Step 5.** We will prove Theorem 3 by assuming

\[
\max_{1 \leq i \leq d} \max\{|\Psi_i|, |\Psi_i|^{-1}\} > \mu^{a/4}.
\]

Using (11), we deduce from our assumption

\[
\frac{a}{4} \log \mu < \kappa_{22} \chi,
\]

hence

\[
a \leq \frac{4 \kappa_{22} (\log^* \lambda_0) (\log^* \lambda)}{\log \mu} \log^* \left( \frac{a \log^* \lambda}{\log^* \lambda_0} \right).
\]

With

\[
U = \frac{a \log^* \lambda}{\log^* \lambda_0} \quad \text{and} \quad V = \frac{4 \kappa_{22} (\log^* \lambda)^2}{\log \mu},
\]

we have \( U \leq V \log^* U \), and we conclude that we can use Lemma \( \[5] \) to deduce

\[
a \leq \frac{8 \kappa_{22} (\log^* \lambda_0) (\log^* \lambda)}{\log \mu} \log \left( \frac{4 \kappa_{22} (\log^* \lambda)^2}{\log \mu} \right),
\]

and the conclusion of Theorem \( \[3] \) follows.

In the rest of the paper, we assume

\[
\max_{1 \leq i \leq d} \max\{|\Psi_i|, |\Psi_i|^{-1}\} \leq \mu^{a/4}.
\]
Step 6. Our next goal is to prove the following results.

(a) Assume $1 \leq i_0 \leq d - 2$ and

$$\frac{|v_d-1|}{|v_i_0|} \geq \sqrt{\mu}.$$ 

Then

$$0 < \left| \frac{\gamma_d-1 \beta_d}{\gamma_d \beta_{d-1}} - 1 \right| \leq 4\lambda_0^2 \mu^{-a/4}. $$

(b) Assume $3 \leq i_0 \leq d$ and

$$\frac{|v_i_0|}{|v_2|} \geq \sqrt{\mu}. $$

Then

$$0 < \left| \frac{\beta_1}{\beta_2} - 1 \right| \leq 2\lambda_0^2 \mu^{-a/4}. $$

(c) Assume $2 \leq i_0 \leq d - 1$ and

$$\min \left\{ \frac{|v_i_0|}{|v_1|}, \frac{|v_d|}{|v_i_0|} \right\} \geq \mu.$$ 

Then

$$\left| \frac{\gamma_i \beta_d}{\gamma_d \beta_1} + 1 \right| \leq 4|m| \lambda_0^4 \mu^{-a/2}. $$

Proof (a) We approximate $\beta_d$ by $-\gamma_d y$, $\beta_{d-1}$ by $-\gamma_{d-1} y$ and we eliminate $y$. Since $\gamma$ has degree $d$, we have

$$\beta_d \gamma_{d-1} - \beta_{d-1} \gamma_d = x(\gamma_{d-1} - \gamma_d) \neq 0. $$

From (17) we deduce $|x| \leq 2|\gamma_i_0 y|$ and

$$|\beta_d \gamma_{d-1} - \beta_{d-1} \gamma_d| \leq 2|\gamma_i_0|(|\gamma_d| + |\gamma_{d-1}|) y.$$ 

Using $\beta_{d-1} = \gamma_{d-1} y \Psi_{d-1}$ together with the assumption

$$|v_d| \geq |v_{d-1}| \geq \sqrt{\mu}|v_i_0|,$$

we deduce

$$\left| \frac{\gamma_d-1 \beta_d}{\gamma_d \beta_{d-1}} - 1 \right| \leq \frac{2|\gamma_i_0|(|\gamma_{d-1}| + |\gamma_d|)}{|\gamma_d-1 \gamma_d|} |\Psi_{d-1}|^{-1} \leq 4\lambda_0^2 \mu^{-a/2}|\Psi_{d-1}|^{-1}.$$ 

The conclusion of (a) follows from (23).

(b) We approximate $\beta_1$ and $\beta_2$ by $x$ and we eliminate $x$. Since $\gamma_1 \neq \gamma_2$, we have

$$|\beta_1 - \beta_2| = |(\gamma_2 - \gamma_1) y| \neq 0.$$
From $\beta_2 = \gamma_{i_0} y \Psi_2$ and the assumption
\[ |v_1| \leq |v_2| \leq \mu^{-1/2} |v_{i_0}|, \]
we deduce
\[ \left| \frac{\beta_1}{\beta_2} - 1 \right| \leq \left| \frac{\gamma_2 + \gamma_{i_0}}{\gamma_{i_0}} \right| |\Psi_2|^{-1} \leq 2 \lambda_0^2 \mu^{-a/2} |\Psi_2|^{-1}. \]

Again, the conclusion of (b) follows from (23).

(c) We approximate $\beta_1$ by $x$, $\beta_d$ by $-y \gamma_d$ and $x$ by $y \gamma_{i_0}$, then we eliminate $x$ and $y$. More precisely we have
\[ \beta_1 \gamma_d + \beta_d \gamma_{i_0} = (\gamma_d + \gamma_{i_0}) \beta_{i_0} + \gamma_{i_0}^2 y - \gamma_1 \gamma_d y. \]
Hence
\[ \frac{\gamma_{i_0} \beta_d}{\gamma_d \beta_1} + 1 = \frac{(\gamma_d + \gamma_{i_0}) \beta_{i_0}}{\gamma_d \beta_1} + \frac{\gamma_{i_0}^2 y}{\gamma_d \beta_1} - \frac{\gamma_1 y}{\beta_1}. \]

We have $\beta_1 = \gamma_{i_0} y \Psi_1$. Therefore we have
\[ \frac{|\gamma_{i_0}|^2 y}{|\gamma_d \beta_1|} = \frac{|\gamma_{i_0}|}{|\gamma_d|} |\Psi_1|^{-1} \leq \lambda_0^2 \frac{|v_{i_0}|}{v_d} |\Psi_1|^{-1} \]
and
\[ \frac{|\gamma_1| y}{|\beta_1|} = \frac{|\gamma_1|}{|\gamma_{i_0}|} |\Psi_1|^{-1} \leq \lambda_0^2 \frac{|v_1|}{v_{i_0}} |\Psi_1|^{-1}. \]

Finally, from
\[ |\beta_{i_0}| \leq \frac{|m|}{y^{d-1}} |\gamma_1 \Psi_{i_0}| \]
we deduce
\[ \frac{|(\gamma_d + |\gamma_{i_0}|)| \beta_{i_0}|}{|\gamma_d \beta_1|} \leq (1 + \lambda_0^2) \frac{|\beta_{i_0}|}{\beta_1} \leq (1 + \lambda_0^2) \frac{|m|}{y^d} |\gamma_1 \Psi_{i_0}| \]
\[ \leq \lambda_0^2 (1 + \lambda_0^2) \frac{|m|}{y^d} \frac{|v_1|}{v_{i_0}} \frac{|\Psi_{i_0}|}{|\Psi_1|}. \]

Hence from the assumptions
\[ |v_1| \leq \mu^{-1} |v_{i_0}| \quad \text{and} \quad |v_{i_0}| \leq \mu^{-1} |v_d|, \]
we deduce
\[ \frac{|\gamma_{i_0} \beta_d}{\gamma_d \beta_1} + 1 \leq 4 |m| \lambda_0^4 \mu^{-a} \frac{|\Psi_{i_0}|}{|\Psi_1|}. \]

The conclusion of (c) follows from (23).

\[ \square \]

Step 7. (a) Assume $|v_{i_0}| = |v_1|$. Since $v_{i_0} \in \mathbb{R}$, we deduce from Lemma 12 that
\[ |v_1| < |v_{d-1}|. \]
If $|v_2| < |v_{d-1}|$, then
\[
\frac{|v_{d-1}|}{|v_0|} \geq \frac{|v_{d-1}|}{|v_2|} = \mu
\]
and we are in the case (a) of the step 6.

If $|v_2| = |v_{d-1}|$, then $i_0 = 1$, we have
\[
\frac{|v_{d-1}|}{|v_1|} \geq \mu
\]
and again we are in the case (a) of the step 6.

(b) Assume $|v_{i_0}| = |v_d|$. Using Lemma 12 we deduce
\[
|v_d| > |v_2|.
\]

If $|v_2| < |v_{d-1}|$, then
\[
\frac{|v_{i_0}|}{|v_1|} \geq \frac{|v_{d-1}|}{|v_2|} = \mu
\]
and we are in the case (b) of the step 6.

If $|v_2| = |v_{d-1}|$, then $i_0 = d$, we have
\[
\frac{|v_d|}{|v_2|} \geq \mu
\]
and again we are in the case (b) of the step 6.

(c) Assume finally $|v_1| < |v_{i_0}| < |v_d|$. In particular we have $2 \leq i_0 \leq d-1$. Assume that we are neither in the case (a) nor in the case (b) of the step 6. From
\[
\frac{|v_{d-1}|}{|v_0|} < \sqrt{\mu} \quad \text{and} \quad \frac{|v_{i_0}|}{|v_2|} < \sqrt{\mu}
\]
we deduce
\[
\frac{|v_{d-1}|}{|v_2|} < \mu.
\]

Given the definition of $\mu$, it follows that we have $|v_2| = |v_{d-1}|$. Since $v_{i_0}$ is real, Lemma 12 implies $d = 3$ and therefore $i_0 = 2$, $|v_1| < |v_2| < |v_3|$ and
\[
\mu = \min \left\{ \frac{|v_3|}{|v_2|}, \frac{|v_2|}{|v_1|} \right\}.
\]

From
\[
|\gamma_1| \leq \lambda_0 |v_1|^a \leq \lambda_0 \lambda^{-a/2} < 1, \quad |\beta_2| = |\beta_{i_0}| < 1
\]
and
\[
|\gamma_2 \beta_3| = |\gamma_2 \gamma_3 \Psi_3| \gamma \geq \frac{\gamma \Psi_3}{|\gamma_1|} \geq \lambda_0^{-1} \lambda^{a/2} |\Psi_3| > 1,
\]
we deduce \(|\gamma_1 \beta_2| < 1 < |\gamma_2 \beta_3|\), hence

\[
\gamma_1 \beta_2 + \gamma_2 \beta_3 \neq 0.
\]

There is an element of the Galois group of the Galois closure of the cubic field \(\mathbb{Q}(\upsilon)\) which maps \(\upsilon_1\) to \(\upsilon_2\), \(\upsilon_2\) to \(\upsilon_3\), \(\upsilon_3\) to \(\upsilon_1\). Therefore,

\[
\gamma_2 \beta_3 + \gamma_3 \beta_1 \neq 0.
\]

From part (c) of the step 6 we deduce

\[
0 < \left| \frac{\gamma_2 \beta_3}{\gamma_3 \beta_1} + 1 \right| \leq 4m \lambda_0^4 \mu^{-a/2}.
\]

**Step 8.** Combining the steps 6 and 7 with the step 4 where we choose

\[
\begin{align*}
  i &= \ell = d - 1, \ j = k = d, \ c = 1 \quad \text{in the case (a),} \\
  i &= k = i_0, \ j = 1 , \ \ell = d, \ c = 1 \quad \text{in the case (b),} \\
  i &= i_0, \ j = k = d, \ \ell = 1, \ c = -1 \quad \text{in the case (c).}
\end{align*}
\]

we deduce

\[
a \log \mu \leq \kappa_{36} R(R + \log |m| + \log \lambda_0 + \log \lambda)(\log \lambda)
\times \log \left( \frac{Ra \log \lambda}{R + \log |m| + \log \lambda_0 + \log \lambda} \right).
\]

For

\[
U = \frac{Ra \log \lambda}{R + \log |m| + \log \lambda_0 + \log \lambda} \quad \text{and} \quad V = \kappa_{37} \frac{R^2(\log \lambda)^2}{\log \mu},
\]

we have \(U \leq V \log^* U\). Therefore we use Lemma 5 to obtain the conclusion of Theorem 3.

### 5 Proofs of Theorem 1 and of Corollary 2

**Proof of Theorem 1.** Since \(d \geq 3\), under the assumptions of Lemma 13 we have

\[
\log \left| \frac{\upsilon''}{\upsilon'} \right| \geq \kappa_{13}(\log \lambda)^2 \lambda^{d^2(d+2)/2}.
\]

From Lemma 12 we deduce that under the assumptions of Theorem 1 and with the notations of Theorem 3 we have

\[
\log \mu \geq \kappa_{36}(\log \lambda)^2 \lambda^{d^2(d+2)/2}.
\]

Hence Theorem 3 implies Theorem 1. \(\Box\)
Proof of Corollary 2. The conclusion of Corollary 2 follows from Theorem 1 thanks to the upper bound (2).

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