Multiple Polylogarithms: An Introduction

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Dedicated to Professor R.P. Bambah on his 75th birthday

Multiple polylogarithms in a single variable are defined by

$$\mathrm{Li}_{(s_1,\ldots,s_k)}(z) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{z^{n_1}}{n_1^{s_1}\cdots n_k^{s_k}},$$

when $s_1,\ldots,s_k$ are positive integers and z a complex number in the unit disk. For $k=1$, this is the classical polylogarithm $\mathrm{Li}_s(z)$. These multiple polylogarithms can be defined also in terms of iterated Chen integrals and satisfy *shuffle relations*. Multiple polylogarithms in several variables are defined for $s_i \geq 1$ and $|z_i| < 1 (1 \leq i \leq k)$ by

$$\mathrm{Li}_{(s_1,\ldots,s_k)}(z_1,\ldots,z_k) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}},$$

and they satisfy not only shuffle relations, but also *stuffle relations*. When one specializes the shuffle relations in one variable at $z=1$ and the stuffle relations in several variables at $z_1=\cdots=z_k=1$, one gets linear or quadratic dependence relations between the Multiple Zeta Values

$$\zeta(s_1,\ldots,s_k) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}}$$

which are defined for $k, s_1,\ldots,s_k$ positive integers with $s_1 \geq 2$. *The Main Diophantine Conjecture* states that one obtains in this way all algebraic relations between these MZV.

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0. Introduction

A long term project is to determine all algebraic relations among the values

$$\pi, \zeta(3), \zeta(5), \ldots, \zeta(2n+1), \ldots$$

of the Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}.$$

So far, one only knows that the first number in this list, π, is transcendental, that the second one, $\zeta(3)$, is irrational, and that the other ones span a $\mathbb{Q}$-vector space of infinite dimension [Ri1], [BR]. See also [Ri2], [Zu1] and [Zu2].

The expected answer is disappointingly simple: it is widely believed that there are no relations, which means that these numbers should be algebraically independent:

(?) *For any $n \geq 0$ and any nonzero polynomial $P \in \mathbb{Z}[X_0, \ldots, X_n]$,*

$$P(\pi, \zeta(3), \zeta(5), \ldots, \zeta(2n+1)) \neq 0.$$

If true, this property would mean that there is no interesting algebraic structure.

The situation changes drastically if we enlarge our set so as to include the so-called Multiple Zeta Values (MZV, also called Euler-Zagier numbers or Polyzêta– see [Eu], [Z] and [C]):

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}},$$

which are defined for $k, s_1, \ldots, s_k$ positive integers with $s_1 \geq 2$. It may be hoped that the initial goal would be reached if one could determine all algebraic relations between the MZV. Now there are plenty of relations between them, providing a rich algebraic structure. One type of such relations arises when one multiplies two such series: it is easy to see that one gets a linear combination of MZV. There is another type of algebraic relations between MZV, coming from their expressions as integrals. Again the product of two such integrals is a linear combination of MZV. Following [B^3], we will use the name *stuffle* for the relations arising from the series, and *stuffle* for those arising from the integrals.

The *Main Diophantine Conjecture* (Conjecture 5.3 below) states that these relations are sufficient to describe all algebraic relations between MZV. One should be careful when stating such a conjecture: it is necessary to include some relations which are deduced from the stuffle and shuffle relations applied to divergent series (i.e. with $s_1 = 1$).

There are several ways of dealing with the divergent case. Here, we use the *multiple polylogarithms*

$$\mathrm{Li}_{(s_1, \ldots, s_k)}(z) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} \cdots n_k^{s_k}}$$

which are defined for $|z| < 1$ when $s_1, \ldots, s_k$ are all ≥ 1, and which are also defined for $|z| = 1$ if $s_1 \geq 2$.

These multiple polylogarithms can be expressed as iterated Chen integrals, and from this representation one deduces shuffle relations. There is no stuffle relations for multiple polylogarithms in a single variable, but one recovers them by introducing the multivariables functions

$$\mathrm{Li}_{(s_1,\dots,s_k)}(z_1,\dots,z_k) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}, \quad (s_i \geq 1,\ 1 \leq i \leq k)$$

which are defined not only for $|z_1| < 1$ and $|z_i| \leq 1$ $(2 \leq i \leq k)$, but also for $|z_i| \leq 1 (1 \leq i \leq k)$ if $s_1 \geq 2$.

1. Multiple Polylogarithms in One Variable and Multiple Zeta Values

Let $k, s_1, \dots, s_k$ be positive integers. Write $\underline{s}$ in place of $(s_1,\dots,s_k)$. One defines a complex function of one variable by

$$\mathrm{Li}_{\underline{s}}(z) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{z^{n_1}}{n_1^{s_1}\cdots n_k^{s_k}}.$$

This function is analytic in the open unit disk, and, in the case $s_1 \geq 2$, it is also continuous on the closed unit disk. In the latter case we have

$$\zeta(\underline{s}) = \mathrm{Li}_{\underline{s}}(1).$$

One can also define in an equivalent way these functions by induction on the number $p = s_1 + \cdots + s_k$ (the *weight* of $\underline{s}$) as follows. Plainly we have

$$z\frac{d}{dz}\mathrm{Li}_{(s_1,\dots,s_k)}(z) = \mathrm{Li}_{(s_1-1,s_2,\dots,s_k)}(z) \quad \text{if} \quad s_1 \geq 2 \tag{1.1}$$

and

$$(1-z)\frac{d}{dz}\mathrm{Li}_{(1,s_2,\dots,s_k)}(z) = \mathrm{Li}_{(s_2,\dots,s_k)}(z) \quad \text{if} \quad k \geq 2. \tag{1.2}$$

Together with the initial conditions

$$\mathrm{Li}_{\underline{s}}(0) = 0, \tag{1.3}$$

the differential equations (1.1) and (1.2) determine all the $\mathrm{Li}_{\underline{s}}$.

Therefore, as observed by M. Kontsevich (cf. [Z]; see also [K] Chap. XIX, § 11 for an early reference to H. Poincaré, 1884), an equivalent definition for $\mathrm{Li}_{\underline{s}}$ is given by integral formulae as follows. Starting(*) with $k = s = 1$, we write

$$\mathrm{Li}_1(z) = -\log(1-z) = \int_0^z \frac{dt}{1-t},$$

(*)This induction could as well be started from $k = 0$, provided that we set $\mathrm{Li}_{\emptyset}(z) = 1$.

where the complex integral is over any path from 0 to z inside the unit circle. From the differential equations (1.1) one deduces, by induction, for $s \geq 2$,

$$\mathrm{Li}_s(z) = \int_0^z \mathrm{Li}_{s-1}(t)\frac{dt}{t} = \int_0^z \frac{dt_1}{t_1}\int_0^{t_1}\frac{dt_2}{t_2}\cdots\int_0^{t_{s-2}}\frac{dt_{s-1}}{t_{s-1}}\int_0^{t_{s-1}}\frac{dt_s}{1-t_s}.$$

In the last formula, the complex integral over t_1, which is written on the left (and which is the last one to be computed), is over any path inside the unit circle from 0 to z, the second one over t_2 is from 0 to $t_1, \ldots,$ and the last one over t_s on the right, which is the first one to be computed, is from 0 to t_{s-1}.

Chen iterated integrals (see [K] Chap. XIX, § 11) provide a compact form for such expressions as follows. For $\varphi_1, \ldots, \varphi_p$ differential forms and x, y complex numbers, define inductively

$$\int_x^y \varphi_1\cdots\varphi_p = \int_x^y \varphi_1(t)\int_x^t \varphi_2\cdots\varphi_p.$$

For $\underline{s} = (s_1, \cdots s_k)$, set

$$\omega_{\underline{s}} = \omega_0^{s_1-1}\omega_1\cdots\omega_0^{s_k-1}\omega_1,$$

where

$$\omega_0(t) = \frac{dt}{t} \quad \text{and} \quad \omega_1(t) = \frac{dt}{1-t}.$$

Then the differential equations (1.1) and (1.2) with initial conditions (1.3) can be written

$$\mathrm{Li}_{\underline{s}}(z) = \int_0^z \omega_{\underline{s}}. \tag{1.4}$$

Example. Given a string $a_1, \ldots, a_k$ of integers, the notation $\{a_1, \ldots, a_k\}_n$ stands for the kn-tuple

$$(a_1, \ldots, a_k, \ldots, a_1, \ldots, a_k),$$

where the string $a_1, \ldots, a_k$ is repeated n times.

For any $n \geq 1$ and $|z| < 1$ we have

$$\mathrm{Li}_{\{1\}_n}(z) = \frac{1}{n!}(\log(1/(1-z)))^n, \tag{1.1}$$

which can be written in terms of generating series as

$$\sum_{n=0}^{\infty}\mathrm{Li}_{\{1\}_n}(z)x^n = (1-z)^{-x}.$$

The constant term $\mathrm{Li}_{\{1\}_0}(z)$ is 1.

2. Shuffle Product and the First Standard Relations

Denote by $X = \{x_0, x_1\}$ the alphabet with two letters and by X^* the set of words on X. A word is nothing else than a non-commutative monomial in the two letters x_0 and x_1. The linear combinations of such words with rational coefficients

$$\sum_u c_u u,$$

where $\{c_u; u \in X^*\}$ is a set of rational numbers with finite support, is the non-commutative ring $\mathfrak{H} = \mathbb{Q}\langle x_0, x_1\rangle$. The product is concatenation, the unit is the empty word e. We are interested with the set X^*x_1 of words which end with x_1. The linear combinations of such words is a left ideal of $\mathfrak{H}$ which we denote by $\mathfrak{H}x_1$. Also we denote by $\mathfrak{H}^1$ the subalgebra $\mathbb{Q}e + \mathfrak{H}x_1$ of $\mathfrak{H}$.

For s a positive integer, set $y_{\underline{s}} y_s = x_0^{s-1}x_1$. Next, for a tuple $\underline{s} = (s_1, \dots, s_k)$ of positive integers, define $y_{\underline{s}} = y_{s_1} \cdots y_{s_k}$. Hence the set X^*x_1 is also the set of words $y_{\underline{s}}$, where $k, s_1, \dots, s_k$ run over the set of all positive integers. We define $\widehat{\mathrm{Li}}_u(z)$ for $u \in X^*x_1$ by $\widehat{\mathrm{Li}}_u(z) = \mathrm{Li}_{\underline{s}}(z)$ when $u = y_{\underline{s}}$. By linearity we extend the definition of $\widehat{\mathrm{Li}}_u(z)$ to $\mathfrak{H}^1$:

$$\widehat{\mathrm{Li}}_u(z) = \sum_u c_u \widehat{\mathrm{Li}}_u(z) \quad \text{for} \quad v = \sum_u c_u u$$

where u ranges over a finite subset of $\{e\} \cup X^*x_1$ and $c_u \in \mathbb{Q}$, while $\widehat{\mathrm{Li}}_e(z) = 1$. The set of *convergent words* is the set, denoted by $\{e\} \cup x_0X^*x_1$, of words which start with x_0 and end with x_1 together with the empty word e. The $\mathbb{Q}$-vector subspace they span in $\mathfrak{H}$ is the subalgebra $\mathfrak{H}^0 = \mathbb{Q}e + x_0\mathfrak{H}x_1$ of $\mathfrak{H}^1$, and for v in $\mathfrak{H}^0$ we set

$$\hat{\zeta}(v) = \widehat{\mathrm{Li}}_v(1)$$

so that $\hat{\zeta} : \mathfrak{H}^0 \to \mathbb{R}$ is a $\mathbb{Q}$-linear map and

$$\hat{\zeta}(y_{\underline{s}}) = \zeta(\underline{s})$$

for $y_{\underline{s}}$ in $x_0X^*x_1$.

Definition. The *shuffle product* of two words in X^* is the element in $\mathfrak{H}$ which is defined inductively as follows:

$$e ш u = u ш e = u$$

for any u in X^*, and

$$(x_i u) ш (x_j v) = x_i (u ш (x_j v)) + x_j ((x_i u) ш v)$$

for u, v in X^* and i, j equal to 0 or 1.

This product is extended by distributivity with respect to the addition to $\mathfrak{H}$ and defines a commutative and associative law. Moreover $\mathfrak{H}^0$ and $\mathfrak{H}^1$ are stable under ш. We denote by $\mathfrak{H}^0_{ш} \subset \mathfrak{H}^1_{ш} \subset \mathfrak{H}_{ш}$ the algebras where the underlying sets are $\mathfrak{H}^0 \subset \mathfrak{H}^1 \subset \mathfrak{H}$ respectively and the product is ш. Radford's Theorem gives the structure of these algebras: they are (commutative) polynomials algebras on the set of Lyndon words (see for instance [R]).

Computing the product $\widehat{\mathrm{Li}}_u(z)\widehat{\mathrm{Li}}_{u'}(z)$ of the two associated Chen iterated integrals yields (see [MPH], Th. 2):

Proposition 2.1. *For u and u' in $\mathfrak{H}^1_{ш}$,*

$$\widehat{\mathrm{Li}}_u(z)\widehat{\mathrm{Li}}_{u'}(z) = \widehat{\mathrm{Li}}_{u ш u'}(z).$$

For instance from

$$x_1 ш (x_0x_1) = x_1x_0x_1 + 2x_0x_1^2$$

we deduce

$$\mathrm{Li}_1(z)\mathrm{Li}_2(z) = \mathrm{Li}_{(1,2)}(z) + 2\mathrm{Li}_{(2,1)}(z). \tag{2.2}$$

Setting $z = 1$, we deduce from Proposition 2.1:

$$\hat{\zeta}(u)\hat{\zeta}(u') = \hat{\zeta}(u ш u') \tag{2.3}$$

for u and u' in $\mathfrak{H}^0_{ш}$.

These are the *first standard relations* between multiple zeta values.

3. Shuffle Product for Multiple Polylogarithms in Several Variables

The functions of k complex variables(*)

$$\mathrm{Li}_{\underline{s}}(z_1, \ldots, z_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{s_1} \ldots n_k^{s_k}}$$

have been considered as early as 1904 by N. Nielsen, and rediscovered later by A.B. Goncharov [G1, G2]. Recently, J. Écalle [É] used them for z_i roots of unity

(*)Our notation for

$$\mathrm{Li}_{(s_1,\ldots,s_k)}(z_1, \ldots, z_k)$$

is the same as in [H], [W] or [C], but for Goncharov's [G2] it corresponds to

$$\mathrm{Li}_{(s_1,\ldots,s_k)}(z_k, \ldots, z_1).$$

(in case $s_1 \geq 2$): these are the *decorated multiple polylogarithms*. Of course one recovers the one variable functions $\mathrm{Li}_{\underline{s}}(z)$ by specializing $z_2 = \cdots = z_k = 1$. For simplicity we write $\mathrm{Li}_{\underline{s}}(\underline{z})$, where $\underline{z}$ stands for $(z_1, \ldots, z_k)$. There is an integral formula which extends (1.4). Define

$$\omega_z(t) = \begin{cases} \frac{zdt}{1-zt} & \text{if } z \neq 0, \\ \frac{dt}{t} & \text{if } z = 0. \end{cases}$$

From the differential equations

$$z_1 \frac{\partial}{\partial z_1} \mathrm{Li}_{\underline{s}}(\underline{z}) = \mathrm{Li}_{(s_1-1, s_2, \ldots, s_k)}(\underline{z}) \quad \text{if} \quad s_1 \geq 2$$

and

$$(1 - z_1) \frac{\partial}{\partial z_1} \mathrm{Li}_{(1, s_2, \ldots, s_k)}(\underline{z}) = \mathrm{Li}_{(s_2, \ldots, s_k)}(z_1 z_2, z_3, \ldots, z_k)$$

generalizing (1.1) and (1.2), we deduce

$$\mathrm{Li}_{\underline{s}}(\underline{z}) = \int_0^1 \omega_0^{s_1-1} \omega_{z_1} \omega_0^{s_2-1} \omega_{z_1 z_2} \cdots \omega_0^{s_k-1} \omega_{z_1 \ldots z_k}.$$

Because of the occurrence of the products $z_1 \ldots z_j$ $(1 \leq j \leq k)$, it is convenient (see for instance [G1] and [B^{3}L]) to perform the change of variables

$$y_j = z_1^{-1} \cdots z_j^{-1} \quad (1 \leq j \leq k) \quad \text{and} \quad z_j = \frac{y_{j-1}}{y_j} \quad (1 \leq j \leq k)$$

with $y_0 = 1$, and to introduce the differential forms

$$\omega'_y(t) = -\omega_{y^{-1}}(t) = \frac{dt}{t - y},$$

so that $\omega'_0 = \omega_0$ and $\omega'_1 = -\omega_1$. Also define

$$\begin{aligned} \lambda \begin{pmatrix} s_1, \ldots, s_k \\ y_1, \ldots, y_k \end{pmatrix} &= \mathrm{Li}_{\underline{s}}(1/y_1, y_1/y_2, \ldots, y_{k-1}/y_k) \\ &= \sum_{\nu_1 \geq 1} \cdots \sum_{\nu_k \geq 1} \prod_{j=1}^{k} y_j^{-\nu_j} \left(\sum_{i=j}^{k} \nu_i \right)^{-s_j} \\ &= (-1)^p \int_{\Delta_p} \omega_0^{s_1-1} \omega'_{y_1} \cdots \omega_0^{s_k-1} \omega'_{y_k}. \end{aligned}$$

With this notation some formulae are simpler. For instance the shuffle relation is easier to write with λ: the shuffle is defined on words on the alphabet $\{\omega'_y\,;\, y \in \mathbb{C}\}$, (including $y = 0$), inductively by

$$(\omega'_y u) ш (\omega'_{y'} v) = \omega'_y (u ш (\omega'_{y'} v)) + \omega'_{y'}((\omega'_y u) ш v).$$

4. Stuffle Product and the Second Standard Relations

The functions $\mathrm{Li}_{\underline{s}}(\underline{z})$ satisfy not only shuffle relations, but also *stuffle relations* arising from the product of two series:

$$\mathrm{Li}_{\underline{s}}(\underline{z})\mathrm{Li}_{\underline{s}'}(\underline{z}') = \sum_{\underline{s}''} \mathrm{Li}_{\underline{s}''}(\underline{z}''), \tag{4.1}$$

where the notation is as follows: $\underline{s}''$ runs over the tuples $(s''_1, \dots, s''_{k''})$ obtained from $\underline{s} = (s_1, \dots, s_k)$ *and* $\underline{s}' = (\underline{s}'_1, \dots, s'_{k'})$ by inserting, in all possible ways, some 0 in the string $(s_1, \dots, s_k)$ as well as in the string $(s'_1, \dots, s'_{k'})$ (including in front and at the end), so that the new strings have the same length k'', with $\max\{k, k'\} \le k'' \le k + k'$, and by adding the two sequences term by term. For each such $\underline{s}''$, the component z''_i of $\underline{z}$ is z_j if the corresponding s''_i is just s_j (corresponding to a 0 in $\underline{s}'$), it is z'_ℓ if the corresponding s''_i is s'_ℓ (corresponding to a 0 in $\underline{s}$), and finally it is $z_j z'_\ell$ if the corresponding s''_i is $s_j + s'_\ell$. For instance

$$\begin{array}{cccccccc}
\underline{s} & s_1 & s_2 & 0 & s_3 & s_4 & \dots & 0 \\
\underline{s}' & 0 & s'_1 & s'_2 & 0 & s'_3 & \dots & s'_{k'} \\
\underline{s}'' & s_1 & s_2 + s'_1 & s'_2 & s_3 & s_4 + s'_3 & \dots & s'_{k'} \\
\underline{z}'' & z_1 & z_2 z'_1 & z'_2 & z_3 & z_4 z'_3 & \dots & z'_{k'}.
\end{array}$$

Of course the 0's are inserted so that no s''_i is zero.

Examples. For $k = k' = 1$ the stuffle relation (4.1) yields

$$\mathrm{Li}_s(z)\mathrm{Li}_{s'}(z') = \mathrm{Li}_{(s,s')}(z, z') + \mathrm{Li}_{s',s}(z', z) + \mathrm{Li}_{s+s'}(zz'), \tag{4.2}$$

while for $k = 1$ and $k' = 2$ we have

$$\begin{aligned} \mathrm{Li}_s(z)\mathrm{Li}_{(s'_1,s'_2)}(z'_1, z'_2) &= \mathrm{Li}_{(s,s'_1,s'_2)}(z, z'_1, z'_2) + \mathrm{Li}_{(s'_1,s,s'_2)}(z'_1, z, z'_2) \\ &+\mathrm{Li}_{(s'_1,s'_2,s)}(z'_1, z'_2, z) + \mathrm{Li}_{(s+s'_1,s'_2)}(zz'_1, z'_2) + \mathrm{Li}_{(s'_1,s+s'_2)}(z'_1, zz'_2). \end{aligned} \tag{4.3}$$

The *stuffle product* $\star$ is defined on X^* inductively by

$$e \star u = u \star e = u$$

for $u \in X^*$,

$$x_0^n \star w = w \star x_0^n = \omega x_0^n$$

for any $n \ge 1$ and $w \in X^*$, and

$$(y_s u) \star (y_t u') = y_s(u \star (y_t u')) + y_t((y_s u) \star u') + y_{s+t}(u \star u')$$

for u and u' in X^*, $s \ge 1$, $t \ge 1$.

This product is extended by distributivity with respect to the addition to $\mathfrak{H}$ and defines a commutative and associative law. Moreover $\mathfrak{H}^0$ and $\mathfrak{H}^1$ are stable under $\star$. We denote by $\mathfrak{H}^0_\star \subset \mathfrak{H}^1_\star \subset \mathfrak{H}_\star$ the corresponding *harmonic* algebras. Their structure has been investigated by M. Hoffman [H]: again they are (commutative) polynomials algebras over Lyndon words.

Specializing (4.1) at $z_1 = \cdots = z_k = z'_1 = \cdots = z'_{k'} = 1$, we deduce

$$\hat{\zeta}(u)\hat{\zeta}(u') = \hat{\zeta}(u \star u') \tag{4.4}$$

for u and u' in $\mathfrak{H}^0_\star$.

These are the *second standard relations* between multiple zeta values. For instance (4.3) with $z = z'_1 = z'_2 = 1$ gives

$$\begin{aligned}\zeta(s)\zeta(s'_1, s'_2) &= \zeta(s, s'_1, s'_2) + \zeta(s'_1, s, s'_2) + \zeta(s'_1, s'_2, s)\\ &\quad + \zeta(s + s'_1, s'_2) + \zeta(s'_1, s + s'_2)\end{aligned}$$

for $s \geq 2$, $s'_1 \geq 2$ and $s'_2 \geq 1$.

5. The Third Standard Relations and the Main Diophantine Conjectures

We start with an example. Combining the stuffle relation (4.2) for $s = s' = 1$ with the shuffle relation (2.2) for $z' = z$, we deduce

$$\mathrm{Li}_{(1,2)}(z, 1) + 2\mathrm{Li}_{(2,1)}(z, 1) = \mathrm{Li}_{(1,2)}(z, z) + \mathrm{Li}_{(2,1)}(z, z) + \mathrm{Li}_3(z^2). \tag{5.1}$$

The two sides are analytic inside the unit circle, but not convergent at $z = 1$. We claim that

$$F(z) = \mathrm{Li}_{(1,2)}(z, 1) - \mathrm{Li}_{(1,2)}(z, z) = \sum_{n_1 > n_2 \geq 1} \frac{z^{n_1}(1 - z^{n_2})}{n_1 n_2^2}$$

tends to 0 as z tends to 1 inside the unit circle. Indeed for $|z| < 1$ we have

$$|1 - z^{n_2}| = |(1 - z)(1 + z + \cdots + z^{n_2 - 1})| < n_2|1 - z|,$$

hence

$$\sum_{n_2=1}^{n_1-1} \frac{|1 - z^{n_2}|}{n_2^2} < |1 - z| \sum_{n_2=1}^{n_1-1} \frac{1}{n_2}.$$

From (1.5) with $n = 2$ we deduce

$$|F(z)| \leq |1 - z|\mathrm{Li}_{(1,1)}(|z|) = \frac{1}{2}|1 - z|(\log(1/(1 - |z|)))^2.$$

Therefore, taking the limit of the relation (5.1) as $z \to 1$ yields Euler's formula

$$\zeta(2, 1) = \zeta(3).$$

This argument works in a quite general setting and yields the relations

$$\hat{\zeta}(x_1 \star v - x_1 ш v) = 0 \tag{5.2}$$

for each $v \in \mathfrak{H}^0$.

These are the *third standard relations* between multiple zeta values.

The Main Diophantine Conjectures below arose after the works of several mathematicians, including D. Zagier, A.B. Goncharov, M. Kontsevich, M. Hoffman, M. Petitot and Hoang Ngoc Minh, K. Ihara and M. Kaneko (see [C]). They imply that the three standard relations (2.3), (4.4) and (5.2) generate the ideal of algebraic relations between all numbers $\zeta(\underline{s})$. Here are precise statements.

We introduce independent variables Z_u, where u ranges over the set $\{e\} \cup X_{x_1}^*$. For $v = \sum_u c_u u$ in $\mathfrak{H}^1$, we set

$$Z_v = \sum_u c_u Z_u.$$

In particular for u_1 and u_2 in $x_0 X_{x_1}^*$, $Z_{u_1 ш u_2}$ and $Z_{u_1 \star u_2}$ are linear forms in $Z_u, u \in x_0 X^* x_1$. Also, for $v \in x_0 X^* x_1$, $Z_{x_1 ш v - x_1 \star v}$ is a linear form in Z_u, $u \in x_0 X^* x_1$.

Denote by R the ring of polynomials with coefficients in $\mathbb{Q}$ in the variables Z_u where u ranges over the set $x_0 X^* x_1$, and by $\mathfrak{J}$ the ideal of R consisting of all polynomials which vanish under the specialization map

$$Z_u \mapsto \hat{\zeta}(u) \ (u \in x_0 X^* x_1).$$

Conjecture 5.3. The polynomials

$$Z_{u_1} Z_{u_2} - Z_{u_1 ш u_2}, \quad Z_{u_1} Z_{u_2} - Z_{u_1 \star u_2} \text{ and } Z_{x_1 \star v - x_1 ш v},$$

where u_1, u_2 and v range over the set of elements in $x_0 X^* x_1$, generate the ideal $\mathfrak{J}$.

This statement is slightly different from the conjecture in § 3 of [IK], where Ihara and Kaneko suggest that all linear relations between MZV's are supplied by the *regularized double shuffle relations*

$$\hat{\zeta}(\mathrm{reg}(u \star v - u ш v)) = 0,$$

where u ranges over $\mathfrak{H}^1$ and v over $\mathfrak{H}^0$. Here, reg is the $\mathbb{Q}$-linear map $\mathfrak{H} \to \mathfrak{H}^0$ which maps w to the constant term of the expression of w as a (commutative) polynomial in x_0 and x_1 with coefficients in $\mathfrak{H}_{ш}^0$. It is proved in [IK] that the linear polynomials $Z_{\mathrm{reg}(u \star v - u ш v)}$ associated to the regularized double shuffle relations belong to the ideal $\mathfrak{J}$. On the other hand, at least for the small weights, one can check that the regularized double shuffle relations follow from the three standard relations. Hence our Conjecture 5.3 seems stronger than the conjecture of [IK], but we expect they are in fact equivalent.

Denote by $\mathfrak{Z}_p$ the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ spanned by the real numbers $\zeta(\underline{s})$ with $\underline{s}$ of weight p, with $\mathfrak{Z}_0 = \mathbb{Q}$ and $\mathfrak{Z}_1 = \{0\}$. Using any of the first two standard relations (2.3) or (4.4), one deduces $\mathfrak{Z}_p \cdot \mathfrak{Z}_{p'} \subset \mathfrak{Z}_{p+p'}$. This means that the $\mathbb{Q}$-vector subspace $\mathfrak{Z}$ of $\mathbb{R}$ spanned by all $\mathfrak{Z}_p$, $p \geq 0$, is a subalgebra of $\mathbb{R}$ over $\mathbb{Q}$ which is graded by the weight. From Conjecture 5.3 one deduces the following conjecture of Goncharov [G1]:

Conjecture 5.4. *As a $\mathbb{Q}$-algebra, $\mathfrak{Z}$ is the direct sum of $\mathfrak{Z}_p$ for $p \geq 0$.*

Conjecture 5.4 reduces the problem of determining all algebraic relations between MZV to the problem of determining all *linear* such relations. The dimension d_p of $\mathfrak{Z}_p$ satisfies $d_0 = 1$, $d_1 = 0$, $d_2 = d_3 = 1$. The expected value for d_p is given by a conjecture of Zagier [Z]:

Conjecture 5.5. *For $p \geq 3$ we have*

$$d_p = d_{p-2} + d_{p-3}.$$

An interesting question is whether Conjecture 5.3 implies Conjecture 5.5. For this question as well as other related problems, see [É] and [C].

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