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Schanuel's Conjecture:
algebraic independence of transcendental numbers

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Abstract. Schanuel's conjecture asserts that given linearly independent complex numbers $x_1, ..., x_n$, there are at least n algebraically independent numbers among the $2n$ numbers

$$x_1, \ldots, x_n,\ e^{x_1}, \ldots, e^{x_n}.$$

This simple statement has many remarkable consequences; we explain some of them. We also present the state of the art on this topic.

1 The origin of Schanuel's Conjecture

To prove that a constant arising from analysis is irrational is most often a difficult task. It was only in 1873 that C. Hermite succeeded to prove the transcendence of $e = 2.718\,281\ldots$ and it took 9 more years before F. Lindemann obtained the transcendence of $\pi = 3.141\,592\ldots$, thereby giving a final negative solution to the Greek problem of squaring the circle. This method produces the so–called Hermite–Lindemann Theorem, which states that *for any nonzero complex number z, one at least of the two numbers z, e^z is transcendental.*

To prove algebraic independence of transcendental numbers is much harder and few results are known. The earliest one is the Lindemann–Weierstrass Theorem, which states that *for $\mathbf{Q}$–linearly independent algebraic numbers $\beta_1, \ldots, \beta_n$, the numbers $e^{\beta_1}, \ldots, e^{\beta_n}$ are algebraically independent.* This is one of the very few statements on algebraic independence of numbers related with the exponential function. Even the quest of a conjectural general statement has been a challenge for many years. A.O. Gel'fond made an attempt in a one page note in the Comptes-Rendus de l'Académie des Sciences de Paris in 1934 (see the appendix), just after he solved the 7th problem of Hilbert on the transcendence of α^β, a problem which was solved by Th Schneider, at the same time, with a different but similar method.

Fourteen years later, A.O. Gel'fond was able to prove a very special case of the first theorem of his note, when he proved that *the two numbers* $2^{\sqrt[3]{2}}$ *and* $2^{\sqrt[3]{4}}$ *are algebraically independent*. This proof is a master piece, which paved the way for a number of later developments (see §2). To find "the right" conjectural statement took 15 more years, until S. Schanuel had a remarkable insight:

Schanuel's Conjecture. *Let* $x_1, ..., x_n$ *be* $\mathbf{Q}$*–linearly independent complex numbers. Then, among the* $2n$ *numbers*

$$x_1, \ldots, x_n,\ e^{x_1}, \ldots, e^{x_n},$$

there are at least n *algebraically independent numbers.*

This statement was proposed by S. Schanuel during a course given by S. Lang at Columbia in the 60's. The reference is Lang's book *Introduction to transcendental numbers*, Addison-Wesley 1966.

2 Related results

A version of Schanuel's conjecture for power series over $\mathbf{C}$ has been proved by J. Ax in 1968 using differential algebra. In his talk at ICM 1970 in Nice *Transcendence and differential algebraic geometry*, Ax describes his contribution in a general setting, where he also quotes Brun's Theorem on the integrals of the three body problem. The stronger version for power series over the field of algebraic numbers is due to R. Coleman (1980), who combined the ideas of Ax with p–adic analysis and the Čebotarev density Theorem.

A former student of Schanuel, W.D. Brownawell, obtained an elliptic analog of Ax's Theorem in a joint work with K. Kubota.

More recently, deep connections between Schanuel's Conjecture and model theory have been investigated by E. Hrushovski, B. Zilber, J. Kirby, A. Macintyre, D.E. Marker, G. Terzo, A.J. Wilkie, D. Bertrand and others.

Under the assumption of Schanuel's Conjecture, "most often", the transcendence degree is $2n$. Indeed, the set of tuples $(x_1, \ldots, x_n)$ in $\mathbf{C}^n$ such that the $2n$ numbers

$$x_1, \ldots, x_n,\ e^{x_1}, \ldots, e^{x_n}$$

are algebraically independent is a G_δ set (countable intersection of dense open sets) in Baire's classification (a *generic set* for dynamical systems) and has full Lebesgue measure. However, this result is not very significant, since it is true for any transcendental function in place of the exponential function.

The transcendence degree can also be as small as n, for instance, when the x_i are algebraic (Lindemann–Weierstrass Theorem), or when the e^{x_i} are algebraic (algebraic independence of logarithms of algebraic numbers — see §3) and also when, for each i, either x_i or e^{x_i} is algebraic, like in the first statement of Gel'fond's 1934 note.

With K. Senthil Kumar and R. Thangadurai, we recently proved that *given two integers m and n with $1 \leq m \leq n$, there exist uncountably many tuples $(x_1, \ldots, x_n)$ in $\mathbf{R}^n$ such that $x_1, \ldots, x_n$ and $e^{x_1}, \ldots, e^{x_n}$ are all Liouville numbers and the transcendence degree of the field*

$$\mathbf{Q}(x_1, \ldots, x_n,\ e^{x_1}, \ldots, e^{x_n})$$

is $n+m$. Whether such a result holds in case $m = 0$ is unclear: for instance (with $n = 1$) we do not even know whether there are Liouville numbers x such that e^x is also a Liouville number and the two numbers x and e^x are algebraically dependent.

For $n = 1$, Schanuel's Conjecture is nothing else than the Hermite–Lindemann Theorem (see §1).

For $n = 2$, Schanuel's Conjecture is not yet known:

Schanuel's Conjecture for $n = 2$: *If x_1, x_2 are $\mathbf{Q}$–linearly independent complex numbers, then among the 4 numbers x_1, x_2, e^{x_1}, e^{x_2}, at least two are algebraically independent.*

A few consequences are the following open problems:

- With $x_1 = 1$, $x_2 = i\pi$: *the number e and π are algebraically independent.*

- With $x_1 = 1$, $x_2 = e$: *the number e and e^e are algebraically independent.*

- With $x_1 = \log 2$, $x_2 = (\log 2)^2$: *the number $\log 2$ and $2^{\log 2}$ are algebraically independent.*

- With $x_1 = \log 2$, $x_2 = \log 3$: *the number $\log 2$ and $\log 3$ are algebraically independent.*

Among many mathematicians who contributed to prove partial results in the direction of Schanuel's Conjecture are Ch. Hermite, F. Lindemann, C.L. Siegel, A.O. Gel'fond, Th. Schneider, A. Baker, S. Lang, W.D. Brownawell, D.W. Masser, D. Bertrand, G.V. Chudnovsky, P. Philippon, G. Diaz, G. Wüstholz, Yu.V. Nesterenko, D. Roy...

An important step, already mentioned in §1, is due to A.O. Gel'fond who proved in 1948 that *the two numbers* $2^{\sqrt[3]{2}}$ *and* $2^{\sqrt[3]{4}}$ *are algebraically independent.* More generally, he proved that *if* α *is an algebraic number,* $\alpha \neq 0$, $\log\alpha \neq 0$ *and if* β *is an algebraic number of degree* $d \geq 3$, *then two at least of the numbers*

$$\alpha^{\beta},\ \alpha^{\beta^2},\ \ldots,\alpha^{\beta^{d-1}}$$

are algebraically independent. Recall that $\alpha^{\beta} = \exp(\beta\log\alpha)$.

The deep method devised by Gel'fond was extended by G.V. Chudnovsky (1978). One of the most remarkable results by Chudnovsky states that π *and* $\Gamma(1/4) = 3.625\,609\ldots$ *are algebraically independent.* Also π *and* $\Gamma(1/3) = 2.678\,938\ldots$ *are algebraically independent.* Until then, the transcendence of $\Gamma(1/4)$ and $\Gamma(1/3)$ was not known. The next important step is due to Yu.V.Nesterenko (1996) who proved the algebraic independence of $\Gamma(1/4)$, $\pi = 3.141\,592$ and $e^{\pi} = 23.140\,692\ldots$ Until then, the algebraic independence of π and e^{π} was not yet known. Nesterenko's proof uses modular functions.

The number e^{π} conceals mysteries.

Open problem: prove that the number e^{π} is not a Liouville number: *there exists a positive absolute constant* κ *such that for any* $p/q \in \mathbf{Q}$ *with* $q \geq 2$,

$$\left|e^{\pi} - \frac{p}{q}\right| > \frac{1}{q^{\kappa}}\cdot$$

An other open problem, consequence of Schanuel's Conjecture, is the algebraic independence of the three numbers e, π and e^{π}. More generally, according to Schanuel's Conjecture, the following numbers are algebraically independent (none of them is known to be irrational!):

$$e+\pi,\ e\pi,\ \pi^{e},\ e^{\pi^2},\ e^{e}, e^{e^2}, \ldots,\ e^{e^{e}}, \ldots,\ \pi^{\pi}, \pi^{\pi^2}, \ldots\ \pi^{\pi^{\pi}}\ldots$$

$$\log\pi,\ \log(\log 2),\ \pi\log 2,\ (\log 2)(\log 3),\ 2^{\log 2},\ (\log 2)^{\log 3}\ldots$$

The proof is an easy exercise using Schanuel's Conjecture. The list of similar exercises is endless, some recent papers pursue this direction with no new idea. A less trivial result has been proved in a joint paper in 2008 by J. Bober,

C. Cheng, B. Dietel, M. Herblot, Jingjing Huang, H. Krieger, D. Marques, J. Mason, M. Mereb and R. Wilson. Define $E_0 = \mathbf{Q}$. Inductively, for $n \geq 1$, define E_n as the algebraic closure of the field generated over E_{n-1} by the numbers $\exp(x) = e^x$, where x ranges over E_{n-1}. Let E be the union of E_n, $n \geq 0$. In a similar way, define $L_0 = \mathbf{Q}$. Inductively, for $n \geq 1$, define L_n as the algebraic closure of the field generated over L_{n-1} by the numbers y, where y ranges over the set of complex numbers such that $e^y \in L_{n-1}$. Let L be the union of L_n, $n \geq 0$. Then Schanuel's Conjecture implies that *the fields E and L are linearly disjoint over $\overline{\mathbf{Q}}$*. As a consequence, π does not belong to E, a statement proposed by S. Lang.

3 Algebraic independence of logarithms of algebraic numbers

Denote by $\mathcal{L}$ the set of complex numbers λ for which e^λ is algebraic. This set $\mathcal{L}$ is the $\mathbf{Q}$–subspace of $\mathbf{C}$ of all logarithms of nonzero algebraic numbers:

$$\mathcal{L} = \{\log \alpha \;\mid\; \alpha \in \overline{\mathbf{Q}}^\times\}.$$

Arguably, the most important special case of Schanuel's Conjecture is:

Conjecture (Algebraic independence of logarithms of algebraic numbers). *Let $\lambda_1, \ldots, \lambda_n$ be $\mathbf{Q}$-linearly independent elements in $\mathcal{L}$. Then the numbers $\lambda_1, \ldots, \lambda_n$ are algebraically independent over $\mathbf{Q}$.*

The homogeneous version is often sufficient for applications:

Conjecture (Homogeneous algebraic independence of logarithms of algebraic numbers). *Let $\lambda_1, \ldots, \lambda_n$ be $\mathbf{Q}$-linearly independent elements in $\mathcal{L}$. Let $P \in \mathbf{Q}[X_1, \ldots, X_n]$ be a* homogeneous *nonzero polynomial. Then*

$$P(\lambda_1, \ldots, \lambda_n) \neq 0.$$

In 1968, A. Baker proved that *if $\lambda_1, \ldots, \lambda_n$ are $\mathbf{Q}$–linearly independent logarithms of algebraic numbers, then the numbers $1, \lambda_1, \ldots, \lambda_n$ are linearly independent over the field $\overline{\mathbf{Q}}$ of algebraic numbers.*

Baker's Theorem is a special case of Schanuel's Conjecture: while Schanuel's Conjecture deals with algebraic independence (over $\mathbf{Q}$ or over $\overline{\mathbf{Q}}$, it is the same), Baker's Theorem deals with linear independence over $\overline{\mathbf{Q}}$.

An open problem is to prove that the transcendence degree over $\mathbf{Q}$ of the field $\mathbf{Q}(\mathcal{L})$ generated by all the logarithms of nonzero algebraic numbers is at least 2. However, even if the answer is not yet known, this does not mean that nothing is known: partial results have been proved, in particular by D. Roy, thanks to a reformulation of the problem. Instead of taking logarithms of algebraic numbers and looking for the algebraic independence relations, he fixes a polynomial and looks at the points, with coordinates logarithms of algebraic numbers, on the corresponding hypersurface. One easily checks that the homogeneous conjecture on algebraic independence of logarithms of algebraic numbers is equivalent to:

Conjecture (D. Roy). *For any algebraic subvariety X of $\mathbf{C}^n$ defined over the field $\overline{\mathbf{Q}}$ of algebraic numbers, the set $X \cap \mathcal{L}^n$ is the union of the sets $\mathcal{V} \cap \mathcal{L}^n$, where $\mathcal{V}$ ranges over the set of vector subspaces of $\mathbf{C}^n$ which are contained in X.*

Special cases of this statement have been proved by D. Roy and S. Fischler.

Even the nonexistence of quadratic relations among logarithms of algebraic numbers is not proved. For instance, Schanuel's Conjecture implies that a relation like

$$\log \alpha_1 \log \alpha_2 = \log \alpha_3$$

among nonzero logarithms of algebraic numbers is not possible. A special case would be the transcendence of the number e^{π^2} – it is not yet proved that this number is irrational.

A special case of the nonexistence of nontrivial homogeneous quadratic relations between logarithms of algebraic numbers is the four exponentials conjecture, which occurs in some works on highly composite numbers by S. Ramanujan, L. Alaoglu and P. Erdős. A very special case, which is open yet, is to prove that *if t is a real number such that 2^t and 3^t are integers, then t is an integer.*

The homogeneous conjecture of algebraic independence of logarithms of algebraic numbers can be stated in an equivalent way as saying that, under suitable assumptions, the determinant of a square matrix having entries in $\mathcal{L}$ does not vanish. To deduce such a statement from Schanuel's Conjecture is easy, the converse relies on the fact, proved by D. Roy, that *any polynomial in variables $X_1, \ldots, X_n$ is the determinant of a matrix having entries which are linear forms in $1, X_1, \ldots, X_n$.*

4 Further consequences of Schanuel's Conjecture

In 1979, P. Bundschuh investigated the transcendence of numbers of the form

$$\sum_{n\geq 1} \frac{A(n)}{B(n)},$$

where $A/B \in \mathbf{Q}(X)$ with $\deg B \geq \deg A + 2$. As an example, he noticed that

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{\pi}{2} \cdot \frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}} = 2.076\,674\,047\,4\ldots$$

while

$$\sum_{n=0}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}$$

(telescoping series).

From the Theorem of Nesterenko it follows that the number

$$\sum_{n=2}^{\infty} \frac{1}{n^s-1}$$

is transcendental over $\mathbf{Q}$ for $s = 4$. The transcendence of this number for even integers $s \geq 4$ would follow as a consequence of Schanuel's Conjecture. The example $A(X)/B(X) = 1/X^3$ shows that it will be hard to achieve a very general result, since :

$$\zeta(3) = \sum_{n\geq 1} \frac{1}{n^3},$$

an irrational number not yet known to be transcendental. Such series of values of rational fractions were studied later by S.D. Adhikari, R. Tijdeman, T.N. Shorey, R. Murty, C. Weatherby and others.

An important recent development is due to the work of R. Murty and several of his collaborators, including K. Murty, N. Saradha, S. Gun, P. Rath, C. Weatherby... They use Schanuel's conjecture to study not only the arithmetic nature of numbers like Euler's constant, Catalan's Constant, values of Euler Gamma function, the digamma function and Barnes's multiple Gamma function, but also the non vanishing of L–series at critical points.

5 Roy's program towards Schanuel's Conjecture

In the Journées Arithmétiques in Roma 1999, D. Roy revealed his ambitious program to prove Schanuel's Conjecture. So far, this is the only approach which is known toward a proof of Schanuel's Conjecture. D. Roy introduces a new conjecture of his own, which bears some similarity with known criteria of algebraic independence and he proves that his new conjecture is equivalent to Schanuel's Conjecture. Both sides of the proof of equivalence are difficult and involve a clever use of the transcendence machinerie.

Let $\mathcal{D}$ denote the derivation

$$\mathcal{D} = \frac{\partial}{\partial X_0} + X_1 \frac{\partial}{\partial X_1}$$

over the ring $\mathbf{C}[X_0, X_1]$. The *height* of a polynomial $P \in \mathbf{C}[X_0, X_1]$ is defined as the maximum of the absolute values of its coefficients.

Roy's Conjecture. *Let k be a positive integer, $y_1, \ldots, y_k$ complex numbers which are linearly independent over $\mathbf{Q}$, $\alpha_1, \ldots, \alpha_k$ non-zero complex numbers and s_0, s_1, t_0, t_1, u positive real numbers satisfying*

$$\max\{1, t_0, 2t_1\} < \min\{s_0, 2s_1\}$$

and

$$\max\{s_0, s_1 + t_1\} < u < \frac{1}{2}(1 + t_0 + t_1).$$

Assume that, for any sufficiently large positive integer N, there exists a non-zero polynomial $P_N \in \mathbf{Z}[X_0, X_1]$ with partial degree $\leq N^{t_0}$ in X_0, partial degree $\leq N^{t_1}$ in X_1 and height $\leq e^N$ which satisfies

$$\left| \left(\mathcal{D}^k P_N\right) \Big(\sum_{j=1}^{k} m_j y_j, \prod_{j=1}^{k} \alpha_j^{m_j} \Big) \right| \leq \exp(-N^u)$$

for any non-negative integers k, $m_1, \ldots, m_k$ with $k \leq N^{s_0}$ and $\max\{m_1, \ldots, m_k\} \leq$ *N^{s_1}. Then*

$$\operatorname{tr}\deg \mathbf{Q}(y_1, \ldots, y_k, \alpha_1, \ldots, \alpha_k) \geq k.$$

D. Roy already obtained partial results for the groups $\mathbf{G}_a$ and $\mathbf{G}_m$; recently he reached the first result for $\mathbf{G}_a \times \mathbf{G}_m$, so far only when the subset is reduced to a single point.

Roy's Conjecture depends on parameters s_0, s_1, t_0, t_1, u in a certain range. He proved that if his conjecture is true for one choice of values of these parameters in the given range, then Schanuel's Conjecture is true and that conversely, if Schanuel's Conjecture is true, then his conjecture is true for all choices of parameters in the same range. Recently, Nguyen Ngoc Ai Van extended the range of these parameters.

6 Ubiquity of Schanuel's Conjecture

Schanuel's Conjecture occurs in many different places; most often only the special case of homogeneous algebraic independence of logarithms of algebraic numbers is required. Some regulators are determinant of matrices with entries logarithms of algebraic numbers; the fact that they do not vanish means that some algebraic relation between these logarithms is not possible. An example, involving the p–adic analog of Schanuel's Conjecture, is Leopoldt's Conjecture on the p–adic rank of the units of an algebraic number field. The nondegenerescence of heights is also sometimes a consequence of Schanuel's Conjecture, as shown by D. Bertrand. Some special cases of a conjecture of B. Mazur on the density of rational points on algebraic varieties can be deduced from Schanuel's Conjecture. Other applications are related with questions on algebraic tori in the works of D. Prasad and of G. Prasad.

Another far-reaching topic is the connection between Schanuel's Conjecture and other conjectures on transcendental number theory, including the Conjecture of A. Grothendieck on the periods of abelian varieties, the conjecture due to Y. André on motives and the Conjecture of M. Kontsevich and D. Zagier on periods.

Appendix:

Comptes rendus hebdomadaires des séances de l'Académie des sciences, Gauthier-Villars (Paris) **199** (1934), p. 259. `http://gallica.bnf.fr/`

SÉANCE DU 23 JUILLET 1934

ARITHMÉTIQUE. — Sur quelques résultats nouveaux dans la théorie des nombres transcendants. Note de M. **A. Gelfond**, présentée par M. Hadamard.

J'ai démontré ([1]) que le nombre ω^r, où $\omega \neq 0, 1$ est un nombre algébrique et r un nombre algébrique irrationnel, doit être transcendant.

Par une généralisation de la méthode qui sert pour la démonstration du théorème énoncé, j'ai démontré les résultats plus généraux suivants:

I. THÉORÈME. — *Soient* $P(x_1, x_2, \ldots, x_n, y_1, \ldots, y_m)$ *un polynôme à coefficients entiers rationnels et* $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ *des nombres algébriques,* $\beta_i \neq 0, 1$.

L'égalité

$$P(e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n},\ \ln \beta_1, \ldots, \ln \beta_m) = 0$$

est impossible; les nombres, $\alpha_1, \alpha_2, \ldots, \alpha_n$, *et aussi les nombres* $\ln \beta_1, \ldots, \ln \beta_m$ *sont linéairement indépendants dans le corps des nombres rationnels.*

Ce théorème contient, comme cas particuliers, le théorème de Hermite et Lindemann, la résolution complète du problème de Hilbert, la transcendance des nombres $e^{\omega_1 e^{\omega_2}}$ (où ω_1 et ω_2 sont des nombres algébriques), le théorème sur la transcendance relative des nombres e et π.

II. THÉORÈME. — *Les nombres*

$$e^{\omega_1 e^{\omega_2 e^{\cdot^{\cdot^{\cdot^{\omega_{n-1} e^{\omega_n}}}}}}} \quad \textit{et} \quad \alpha_1^{\alpha_2^{\alpha_3^{\cdot^{\cdot^{\cdot^{\alpha_m}}}}}},$$

où $\omega_1 \neq 0, \omega_2, \ldots, \omega_n$ *et* $\alpha_1 \neq 0, 1,\ \alpha_2 \neq 0, 1,\ \alpha_3 \neq 0,\ \alpha_4, \ldots, \alpha_m$ *sont des nombres algébriques, sont des nombres transcendants et entre les nombres de cette forme n'existent pas de relations algébriques, à coefficients entiers rationnels* (non triviales).

La démonstration de ces résultats et de quelques autres résultats sur les nombres transcendants sera donnée dans un autre Recueil.

[1] *Sur le septième problème de D. Hilbert (C.R. de l'Acad. des Sciences de l'U.R.S.S.*, 2, I, 1er avril 1934, et *Bull. de l'Acad. des Sciences de l'U.R.S.S.*, 7e série, 4, 1934, p. 623).