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Université Polytechnique HAUTS-DE-FRANCE

Convexity of the fundamental domain of a binary form: the cyclotomic case

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Abstract

Let $\Phi_n(X,Y) \in \mathbb{Z}[X,Y]$, $n \geqslant 1$, denote the sequence of binary cyclotomic forms, so that $\Phi_n(T,1) \in \mathbb{Z}[T]$, $n \geqslant 1$, is the sequence of cyclotomic polynomials. For $n \geqslant 3$, the fundamental domain

$$\mathcal{O}_n := \{(x,y) \in \mathbb{R}^2 \mid \Phi_n(x,y) \leqslant 1\}$$

of Φ_n is a bounded subset of $\mathbb{R}^2$. In a forthcoming joint work with Étienne Fouvry we answer the question :

For which values of n is $\mathcal{O}_n$ convex ?

Convex subset of $\mathbb{R}^2$

A set $C \subset \mathbb{R}^2$ is convex if the line segment between any two points in C lies in C, i.e., if for any (x_1, y_1) and (x_2, y_2) in C and any λ with $0 \leqslant \lambda \leqslant 1$, we have

$$(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in C.$$

Peter M. Gruber. *Convex and discrete geometry*. Grundlehren der Mathematischen Wissenschaften **336**. Springer, Berlin, 2007.

Stephen Boyd & Lieven Vandenberghe. *Convex Optimization* Cambridge Univ. Press 2004.

Fundamental domain of a binary form

Let $F(X,Y) \in \mathbb{R}[X,Y]$ be a binary form of degree $\geqslant 2$ and nonzero disscriminant. We denote by $\mathcal{O}_F$ the fundamental domain of F :

$$\mathcal{O}_F = \{(x,y) \in \mathbb{R}^2 \mid |F(x,y)| \leqslant 1\}.$$

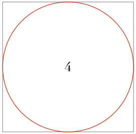
We are interested in the question of convexity of $\mathcal{O}_F$.

The domain $\mathcal{O}_F$ is bounded if and only if the polynomial $F(T,1) \in \mathbb{R}[T]$ has no real root.

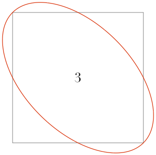
We will assume that the form F is definite positive - hence of even degree.

Quadratic forms

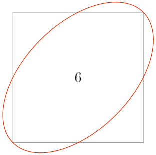
If F is a positive definite quadratic form, then after a change of variables $F(X,Y) = aX^2 + bY^2$ with a and b positive, hence $F(x,y) = 1$ is an ellipse, and therefore $\mathcal{O}_F$ is convex.



$\Phi_4(X,Y) = X^2 + Y^2$



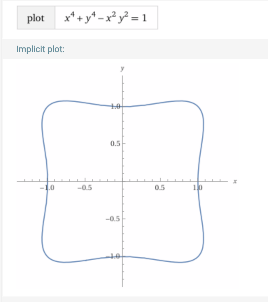
$\Phi_3(X,Y) = X^2 + XY + Y^2$

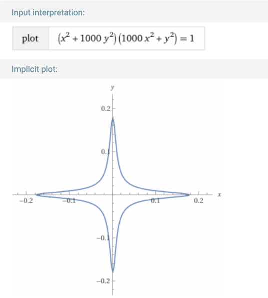


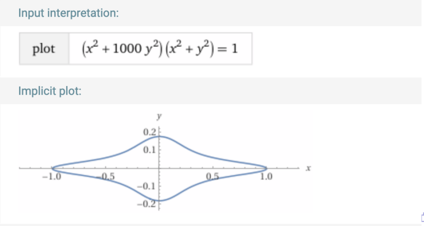
$\Phi_6(X,Y) = X^2 - XY + Y^2$

Not convex fundamental domains

Here are examples in degree 4 showing that the fundamental domain of a positive definite binary form may not be convex :





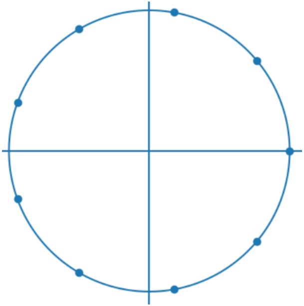


Cyclo–tomy

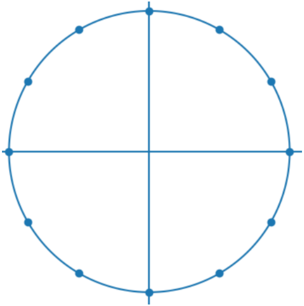
Cyclo–tomy : cut the circle

Carl Friedrich Gauss

1777 – 1855



ζ_9^m



ζ_{12}^m

Cyclotomy

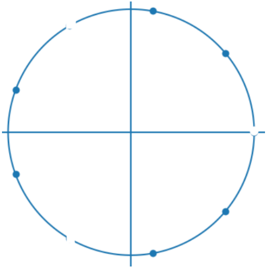
n equidistributed points on the circle : roots of $T^n - 1$

$$1, \zeta, \zeta^2, \ldots, \zeta^{n-1}, \quad \zeta = \mathrm{e}^{2\mathrm{i}\pi/n}.$$

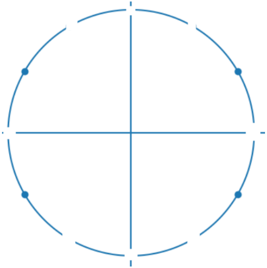
If d divides n, say $n = kd$, then $T^d - 1$ divides $T^n - 1$:

$$\frac{Z^k - 1}{Z - 1} = Z^{k-1} + \cdots + Z + 1, \quad Z = T^d.$$

New points : $\mathrm{e}^{2\mathrm{i}\pi\ell/n}$, $\gcd(\ell, n) = 1$: primitive roots of unity.



ζ_9^ℓ, $\ell = 1, 2, 4, 5, 7, 8$



ζ_{12}^ℓ, $\ell = 1, 5, 7, 11$

The sequence of cyclotomic polynomials

$T^n - 1 = \prod_{d|n} \phi_d(T)$

$T - 1 = \phi_1(T)$, $\quad \boxed{\phi_1(T) = T - 1}$

$T^2 - 1 = (T-1)(T+1) = \phi_1(T)\phi_2(T)$ $\quad \boxed{\phi_2(T) = T + 1}$

$T^3 - 1 = (T-1)(T^2+T+1) = \phi_1(T)\phi_3(T)$

$\boxed{\phi_3(T) = T^2 + T + 1}$

$T^4 - 1 = (T-1)(T+1)(T^2+1) = \phi_1(T)\phi_2(T)\phi_4(T)$

$\boxed{\phi_4(T) = T^2 + 1}$

$T^5 - 1 = (T-1)(T^4+T^3+T^2+T+1) = \phi_1(T)\phi_5(T)$

$\boxed{\phi_5(T) = T^4 + T^3 + T^2 + T + 1}$

$T^6 - 1 = \phi_1(T)\phi_2(T)\phi_3(T)\phi_6(T)$

$\boxed{\phi_6(T) = T^2 - T + 1}$

Roots of the cyclotomic polynomials

For any positive integer n, the polynomial $\phi_n(T)$ has its coefficients in $\mathbb{Z}$. Moreover, $\phi_n(T)$ is irreducible in $\mathbb{Z}[T]$.

$$T^n - 1 = \prod_{j=0}^{n-1} (T - \zeta_n^j), \quad \zeta_n = \mathrm{e}^{2i\pi/n}.$$

$$\phi_n(T) = \prod_{\gcd(j,n)=1} (T - \zeta_n^j), \qquad T^n - 1 = \prod_{d|n} \phi_d(T).$$

Let K be a field of characteristic 0 and let n be a positive integer. Then the roots of the polynomial $\phi_n(T)$ are simple and are exactly the primitive n–th roots of unity which belong to K.

Elementary properties of cyclotomic polynomials

If $n = p$ is a prime number, then from

$$T^p - 1 = (T - 1)(T^{p-1} + \cdots + T + 1) = \phi_1(T)\phi_p(T)$$

we deduce

$$\phi_p(T) = T^{p-1} + \cdots + T + 1.$$

Let m be odd.
For $a \geqslant 2$, we have $\varphi(2^a m) = 2^{a-1}\varphi(m)$ and

$\phi_{2^a}(T) = T^{2^{a-1}} + 1$ and $\phi_{2^a m}(T) = \phi_m(-T^{2^{a-1}})$ for $m \geqslant 3$.

Example : since m is odd, we have $\varphi(2m) = \varphi(m)$, and if $m \geqslant 3$,

$$\phi_{2m}(T) = \phi_m(-T).$$

Elementary properties of cyclotomic polynomials

For $n \geqslant 2$ the polynomial ϕ_n is reciprocal :

$$\phi_n(T) = T^{\varphi(n)}\phi_n(1/T).$$

When q is the radical of n (i.e. the product of all primes dividing n), we have

$$\phi_n(T) = \phi_q(T^{n/q}).$$

For p prime and m prime to p, $\varphi(pm) = (p-1)\varphi(m)$ and

$$\phi_m(T)\phi_{pm}(T) = \phi_m(T^p)$$

For instance, when m is odd, we have

$$\phi_m(T)\phi_{2m}(T) = \phi_m(T)\phi_m(-T) = \phi_m(T^2)$$

Cyclotomic binary forms

$$\Phi_n(X,Y) = Y^{\varphi(n)}\phi_n(X/Y) = X^{\varphi(n)}\phi_n(Y/X)$$

$$X^n - Y^n = \prod_{d|n} \Phi_d(X,Y).$$

$$\begin{aligned}
\Phi_1(X,Y) &= X - Y,\\
\Phi_2(X,Y) &= X + Y,\\
\Phi_3(X,Y) &= X^2 + XY + Y^2,\\
\Phi_4(X,Y) &= X^2 + Y^2,\\
\Phi_5(X,Y) &= X^4 + X^3Y + X^2Y^2 + XY^3 + Y^4,\\
\Phi_6(X,Y) &= X^2 - XY + Y^2,\\
\Phi_7(X,Y) &= X^6 + X^5Y + X^4Y^2 + X^3Y^3 + X^2Y^4 + XY^5 + Y^6,\\
\Phi_8(X,Y) &= X^4 + Y^4,\\
\Phi_9(X,Y) &= X^6 + X^3Y^3 + Y^6,\\
\Phi_{10}(X,Y) &= X^4 - X^3Y + X^2Y^2 - XY^3 + Y^4.
\end{aligned}$$

Fundamental domain of cyclotomic forms

For $n \geqslant 3$, let

$$\mathcal{O}_n := \{(x, y) \in \mathbb{R}^2 \mid \Phi_n(x, y) \leqslant 1\}$$

denote the fundamental domain of the cyclotomic polynomial of index n.

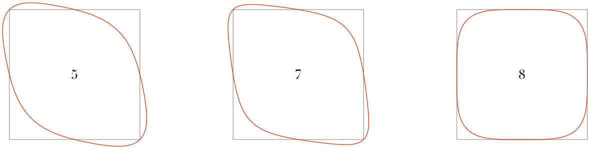
Question :

When is $\mathcal{O}_n$ convex ?

$\mathcal{O}_n$ for small values of $n \geqslant 3$: convex

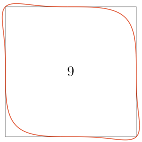
Recall : $n = 3, 4, 6$: ellipses

$$\Phi_5(X,Y) = X^4 + X^3Y + X^2Y^2 + XY^3 + Y^4$$
$$\Phi_7(X,Y) = X^6 + X^5Y + X^4Y^2 + X^3Y^3 + X^2Y^4 + XY^5 + Y^6$$
$$\Phi_8(X,Y) = X^4 + Y^4$$

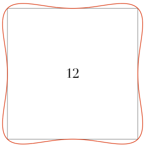


Φ_5 Φ_7 Φ_8

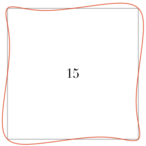
$\mathcal{O}_n$ – further values of n : not convex



Φ_9



Φ_{12}



Φ_{15}

On the convexity of the fundamental domain of binary cyclotomic forms – submitted

Étienne Fouvry

With Étienne Fouvry we prove that, for $n \geqslant 3$, the fundamental domain

$$\mathcal{O}_n := \{(x, y) \in \mathbb{R}^2 \mid \Phi_n(x, y) \leqslant 1\}$$

of Φ_n is a convex subset of $\mathbb{R}^2$ if and only if n is equal to p, $2p$ or 2^a with p an odd prime and $a \geqslant 2$.

$$\mathcal{O}_{2^a} = \{(x,y) \in \mathbb{R}^2 \mid x^{2^{a-1}} + y^{2^{a-1}} \leqslant 1\},\ a \geqslant 1$$

$$\Phi_2(X,Y) = X + Y,$$
$$\Phi_4(X,Y) = X^2 + Y^2,$$
$$\Phi_8(X,Y) = X^4 + Y^4.$$

$$\varphi(2^a) = 2^{a-1}, \qquad \Phi_{2^a}(X,Y) = X^{2^{a-1}} + Y^{2^{a-1}}.$$

Result :

For $a \geqslant 2$, *the fundamental domain*

$$\mathcal{O}_{2^a} = \{(x,y) \in \mathbb{R}^2 \mid x^{2^{a-1}} + y^{2^{a-1}} \leqslant 1\}$$

of $\Phi_{2^a}(X,Y)$ *is convex.*

More generally,

For even $k \geqslant 2$, *the fundamental domain*

$$\{(x,y) \in \mathbb{R}^2 \mid x^k + y^k \leqslant 1\}$$

of $X^k + Y^k$ *is convex.*

$x^k + y^k \leqslant 1$ is convex for even $k \geqslant 2$

Let $k \geqslant 2$ be even and let $F(x, y) = x^k + y^k$.
The boundary of $\mathcal{O}(F)$ in the quadrant $\{(x, y) \mid x \geq 0,\ y \geq 0\}$ is given by the function $x \in [0, 1] \mapsto (x, y(x))$ with

$$y(x) = (1 - x^k)^{1/k} = \sqrt[k]{1 - x^k}.$$

By elementary calculus we have

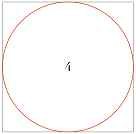
$$y' = -x^{k-1}(1 - x^k)^{(1/k)-1} = x^{k-1}\frac{y(x)}{x^k - 1}$$

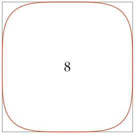
and

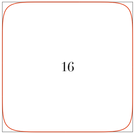
$$y'' = -(k-1)x^{k-2}(1 - x^k)^{(1/k)-2} = -(k-1)x^{k-2}\frac{y(x)}{(x^k - 1)^2}$$

which is negative for $0 < x < 1$. Hence $\mathcal{O}(F)$ is convex.

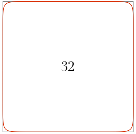
$\mathcal{O}_{2^a}$ for $a = 2, 3, 4, 5, 6, 7$

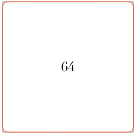


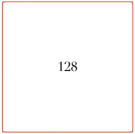




$\Phi_4(X, Y) = X^2 + Y^2$ $\Phi_8(X, Y) = X^4 + Y^4$ $\Phi_{16}(X, Y) = X^8 + Y^8$



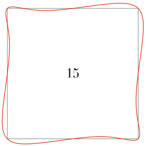




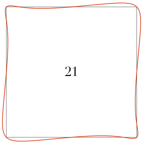
$\Phi_{32}(X, Y) = X^{16} + Y^{16}$ $\Phi_{64}(X, Y) = X^{32} + Y^{32}$ $\Phi_{128}(X, Y) = X^{64} + Y^{64}$

Not convex

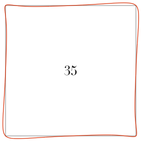
If n has two or more distinct odd prime divisors, then $\mathcal{O}_n$ is not convex.



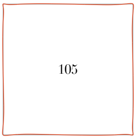
Φ_{15}



Φ_{21}



Φ_{35}



Φ_{105}

Not convex

Assume n has two or more distinct odd prime divisors. Then

$$\phi_n(-1) = \phi_n(0) = \phi_n(1) = 1,$$

hence

$$\Phi_n(-1, 1) = \Phi_n(0, 1) = \Phi_n(1, 1) = 1,$$

and therefore the line $y = 1$ has three intersection points with the boundary of $\mathcal{O}_n$ and $\mathcal{O}_n$ is not convex.

If $n = 2m$ with m odd having two or more distinct odd prime divisors, then $\mathcal{O}_n$ is not convex, since

$$\Phi_{2m}(X, Y) = \Phi_m(X, -Y).$$

Special values of the cyclotomic polynomials

We have $\phi_1(0) = -1$, $\phi_n(0) = 1$ for $n \geqslant 2$,

$$\phi_n(1) = \begin{cases} 0 & \text{if } n = 1, \\ p & \text{if } n = p^k \ (k \geq 1), \\ 1 & \text{if } \omega(n) \geq 2, \end{cases}$$

and

$$\phi_n(-1) = \begin{cases} -2 & \text{if } n = 1, \\ 0 & \text{if } n = 2, \\ p & \text{if } n = 2p^k \text{ with } p \text{ a prime and } k \geqslant 1, \\ 1 & \text{if } n \text{ is odd} \geqslant 3 \text{ or if } n = 2m \text{ where } m \\ & \text{has at least two distinct prime divisors.} \end{cases}$$

Contributions à la théorie des corps et des polynômes cyclotomiques

Trygve Nagell
1895–1988

Trygve Nagell
Contributions à la théorie des corps et des polynômes cyclotomiques.
Ark. Mat. **5,** 153–192 (1964).
Zbl 0119.27602

Special values of cyclotomic polynomials

[Herrera-Poyatos–Moree 2021, Lemmas 2.2 and 2.3]

Andrés Herrera–Poyatos

Pieter Moree

Andrés Herrera-Poyatos & Pieter Moree.
Coefficients and higher order derivatives of cyclotomic polynomials : old and new (with an appendix by Pedro García-Sánchez).
Expo. Math. 39, No. 3, 309-343 (2021).
Zbl 1486.11041 https://arxiv.org/abs/1805.05207
https://doi.org/10.1016/j.exmath.2019.07.003

Nonconvexity : end of the proof

Let $n \geqslant 3$ be an integer not of the form $n = p$, $2p$ nor $n = 2^a$ with p an odd prime and $a \geqslant 2$. Then n satisfies one of the following conditions :

1. n has at least two different odd prime divisors,
2. $n = p^\ell$ with p an odd prime and $\ell \geqslant 2$,
3. $n = 2^a p^\ell$ with p an odd prime, $a \geqslant 1$, $\ell \geqslant 1$ and $(a, \ell) \neq (1, 1)$.

In case (1), and also in case (3) when $a \geqslant 2$, the line $y = 1$ has three points of intersection with $\mathcal{O}_n$, and therefore $\mathcal{O}_n$ is not convex.

The case (3) with $a = 1$ reduces to case (2) since $\Phi_{2m}(X, Y) = \Phi_m(X, -Y)$ for m odd.

Proof of nonconvexity near $(0, 1)$

For the nonconvexity part of our result, it remains to show that in case

$$n = p^\ell \text{ with } p \text{ an odd prime and } \ell \geqslant 2,$$

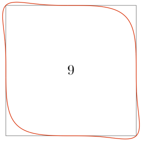
the set $\mathcal{O}_n$ is not convex.

We have

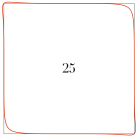
$$\phi_n(t) = \phi_{p^\ell}(t) = \phi_p(t^{p^{\ell-1}}) = 1 + t^{p^{\ell-1}} + t^{2p^{\ell-1}} + \cdots + t^{(p-1)p^{\ell-1}},$$

and the first nonzero derivative or order $\geqslant 2$ at 0 of ϕ_n has an odd order, precisely $p^{\ell-1}$. Therefore, near $t = 0$, we have $\phi_n''(t) > 0$ for $t > 0$, $\phi_n''(t) < 0$ for $t < 0$, hence the curve $\Phi_n(x, y) = 1$ has an inflexion point at $(0, 1)$, and therefore is not convex near this point.

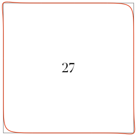
$\mathcal{O}_{p^\ell}$



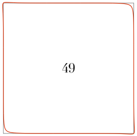
Φ_9



Φ_{25}



Φ_{27}



Φ_{49}

End of the proof

It remains to prove that for any odd prime p, the fundamental domain of $\Phi_p(X, Y)$ is convex. This result follows from the next statement with $m = p$.

Lemma.

Let $m \geqslant 2$ be an odd integer. Then the fundamental domain of the binary form

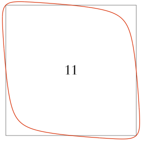
$$F(X, Y) = X^{m-1} + X^{m-2}Y + \cdots + XY^{m-2} + Y^{m-1} = \frac{X^m - Y^m}{X - Y},$$

namely

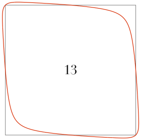
$$\mathcal{O}(F) = \{(x, y) \in \mathbb{R}^2 \ \mid \ |x^m - y^m| \leqslant |x - y|\},$$

is convex.

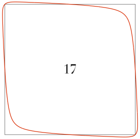
$\mathcal{O}_p$



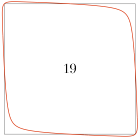
Φ_{11}



Φ_{13}



Φ_{17}



Φ_{19}

Convex function

A real valued function $F : \mathbb{R}^2 \to \mathbb{R}$ is convex if

$$F\big(\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)\big) \leq \lambda F(x_1, y_1) + (1-\lambda) F(x_2, y_2)$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and $0 \leqslant \lambda \leqslant 1$.

If F is convex, then F is continuous.

For a continuous function, the condition

$$F\left(\frac{1}{2}(x_1, y_1) + \frac{1}{2}(x_2, y_2)\right) \leqslant \frac{1}{2}\left(F(x_1, y_1) + F(x_2, y_2)\right)$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ implies that f is convex.

Quasiconvexity

A positive real valued function $F : \mathbb{R}^2 \to \mathbb{R}_{>0}$ is quasiconvex if all its sublevels

$$\mathcal{O}_\alpha := \{(x, y) \in \mathbb{R}^2 \mid F(x, y) \leqslant \alpha\}, \quad \alpha \in \mathbb{R}_{>0}$$

are convex.

Lemma.

If F is convex, then F is quasiconvex.

Proof.

Let $(x_1, y_1) \in \mathcal{O}_\alpha$ and $(x_2, y_2) \in \mathcal{O}_\alpha$. Then $F(x_1, y_1) \leqslant \alpha$ and $F(x_2, y_2) \leqslant \alpha$, hence

$$F\big(\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)\big) \leq \lambda F(x_1, y_1) + (1-\lambda)F(x_2, y_2) \leqslant \alpha$$

and therefore

$$\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) \in \mathcal{O}_\alpha.$$

Quasiconvex and Not Convex

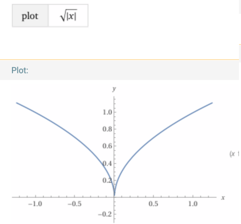
A positive real valued function may be quasiconvex and not convex. An example is

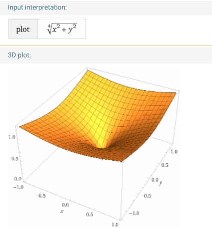
$$F(x, y) = (x^2 + y^2)^{1/4}.$$

The sublevels are discs $x^2 + y^2 \leqslant \alpha^4$, but

$$F(0,0) = 0, \quad F(1,0) = 1, \quad \frac{1}{2}(0,0) + \frac{1}{2}(1,0) = (1/2,0)$$

$$F(1/2,0) = \frac{1}{\sqrt{2}} > \frac{1}{2} = \frac{1}{2}F(0,0) + \frac{1}{2}F(1,0).$$





Quasiconvexity = convexity for homogeneous polynomials

Let $F \in \mathbb{R}[X, Y]$ *be a positive definite binary form of degree* $d \geqslant 2$*. If* F *is quasiconvex, then* F *is convex.*

Amir Ali Ahmadi & Pablo A. Parrilo. *On the Equivalence of Algebraic Conditions for Convexity and Quasiconvexity of Polynomials.* Proceedings of the 49th IEEE Conference on Decision and Control, CDC 2010, December 15-17, 2010, Atlanta, Georgia, USA.
doi CDC.2010.5717510

Sketch of proof.

Let F be a quasiconvex positive definite form of degree $d \geqslant 2$. Define $G(x,y) = F(x,y)^{1/d}$. The sublevel 1 of G,

$$S := \{(x,y) \in \mathbb{R}^2 \ \mid \ G(x,y) \leqslant 1\},$$

is the same as the sublevel 1 of F, hence is convex by assumption. Since G is continuous and homogeneous of weight 1, to prove that G is convex, it suffices to prove

$$G((x_1,y_1)+(x_2,y_2)) \leqslant G(x_1,y_1) + G(x_2,y_2)$$

for all (x_1,y_1), (x_2,y_2) in $\mathbb{R}^2$. For $(x_i,y_i) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $i=1,2$, set $\xi_i = (x_i,y_i)/G(x_i,y_i)$, so that $\xi_i \in S$, and therefore

$$\frac{G(x_1,y_1)}{G(x_1,y_1)+G(x_2,y_2)}\xi_1 + \frac{G(x_2,y_2)}{G(x_1,y_1)+G(x_2,y_2)}\xi_2 \in S.$$

Hence G is convex, and since $d \geqslant 2$ it follows that F is also convex.

Brunn–Hadamard criterion

Lemma.

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^2$–function. Then F is convex if and only if the Hessian matrix of F is semi–definite positive. This condition is equivalent to the three inequalities

$$\begin{cases} F''_{xx}(x,y) \geqslant 0 \\ F''_{yy}(x,y) \geqslant 0 \\ F''_{xx}(x,y)F''_{yy}(x,y) - F''_{xy}(x,y)^2 \geqslant 0, \end{cases}$$

for all $(x,y) \in \mathbb{R}^2$.

References :

[Ahmadi–Parrilo 2010, Th. 2.3]
[Gruber 2007, § 2.3, Th. 2.10 p. 32].

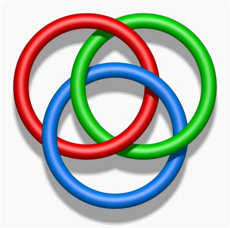
Jacques Hadamard (1865–1963)

Major contributions in number theory, complex analysis, differential geometry, and partial differential equations.

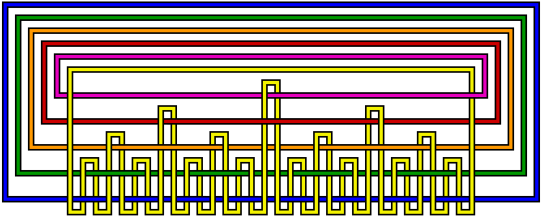
In 1896 he proved the prime number theorem, using complex function theory (also proved independently by Charles Jean de la Vallée-Poussin).

https://en.wikipedia.org/wiki/Jacques_Hadamard

Hermann Brunn (1862–1939)



Borromean rings



Brunnian links

https://en.wikipedia.org/wiki/Hermann_Brunn

Corollary of the Brunn–Hadamard criterion

Lemma.
Let n be a positive even integer and $f \in \mathbb{R}[t]$ a monic polynomial of degree n without real root. Denote by $F(X,Y) = Y^n f(X/Y)$ the associated definite positive binary form. Then $\mathcal{O}(F)$ is convex if and only if, for all $(x,y) \in \mathcal{O}(F)$, we have

$$\begin{cases} F''_{xx}(x,y) \geqslant 0 \\ F''_{yy}(x,y) \geqslant 0 \\ F''_{xx}(x,y)F''_{yy}(x,y) \geqslant F''_{xy}(x,y)^2. \end{cases}$$

Convexity proof of $\mathcal{O}_{(X^m - Y^m)/(X-Y)}$

Let $m \geqslant 3$ be odd and

$$F(X,Y) = \frac{X^m - Y^m}{X - Y} = X^{m-1} + X^{m-2} + \cdots + X + 1.$$

Let $(x,y) \in \mathbb{R}^2$ satisfy $|F(x,y)| \leqslant 1$. For $0 \leqslant s \leqslant 1$ define the linear function $L(s) = sx + (1-s)y$. The derivative of $L(s)^m$ is $m(x-y)L(s)^{m-1}$. Hence

$$F(x,y) = m \int_0^1 L(s)^{m-1} \mathrm{d}s.$$

We deduce

$$F''_{xx}(x,y) = m(m-1)(m-2) \int_0^1 s^2 L(s)^{m-3} \mathrm{d}s,$$

$$F''_{xy}(x,y) = m(m-1)(m-2) \int_0^1 s(1-s) L(s)^{m-3} \mathrm{d}s,$$

$$F''_{yy}(x,y) = m(m-1)(m-2) \int_0^1 (1-s)^2 L(s)^{m-3} \mathrm{d}s.$$

$\mathcal{O}_{(X^m - Y^m)/(X-Y)}$ is convex when m is odd

As a consequence, for $t \in \mathbb{R}$, we have

$$F''_{xx}(x,y)t^2 + 2F''_{xy}(x,y)t + F''_{yy}(x,y) = \\ m(m-1)(m-2)\int_0^1 (st+1-s)^2 L(s)^{m-3} \mathrm{d}s,$$

which is $\geqslant 0$. Therefore the above quadratic form in t is definite positive.
We deduce that equation

$$F''_{xx}(x,y)F''_{yy}(x,y) \geqslant F''_{xy}(x,y)^2$$

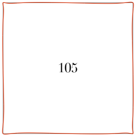
holds and we can apply the Brunn–Hadamard criterion. □

Limit of $\mathcal{O}_n$

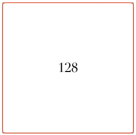
Let $\epsilon > 0$. There exists $n_0 = n_0(\epsilon)$ such that, for $n \geqslant n_0$, the cyclotomic fundamental domain $\mathcal{O}_n$ of index n contains the square centered at 0 and side length $2 - n^{-1+\epsilon}$ and is contained in the square centered at 0 and side length $2 + n^{-1+\epsilon}$.

As a consequence, its area A_{Φ_n} satifies

$$\lim_{n\to\infty} A_{\Phi_n} = 4.$$



Φ_{105}



Φ_{128}

5th International Congress on Algebra, Number Theory and Their Applications (ICANTA'5)

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Convexity of the fundamental domain of a binary form: the cyclotomic case

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