A course on interpolation

Second Course : Two Points. Lidstone, Whittaker

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Abstract

A polynomial is determined by its derivatives of even order at 0 and 1. Indeed, there exists a unique sequence of polynomials $\Lambda_0(z), \Lambda_1(z), \Lambda_2(z), \ldots$ (Lidstone polynomials) such that any polynomial f can be written as a finite sum

$$f(z) = \sum_{n\geq 0} f^{(2n)}(0)\Lambda_n(1-z) + \sum_{n\geq 0} f^{(2n)}(1)\Lambda_n(z).$$

Such an expansion into an infinite series holds for functions of exponential type $< \pi$ (Poritsky).

We also investigate the analogous problem for odd derivatives at 0 and even derivatives at 1 (Whittaker interpolation):

$$f(z) = \sum_{n\geq 0} f^{(2n)}(1)M_n(z) - \sum_{n=0}^{\infty} f^{(2n+1)}(0)M'_{n+1}(1-z).$$

Two interpolation problems

We are going to consider the following interpolation problems:

- (Lidstone):

$$f^{(2n)}(0) = a_n, \quad f^{(2n)}(1) = b_n \text{ for } n \geq 0.$$

- (Whittaker):

$$f^{(2n+1)}(0) = a_n, \quad f^{(2n)}(1) = b_n \text{ for } n \geq 0.$$

We also introduce Whittacker classification of complete, indeterminate and redundant sequences, involving standard sets of polynomials.

Lidstone interpolation problem

The following interpolation problem was considered by G.J. Lidstone in 1930.
Given two sequences of complex numbers $(a_n)_{n\geq 0}$ *and* $(b_n)_{n\geq 0}$*, does there exist an entire function* f *satisfying*

$$f^{(2n)}(0) = a_n, \quad f^{(2n)}(1) = b_n \text{ for } n \geq 0 \quad ?$$

Is such a function f *unique?*

The answer to unicity is plain and negative in general: the transcendental entire function $\sin(\pi z)$ satisfies these conditions with $a_n = b_n = 0$, hence there is no unicity, unless we restrict to entire functions satisfying some extra condition. Such a condition is a bound on the growth of f.

We start with unicity ($a_n = b_n = 0$) and polynomials.

Even derivatives at 0 and 1: first proof

Lemma. *Let f be a polynomial satisfying*

$$f^{(2n)}(0) = f^{(2n)}(1) = 0 \text{ for all } n \geq 0.$$

Then $f = 0$.

First proof.

By induction on the degree of the polynomial f.
If f has degree ≤ 1, say $f(z) = a_0 z + a_1$, the conditions $f(0) = f(1) = 0$ imply $a_0 = a_1 = 0$, hence $f = 0$.
If f has degree $\leq n$ with $n \geq 2$ and satisfies the hypotheses, then f'' also satisfies the hypotheses and has degree $< n$, hence by induction $f'' = 0$ and therefore f has degree ≤ 1.
The result follows. □

Even derivatives at 0 and 1: second proof

Second proof.

Let f be a polynomial satisfying

$$f^{(2n)}(0) = f^{(2n)}(1) = 0 \text{ for all } n \geq 0.$$

The assumption $f^{(2n)}(0) = 0$ for all $n \geq 0$ means that f is an odd function: $f(-z) = -f(z)$. The assumption $f^{(2n)}(1) = 0$ for all $n \geq 0$ means that $f(1-z)$ is an odd function: $f(1-z) = -f(1+z)$. We deduce $f(z+2) = f(1+z+1) = -f(1-z-1) = -f(-z) = f(z)$, hence the polynomial f is periodic, and therefore it is a constant. Since $f(0) = 0$, we conclude $f = 0$. □

Even derivatives at 0 and 1: third proof

Third proof.

Assume

$$f^{(2n)}(0) = f^{(2n)}(1) = 0 \text{ for all } n \geq 0.$$

Write

$$f(z) = a_1 z + a_3 z^3 + a_5 z^5 + a_7 z^7 + \cdots + a_{2n+1} z^{2n+1} + \cdots$$

(finite sum). We have $f(1) = f''(1) = f^{(\text{iv})}(1) = \cdots = 0$:

$$\begin{array}{lllllll}
a_1 & +a_3 & +a_5 & +a_7 & +\cdots & +a_{2n+1} & +\cdots = 0 \\
 & 6a_3 & +20a_5 & +42a_7 & +\cdots & +2n(2n+1)a_{2n+1} & +\cdots = 0 \\
 & & 120a_5 & +840a_7 & +\cdots & +\frac{(2n+1)!}{(2n-3)!}a_{2n+1} & +\cdots = 0 \\
 & & & & & \ddots & \vdots
\end{array}$$

The matrix of this system is triangular with maximal rank. □

Even derivatives at 0 and 1

The fact that this matrix has maximal rank means that a polynomial f is uniquely determined by the numbers

$$f^{(2n)}(0) \quad \text{and} \quad f^{(2n)}(1) \text{ for } n \geq 0.$$

Given numbers a_n and b_n, all but finitely many of them are 0, there is a unique polynomial f such that

$$f^{(2n)}(0) = a_n \quad \text{and} \quad f^{(2n)}(1) = b_n \text{ for all } n \geq 0.$$

Involution: $z \mapsto 1 - z$:

$$0 \mapsto 1, \quad 1 \mapsto 0, \quad 1 - z \mapsto z.$$

Lidstone expansion of a polynomial

G. J. Lidstone (1930). There exists a unique sequence of polynomials $\Lambda_0(z), \Lambda_1(z), \Lambda_2(z), \dots$ such that any polynomial f can be written as a finite sum

$$f(z) = \sum_{n\geq 0} f^{(2n)}(0)\Lambda_n(1-z) + \sum_{n\geq 0} f^{(2n)}(1)\Lambda_n(z).$$

This is equivalent to

$$\Lambda_n^{(2k)}(0) = 0 \quad \text{and} \quad \Lambda_n^{(2k)}(1) = \delta_{nk} \text{ for } n \geq 0 \quad \text{and} \quad k \geq 0.$$

(Kronecker symbol).
A basis of the $\mathbb{Q}$–space of polynomials in $\mathbb{Q}[z]$ of degree $\leq 2n+1$ is given by the $2n+2$ polynomials

$$\Lambda_0(z), \Lambda_1(z), \dots, \Lambda_n(z), \quad \Lambda_0(1-z), \Lambda_1(1-z), \dots, \Lambda_n(1-z).$$

Analogy with Taylor series

Given a sequence $(a_n)_{n\geq 0}$ of complex numbers, the unique analytic solution (if it exists) f of the interpolation problem

$$f^{(n)}(0) = a_n \text{ for all } n \geq 0$$

is given by the Taylor expansion

$$f(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}.$$

The polynomials $z^n/n!$ satisfy

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\left(\frac{z^n}{n!}\right)_{z=0} = \delta_{nk} \text{ for } n \geq 0 \quad \text{and} \quad k \geq 0.$$

Lidstone polynomials

$\Lambda_0(z) = z$:

$$\Lambda_0(0) = 0, \quad \Lambda_0(1) = 1, \quad \Lambda_0^{(2k)} = 0 \text{ for } k \geq 1.$$

Induction: the sequence of Lidstone polynomials is determined by $\Lambda_0(z) = z$ and

$$\Lambda_n'' = \Lambda_{n-1} \text{ for } n \geq 1$$

with the initial conditions $\Lambda_n(0) = \Lambda_n(1) = 0$ for $n \geq 1$.
Let $L_n(z)$ be any solution of

$$L_n''(z) = \Lambda_{n-1}(z).$$

Define

$$\Lambda_n(z) = -L_n(1)z + L_n(z).$$

Lidstone polynomials

$\Lambda_0(z) = z$,

$$\Lambda_n'' = \Lambda_{n-1}, \quad \Lambda_n(0) = \Lambda_n(1) = 0 \text{ for } n \geq 1.$$

For $n \geq 0$, the polynomial Λ_n is odd, it has degree $2n+1$ and leading term $\frac{1}{(2n+1)!} z^{2n+1}$.

For instance

$$\Lambda_1(z) = \frac{1}{6}(z^3 - z)$$

and

$$\Lambda_2(z) = \frac{1}{120} z^5 - \frac{1}{36} z^3 + \frac{7}{360} z = \frac{1}{360} z(z^2-1)(3z^2-7).$$

Lidstone polynomials

The polynomial $f(z) = z^{2n+1}$ satisfies

$$f^{(2k)}(0) = 0 \text{ for } k \geq 0, \quad f^{(2k)}(1) = \begin{cases} \frac{(2n+1)!}{(2n-2k+1)!} & \text{for } 0 \leq k \leq n, \\ 0 & \text{for } k \geq n+1. \end{cases}$$

One deduces

$$z^{2n+1} = \sum_{k=0}^{n-1} \frac{(2n+1)!}{(2n-2k+1)!} \Lambda_k(z) + (2n+1)! \Lambda_n(z),$$

which yields the induction formula

$$\Lambda_n(z) = \frac{1}{(2n+1)!} z^{2n+1} - \sum_{k=0}^{n-1} \frac{1}{(2n-2k+1)!} \Lambda_k(z).$$

Lidstone series : exponential type $< \pi$

Theorem (H. Poritsky, 1932).
Let f be an entire function of exponential type $< \pi$ satisfying $f^{(2n)}(0) = f^{(2n)}(1) = 0$ for all sufficiently large n. Then f is a polynomial.

This is best possible: the entire function $\sin(\pi z)$ has exponential type π and satisfies $f^{(2n)}(0) = f^{(2n)}(1) = 0$ for all $n \geq 0$.

Lidstone series : exponential type $< \pi$

Let f be an entire function of exponential type $< \pi$ satisfying $f^{(2n)}(0) = f^{(2n)}(1) = 0$ for all sufficiently large n. Then f is a polynomial.

Proof.

Let $\tilde{f} = f - P$, where P is the polynomial satisfying

$$P^{(2n)}(0) = f^{(2n)}(0) \quad \text{and} \quad P^{(2n)}(1) = f^{(2n)}(1) \text{ for } n \geq 0.$$

We have $\tilde{f}^{(2n)}(0) = \tilde{f}^{(2n)}(1) = 0$ for all $n \geq 0$. The functions $\tilde{f}(z)$ and $\tilde{f}(1-z)$ are odd, hence $\tilde{f}(z)$ is periodic of period 2. Therefore there exists a function g analytic in $\mathbb{C}^\times$ such that $\tilde{f}(z) = g(e^{i\pi z})$. Hence $g(1) = 0$. Since $\tilde{f}(z)$ has exponential type $< \pi$, we deduce $g = 0$, $\tilde{f} = 0$ and $f = P$. □

Some results on entire functions

Lemma. An entire function f is periodic of period $\omega \neq 0$ if and only if there exists a function g analytic in $\mathbb{C}^\times$ such that $f(z) = g(e^{2i\pi z/\omega})$.

Lemma. If g is an analytic function in $\mathbb{C}^\times$ and if the entire function $g(e^{2i\pi z/\omega})$ has a type $< 2(N+1)\pi/|\omega|$, then $t^N g(t)$ is a polynomial of degree $\leq 2N$.

If $g(e^{2i\pi z/\omega})$ has a type $< 2\pi/|\omega|$, then g is constant.

Exponential type $< \pi$: Poritsky's expansion

Theorem (H. Poritsky, 1932).
The expansion

$$f(z) = \sum_{n=0}^{\infty} f^{(2n)}(0)\Lambda_n(1-z) + \sum_{n=0}^{\infty} f^{(2n)}(1)\Lambda_n(z)$$

holds for any entire function f of exponential type $< \pi$.

We will check Poritsky's formula for $f_t(z) = e^{tz}$ with $|t| < \pi$, then deduce the general case.

Special case: e^{tz} for $|t| < \pi$

Consider Poritsky's expansion formula

$$f(z) = \sum_{n=0}^{\infty} f^{(2n)}(0)\Lambda_n(1-z) + \sum_{n=0}^{\infty} f^{(2n)}(1)\Lambda_n(z)$$

for the function $f_t(z) = \mathrm{e}^{tz}$ where $|t| < \pi$. Since $f_t^{(2n)}(0) = t^{2n}$ and $f_t^{(2n)}(1) = t^{2n}\mathrm{e}^{t}$ it gives

$$\mathrm{e}^{tz} = \sum_{n=0}^{\infty} t^{2n}\Lambda_n(1-z) + \mathrm{e}^{t}\sum_{n=0}^{\infty} t^{2n}\Lambda_n(z).$$

Replacing t with $-t$ yields

$$\mathrm{e}^{-tz} = \sum_{n=0}^{\infty} t^{2n}\Lambda_n(1-z) + \mathrm{e}^{-t}\sum_{n=0}^{\infty} t^{2n}\Lambda_n(z).$$

Hence

$$\mathrm{e}^{tz} - \mathrm{e}^{-tz} = (\mathrm{e}^{t} - \mathrm{e}^{-t})\sum_{n=0}^{\infty} t^{2n}\Lambda_n(z).$$

Generating series

Let $t \in \mathbb{C}$, $t \notin i\pi\mathbb{Z}$. The entire function

$$f(z) = \frac{\sinh(tz)}{\sinh(t)} = \frac{\mathrm{e}^{tz} - \mathrm{e}^{-tz}}{\mathrm{e}^{t} - \mathrm{e}^{-t}}$$

satisfies

$$f'' = t^2 f, \quad f(0) = 0, \quad f(1) = 1,$$

hence $f^{(2n)}(0) = 0$ and $f^{(2n)}(1) = t^{2n}$ for all $n \geq 0$.
For $0 < |t| < \pi$ and $z \in \mathbb{C}$, we deduce

$$\boxed{\frac{\sinh(tz)}{\sinh(t)} = \sum_{n=0}^{\infty} t^{2n} \Lambda_n(z).}$$

Notice that

$$\mathrm{e}^{tz} = \frac{\sinh((1-z)t)}{\sinh(t)} + \mathrm{e}^{t}\,\frac{\sinh(tz)}{\sinh(t)}.$$

Special case: e^{tz}

From Poritsky's expansion of an entire function of exponential type $< \pi$ we deduced the formula

$$\frac{\sinh(tz)}{\sinh(t)} = \sum_{n=0}^{\infty} t^{2n} \Lambda_n(z).$$

Let us prove this formula directly.
We will deduce

$$e^{tz} = \sum_{n=0}^{\infty} t^{2n} \Lambda_n(1-z) + e^{t} \sum_{n=0}^{\infty} t^{2n} \Lambda_n(z)$$

for $|t| < \pi$.

Expansion of $F(z,t) = \sinh(tz)/\sinh(t)$

For $z \in \mathbb{C}$ and $|t| < \pi$ let

$$F(z,t) = \frac{\sinh(tz)}{\sinh(t)}$$

with $F(z,0) = z$.

Fix $z \in \mathbb{C}$. The function $t \mapsto F(z,t)$ is analytic in the disc $|t| < \pi$ and is an even function: $F(z,-t) = F(z,t)$. Consider its Taylor series at the origin:

$$F(z,t) = \sum_{n\geq 0} c_n(z) t^{2n}$$

with $c_0(z) = z$.

We have $F(0,t) = 0$ and $F(1,t) = 1$.

Expansion of $F(z,t) = \sinh(tz)/\sinh(t)$

$$F(z,t) = \frac{\mathrm{e}^{tz} - \mathrm{e}^{-tz}}{\mathrm{e}^{t} - \mathrm{e}^{-t}} = \sum_{n \geq 0} c_n(z) t^{2n}.$$

From

$$c_n(z) = \frac{1}{(2n)!} \left(\frac{\partial}{\partial t} \right)^{2n} F(z,0)$$

it follows that $c_n(z)$ is a polynomial.

From

$$\left(\frac{\partial}{\partial z} \right)^2 F(z,t) = t^2 F(z,t)$$

we deduce

$$c_n''(z) = c_{n-1}(z) \text{ for } n \geq 1.$$

Since $c_n(0) = c_n(1) = 0$ for $n \geq 1$ we conclude $c_n(z) = \Lambda_n(z)$.

From e^{tz} to exponential type $< \pi$

Hence a special case of the Poritsky's expansion formula

$$f(z) = \sum_{n=0}^{\infty} f^{(2n)}(0)\Lambda_n(1-z) + \sum_{n=0}^{\infty} f^{(2n)}(1)\Lambda_n(z),$$

which holds for any entire function f of exponential type $< \pi$, is

$$\mathrm{e}^{tz} = \sum_{n=0}^{\infty} t^{2n}\Lambda_n(1-z) + \mathrm{e}^{t} \sum_{n=0}^{\infty} t^{2n}\Lambda_n(z)$$

for $|t| < \pi$.

Conversely, from this special case (that we proved directly) we are going to deduce the general case by means of Laplace transform (R.C. Buck, 1955, *kernel expansion method)*.

Recall Laplace transform

Let

$$f(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n$$

be an entire function of exponential type $\tau(f)$. The Laplace transform of f, viz.

$$F(t) = \sum_{n \geq 0} a_n t^{-n-1},$$

is analytic in the domain $|t| > \tau(f)$. The inverse Laplace transform is given, for $r > \tau(f)$, by

$$f(z) = \frac{1}{2\pi i} \int_{|t|=r} \mathrm{e}^{tz} F(t) \mathrm{d}t.$$

Hence

$$f^{(2n)}(z) = \frac{1}{2\pi i} \int_{|t|=r} t^{2n} \mathrm{e}^{tz} F(t) \mathrm{d}t.$$

Laplace transform

Assume $\tau(f) < \pi$. Let r satisfy $\tau(f) < r < \pi$. For $|t| = r$ we have

$$e^{tz} = \sum_{n=0}^{\infty} t^{2n} \Lambda_n(1-z) + e^t \sum_{n=0}^{\infty} t^{2n} \Lambda_n(z).$$

We deduce

$$f(z) = \sum_{n\geq 0} \Lambda_n(1-z) \left(\frac{1}{2\pi i} \int_{|t|=r} t^{2n} F(t) \mathrm{d}t \right) + \sum_{n\geq 0} \Lambda_n(z) \left(\frac{1}{2\pi i} \int_{|t|=r} t^{2n} e^t F(t) \mathrm{d}t \right)$$

and therefore

$$f(z) = \sum_{n\geq 0} f^{(2n)}(0) \Lambda_n(1-z) + \sum_{n\geq 0} f^{(2n)}(1) \Lambda_n(z),$$

where the last series are absolutely and uniformly convergent for z on any compact in $\mathbb{C}$.

Integral formula for Lidstone polynomials

Using Cauchy's residue Theorem, we deduce the integral formula

$$\Lambda_n(z) = (-1)^n \frac{2}{\pi^{2n+1}} \sum_{s=1}^{S} \frac{(-1)^s}{s^{2n+1}} \sin\bigl(s\pi z\bigr) + \frac{1}{2\pi i} \int_{|t|=(2S+1)\pi/2} t^{-2n-1} \frac{\sinh(tz)}{\sinh(t)} \mathrm{d}t$$

for $S = 1, 2, \ldots$ and $z \in \mathbb{C}$.
In particular, with $S = 1$ we have

$$\Lambda_n(z) = (-1)^n \frac{2}{\pi^{2n+1}} \sin(\pi z) + \frac{1}{2\pi i} \int_{|t|=3\pi/2} t^{-2n-1} \frac{\sinh(tz)}{\sinh(t)} \mathrm{d}t.$$

One deduces that there exists an absolute constant $\gamma > 0$ such that

$$|\Lambda_n|_r \le \gamma \pi^{-2n} \mathrm{e}^{3\pi r/2}.$$

Further estimates on Lidstone polynomials

There exist positive absolute constants γ_1, γ_2, γ_3 and γ_4 such that the following holds.

(i) For $r \geq 0$ and $n \geq 0$, we have

$$|\Lambda_n|_r \leq \frac{\gamma_1}{(2n+1)!} \max\{r, 2n+1\}^{2n+1}.$$

(ii) For sufficiently large r, we have, for all $n \geq 0$,

$$|\Lambda_n|_r \leq \gamma_2 \frac{\mathrm{e}^{r+1/(4r)}}{\sqrt{2\pi r}}.$$

(iii) For $r \geq 0$ and $n \geq 0$,

$$|\Lambda_n|_r \leq \gamma_3 \pi^{-2n} \mathrm{e}^{3\pi r/2}.$$

(iv) There exists a constant $\gamma_4 > 0$ such that, for r sufficiently large,

$$\sum_{n \geq \gamma_4 r} |\Lambda_n|_r < 1.$$

Solution of the Lidstone interpolation problem

Consequence of Poritsky's expansion formula:
Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of complex numbers satisfying

$$\limsup_{n\to\infty} |a_n|^{1/n} < \pi^2 \quad \text{and} \quad \limsup_{n\to\infty} |b_n|^{1/n} < \pi^2.$$

Then the function

$$f(z) = \sum_{n=0}^{\infty} a_n \Lambda_n(1-z) + \sum_{n=0}^{\infty} b_n \Lambda_n(z)$$

is the unique entire function of exponential type $< \pi$ satisfying

$$f^{(2n)}(0) = a_n \quad \text{and} \quad f^{(2n)}(1) = b_n \text{ for all } n \geq 0.$$

Entire functions of finite exponential type

Proposition (I.J. Schoenberg, 1936).

Let f be an entire function of finite exponential type $\tau(f)$. Then the two following conditions are equivalent.
(i) $f^{(2n)}(0) = f^{(2n)}(1) = 0$ *for all* $n \geq 0$.
(ii) *There exist complex numbers* $c_1, \ldots, c_L$ *with* $L \leq \tau(f)/\pi$ *such that*

$$f(z) = \sum_{\ell=1}^{L} c_\ell \sin(\ell \pi z).$$

Entire functions of finite exponential type

Proposition (R.C. Buck, 1954).

An entire function f of finite exponential type $\tau(f)$ can be written

$$f(z) = \sum_{k=0}^{\infty} \left(f^{(2k)}(0) g_k(1-z) + f^{(2k)}(1) g_k(z) \right) + \sum_{j=1}^{m-1} a_j \sin(\pi j z)$$

with $m\pi \le \tau$, while g_k is the sum of Λ_k and a finite trigonometric sum.

For $|t| < (N+1)\pi$,

$$\frac{\sinh(tz)}{\sinh(t)} = \pi \sum_{n=1}^{N} \frac{(-1)^{n+1} n \sin(n\pi z)}{t^2 + n^2\pi^2} + \sum_{n=0}^{\infty} g_n(z) t^{2n}.$$

An expansion of entire functions

Proposition.

Let f *be an entire function. The two following conditions are equivalent.*
(i) $f^{(2n)}(0) = f^{(2n)}(1) = 0$ *for all* $n \geq 0$.
(ii) f *is the sum of a series*

$$\sum_{n \geq 1} a_n \sin(n\pi z)$$

which converges normally on any compact.

Odd derivatives at 0 and 1

A polynomial f is determined *up to the addition of a constant* by the numbers

$$f^{(2n+1)}(0) \quad \text{and} \quad f^{(2n+1)}(1).$$

The interpolation problem related with odd derivatives at 0 and 1 is solved by using Lidstone interpolation for the derivative of f.

Odd derivatives at 0 and even derivatives at 1

Lemma. *Let f be a polynomial satisfying*

$$f^{(2n+1)}(0) = f^{(2n)}(1) = 0 \text{ for all } n \geq 0.$$

Then $f = 0$.

Proofs.

1. By induction.
2. $f(z+4) = f(z)$.
3. Triangular system. □

Whittaker expansion of a polynomial

The Lemma means that a polynomial f is uniquely determined by the numbers

$$f^{(2n+1)}(0) \quad \text{and} \quad f^{(2n)}(1) \text{ for } n \geq 0.$$

Any polynomial $f \in \mathbb{C}[z]$ has the finite expansion

$$f(z) = \sum_{n=0}^{\infty} \left(f^{(2n)}(1) M_n(z) - f^{(2n+1)}(0) M'_{n+1}(1-z) \right),$$

with only finitely many nonzero terms in the series.
A basis of the $\mathbb{Q}$–space of polynomials in $\mathbb{Q}[z]$ of degree $\leq 2n$ is given by the $2n+1$ polynomials

$$M_0(z), M_1(z), \ldots, M_n(z), \quad M'_1(1-z), \ldots, M'_n(1-z).$$

Whittaker polynomials

Following J.M. Whittaker (1935), one defines a sequence $(M_n)_{n\geq 0}$ of even polynomials by induction on n with $M_0 = 1$,

$$M_n'' = M_{n-1}, \quad M_n(1) = M_n'(0) = 0 \text{ for all } n \geq 1.$$

This is equivalent to

$$M_n^{(2k+1)}(0) = 0, \quad M_n^{(2k)}(1) = \delta_{nk} \text{ for } n \geq 0 \quad \text{and} \quad k \geq 0.$$

For instance

$$M_1(z) = \frac{1}{2}(z^2 - 1), \quad M_2(z) = \frac{1}{24}(z^2 - 1)(z^2 - 5),$$

$$M_3(z) = \frac{1}{720}(z^2 - 1)(z^4 - 14z^2 + 61).$$

Induction formula for Whittaker polynomials

The polynomial $f(z) = z^{2n}$ satisfies

$$f^{(2k+1)}(0) = 0 \text{ for } k \geq 0, \quad f^{(2k)}(1) = \begin{cases} \frac{(2n)!}{(2n-2k)!} & \text{for } 0 \leq k \leq n, \\ 0 & \text{for } k \geq n+1. \end{cases}$$

One deduces

$$z^{2n} = \sum_{k=0}^{n-1} \frac{(2n)!}{(2n-2k)!} M_k(z) + (2n)! M_n(z),$$

which yields the following induction formula

$$M_n(z) = \frac{1}{(2n)!} z^{2n} - \sum_{k=0}^{n-1} \frac{1}{(2n-2k)!} M_k(z).$$

Exponential type $< \pi/2$

Theorem (J.M. Whittaker, 1935).

Any entire function f of exponential type $< \pi/2$ has a unique convergent expansion

$$f(z) = \sum_{n=0}^{\infty} \left(f^{(2n)}(1) M_n(z) - f^{(2n+1)}(0) M'_{n+1}(1-z) \right).$$

Hence, if such a function satisfies $f^{(2n+1)}(0) = f^{(2n)}(1) = 0$ for all sufficiently large n, then it is a polynomial.

This is best possible: the entire function $\cos(\frac{\pi}{2} z)$ has exponential type $\pi/2$ and satisfies $f^{(2n+1)}(0) = f^{(2n)}(1) = 0$ for all $n \geq 0$.

Generating series

For $t \in \mathbb{C}$, $t \notin i\pi + 2i\pi\mathbb{Z}$, the entire function

$$f(z) = \frac{\cosh(tz)}{\cosh(t)} = \frac{e^{tz} + e^{-tz}}{e^{t} + e^{-t}}$$

satisfies

$$f'' = t^2 f, \quad f(1) = 1, \quad f'(0) = 0,$$

hence $f^{(2n)}(1) = t^{2n}$ and $f^{(2n+1)}(0) = 0$ for all $n \geq 0$.

The sequence $(M_n)_{n \geq 0}$ is also defined by the expansion

$$\boxed{\frac{\cosh(tz)}{\cosh(t)} = \sum_{n=0}^{\infty} t^{2n} M_n(z)}$$

for $|t| < \pi/2$ and $z \in \mathbb{C}$.

Integral formula for Whittaker polynomials

Using Cauchy's residue Theorem, we deduce the integral formula

$$M_n(z) = (-1)^n \frac{2^{2n+2}}{\pi^{2n+1}} \sum_{s=0}^{S-1} \frac{(-1)^s}{(2s+1)^{2n+1}} \cos\left(\frac{(2s+1)\pi}{2} z\right) + \frac{1}{2\pi i} \int_{|t|=S\pi} t^{-2n-1} \frac{\cosh(tz)}{\cosh(t)} \mathrm{d}t$$

for $S = 1, 2, \ldots$ and $z \in \mathbb{C}$.

In particular, with $S = 1$ we obtain

$$M_n(z) = (-1)^n \frac{2^{2n+2}}{\pi^{2n+1}} \cos(\pi z/2) + \frac{1}{2\pi i} \int_{|t|=\pi} t^{-2n-1} \frac{\cosh(tz)}{\cosh(t)} \mathrm{d}t.$$

Further estimates on Whittaker polynomials

There exist positive contants γ_1', γ_2', γ_3' and γ_4' such that the following holds.

(i) For $r \geq 0$ and $n \geq 0$, we have

$$|M_n|_r \leq \frac{\gamma_1'}{(2n)!} \max\{r, 2n\}^{2n}.$$

(ii) For sufficiently large r and for all $n \geq 0$,

$$|M_n|_r \leq \gamma_2' \frac{\mathrm{e}^{r+1/(4r)}}{\sqrt{2\pi r}}.$$

(iii) For $r \geq 0$ and $n \geq 0$,

$$|M_n|_r \leq \gamma_3' 2^{2n} \pi^{-2n} \mathrm{e}^{\pi r}.$$

(iv) For r sufficiently large,

$$\sum_{n \geq \gamma_4' r} |M_n|_r < 1.$$

Solution of the Whittaker interpolation problem

Consequence of Whittaker's expansion formula:
Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of complex numbers satisfying

$$\limsup_{n\to\infty} |a_n|^{1/n} < \frac{\pi^2}{4} \quad \text{and} \quad \limsup_{n\to\infty} |b_n|^{1/n} < \frac{\pi^2}{4}.$$

Then the function

$$f(z) = \sum_{n=0}^{\infty} a_n M_n(z) - \sum_{n=0}^{\infty} b_n M'_{n+1}(1-z)$$

is the unique entire function of exponential type $< \frac{\pi}{2}$ satisfying

$$f^{(2n)}(1) = a_n \quad \text{and} \quad f^{(2n+1)}(0) = b_n \text{ for all } n \geq 0.$$

Finite exponential type

Theorem (I.J. Schoenberg, 1936).
Let f be an entire function of finite exponential type $\tau(f)$ satisfying $f^{(2n+1)}(0) = f^{(2n)}(1) = 0$ for all $n \geq 0$. Then there exist complex numbers $c_1, \ldots, c_L$ with $L \leq 2\tau(f)/\pi$ such that

$$f(z) = \sum_{\ell=0}^{L} c_\ell \cos\left(\frac{(2\ell+1)\pi}{2} z\right).$$

Whittaker classification

Given two sequences $\underline{p} = (p_n)_{n\geq 0}$ and $\underline{q} = (q_n)_{n\geq 0}$ of nonnegative integers, does there exist two sequences $\underline{\pi} = (\pi_n)_{n\geq 0}$ and $\underline{\zeta} = (\zeta_n)_{n\geq 0}$ of polynomials such that, for $n, k \geq 0$,

$$\pi_n^{(p_k)}(1) = \delta_{nk}, \quad \pi_n^{(q_k)}(0) = 0, \quad \text{and} \quad \zeta_n^{(p_k)}(1) = 0, \quad \zeta_n^{(q_k)}(0) = \delta_{nk}?$$

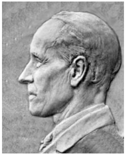
Such a pair $(\underline{\pi}, \underline{\zeta})$ is called a *standard set of polynomials* for $(\underline{p}, \underline{q})$.

If the answer is yes and if the solution $(\underline{\pi}, \underline{\zeta})$ is unique, then $(\underline{p}, \underline{q})$ is called *complete*, and any polynomial f can be written in a unique way as a finite sum

$$f(z) = \sum_{n\geq 0} f^{(p_n)}(1)\pi_n(z) + \sum_{n\geq 0} f^{(q_n)}(0)\zeta_n(z).$$

If there are several solutions $(\underline{\pi}, \underline{\zeta})$, then $(\underline{p}, \underline{q})$ is called *indeterminate*.

If there is no solution $(\underline{\pi}, \underline{\zeta})$, then $(\underline{p}, \underline{q})$ is called *redundant*.



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A course on interpolation

Second Course : Two Points. Lidstone, Whittaker

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