The role of complex conjugation in transcendental number theory

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Hermite–Lindemann's Theorem

- *Let α be a nonzero algebraic number and let $\log\alpha$ be any nonzero logarithm of α. Then $\log\alpha$ is transcendental.*
- Notations. Denote by $\overline{\mathbb{Q}}$ the field of algebraic numbers and by $\mathcal{L}$ the $\mathbb{Q}$-vector space of logarithms of algebraic numbers :

$$\mathcal{L} = \{\lambda \in \mathbb{C}\ ;\ e^\lambda \in \overline{\mathbb{Q}}^\times\} = \exp^{-1}(\overline{\mathbb{Q}}^\times) = \{\log\alpha\ ;\ \alpha \in \overline{\mathbb{Q}}^\times\}.$$

- Alternative statement of Hermite–Lindemann's Theorem :

$$\mathcal{L} \cap \overline{\mathbb{Q}} = \{0\}.$$

Hermite–Lindemann's Theorem (continued)

- Another alternative statement of Hermite–Lindemann's Theorem : *Let β be a nonzero algebraic number. Then e^β is transcendental.*
- Question (G. Diaz) : *Let t be a non-zero real number and β a non-zero algebraic number. Is it true that $e^{t\beta}$ is transcendental ?*
- Answer (G. Diaz) : **No !**
- First example : assume $\beta \in \mathbb{R}$. Take $t = (\log 2)/\beta$.
- Second example : assume $\beta \in i\mathbb{R}$. Take $t = i\pi/\beta$.

Diaz' Theorem

- *Let $\beta \in \overline{\mathbb{Q}}$ and $t \in \mathbb{R}^\times$. Assume $\beta \not\in \mathbb{R} \cup i\mathbb{R}$. Then $e^{t\beta}$ is transcendental.*
- Equivalently : for $\lambda \in \mathcal{L}$ with $\lambda \not\in \mathbb{R} \cup i\mathbb{R}$,
$$\mathbb{R}\lambda \cap \overline{\mathbb{Q}} = \{0\}.$$
- Proof. Set $\alpha = e^{t\beta}$. The complex conjugate $\overline{\alpha}$ of α is $e^{t\overline{\beta}} = \alpha^{\overline{\beta}/\beta}$. Since $\beta \not\in \mathbb{R} \cup i\mathbb{R}$, the algebraic number $\overline{\beta}/\beta$ is not real (its modulus is 1 and it is not ± 1), hence not rational. Gel'fond-Schneider's Theorem implies that α and $\overline{\alpha}$ cannot be both algebraic. Hence they are both transcendental. □

Gel'fond-Schneider implies Hermite-Lindemann (almost)

- Gel'fond-Schneider's Theorem implies : *there exists* $\beta_0 \in \mathbb{R} \cup i\mathbb{R}$ *such that*
$$\{\beta \in \overline{\mathbb{Q}}\; ;\; e^\beta \in \overline{\mathbb{Q}}\} = \mathbb{Q}\beta_0.$$

- Remark. Hermite–Lindemann's Theorem tells us that in fact $\beta_0 = 0$.
- Proof. From Gel'fond-Schneider's Theorem one deduces that the $\mathbb{Q}$-vector–space $\{\beta \in \overline{\mathbb{Q}}\; ;\; e^\beta \in \overline{\mathbb{Q}}\}$ has dimension ≤ 1 and is contained in $\mathbb{R} \cup i\mathbb{R}$. □
- Schneider's method : proof without derivatives.

Reference

G. Diaz – « Utilisation de la conjugaison complexe dans l'étude de la transcendance de valeurs de la fonction exponentielle », *J. Théor. Nombres Bordeaux* **16** (2004), p. 535–553.

The Six Exponentials Theorem

- Selberg, Siegel, Lang, Ramachandra.
- Theorem : *If x_1, x_2 are $\mathbb{Q}$–linearly independent complex numbers and y_1, y_2, y_3 are $\mathbb{Q}$–linearly independent complex numbers, then one at least of the six numbers*

$$e^{x_1y_1},\ e^{x_1y_2},\ e^{x_1y_3},\ \ e^{x_2y_1},\ e^{x_2y_2},\ e^{x_2y_3}$$

is transcendental.

The Six Exponentials Theorem

References :

- S. Lang – *Introduction to transcendental numbers*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966.
- K. Ramachandra – « Contributions to the theory of transcendental numbers. I, II », *Acta Arith. 14 (1967/68), 65-72 ; ibid.* **14** (1967/1968), p. 73–88.

Corollary

- Example : Take $x_1 = 1$, $x_2 = \pi$, $y_1 = \log 2$, $y_2 = \pi \log 2$, $y_3 = \pi^2 \log 2$, the six exponentials are respectively
$$2,\; 2^\pi,\; 2^{\pi^2},\; 2^\pi,\; 2^{\pi^2},\; 2^{\pi^3},$$
hence one at least of the three numbers
$$2^\pi,\; 2^{\pi^2},\; 2^{\pi^3}$$
is transcendental
- Shorey : lower bound for
$$|2^\pi - \alpha_1| + |2^{\pi^2} - \alpha_2| + |2^{\pi^3} - \alpha_3|$$
for algebraic α_1, α_2, α_3. The estimate depends on the heights and degrees of these algebraic numbers.

Relevant references

References :

- T. N. Shorey – « On a theorem of Ramachandra », *Acta Arith.* **20** (1972), p. 215–221.
- T. N. Shorey – « On the sum $\sum_{k=1}^{3} 2^{\pi^k} - \alpha_k$, α_k algebraic numbers », *J. Number Theory* **6** (1974), p. 248–260.

S. Srinivasan contributions

Further references :

- S. SRINIVASAN – « On algebraic approximations to $2^{\pi^k}(k = 1, 2, 3, \ldots)$ », *Indian J. Pure Appl. Math.* **5** (1974), no. 6, p. 513–523.
- S. SRINIVASAN – « On algebraic approximations to 2^{π^k} $(k = 1,\ 2,\ 3, \cdots)$. II », *J. Indian Math. Soc. (N.S.)* **43** (1979), no. 1-4, p. 53–60 (1980).
- K. RAMACHANDRA & S. SRINIVASAN – « A note to a paper : "Contributions to the theory of transcendental numbers. I, II" by Ramachandra on transcendental numbers », *Hardy-Ramanujan J.* **6** (1983), p. 37–44.

Conjectures

- Remark : It is unknown whether one of the two numbers
$$2^{\pi},\ 2^{\pi^2}$$
is transcendental. One conjectures (Schanuel) that each of the three numbers $2^{\pi},\ 2^{\pi^2},\ 2^{\pi^3}$ is transcendental
- and that the numbers
$$\pi,\ \log 2,\ 2^{\pi},\ 2^{\pi^2},\ 2^{\pi^3}$$
are algebraically independent.

The Four Exponentials Conjecture

- Selberg, Siegel, Schneider, Lang, Ramachandra.
- Conjecture. *If* x_1, x_2 *are* $\mathbb{Q}$*–linearly independent complex numbers and* y_1, y_2 *are* $\mathbb{Q}$*–linearly independent complex numbers, then one at least of the four numbers*

$$e^{x_1 y_1},\ e^{x_1 y_2},\ \ e^{x_2 y_1},\ e^{x_2 y_2}$$

is transcendental.

Ramachandra's trick

- Remark : Let x and y be two real numbers.
  The following properties are equivalent :
  (i) one at least of the two numbers x, y is transcendental.
  (ii) the complex number $x+iy$ is transcendental.
- Example : (H.W. Lenstra) if γ is Euler's constant, then the number $\gamma + ie^\gamma$ is transcendental.
- Proof : check $\gamma \neq 0$ and use Hermite–Lindemann's Theorem. □

Ramachandra's trick

Other example.

- *Let x_1, x_2 be two elements in $\mathbb{R} \cup i\mathbb{R}$ which are $\mathbb{Q}$–linearly independent. Let y_1, y_2 be two complex numbers. Assume that the three numbers $y_1, y_2, \overline{y_2}$ are $\mathbb{Q}$–linearly independent. Then one at least of the four numbers*

$$e^{x_1y_1},\ e^{x_1y_2},\ \ e^{x_2y_1},\ e^{x_2y_2}$$

is transcendental.

- Proof : Set $y_3 = \overline{y_2}$. Then $e^{x_jy_3} = \overline{e^{\pm x_jy_2}}$ for $j = 1, 2$ and $\overline{\mathbb{Q}}$ is stable under complex conjugation. □

Logarithms of algebraic numbers

Rank of matrices. An alternate form of the Six Exponentials Theorem (resp. the Four Exponentials Conjecture) is the fact that *a* 2×3 *(resp.* 2×2*) matrix with entries in* $\mathcal{L}$

$$\begin{pmatrix}\lambda_{11} & \lambda_{12} & \lambda_{13}\\ \lambda_{21} & \lambda_{22} & \lambda_{23}\end{pmatrix} \qquad \textit{(resp. } \begin{pmatrix}\lambda_{11} & \lambda_{12}\\ \lambda_{21} & \lambda_{22}\end{pmatrix} \quad \textit{)},$$

the rows of which are linearly independent over $\mathbb{Q}$ *and the columns of which are also linearly independent over* $\mathbb{Q}$*, has maximal rank* 2*.*

A lemma on the rank of matrices

Remark. A $d \times \ell$ matrix M has rank ≤ 1 if and only if there exist $x_1, \ldots, x_d$ and $y_1, \ldots, y_\ell$ such that

$$M = \begin{pmatrix} x_1y_1 & x_1y_2 & \dots & x_1y_\ell \\ x_2y_1 & x_2y_2 & \dots & x_2y_\ell \\ \vdots & \vdots & \ddots & \vdots \\ x_dy_1 & x_dy_2 & \dots & x_dy_\ell \end{pmatrix}.$$

Linear combinations of logarithms of algebraic numbers

Denote by $\widetilde{\mathcal{L}}$ the $\overline{\mathbb{Q}}$-vector space spanned by 1 and $\mathcal{L}$: hence $\widetilde{\mathcal{L}}$ is the set of linear combinations with algebraic coefficients of logarithms of algebraic numbers :

$$\widetilde{\mathcal{L}} = \{\beta_0 + \beta_1\lambda_1 + \cdots + \beta_n\lambda_n \; ; \; n \geq 0, \beta_i \in \overline{\mathbb{Q}}, \; \lambda_i \in \mathcal{L}\}.$$

The strong Six Exponentials Theorem

Theorem (D.Roy). *If x_1, x_2 are $\overline{\mathbb{Q}}$–linearly independent complex numbers and y_1, y_2, y_3 are $\overline{\mathbb{Q}}$–linearly independent complex numbers, then one at least of the six numbers*

$$x_1y_1,\ x_1y_2,\ x_1y_3,\ \ x_2y_1,\ x_2y_2,\ x_2y_3$$

is not in $\widetilde{\mathcal{L}}$.

The strong Four Exponentials Conjecture

Conjecture. *If x_1, x_2 are $\overline{\mathbb{Q}}$–linearly independent complex numbers and y_1, y_2 are $\overline{\mathbb{Q}}$–linearly independent complex numbers, then one at least of the four numbers*

$$x_1y_1,\ x_1y_2,\ \ x_2y_1,\ x_2y_2$$

is not in $\widetilde{\mathcal{L}}$.

Lower bound for the rank of matrices

- Rank of matrices. An alternate form of the strong Six Exponentials Theorem (resp. the strong Four Exponentials Conjecture) is the fact that *a* 2×3 *(resp.* 2×2*) matrix with entries in* $\widetilde{\mathcal{L}}$

$$\begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix} \qquad \textit{(resp.} \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \quad \textit{)},$$

  the rows of which are linearly independent over $\overline{\mathbb{Q}}$ *and the columns of which are also linearly independent over* $\overline{\mathbb{Q}}$*, has maximal rank* 2*.*
- Remark : Under suitable conditions one can show that a $d \times \ell$ matrix with entries in $\widetilde{\mathcal{L}}$ has rank $\geq d\ell/(d+\ell)$. This is a consequence of the *Linear Subgroup Theorem.*

The strong Six Exponentials Theorem

References :

- D. Roy – « Matrices whose coefficients are linear forms in logarithms », *J. Number Theory* **41** (1992), no. 1, p. 22–47.
- M. Waldschmidt – *Diophantine approximation on linear algebraic groups*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. **326**, Springer-Verlag, Berlin, 2000.

Alternate form of the strong Four Exponentials Conjecture

- Conjecture. *Let Λ_1, Λ_2, Λ_3 be nonzero elements in $\widetilde{\mathcal{L}}$. Assume the numbers Λ_2/Λ_1 and Λ_3/Λ_1 are both transcendental. Then the number $\Lambda_2\Lambda_3/\Lambda_1$ is not in $\widetilde{\mathcal{L}}$.*
- Equivalence between both statements : the matrix

$$\begin{pmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_3 & \Lambda_2\Lambda_3/\Lambda_1 \end{pmatrix}$$

has rank 1. □

Consequences of the strong Four Exponentials Conjecture

Assume the strong Four Exponentials Conjecture.

- If Λ is in $\widetilde{\mathcal{L}} \setminus \overline{\mathbb{Q}}$ then the quotient $1/\Lambda$ is not in $\widetilde{\mathcal{L}}$.
- If Λ_1 and Λ_2 are in $\widetilde{\mathcal{L}} \setminus \overline{\mathbb{Q}}$, then the product $\Lambda_1\Lambda_2$ is not in $\widetilde{\mathcal{L}}$.
- If Λ_1 and Λ_2 are in $\widetilde{\mathcal{L}}$ with Λ_1 and Λ_2/Λ_1 transcendental, then this quotient Λ_2/Λ_1 is not in $\widetilde{\mathcal{L}}$.

Example where the strong Four Exponentials Conjecture is true

- Theorem (G. Diaz). *Let* x_1 *and* x_2 *be two elements of* $\mathbb{R} \cup i\mathbb{R}$ *which are* $\overline{\mathbb{Q}}$*–linearly independent. Let* y_1, y_2 *be two complex numbers such that the three numbers* y_1, y_2, $\overline{y_2}$ *are* $\overline{\mathbb{Q}}$*-linearly independent. Then one at least of the four numbers*

$$x_1y_1,\ x_1y_2,\ \ x_2y_1,\ x_2y_2$$

is not in $\widetilde{\mathcal{L}}$.

- Proof : Set $y_3 = \overline{y_2}$. Then $e^{x_jy_3} = \overline{e^{\pm x_jy_2}}$ for $j = 1, 2$ and $\widetilde{\mathcal{L}}$ is stable under complex conjugation. □

Example where the strong Four Exponentials Conjecture is true

- Corollary of Diaz' Theorem. *Let* Λ_1, Λ_2, Λ_3 *be three elements in* $\widetilde{\mathcal{L}}$. *Assume that the three numbers* Λ_1, Λ_2, $\overline{\Lambda_2}$ *are linearly independent over* $\overline{\mathbb{Q}}$. *Further assume* $\Lambda_3/\Lambda_1 \in (\mathbb{R} \cup i\mathbb{R}) \setminus \overline{\mathbb{Q}}$. *Then*

$$\Lambda_2\Lambda_3/\Lambda_1 \notin \overline{\mathbb{Q}}.$$

- Proof : set $x_1 = 1$, $x_2 = \Lambda_3/\Lambda_1$, $y_1 = \Lambda_1$, $y_2 = \Lambda_2$. $\square$

Example where the strong Four Exponentials Conjecture is true

Consequence : one deduces examples where one can actually prove that numbers like

$$1/\Lambda, \quad \Lambda_1\Lambda_2, \quad \Lambda_2/\Lambda_1$$

(with Λ, Λ_1, Λ_2 in $\widetilde{\mathcal{L}}$) are not in $\widetilde{\mathcal{L}}$.

Transcendence of e^{π^2}

- Open problem : *is the number* e^{π^2} *transcendental ?*
- More generally : *for* $\lambda \in \mathcal{L} \setminus \{0\}$, *is it true that* $\lambda\overline{\lambda} \not\in \mathcal{L}$ *?*
- More generally : *for* λ_1 *and* λ_2 *in* $\mathcal{L} \setminus \{0\}$, *is it true that* $\lambda_1\lambda_2 \not\in \mathcal{L}$ *?*
- *For* λ_1 *and* λ_2 *in* $\mathcal{L} \setminus \{0\}$, *is it true that* $\lambda_1\lambda_2 \not\in \widetilde{\mathcal{L}}$ *?*

Product of logarithms of algebraic numbers

- Theorem (Diaz). *Let* λ_1 *and* λ_2 *be in* $\mathcal{L}\setminus\{0\}$. *Assume* $\lambda_1\in\mathbb{R}\cup i\mathbb{R}$ *and* $\lambda_2\notin\mathbb{R}\cup i\mathbb{R}$. *Then* $\lambda_1\lambda_2\notin\widetilde{\mathcal{L}}$.
- Proof. Apply the strong Six Exponentials Theorem to

$$\begin{pmatrix} 1 & \lambda_2 & \overline{\lambda_2}\\ \lambda_1 & \Lambda & \overline{\Lambda}\end{pmatrix}$$

  with $\Lambda\in\widetilde{\mathcal{L}}$. □
- Diaz' Conjecture. *Let* $u\in\mathbb{C}^\times$. *Assume* $|u|$ *is algebraic. Then* e^u *is transcendental.*

Recent results

- G. Diaz – « Utilisation de la conjugaison complexe dans l'étude de la transcendance de valeurs de la fonction exponentielle », *J. Théor. Nombres Bordeaux* **16** (2004), p. 535–553.
- G. Diaz – « Produits et quotients de combinaisons linéaires de logarithmes de nombres algébriques : conjectures et résultats partiels », Submitted (2005), 19 p.

Recent results

- M. Waldschmidt – « Transcendence results related to the six exponentials theorem », *Hindustan Book Agency* (2005), p. 338–355.
  Appendix by H. Shiga : Periods of the Kummer surface, p. 356–358.

- M. Waldschmidt – « Further variations on the Six Exponentials Theorem », The Hardy-Ramanujan Journal, vol. **28**, to appear on december 22, 2005.

Further result

Let M be a 2×3 matrix with entries in $\widetilde{\mathcal{L}}$:

$$M = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix}.$$

Assume that the five rows of the matrix

$$\begin{pmatrix} M \\ I_3 \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are linearly independent over $\overline{\mathbb{Q}}$ and that the five columns of the matrix

$$(I_2, M) = \begin{pmatrix} 1 & 0 & \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ 0 & 1 & \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix}$$

are linearly independent over $\overline{\mathbb{Q}}$.

Further result

Then one at least of the three numbers

$$\Delta_1 = \begin{vmatrix} \Lambda_{12} & \Lambda_{13} \\ \Lambda_{22} & \Lambda_{23} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} \Lambda_{13} & \Lambda_{11} \\ \Lambda_{23} & \Lambda_{21} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{vmatrix}$$

is not in $\widetilde{\mathcal{L}}$.

Higher rank : an example

Let $M = (\Lambda_{ij})_{1\le i\le m; 1\le j\le \ell}$ be a $m\times\ell$ matrix with entries in $\widetilde{\mathcal{L}}$. Denote by I_m the identity $m\times m$ matrix and assume that the $m+\ell$ column vectors of the matrix (I_m, M) are linearly independent over $\overline{\mathbb{Q}}$. Let $\Lambda_1,\dots,\Lambda_m$ be elements of $\widetilde{\mathcal{L}}$. Assume that the numbers $1, \Lambda_1,\dots,\Lambda_m$ are $\overline{\mathbb{Q}}$-linearly independent. Assume further $\ell > m^2$. Then one at least of the ℓ numbers

$$\Lambda_1\Lambda_{1j}+\cdots+\Lambda_m\Lambda_{mj}\quad (j=1,\dots,\ell)$$

is not in $\widetilde{\mathcal{L}}$.

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Tarlok !

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