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Number Theory Days in KKU

http ://202.28.94.202/math/thai/

Discrete mathematics and Diophantine Problems

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Abstract

One of the first goals of Diophantine Analysis is to decide whether a given number is rational, algebraic or else transcendental. Such a number may be given by its binary or decimal expansion, by its continued fraction expansion, or by other limit process (sum of a series, infinite product, integrals...). Language theory provides sometimes convenient tools for the study of numbers given by expansions. We survey some of the main recent results on Diophantine problems related with the complexity of words.

Émile Borel (1871–1956)

- *Les probabilités dénombrables et leurs applications arithmétiques,*
  Palermo Rend. **27**, 247-271 (1909).
  Jahrbuch Database JFM 40.0283.01
  http ://www.emis.de/MATH/JFM/JFM.html
- *Sur les chiffres décimaux de $\sqrt{2}$ et divers problèmes de probabilités en chaînes,*
  C. R. Acad. Sci., Paris **230**, 591-593 (1950).
  Zbl 0035.08302

Émile Borel : 1950

Let $g \geq 2$ be an integer and x a real irrational algebraic number. *The expansion in base g of x should satisfy some of the laws which are valid for almost all real numbers (for Lebesgue's measure).*

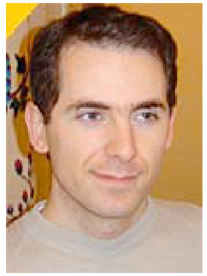
First decimals of $\sqrt{2}$ http://wims.unice.fr/wims/wims.cgi

1.4142135623730950488016887242096980785696718753769480731766797379907324784621070388503875343276415727350138462309122970249248360558507372126441214970999358314132226659275055927557999505011527820605714701095599716059702745345968620147285174186408891986095523292304843087143214508397626036279952514079896872533965463318088296406206152583523950547457502877599617298355752203375318570113543746034084988471603868999706990048150305440277903164542478230684929369186215805784631115966687130130156185689872372352885092648612494977154218334204285686060146824720771435854874155657069677653720226485447015858801620758474922657226002085584466521458398893944370926591800311388246468157082630100594858704003186480342194897278290641045072636881313739855256117322040245091227700226941127573627280495738108967504018369868368450725799364729060762996941380475654823728997180326802474420629269124859052181004459842150591120249441341728531478105803603371077309182869314710171111683916581726889419758716582152128229518488472 . . .

First binary digits of $\sqrt{2}$ http://wims.unice.fr/wims/wims.cgi

1.0110101000001001111001100110011111110011101111001100100100001000101100101111101100010011011001101110101010010101011111010011111000111010110111101100000101110101000100100111011101010000100110011101101000101111010110010000101100000110011001110011001000101010100101011111100100000110000010000111010101110001010001011000011101010001011000111111110011011111101110010000011110110110011100100001111011101001010100001011110010000111001110001111011010010100111100000000100100001110011011000111101111110100010011101101000110100100010000000101110100001110100001010101111000111110100111001010011000001011001110001100000000100011011110000011001101111011110010101011000110111100100100010001011010001000010001011000101001000110000010101011110001110010001011110111111000100111000110011110001101101010110101000101000111000101110110111111010011101110011001011001010100110001101000011001100011111100111100100001001101111110101001011110001001000000111110000011011011100101100000101110111010101010010010100000010001001100100000010000000110010100100101010000000100111001010010 . . .

The fabulous destiny of $\sqrt{2}$



- Benoît Rittaud, Éditions *Le Pommier* (2006).

http://www.math.univ-paris13.fr/∼rittaud/RacineDeDeux

Computation of decimals of $\sqrt{2}$

1 542 computed by hand by Horace Uhler in 1951

14 000 decimals computed in 1967

1 000 000 decimals in 1971

$137 \cdot 10^9$ decimals computed by Yasumasa Kanada and Daisuke Takahashi in 1997 with Hitachi SR2201 in 7 hours and 31 minutes.

- Motivation : computation of π.

Expansion in base g of a real number

Let g be an integer ≥ 2. Any real number x has an expansion *which is unic if x is irrational*

$$x = a_{-k}g^k + \cdots + a_{-1}g + a_0 + a_1 g^{-1} + a_2 g^{-2} + \cdots$$

where k is an integer ≥ 0 and where the a_i for $i \geq -k$ (digits of x in the base g expansion of x) belong to the set $\{0, 1, \ldots, g-1\}$.
We write

$$x = a_{-k} \cdots a_{-1} a_0, a_1 a_2 \cdots$$

Examples : in base 10 (*decimal expansion*) :

$$\sqrt{2} = 1,4142135623730950488016887242 0 \ldots$$

and in base 2 (*binary expansion*) :

$$\sqrt{2} = 1,011010100000100111100110011001111111110 \ldots$$

Complexity of the g–ary expansion of an irrational algebraic real number

Let $g \geq 2$ be an integer.

• É. Borel (1909 and 1950) : *the g–ary expansion of an algebraic irrational number should satisfy some of the laws shared by almost all numbers (with respect to Lebesgue's measure).*

• **Remark** : no number satisfies **all** the laws which are shared by all numbers outside a set of measure zero, because the intersection of all these sets of full measure is empty !

$$\bigcap_{x \in \mathbf{R}} \mathbf{R} \setminus \{x\} = \emptyset.$$

• More precise statements by B. Adamczewski and Y. Bugeaud.

First conjecture of Émile Borel

Conjecture 1 (É. Borel). *Let x be an irrational algebraic real number, $g \geq 3$ a positive integer and a an integer in the range $0 \leq a \leq g-1$. Then the digit a occurs at least once in the g–ary expansion of x.*
Corollary. • *Each given sequence of digits should occur infinitely often in the g–ary expansion of any real irrational algebraic number.*
(consider powers of g).
• For instance, Borel's Conjecture 1 with $g = 4$ implies that each of the four sequences $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ should occur infinitely often in the binary expansion of each irrational algebraic real number x.

The state of the art

There is no explicitly known example of a triple (g, a, x), where $g \geq 3$ is an integer, a a digit in $\{0, \ldots, g-1\}$ and x an algebraic irrational number, for which one can claim that the digit a occurs infinitely often in the g–ary expansion of x.

Kurt Mahler

Kurt Mahler (1903 - 1988)

For any $g \geq 2$ and any $n \geq 1$, there exist algebraic irrational numbers x such that any block of n digits occurs infinitely often in the g–ary expansion of x.

Simply normal numbers in base g

• A real number x is called *simply normal in base g* if each digit occurs with frequency $1/g$ in its g–ary expansion.
• For instance the decimal number

$$0,123456789012345678901234567890\ldots$$

is simply normal in base 10. This number is rational :

$$= \frac{1\,234\,567\,890}{9\,999\,999\,999} = \frac{137\,174\,210}{1\,111\,111\,111}.$$

Normal numbers in base g

• A real number x is called *normal in base g* or *g–normal* if it is simply normal in base g^m for all $m \geq 1$.

Hence a real number x is normal in base g if and only if, for any $m \geq 1$, each sequence of m digits occurs with frequency $1/g^m$ in its g–ary expansion.

Normal numbers

• A real number is called *normal* if it is normal in any base $g \geq 2$. Hence a real number is normal if and only if it is simply normal in any base $g \geq 2$.

Conjecture 2 (É. Borel). *Any irrational algebraic real number is normal.*

• Almost all real numbers (for Lebesgue's measure) are normal.

• Examples of computable normal numbers have been constructed (W. Sierpinski 1917, H. Lebesgue 1917, V. Becher and S. Figueira 2002), but the known algorithms to compute such examples are fairly complicated (*"ridiculously exponential"*, according to S. Figueira).

Example of normal numbers

An example of a 2–normal number (Champernowne 1933, Bailey and Crandall 2001) is the *binary Champernowne number*, obtained by the concatenation of the sequence of integers

$$0.\,1\,10\,11\,100\,101\,110\,111\,1000\,1001\,1010\,1011\,1100\,\ldots$$

$$=\sum_{k\ge 1} k2^{-c_k} \quad \text{with} \quad c_k = k+\sum_{j=1}^{k}[\log_2 j].$$

• S. S. Pillai (1940), Collected papers edited by R. Balasubramanian and R. Thangadurai, (2009 or 2010).

Further examples of normal numbers

• (Stoneham Numbers . . .) : *if a and g are coprime integers >1, then*

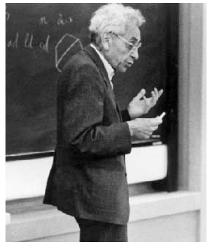
$$\sum_{n\ge 0} a^{-n} g^{-a^n}$$

is normal in base g.

Reference : R. Stoneham (1973), D.H. Bailey, J.M. Borwein, R.E. Crandall and C. Pomerance (2004).

Copeland – Erdős

$$0.2\,3\,5\,7\,11\,13\,17\,19\,23\,29\,31\,37\,41\,43\,47\,53\,59\,61\,67\ \ldots$$



Paul Erdös
(1913 - 1996)

A.H. Copeland and P. Erdős (1946) : a normal number in base 10 is obtained by concatenation of the sequence of prime numbers

Infinite words

Let $\mathcal{A}$ be a finite alphabet with g elements.

• We shall consider *infinite words* $w = a_1 \ldots a_n \ldots$
A *factor of length* m of w is a word of the form $a_k a_{k+1} \ldots a_{k+m-1}$ for some $k \ge 1$.

• The *complexity* $p = p_w$ of w is the function which counts, for each $m \ge 1$, the number $p(m)$ of distinct factors of w of length m.

• Hence $1 \le p(m) \le g^m$ and the function $m \mapsto p(m)$ is non–decreasing.

• According to Borel's Conjecture 1, the complexity of the sequence of digits in base g of an irrational algebraic number should be $p(m) = g^m$.

Sturmian words

Assume $g = 2$, say $\mathcal{A} = \{0, 1\}$.

• A word is periodic if and only if its complexity is bounded.

• If the complexity $p(m)$ a word w satisfies $p(m) = p(m+1)$ for one value of m, then $p(m+k) = p(m)$ for all $k \geq 0$, hence the word is periodic. It follows that *a non–periodic* w *has a complexity* $p(m) \geq m+1$.

• An infinite word of minimal complexity $p(m) = m+1$ is called *Sturmian* (Morse and Hedlund, 1938).

• Examples of Sturmian words are given by 2–dimensional billiards.

Sturm and Morse

Jacques Charles François Sturm (1803 - 1855)

Harold Calvin Marston Morse (1892 - 1977)

The Fibonacci word

• Define $f_1 = 1$, $f_2 = 0$
and, for $n \geq 3$ (concatenation) : $f_n = f_{n-1}f_{n-2}$.

Leonardo Pisano Fibonacci (1170 - 1250)

$f_3 = 01$, $f_4 = 010$, $f_5 = 01001$, $f_6 = 01001010, \ldots$

The *Fibonacci word*

$$w = 0100101001001010010100100101001001\ldots$$

is Sturmian.

• *For each* $m \geq 1$, *there is exactly one factor* v *of* w *of length* m *such that both* $v0$ *and* $v1$ *are factors of* w *of length* $m+1$.

The Fibonacci word

0100101001001010010100100101001001 ... is Sturmian

			↗	00100
0 →	00 →	001 →	0010 →	00101
↘	01 →	010 →	0100 →	01001
		↘	0101 →	01010
1 →	10 →	100 →	1001 →	10010
	↘	101 →	1010 →	10100

Transcendence and Sturmian words

• S. Ferenczi, C. Mauduit, 1997 : *A number whose sequence of digits is Sturmian is transcendental.*
Combinatorial criterion : *the complexity of the g–ary expansion of every irrational algebraic number satisfies*

$$\liminf_{m\to\infty}(p(m)-m)=+\infty.$$

• *Tool :* a p–adic version of the Thue–Siegel–Roth-Schmidt Theorem due to Ridout (1957).
• **Reference :** Yuri Bilu's Lecture in the Bourbaki Seminar, November 2006 :
The many faces of the Subspace Theorem [after Adamczewski, Bugeaud, Corvaja, Zannier. . .]
http://www.math.u-bordeaux.fr/~yuri/publ/subspace.pdf

Complexity of the g-ary expansion of an algebraic number

• **Theorem** (B. Adamczewski, Y. Bugeaud, F. Luca 2004).
The binary complexity p of a real irrational algebraic number x satisfies

$$\liminf_{m\to\infty}\frac{p(m)}{m}=+\infty.$$

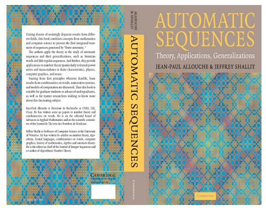
• **Corollary** (Conjecture of A. Cobham, 1968). *If the sequence of digits of an irrational real number x is* **automatic**, *then x is transcendental.*

Automata

A *finite automaton* consists of
• the *input alphabet* $\mathcal{A}$, usually the set of digits $\{0, 1, 2, \ldots, g-1\}$;
• the set $\mathcal{Q}$ of states, a finite set of 2 or more elements, with one element called the *initial state* i singled out ;
• the *transition map* $\mathcal{Q}\times\mathcal{A}\to\mathcal{Q}$, which associates to every state a new state depending on the current input ;
• the *output alphabet* $\mathcal{B}$, together with the *output map* $f:\mathcal{Q}\to\mathcal{B}$.

Automata : reference

Jean-Paul Allouche and Jeffrey Shallit
Automatic Sequences : Theory, Applications, Generalizations,
Cambridge University Press (2003).



http ://www.cs.uwaterloo.ca/~shallit/asas.html

Example : powers of 2

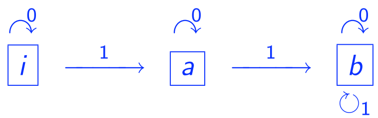
The sequence of binary digits of the number

$$\sum_{n \geq 0} 2^{-2^n} = 0.1101000100000001000 \cdots = 0.a_1 a_2 a_3 \ \ldots$$

with

$$a_n = \begin{cases} 1 & \text{if } n \text{ is a power of 2,} \\ 0 & \text{otherwise} \end{cases}$$

is automatic : $\mathcal{A} = \mathcal{B} = \{0, 1\}$, $\mathcal{Q} = \{i, a, b\}$,
$f(i) = 0$, $f(a) = 1$, $f(b) = 0$,



Automatic sequences

• Let $g \geq 2$ be an integer. An infinite sequence $(a_n)_{n \geq 0}$ is said to be *g–automatic* if a_n is a finite-state function of the base g representation of n : this means that there exists a finite automaton starting with the g–ary expansion of n as input and producing the term a_n as output.

• A. Cobham, 1972 : *Automatic sequences have a complexity* $p(m) = O(m)$.

Automatic sequences are between periodicity and chaos. They occur in connection with harmonic analysis, ergodic theory, fractals, Feigenbaum cascades, quasi–crystals.

Automatic sequences and theoretical physics

J.P. Allouche and M. Mendes-France : computation of physical constants of an Ising model in one dimension involving an automatic distribution.
Reference : J-P. Allouche and M. Mignotte, *Arithmétique et Automates*, Images des Mathématiques 1988, Courrier du CNRS Supplément au N° 69, 5–9.

Ising model : to study phase transition in statistical mechanics :
Reference : Raphaël Cerf, *Le modèle d'Ising et la coexistence des phases*, Images des Mathématiques (2004), 47–51.
`http ://www.spm.cnrs-dir.fr/actions/publications/IdM.htm`

Powers of 2 *(continued)*

The complexity $p(m)$ of the automatic sequence of binary digits of the number

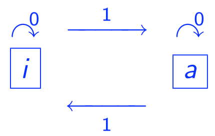
$$\sum_{n \geq 0} 2^{-2^n} = 0.1101000100000001000 \cdots$$

is at most $2m$:

$$\begin{array}{rcccccccc} m = & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ p(m) = & 2 & 4 & 6 & 7 & 9 & 11 & \cdots \end{array}$$

Prouhet–Thue–Morse sequence

- The automaton



avec $f(i) = 0$, $f(a) = 1$

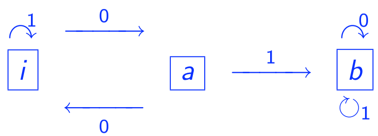
produces the sequence $a_0a_1a_2\dots$ where, for instance, a_9 is $f(i) = 0$, since $1001[i] = 100[a] = 10[a] = 1[a] = i$. This is the Prouhet–Thue–Morse sequence, where the $n+1$-ème term a_n is 1 if the number of 1 in the binary expansion of n is odd, 0 if it is even.
The Prouhet–Thue–Morse number is $\sum_{n\geq 0} a_n 2^{-n}$.

The Baum–Sweet sequence

- The Baum–Sweet sequence. For $n \geq 0$ define $a_n = 1$ if the binary expansion of n contains no block of consecutive 0's of odd length, $a_n = 0$ otherwise : the sequence $(a_n)_{n\geq 0}$ starts with

$$1\,1\,0\,1\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,0\,1\,1\,0\,0\,1\,0\dots$$

- This sequence is automatic, associated with the automaton



with $f(i) = 1$, $f(a) = 0$, $f(b) = 0$.

The Rudin–Shapiro sequence

- The Rudin–Shapiro word $aaabaabaaaabbbab\dots$. For $n \geq 0$ define $r_n \in \{a, b\}$ as being equal to a (respectively b) if the number of occurrences of the pattern 11 in the binary representation of n is even (respectively odd).
- Let σ be the morphism defined from the monoid B^* on the alphabet $B = \{1, 2, 3, 4\}$ into B^* by : $\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 42$ and $\sigma(4) = 43$. Let

$$\mathbf{u} = 121312421213\dots$$

be the fixed point of σ begining with 1 and let φ be the morphism defined from B^* to $\{a, b\}^*$ by : $\varphi(1) = aa$, $\varphi(2) = ab$ and $\varphi(3) = ba$, $\varphi(4) = bb$. Then the Rudin-Shapiro word is $\varphi(\mathbf{u})$.

Paper folding sequence

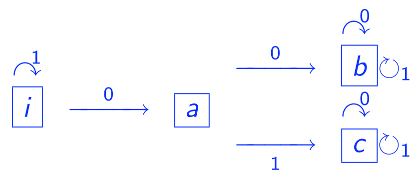
If you fold a long piece of paper, always in the same direction, and then you unfold it, you get two kind of edges, which you encode with 0 or 1. This gives rise to a sequence

$$1101100111001001\ \dots$$

which satisfies

$$u_{4n} = 1, \quad u_{4n+2} = 0, \quad u_{2n+1} = u_n$$

and which is produced by the automaton



with $f(i) = f(a) = f(b) = 1$, $f(c) = 0$.

The Fibonacci number is not automatic

- Cobham (1972) : the frequency of each letter in an automatic word is a rational number.

- Consequence : the Fibonacci word

$$0100101001001010010 10\dots$$

is not automatic.

The frequency of the letter 0 (resp. of the letter 1) is $1/\Phi$ (resp. $1/\Phi^2$), where $\Phi = (1+\sqrt{5})/2$ is the Golden Ratio an irrational number.

Complexity of the expansion in base g of a real irrational algebraic number

Theorem (B. Adamczewski, Y. Bugeaud, F. Luca 2004). *The binary complexity p of a real algebraic irrational number x satisfies*

$$\liminf_{m\to\infty} \frac{p(m)}{m} = +\infty.$$

Corollary (conjecture of A. Cobham (1968)) : *If the sequence of binary digits of a real irrational number x is automatic, then x is a transcendental number.*

Transcendence of automatic numbers

In other terms
Theorem (B. Adamczewski, Y. Bugeaud, F. Luca, 2004 – conjecture of A. Cobham, 1968) : *The sequence of digits of a real algebraic irrational number is not automatic.*
Tool : W.M. Schmidt Subspace Theorem.

Liouville numbers and exponent of irrationality

- An *exponent of irrationality* for $\xi \in \mathbf{R}$ is a number $\kappa \geq 2$ such that there exists $C > 0$ with

$$\left|\xi - \frac{p}{q}\right| \geq \frac{C}{q^\kappa} \quad \text{for all} \quad \frac{p}{q} \in \mathbf{Q}.$$

- A *Liouville number* is a real number with no finite exponent of irrationality.
- **Liouville's Theorem.** *Any Liouville number is transcendental.*
- In the theory of *dynamical systems*, a *Diophantine number* (or *a number satisfying a Diophantine condition*) is a real number which is not Liouville.

References : M. Herman, J.C. Yoccoz.

Irrationality measures for automatic numbers

- B. Adamczewski and J. Cassaigne (2006) – Solution to a Conjecture of J. Shallit (1999) : *A Liouville number cannot be generated by a finite automaton.*

- For instance for the Prouhet–Thue–Morse–Mahler numbers

$$\xi_g = \sum_{n\geq 0} \frac{a_n}{g^n}$$

(where $a_n = 0$ if the sum of the binary digits in the expansion of n is even, $a_n = 1$ if this sum is odd) the exponent of irrationality is ≤ 5.

Independence of expansions of algebraic numbers

Following Borel, the sequences of binary digits of two numbers like $\sqrt{2}$ and $\sqrt{3}$ should look like random sequences. One may ask whether these sequences of digits behave like independent random sequences.
B. Adamczewski and Y. Bugeaud remark that this is true for almost all pairs of real numbers (using the Borel-Cantelli Lemma), they suggest that this property should hold for any base g and pair of irrational numbers, unless they have ultimately the same sequences of digits.

Further transcendence results on g–ary expansions of real numbers

- J-P. Allouche and L.Q. Zamboni (1998).

- R.N. Risley and L.Q. Zamboni (2000).

- B. Adamczewski and J. Cassaigne (2003).

Christol, Kamae, Mendes-France, Rauzy

The result of B. Adamczewski, Y. Bugeaud and F. Luca implies the following statement related to the work of G. Christol, T. Kamae, M. Mendès-France and G. Rauzy (1980) :
Corollary. *Let $g \geq 2$ be an integer, p be a prime number and $(u_k)_{k\geq 1}$ a sequence of integers in the range $\{0, \dots, p-1\}$. The formal power series*

$$\sum_{k\geq 1} u_k X^k$$

and the real number

$$\sum_{k\geq 1} u_k g^{-k}$$

are both algebraic (over $\mathbf{F}_p(X)$ and over $\mathbf{Q}$, respectively) if and only if they are rational.

The Prouhet–Thue–Morse sequence

Let $(a_n)_{n\geq 0}$ be the Prouhet–Thue–Morse sequence. The series

$$F(X) = \sum_{n\geq 0} a_n X^n$$

is algebraic over the field $\mathbf{F}_2(X)$:

$$(1+X)^3F^2 + (1+X)^2F + X = 0.$$

This produces a new proof of Mahler's result on the transcendence of the number

$$\sum_{n\geq 0} a_n g^{-n}.$$

Further transcendence results

Consequences of Nesterenko 1996 result on the transcendence of values of theta series at rational points.

- The number $\sum_{n\geq 0} 2^{-n^2}$ is transcendental (D. Bertrand 1997 ; D. Duverney, K. Nishioka, K. Nishioka and I. Shiokawa 1998)

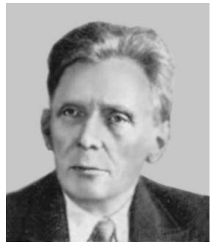
- For the word

$$\mathbf{u} = 0121221222122221222221222222122 2\ldots$$

generated by $0 \mapsto 012$, $1 \mapsto 12$, $2 \mapsto 2$, the number $\eta = \sum_{k\geq 1} u_k 3^{-k}$ is transcendental.

Complexity of the continued fraction expansion of an algebraic number

Similar questions arise by considering the continued fraction expansion of a real number instead of its g–ary expansion.



Aleksandr Yakovlevich Khinchin (1894 - 1959)

- Open question – A.Ya. Khintchin (1949) : *are the partial quotients of the continued fraction expansion of a non–quadratic irrational algebraic real number bounded ?*

Transcendence of continued fractions

- J. Liouville, 1844
- É. Maillet, 1906, O. Perron, 1929
- H. Davenport and K.F. Roth, 1955
- A. Baker, 1962
- J.L. Davison, 1989

Transcendence of continued fractions (continued)

- M. Queffélec, 1998 : *transcendence of the Prouhet–Thue–Morse continued fraction*.

- P. Liardet and P. Stambul, 2000.

- J-P. Allouche, J.L. Davison, M Queffélec and L.Q. Zamboni, 2001 : *transcendence of Sturmian or morphic continued fractions*.

- B. Adamczewski, Y. Bugeaud, J.L. Davison, 2005 : *transcendence of the Rudin–Shapiro and of the Baum–Sweet continued fractions*.

Open Problems

- Give an example of a real automatic number $x > 0$ such that $1/x$ is not automatic.

- Show that

$$\log 2 = \sum_{n\geq 1} \frac{1}{n} 2^{-n}$$

is not 2-automatic.

- Show that

$$\pi = \sum_{n\geq 0} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right) 2^{-4n}$$

is not 2-automatic.

Problems dealing with normal numbers (T. Rivoal)

- Give an explicit example of an irrational real number which is simply normal in base g and such that $1/x$ is not simply normal in base g .

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- Give an explicit example of an irrational real number which is normal and such that $1/x$ is not simply normal.

Other open problem

- Let $(e_n)_{n\geq 1}$ be an infinite sequence on $\{0, 1\}$ which is not ultimately periodic. Is–it true that *one at least of the two numbers*

$$\sum_{n\geq 1} e_n 2^{-n}, \qquad \sum_{n\geq 1} e_n 3^{-n}$$

is transcendental ?

According to Borel, the second number should be transcendental, since it is irrational and has no digit 2 in its base 3 expansion.

Liouville numbers

• **Liouville's Theorem.** *for any real algebraic number α there exists a constant $c > 0$ such that the set of $p/q \in \mathbf{Q}$ with $|\alpha - p/q| < q^{-c}$ is finite.*

• Liouville's Theorem yields the transcendence of the value of a series like $\sum_{n\geq 0} 2^{-u_n}$, provided that the sequence $(u_n)_{n\geq 0}$ is increasing and satisfies

$$\limsup_{n\to\infty} \frac{u_{n+1}}{u_n} = +\infty.$$

• For instance $u_n = n!$ satisfies this condition : hence the number $\sum_{n\geq 0} 2^{-n!}$ is transcendental.

Thue–Siegel–Roth Theorem

Axel Thue (1863 - 1922)

Carl Ludwig Siegel

Klaus Friedrich Roth (1925 –)

For any real algebraic number α, for any $\epsilon > 0$, the set of $p/q \in \mathbf{Q}$ with $|\alpha - p/q| < q^{-2-\epsilon}$ is finite.

Consequences of Roth's Theorem

• Roth's Theorem yields the transcendence of $\sum_{n\geq 0} 2^{-u_n}$ under the weaker hypothesis

$$\limsup_{n\to\infty} \frac{u_{n+1}}{u_n} > 2.$$

• The sequence $u_n = [2^{\theta n}]$ satisfies this condition as soon as $\theta > 2$. For example the number

$$\sum_{n\geq 0} 2^{-3^n}$$

is transcendental.

Transcendence of $\sum_{n\geq 0} 2^{-2^n}$

• A stronger result follows from Ridout's Theorem, using the fact that the denominators 2^{u_n} are powers of 2 : the condition

$$\limsup_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$$

suffices to imply the transcendence of the sum of the series $\sum_{n\geq 0} 2^{-u_n}$.

• Since $u_n = 2^n$ satisfies this condition, the transcendence of $\sum_{n\geq 0} 2^{-2^n}$ follows (Kempner 1916).

• **Ridout's Theorem.** *for any real algebraic number α, for any $\epsilon > 0$, the set of $p/q \in \mathbf{Q}$ with $q = 2^k$ and $|\alpha - p/q| < q^{-1-\epsilon}$ is finite.*

Consequence of Ridout's Theorem

• Let $x = 0.a_1a_2\dots$ be the binary expansion of a real algebraic irrational number $x \in (0,1)$. For $n \geq 0$ set

$$\ell(n) = \min\{\ell \geq 0 \,;\; a_{n+\ell} \neq 0\}.$$

Then $\ell(n) = o(n)$

• For the number $\sum_{n\geq 0} 2^{-2^n}$ the sequence of digits has $\ell(2^n) = 2^n$.

• Main tool of Adamczewski and Bugeaud : Schmidt's subspace Theorem.

Schmidt's subspace Theorem (simplest version)

For $\mathbf{x} = (x_0, \dots, x_{m-1}) \in \mathbf{Z}^m$, define $|\mathbf{x}| = \max\{|x_0|, \dots, |x_{m-1}|\}$.

• W.M. Schmidt (1970) : *For $m \geq 2$ let $L_0, \dots, L_{m-1}$ be m independent linear forms in m variables with complex algebraic coefficients. Let $\epsilon > 0$. Then the set*

$$\{\mathbf{x} = (x_0, \dots, x_{m-1}) \in \mathbf{Z}^m \,;\; |L_0(\mathbf{x}) \cdots L_{m-1}(\mathbf{x})| \leq |\mathbf{x}|^{-\epsilon}\}$$

is contained in the union of finitely many proper subspaces of $\mathbf{Q}^m$.

• Example : $m = 2$, $L_0(x_0, x_1) = x_0$, $L_1(x_0, x_1) = \alpha x_0 - x_1$.
Roth's Theorem. *for any real algebraic number α, for any $\epsilon > 0$, the set of $p/q \in \mathbf{Q}$ with $|\alpha - p/q| < q^{-2-\epsilon}$ is finite.*

Schmidt's subspace Theorem – Several places

For $\mathbf{x} = (x_0, \dots, x_{m-1}) \in \mathbf{Z}^m$, define $|\mathbf{x}| = \max\{|x_0|, \dots, |x_{m-1}|\}$.
W.M. Schmidt (1970) : *Let $m \geq 2$ be a positive integer, S a finite set of places of $\mathbf{Q}$ containing the infinite place. For each $v \in S$ let $L_{0,v}, \dots, L_{m-1,v}$ be m independent linear forms in m variables with algebraic coefficients in the completion of $\mathbf{Q}$ at v. Let $\epsilon > 0$. Then the set of $\mathbf{x} = (x_0, \dots, x_{m-1}) \in \mathbf{Z}^m$ such that*

$$\prod_{v\in S} |L_{0,v}(\mathbf{x}) \cdots L_{m-1,v}(\mathbf{x})|_v \leq |\mathbf{x}|^{-\epsilon}$$

is contained in the union of finitely many proper subspaces of $\mathbf{Q}^m$.

Consequence : Ridout's Theorem

• **Ridout's Theorem.** *For any real algebraic number α, for any $\epsilon > 0$, the set of $p/q \in \mathbf{Q}$ with $q = 2^k$ and $|\alpha - p/q| < q^{-1-\epsilon}$ is finite.*

• In Schmidt's Theorem take $m = 2$, $S = \{\infty,\ 2\}$,
$L_{0,\infty}(x_0, x_1) = L_{0,2}(x_0, x_1) = x_0$,
$L_{1,\infty}(x_0, x_1) = \alpha x_0 - x_1$, $\quad L_{1,2}(x_0, x_1) = x_1$.

For $(x_0, x_1) = (q, p)$ with $q = 2^k$, we have
$|L_{0,\infty}(x_0, x_1)|_\infty = q$, $\quad |L_{1,\infty}(x_0, x_1)|_\infty = |q\alpha - p|$,
$|L_{0,2}(x_0, x_1)|_2 = q^{-1}$, $\quad |L_{1,2}(x_0, x_1)|_2 = |p|_2 \leq 1$.

Mahler's method for the transcendence of $\sum_{n\geq 0} 2^{-2^n}$

• Mahler (1930, 1969) : the function $f(z) = \sum_{n\geq 0} z^{-2^n}$ satisfies $f(z^2) + z = f(z)$ for $|z| < 1$.

• J.H. Loxton and A.J. van der Poorten (1982–1988).

• P.G. Becker (1994) : *for any given non–eventually periodic automatic sequence* $\mathbf{u} = (u_1, u_2, \dots)$, *the real number*

$$\sum_{k\geq 1} u_k g^{-k}$$

is transcendental, provided that the integer g *is sufficiently large (in terms of* $\mathbf{u}$*).*

More on Mahler's method

• K. Nishioka (1991) : algebraic independence measures for the values of Mahler's functions.

• For any integer $d \geq 2$,

$$\sum_{n\geq 0} 2^{-d^n}$$

is a S–number in the classification of transcendental numbers due to… Mahler.

• **Reference** : K. Nishioka, *Mahler functions and transcendence*, Lecture Notes in Math. **1631**, Springer Verlag, 1996.

• Conjecture – P.G. Becker, J. Shallitt : *more generally any automatic irrational real number is a* S*–number.*