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King Khalid University,
Wams school on Introductory topics
in Number Theory and Differential Geometry

Diophantine Equations

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Elliptic binomial Diophantine equations

Problem. To determine all natural numbers that occur at least twice in Pascal's Triangle as binomial coefficient $\binom{n}{k}$ with $2 \leq k \leq n/2$.

Known non trivial solutions :

$$\binom{16}{2} = \binom{10}{3} = 120, \; \binom{21}{2} = \binom{10}{4} = 210, \; \binom{56}{2} = \binom{22}{3} = 1540,$$

$$\binom{120}{2} = \binom{36}{3} = 7140, \; \binom{153}{2} = \binom{19}{5} = 11628,$$

$$\binom{221}{2} = \binom{17}{8} = 24310, \; \binom{78}{2} = \binom{15}{5} = \binom{14}{6} = 3003,$$

$$\binom{F_{2i+2}F_{2i+3}}{F_{2i}F_{2i+3}} = \binom{F_{2i+2}F_{2i+3} - 1}{F_{2i}F_{2i+3} + 1}.$$

Elliptic binomial Diophantine equations

Complete solutions for 8 equations :

$$\binom{n}{2} = \binom{m}{3}, \quad \binom{n}{2} = \binom{m}{4},$$

$$\binom{n}{2} = \binom{m}{6}, \quad \binom{n}{2} = \binom{m}{8},$$

$$\binom{n}{3} = \binom{m}{4}, \quad \binom{n}{3} = \binom{m}{6},$$

$$\binom{n}{4} = \binom{m}{6}, \quad \binom{n}{4} = \binom{m}{8}.$$

R.J. Stroeker and N. Tzanakis. — *Solving elliptic diophantine equations by estimating linear forms in elliptic logarithms*, Acta Arithmetica **67** (1994), 177 – 196.

$$\binom{n}{3} = \binom{m}{4} \quad \text{and} \quad y^2 + y = x^3 - x$$

$$\binom{n}{3} = \frac{x^3 - x}{3!}, \quad \boxed{x = n - 1}.$$

$$\binom{m}{4} = \frac{m(m-1)(m-2)(m-3)}{4!}$$

$$k = \left(m + \frac{3}{2}\right)^2 = m^2 + 3m + \frac{9}{4}$$

$$m(m-1)(m-2)(m-3) = k^2 - \frac{5}{2}k + \frac{9}{16}$$

$$\boxed{y = \frac{k}{2} + \frac{1}{4}}$$

Elliptic linear forms method : examples

$y^2 + y = x^3 - x$,

$x = -1, 0, 1, 2, 6$

Rank 1, basis $(0, 0)$.

$6y^2 = (x+1)(x^2 - x + 6)$,

Mordell (1969) : $x = -1, 0, 2, 7, 15, 74$

Ljunggreen (1971) : $x = 767$

Weierstrass form : $Y^2 = X^3 + 180X + 1296$

$Y = 36y$, $X = 6x$ — Extra solution : $(X, Y) = (69, 585)$

Torsion subgroup order 2 : $(-6, 0)$

Rank 2 ; generators modulo torsion $(-3, 27)$, $(10, 64)$.

References

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