Phnom Penh, RUPP *Final Exam: April 6, 2012*

RUPP Master in Mathematics Program: Number Theory

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1. Prove by induction that $4^{2n+1} + 3^{n+2}$ is a multiple of 13 for $n \geq 0$.

2. If $m > 1$ and a is prime to m, show that the remainders obtained by dividing $a, 2a, \ldots, (m-1)a$ by m are the numbers $1, 2, \ldots, m-1$ in some order.

3. If m is any odd integer, prove that

$$1^m + 2^m + \cdots + (m-1)^m \equiv 0 \pmod{m}.$$

4. Let G be a finite multiplicative group with p elements and p is prime. Show that G is cyclic and that any element $\neq 1$ is a generator.

5. Show that a finite ring without zero divisor is a field.

6.
a) Find all solutions (x, y) in positive integers of the equation $x^2 - 7y^2 = -1$.
b) Find two solutions (x_1, y_1) and (x_2, y_2) in positive integers with $1 < x_1 < x_2$ of the equation $x^2 - 7y^2 = 1$.
`Hint:` *the following computations can be used:*

$$\sqrt{7} = 2.6457\cdots = 2 + \frac{1}{x_1}, \qquad x_1 = 1.5485\cdots = 1 + \frac{1}{x_2},$$

$$x_2 = 1.8228\cdots = 1 + \frac{1}{x_3}, \qquad x_3 = 1.2152\cdots = 1 + \frac{1}{x_4},$$

$$x_4 = 4.6457\cdots = 2 + \sqrt{7}.$$