Master Training Program : Royal Academy of Cambodia/CIMPA

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Timing: 3 hours

No document, no calculator

All answers require a proof.

1. Recall that the continued fraction expansion of a real irrational number t, namely

$$t = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

with $a_j \in \mathbf{Z}$ for all $j \geq 0$ and $a_j \geq 1$ for $j \geq 1$, is denoted by $[a_0; a_1, a_2, a_3, \ldots]$.

Let t be the real number whose continued fraction expansion is $[1; 3, 1, 3, 1, 3, 1, \ldots]$, which means $a_{2n} = 1$ and $a_{2n+1} = 3$ for $n \geq 0$. Write a quadratic polynomial with rational coefficients vanishing at t.

2. Solve the equation $y^2 - y = x^2$
a) in $\mathbf{Z} \times \mathbf{Z}$,
b) in $\mathbf{Q} \times \mathbf{Q}$.

3. Solve the equation $x^{15} = y^{21}$ in $\mathbf{Z} \times \mathbf{Z}$.

4. Let $A = \mathbf{Z}[1/2]$ be the subring of $\mathbf{Q}$ spanned by $1/2$.
a) Is A a finitely generated $\mathbf{Z}$–module?
b) Which are the units of A?

5. Which are the finitely generated sub–$\mathbf{Z}$–modules of the additive group $\mathbf{Q}$?

6. Find the rational roots of the polynomial $X^7 - X^6 + X^5 - X^4 - X^3 + X^2 - X + 1$.

7. Let k be the number field $\mathbf{Q}(i, \sqrt{2})$.
a) What is the degree of k over $\mathbf{Q}$? Give a basis of k over $\mathbf{Q}$. Find $\gamma \in k$ such that $k = \mathbf{Q}(\gamma)$. Which are the conjugates of γ over $\mathbf{Q}$?
b) Show that k is a Galois extension of $\mathbf{Q}$. What is the Galois group? Which are the subfields of k?

8. Let $\zeta \in \mathbf{C}$ satisfy $\zeta^5 = 1$ and $\zeta \neq 1$. Let $K = \mathbf{Q}(\zeta)$.
a) What is the monic irreducible polynomial of ζ over $\mathbf{Q}$? Which are the conjugates of ζ over $\mathbf{Q}$? What is the Galois group G of K over $\mathbf{Q}$? Which are the subgroups of G?
b) Show that K contains a unique subfield L of degree 2 over $\mathbf{Q}$. What is the ring of integers of L? What is its discriminant? What is the group of units?

`The solution will soon be available on the web site`
http://www.math.jussieu.fr/~miw/coursCambodge2006.html