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Exercices on the third course.

1. Let s_0, s_1, s_2 be three complex numbers. Give a necessary and sufficient condition for the following to hold. There exist three sequences of polynomials $(\Lambda_{n,0}(z))_{n\geq 0}$, $(\Lambda_{n,1}(z))_{n\geq 0}$, $(\Lambda_{n,2}(z))_{n\geq 0}$ such that any polynomial $f \in \mathbb{C}[z]$ can be written in a unique way as a finite sum

$$f(z) = \sum_{n\geq 0} \left(f^{(3n)}(s_0)\Lambda_{n,0}(z) + f^{(3n)}(s_1)\Lambda_{n,1}(z) + f^{(3n)}(s_2)\Lambda_{n,2}(z) \right).$$

What is the degree of $\Lambda_{n,j}(z)$? The leading term? Write the six polynomials

$$\Lambda_{0,0}(z), \Lambda_{0,1}(z), \Lambda_{0,2}(z), \Lambda_{1,0}(z), \Lambda_{1,1}(z), \Lambda_{1,2}(z).$$

2. Let s_0, s_1, s_2 be three complex numbers. Give a necessary and sufficient condition for the following to hold. There exist three sequences of polynomials $(M_{n,0}(z))_{n\geq 0}$, $(M_{n,1}(z))_{n\geq 0}$, $(M_{n,2}(z))_{n\geq 0}$ such that any polynomial $f \in \mathbb{C}[z]$ can be written in a unique way as a finite sum

$$f(z) = \sum_{n\geq 0} \left(f^{(3n)}(s_0)M_{n,0}(z) + f^{(3n+1)}(s_1)M_{n,1}(z) + f^{(3n+2)}(s_2)M_{n,2}(z) \right).$$

What is the degree of $M_{n,j}(z)$? The leading term? Write the six polynomials

$$M_{0,0}(z), M_{0,1}(z), M_{0,2}(z), M_{1,0}(z), M_{1,1}(z), M_{1,2}(z).$$

3. Let s_0, s_1, s_2 be three complex numbers. Give a necessary and sufficient condition for the following to hold. There exist three sequences of polynomials $(N_{n,0}(z))_{n\geq 0}$, $(N_{n,1}(z))_{n\geq 0}$, $(N_{n,2}(z))_{n\geq 0}$ such that any polynomial $f \in \mathbb{C}[z]$ can be written in a unique way as a finite sum

$$f(z) = \sum_{n\geq 0} \left(f^{(3n)}(s_0)N_{n,0}(z) + f^{(3n)}(s_1)N_{n,1}(z) + f^{(3n+1)}(s_2)N_{n,2}(z) \right).$$

What is the degree of $N_{n,j}(z)$? The leading term? Write the six polynomials

$$N_{0,0}(z), N_{0,1}(z), N_{0,2}(z), N_{1,0}(z), N_{1,1}(z), N_{1,2}(z).$$

4. On p. 11, check that if the determinant $D(\mathbf{s})$ does not vanish, then $r_j \leq j$ for all $j = 0, 1, \ldots, m-1$.

5. Prove the proposition p. 11.

6. Poritsky's interpolation p. 31. Prove that the condition $\mathrm{D}(\mathbf{s}) = 0$ means that $s_0, s_1, \ldots, s_{m-1}$ are pairwise distinct.
Prove also that the function $\Delta(t)$ has a zero at the origin of multiplicity at least $m(m-1)/2$.
N.B. *The fact that the multiplicity is exactly $m(m-1)/2$ follows from the fact that the coefficient of*

$t^{m(m-1)/2}$ *in the Taylor expansion at the origin of* $\Delta(t)$ *is given by a product of two Vandermonde determinants*

$$\frac{1}{1!2!\cdots(m-1)!}\det\begin{pmatrix}1 & 1 & \cdots & 1\\ 1 & \zeta & \cdots & \zeta^{m-1}\\ 1 & \zeta^2 & \cdots & \zeta^{2(m-1)}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & \zeta^{m-1} & \cdots & \zeta^{(m-1)^2}\end{pmatrix}\det\begin{pmatrix}1 & 1 & \cdots & 1\\ s_0 & s_1 & \cdots & s_{m-1}\\ s_0^2 & s_1^2 & \cdots & s_{m-1}^2\\ \vdots & \vdots & \ddots & \vdots\\ s_0^{m-1} & s_1^{m-1} & \cdots & s_{m-1}^{m-1}\end{pmatrix}.$$

But this is not so easy to prove [Macintyre 1954, §3].

7. Let $\mathbf{w}=(w_n)_{n\geq 0}$ be a sequence of complex numbers. Prove that the sequence of polynomials $(\Omega_{w_0,w_1,\dots,w_{n-1}}(z))_{n\geq 0}$ defined by $\Omega_\emptyset=1$ and

$$\Omega_{w_0,w_1,\dots,w_{n-1}}(z)=\int_{w_0}^{z}\mathrm{d}t_1\int_{w_1}^{t_1}\mathrm{d}t_2\cdots\int_{w_{n-1}}^{t_{n-1}}\mathrm{d}t_n$$

for $n\geq 1$ satisfy $\Omega_{w_0}(z)=z-w_0$ and for $n\geq 0$, $\Omega_{w_0,w_1,w_2,\dots,w_n}(w_0)=0$,

$$\Omega'_{w_0,w_1,w_2,\dots,w_n}(z)=\Omega_{w_1,w_2,\dots,w_n}(z).$$

What are the degree and the leading term of $\Omega_{w_0,w_1,w_2,\dots,w_n}(z)$? Check

$$\Omega^{(k)}_{w_0,w_1,w_2,\dots,w_n}(w_k)=\delta_{kn}$$

for $n\geq 0$ and $k\geq 0$. Deduce that any polynomial is a finite sum

$$f(z)=\sum_{n\geq 0}f^{(n)}(w_n)\Omega_{w_0,w_1,w_2,\dots,w_n}(z).$$

Check the formula for the Gontcharoff determinant p. 39.
Give a close formula for these polynomials $\Omega_{w_0,w_1,\dots,w_{n-1}}(z)$ when
- $w_n=0$ for all $n\geq 0$.
- $w_n=1$ for even $n\geq 0$, $w_n=0$ for odd $n\geq 1$.
- $w_n=n$ for all $n\geq 0$.

References

[Macintyre 1954] A. J. Macintyre, "Interpolation series for integral functions of exponential type", *Trans. Amer. Math. Soc.* **76** (1954), 1–13. MR Zbl