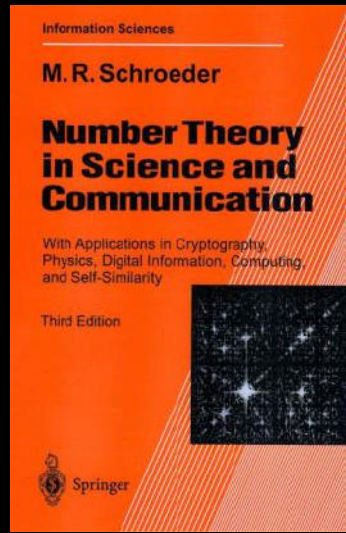
December 19, 2005 Institute of Science

Mathematics in the real life: The Fibonacci Sequence and the Golden Number

Michel Waldschmidt
Université P. et M. Curie (Paris VI)

Mumbai http://www.math.jussieu.fr/~miw/



Manfred R. SCHROEDER

Number Theory in Science and Communication

With application in Cryptography, Physics, Digital Information, Computing and Self-Similarity

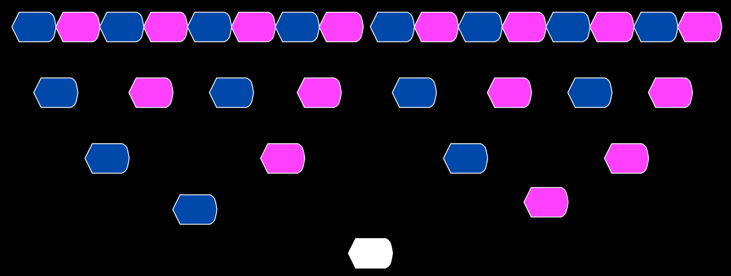
Springer Series in Information Sciences 1985

Some applications of Number Theory

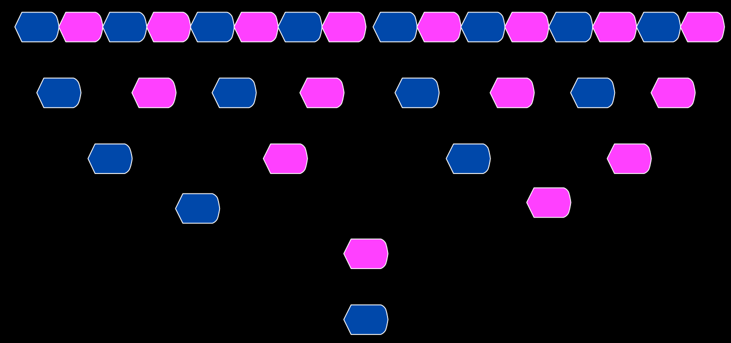
- Cryptography, security of computer systems
- Data transmission, error correcting codes
- Interface with theoretical physics
- Musical scales
- Numbers in nature

How many ancesters do we have?

Sequence: 1, 2, 4, 8, 16 … $E_{n+1} = 2E_n$ $E_n = 2^n$



Bees genealogy



Bees genealogy

Number of females at level *n+1* =
total population at level *n*

Number of males at level *n+1* =
number of females at level *n*

Sequence: 1, 1, 2, 3, 5, 8, … $F_{n+1} = F_n + F_{n-1}$

3 + 5 = 8

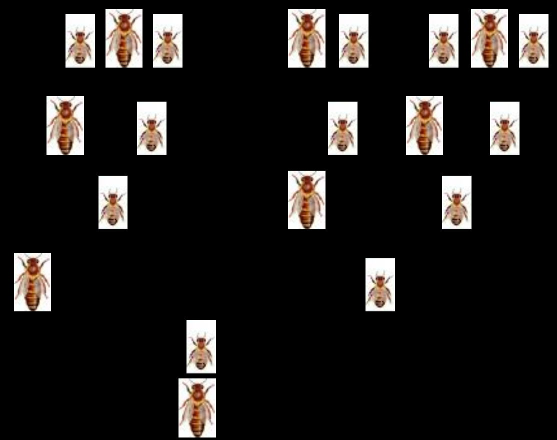
2 + 3 = 5

1 + 2 = 3

1 + 1 = 2

0 + 1 = 1

1 + 0 = 1



Fibonacci (Leonardo di Pisa)

- Pisa ≈1175, ≈1250
- Liber Abaci ≈ 1202

$F_0 = 0$, $F_1 = 1$, $F_2 = 1$,
$F_3 = 2$, $F_4 = 3$, $F_5 = 5$,…

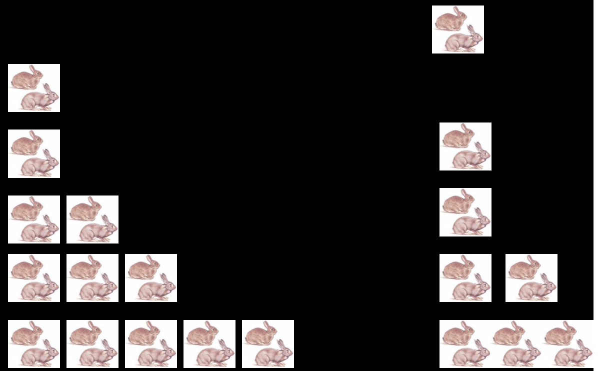
The Fibonacci Quarterly

Official Publication of The Fibonacci Association

Modelization of a population

Adult pairs **Young pairs**

- First year
- Second year
- Third year
- Fourth year
- Fifth year
- Sixth Year



Sequence: 1, 1, 2, 3, 5, 8, … $F_{n+1} = F_n + F_{n-1}$

Theory of stable populations (Alfred Lotka)

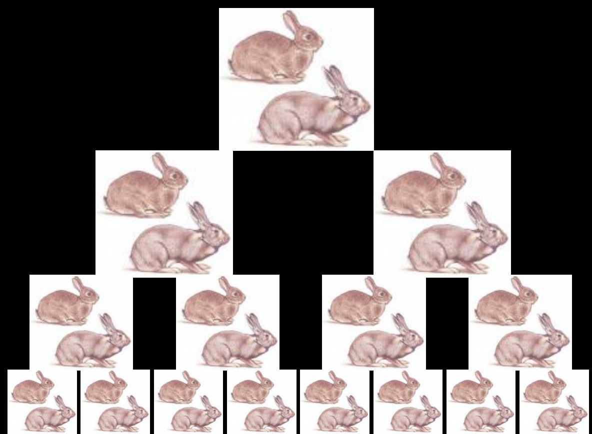
Assume each pair generates a new pair the first two years only. Then the number of pairs who are born each year again follow the Fibonacci rule.

Arctic trees

In cold countries, each branch of some trees gives rise to another one after the second year of existence only.

Exponential Sequence

- First year
- Second year
- Third year
- Fourth year



Number of pairs: 1, 2, 4, 8, … $E_n = 2^n$

Representation of a number as a sum of distinct powers of 2

- $51 = 32 + 19$, $32 = 2^5$
- $19 = 16 + 3$, $16 = 2^4$
- $3 = 2 + 1$, $2 = 2^1$, $1 = 2^0$
- $51 = 2^5 + 2^4 + 2^1 + 2^0$

Binary expansion

Decimal expansion of an integer

- *51 = 5 × 10 + 1*
- *2005 = 20 × 10 + 5*

Representation of an integer as a sum of Fibonacci numbers

- N a positive integer
- F_n the largest Fibonacci number ≤ N
- Hence N= F_n + remainder which is < F_{n-1}
- Repeat with the remainder

Example

$F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $F_7 = 13$, $F_8 = 21$, $F_9 = 34$, $F_{10} = 55$, $F_{11} = 89$, $F_{12} = 144$, $F_{13} = 233$, $F_{14} = 377$, $F_{15} = 610$

51= F_9 + F_7 + F_4 + F_2

$F_9 = 34$

51= F_9 +17

$F_7 = 13$

17= F_7 +4

$F_4 = 3$

4= F_4 +1

$F_2 = 1$

In this representation
there is no two consecutive Fibonacci numbers

The Fibonacci sequence

$F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$,
$F_6 = 8$, $F_7 = 13$ $F_8 = 21$, $F_9 = 34$, $F_{10} = 55$,
$F_{11} = 89$, $F_{12} = 144$, $F_{13} = 233$, $F_{14} = 377$, $F_{15} = 610$,
…

The sequence of integers

1=F_2,
2=F_3,
3=F_4, 4= F_4 + F_2,
5=F_5, 6= F_5 + F_2, 7= F_5 + F_3,
8= F_6, 9= F_6 + F_2, 10= F_6 + F_3, 11= F_6 + F_4, 12= F_6+F_4+F_2
…

The Fibonacci sequence

$F_1=1$, $F_2=1$, $F_3=2$, $F_4=3$, $F_5=5$,
$F_6=8$, $F_7=13$, $F_8=21$, $F_9=34$, $F_{10}=55$,
$F_{11}=89$, $F_{12}=144$, $F_{13}=233$, $F_{14}=377$, $F_{15}=610$,
. . .

Divisibility (Lucas, 1878)

If $b \geq 3$, then b divides a if and only if F_b divides F_a.

Examples:

$F_{12}=144$ is divisible by $F_3=2$, $F_4=3$, $F_6=8$,
$F_{14}=377$ by $F_7=13$,
$F_{16}=987$ by $F_8=21$.

Analogy with the sequence 2^n

2^b divides 2^a if and only if $b \leq a$.

Sequence $u_n = 2^n - 1$

$2^b - 1$ divides $2^a - 1$ if and only if b divides a.

If $a=kb$ set $x=2^b$ so that $2^a = x^k$ and write
$x^k - 1 = (x-1)(x^{k-1} + x^{k-2} + \dots + x + 1)$

Recurrence relation :

$$u_{n+1} = 2u_n + 1$$

Exponential Diophantine equations

Y. Bugeaud, M. Mignotte, S. Siksek (2004):
The only perfect powers in the Fibonacci sequence are 1, 8 and 144.

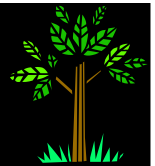
Equation: $F_n = a^b$
Unknowns: n, a *and* b
with $n \geq 1$, $a \geq 1$ *and* $b \geq 2$.

Exponential Diophantine equations

T.N. Shorey, TIFR (2005):
The product of 2 or more consecutive Fibonacci numbers other than $F_1 F_2$ *is never a perfect power.*

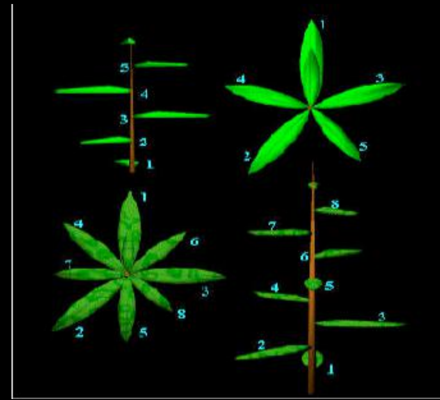
Conference DION2005, TIFR Mumbai, december 16-20, 2005

Phyllotaxy



- Study of the position of leaves on a stem and the reason for them
- Number of petals of flowers: daisies, sunflowers, aster, chicory, asteraceae,…
- Spiral patern to permit optimum exposure to sunlight
- Pine-cone, pineapple, Romanesco cawliflower, cactus

Leaf arrangements

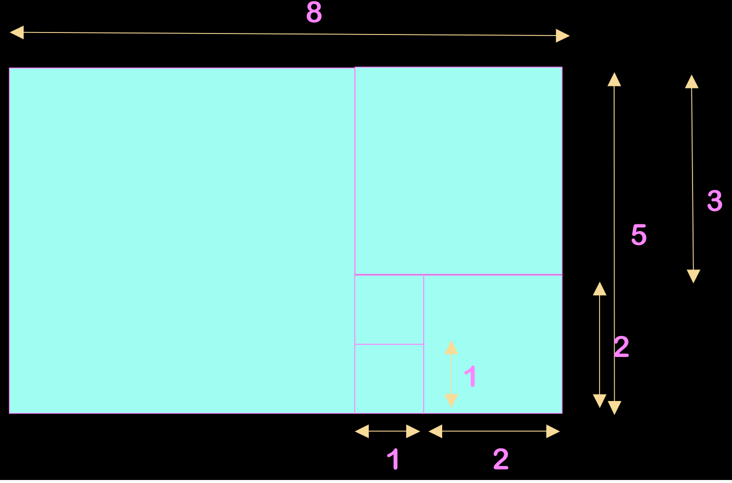


- Université de Nice,
  Laboratoire Environnement Marin Littoral,
  Equipe d'Accueil "Gestion de la Biodiversité"

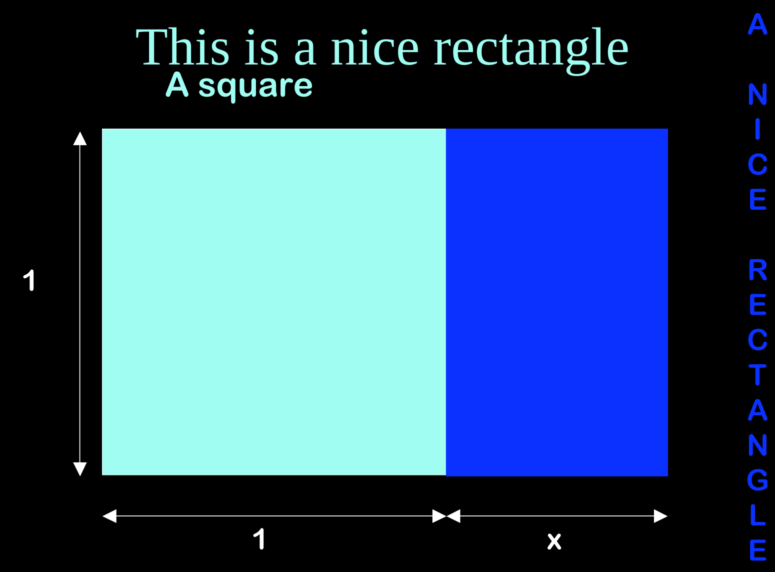
http://www.unice.fr/LEML/coursJDV/tp/tp3.htm

Phyllotaxy

Geometric construction of the Fibonacci sequence



This is a nice rectangle



1+x=1/x

Golden Rectangle

Sides 1 and $1+x$ with $x>0$.

Condition: **the two rectangles of sides $1+x, 1$ and $1, x$ have the same proportion**

$$1+x=\frac{1}{x}.$$

Hence

$$x^2+x=1 \quad \text{and} \quad x=\frac{-1+\sqrt{5}}{2}.$$

The number

$$1+x=\frac{1}{x}=\frac{1+\sqrt{5}}{2}=2\cos(\pi/5)$$

is the root >1 of the equation $\Phi^2=\Phi+1$.

This is the Golden Number

$$\Phi=1,6180339887499\ldots$$

The Golden Number

$$\Phi^2 = 1 + \Phi$$

$$\Phi = 1 + \frac{1}{\Phi} = 1 + \frac{1}{1+\frac{1}{\Phi}} = \sqrt{1+\Phi}$$

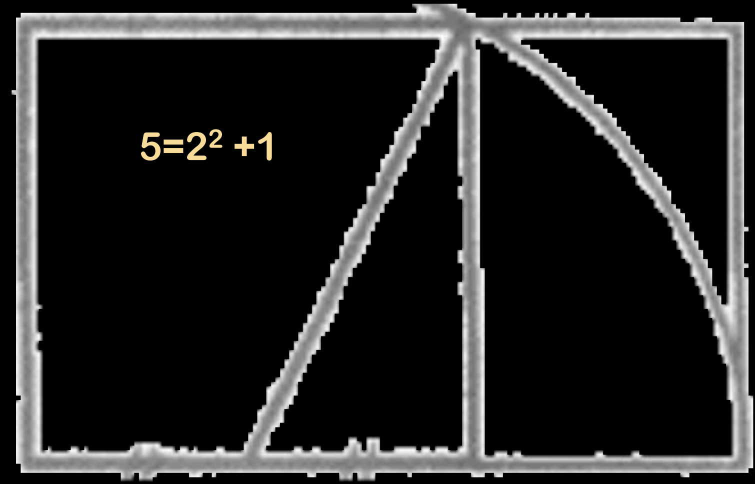
Fra Luca Pacioli (1509)

De Divina Proportione

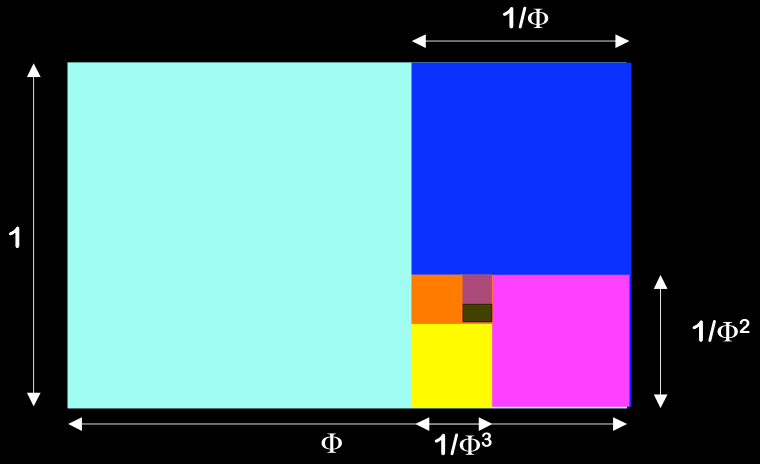
$$\Phi = 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{\ddots}}} = \sqrt{1+\sqrt{1+\cdots}}$$

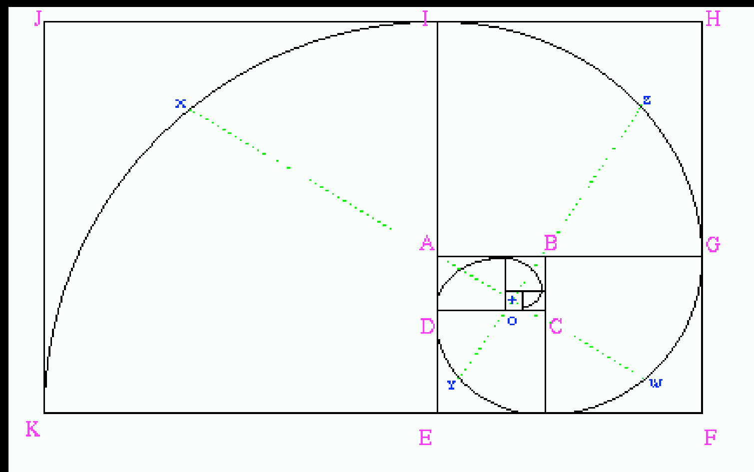
Exercise:

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{\cdots}}}} = 3$$

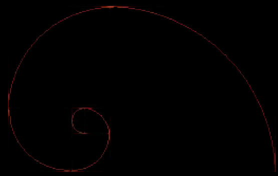


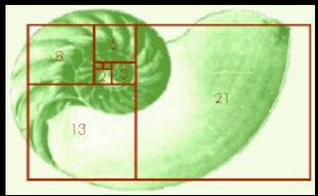
The Golden Rectangle

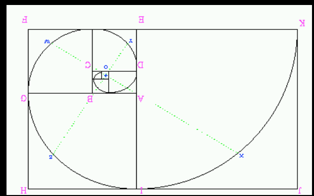


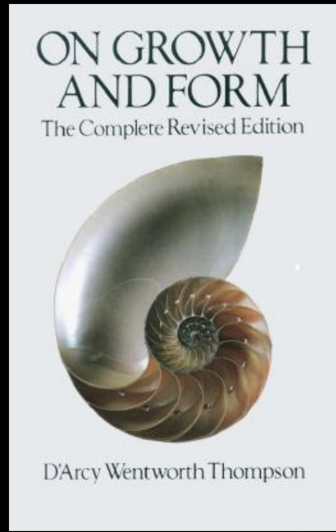


Ammonite (Nautilus shape)

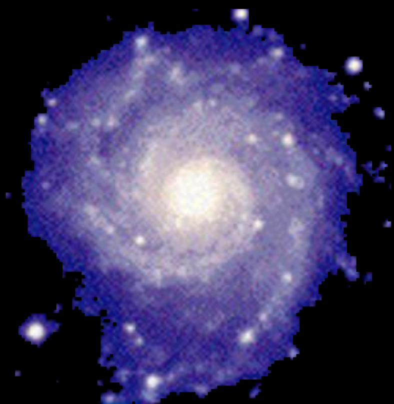








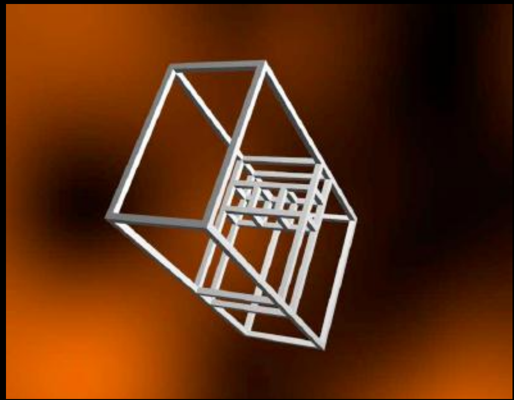
Spirals in the Galaxy



The Golden Number in art, architecture,… aesthetic

Kees van Prooijen

http://www.kees.cc/gldsec.html



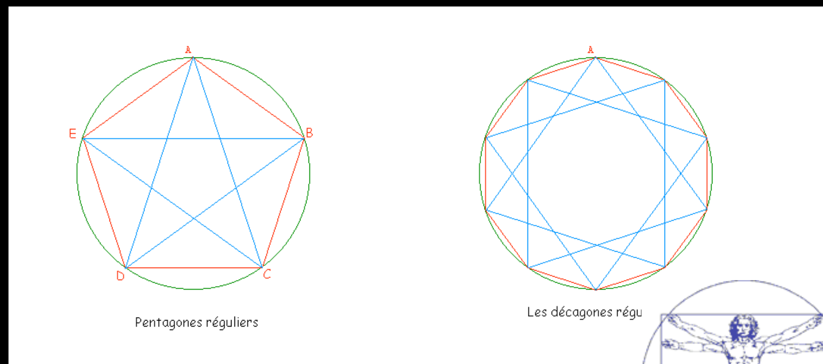
Music and the Fibonacci sequence

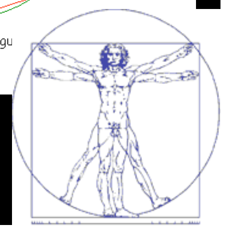
- Dufay, XV^{ème} siècle
- Roland de Lassus
- Debussy, Bartok, Ravel, Webern
- Stoskhausen
- Xenakis
- **Tom Johnson** *Automatic Music for six percussionists*

Phyllotaxy

- J. Kepler (1611) uses the Fibonacci sequence in his study of the dodecahedron and the icosaedron, and then of the symmetry of order 5 of the flowers
- Stéphane Douady et Yves Couder
  Les spirales végétales
  La Recherche *250* (janvier 1993) vol. **24.**

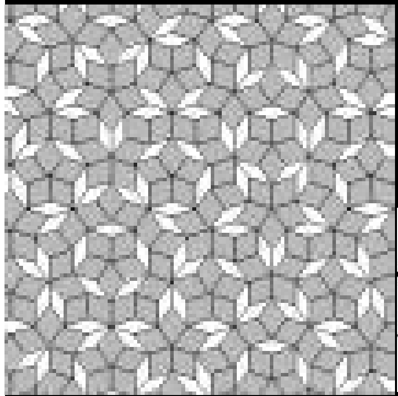
Regular pentagons and dodecagons

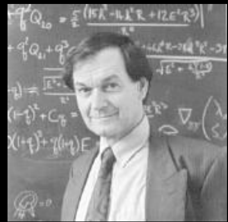


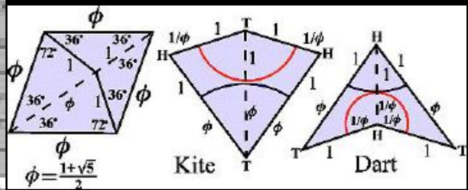


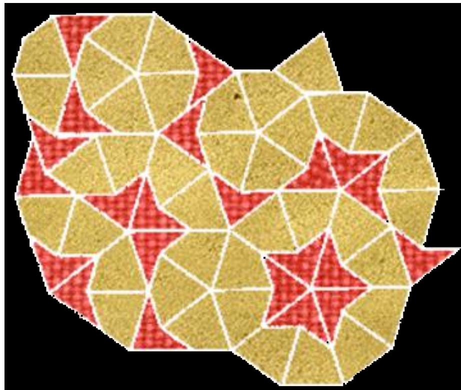
$\Phi = 2 \cos(\pi/5)$

Penrose non-periodic tiling patterns and quasi-crystals

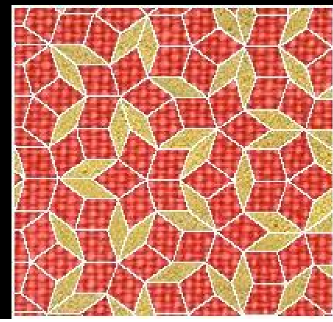




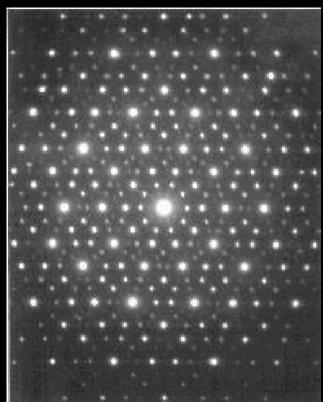




proportion=Φ

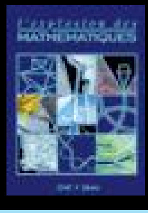


Diffraction of quasi-crystals



Doubly periodic tessalation (lattices) - cristallography

Géométrie d'un champ de lavande
http://math.unice.fr/~frou/lavande.html
François Rouvière (Nice)

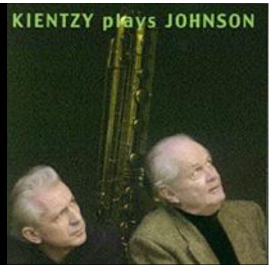


L'explosion des Mathématiques

http://smf.emath.fr/Publication/ExplosionDesMathematiques/Presentation.html

http://www.math.jussieu.fr/~miw/

http://www.pogus.com/21033.html



Narayana's Cows

Music: Tom Johnson
Saxophones: Daniel Kientzy
Realization: Michel Waldschmidt

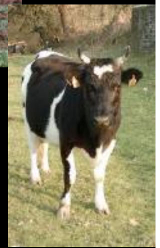
Update: august 8, 2005

http://www.math.jussieu.fr/~miw/

Narayana was an Indian mathematician in the 14th. century, who proposed the following problem:
A cow produces one calf every year.
Begining in its fourth year, each calf produces one calf at the begining of each year.
How many cows are there altogether after, for example, 17 years?

While you are working on that,
let us give you a musical demonstration.

The first year there is only the original cow and her first calf.



Year	1
Original Cow	1
Second generation	1
Total	2

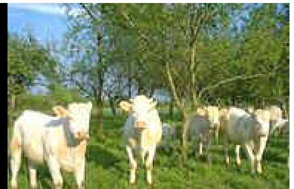
long-short

The second year there is the original cow and 2 calves.

Year	1	2
Original Cow	1	1
Second generation	1	2
Total	2	3

long -short -short

The third year there is the original cow and 3 calves.



Year	1	2	3
Original Cow	1	1	1
Second generation	1	2	3
Total	2	3	4

long -short -short -short